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MULTIPLE POSITIVE SOLUTIONS FOR A SCHRÖDINGER-NEWTON SYSTEM WITH SINGULARITY AND CRITICAL GROWTH

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Abstract. In this work, we study a class of Schrödinger-Newton systems with singular and critical growth terms in unbounded domains. By using the variational methods and the Brézis-Lieb [6] classical technique, the existence and multiplicity of positive solutions are established.

1. Introduction and statement of main result

In this work, we are concerned with the existence and multiplicity of positive solutions to the Schrödinger-Newton system

$$\begin{gathered} -\Delta u = \lambda g(x) u^{-\gamma} + \phi |u|^{2^*-3} u, \quad \text{in } \mathbb{R}^N, \\ -\Delta \phi = |u|^{2^*-1}, \quad \text{in } \mathbb{R}^N, \\ u > 0, \quad \text{in } \mathbb{R}^N, \end{gathered} \tag{1.1}$$

where $N \geq 3$, $\gamma \in (0,1)$ and $\lambda > 0$ is a real parameter and $g \in L^{\frac{2^*}{2^*+\gamma-1}}(\mathbb{R}^N)$ is a nonnegative function.

This system is derived from the Schrödinger-Poisson system

$$\begin{gathered} -\Delta u + V(x) u + \eta \phi f(u) = h(x,u), \quad \text{in } \mathbb{R}^3, \\ -\Delta \phi = 2F(u), \quad \text{in } \mathbb{R}^3. \end{gathered} \tag{1.2}$$

Systems as (1.2) have been studied extensively by many researchers because (1.2) has a strong physical meaning, which describes quantum particles interacting with the electromagnetic field generated by the motion. For more details as regards the physical relevance of the Schrödinger-Poisson system, we refer to [1, 4, 20]. System (1.2) has been extensively studied after the seminal work of Benci and Fortunato [4]. Many important results concerning existence of positive solutions, ground state solutions and multiplicity of solutions, least energy solutions, and so on, have been reported; see for instance [2, 3, 5, 7, 8, 9, 10, 14, 15, 16, 17, 18, 19, 21, 22, 23, 30, 31, 32] and the references therein.

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There are some references which investigated Schrödinger-Poisson systems involving the critical growing nonlocal term, such as [2, 3, 15, 18]. Precisely, in bounded domains, the system

$$
\begin{gathered}
-\Delta u+\varepsilon q\phi f(u)=\eta|u|^{p-1}u, \quad \text{in } \Omega,\\
-\Delta\phi=2qF(u), \quad \text{in } \Omega,\\
u=\phi=0, \quad \text{in } \partial\Omega,
\end{gathered}
$$

was considered in [15], where $\Omega\subset\mathbb{R}^N$ ($N\geq 3$) is a smooth bounded domain, and the existence and multiplicity results were established when f a subcritical growth condition or the critical growth case by using the methods of a cut-off function and the variational arguments. In [3], the following system involving the critical growing nonlocal term was also considered

$$
\begin{gathered}
-\Delta u=\lambda u+\phi|u|^{2^*-3}u, \quad \text{in } \Omega,\\
-\Delta\phi=|u|^{2^*-1}, \quad \text{in } \Omega,\\
u=\phi=0, \quad \text{in } \partial\Omega.
\end{gathered}
$$

They proved the existence and nonexistence results of positive solutions when $N=3$ and existence of solutions in both the resonance and the non-resonance case for higher dimensions.

Specially, in unbounded domains, Liu [18] studied the system

$$
\begin{gathered}
-\Delta u+V(x)u=K(x)\phi|u|^3u+h(x,u), \quad \text{in } \mathbb{R}^3,\\
-\Delta\phi=K(x)|u|^5, \quad \text{in } \mathbb{R}^3,
\end{gathered}
$$

where V,K,h are asymptotically periodic functions, and a positive solution was obtained by using variational methods.

Recently, in a bounded domain, in [31], the following system involving weak singularity was studied

$$
\begin{gathered}
-\Delta u+\eta\phi u=\mu u^{-\gamma}, \quad \text{in } \Omega,\\
-\Delta\phi=u^2, \quad \text{in } \Omega,\\
u>0, \quad \text{in } \Omega,\\
u=\phi=0, \quad \text{on } \partial\Omega.
\end{gathered}
$$

The existence, uniqueness and multiplicity of positive solutions for the above system are obtained in the case when $\eta=\pm 1$ by employing the Nehari manifold.

In bounded domains, the singular semilinear elliptic problem

$$
\begin{gathered}
-\Delta u=\lambda f(x)u^{-\gamma}+\mu h(x)u^p, \quad \text{in } \Omega,\\
u>0, \quad \text{in } \Omega,\\
u=0, \quad \text{on } \partial\Omega,
\end{gathered}
$$

has been extensively studied. For example, Yang [29] obtained the multiplicity positive solutions by combining variational and sub-supersolution methods when $0<\gamma<1<p\leq\frac{N+2}{N-2}$, $f=\mu h=1$, λ enough small. In the case when $0<\gamma<1<p\leq\frac{N+2}{N-2}$ and $h=1,\lambda=1$ and μ enough small, Sun and Wu [25] also got two positive solutions by employing the Nehari manifold provided μ enough small. In [13], Hirano et al. studied the existence of multiple positive solutions in the case of $0<\gamma\leq 1<p\leq\frac{N+2}{N-2}$, $\mu>0$. When $\Omega=\mathbb{R}^N$, we should mention that semilinear elliptic equations involving singular and subcritical growth terms have been dealt with by a number of authors, see for example, [12, 24] and the references therein.

Motivated by the above facts, to the best of our knowledge, there are no results on the multiplicity of positive solutions for Schrödinger-Newton system involving critical and weak singular nonlinearities on unbounded domains. We shall give a positive answer to this question. Our main result reads as follows.

Theorem 1.1. *Assume that $\gamma \in (0,1)$. Then there exists $\lambda_* > 0$ such that for any $\lambda \in (0, \lambda_*)$, system (1.1) has at least two positive solutions $(u, \phi_u) \in D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$, and one of the solutions is a positive ground state solution.*

Throughout this paper, we use the following notation:

- The space $D^{1,2}(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N) : \frac{\partial u}{\partial x_i} \in L^2(\mathbb{R}^N)\}$ endowed with the norm $\|u\|^2 = \int_{\mathbb{R}^N} |\nabla u|^2 dx$. The norm in $L^p(\mathbb{R}^N)$ is denoted by $|\cdot|_p$;
- $C, C_1, C_2, \dots$ denote various positive constants, which may vary from line to line;
- Let S be the best constant for Sobolev embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$, namely
$$S = \inf_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{(\int_{\mathbb{R}^N} |u|^{2^*} dx)^{\frac{2}{2^*}}}.$$

2. Existence of the first positive solution of system (1.1)

The energy functional associated with system (1.1) is defined as
$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\|u\|^2 - \frac{\lambda}{1-\gamma}\int_{\mathbb{R}^N} g(x)|u|^{1-\gamma}dx - \frac{1}{2(2^*-1)}\int_{\mathbb{R}^N} \phi_u |u|^{2^*-1}dx \\ &= \frac{1}{2}\|u\|^2 - \frac{\lambda}{1-\gamma}\int_{\mathbb{R}^N} g(x)|u|^{1-\gamma}dx - \frac{1}{2(2^*-1)}\int_{\mathbb{R}^N} |\nabla \phi_u|^2 dx. \end{aligned}$$
In general, a function $u \in D^{1,2}(\mathbb{R}^N)$ is called a solution of system (1.1), that is (u, ϕ_u) is a solution of system (1.1) and $u > 0$ enjoying
$$\int_{\mathbb{R}^N} (\nabla u, \nabla v)dx - \lambda \int_{\mathbb{R}^N} g(x)u^{-\gamma} v dx - \int_{\mathbb{R}^N} \phi_u |u|^{2^*-3} uv dx = 0, \quad \forall v \in D^{1,2}(\mathbb{R}^N).$$

It is well known that the singular term leads to the non-differentiability of the functional I_λ on $D^{1,2}(\mathbb{R}^N)$, therefore system (1.1) cannot be considered by using critical point theory directly. In order to obtain the multiple positive solutions of system (1.1), we consider a set
$$\mathcal{N}_\lambda = \Big\{u \in D^{1,2}(\mathbb{R}^N) : \|u\|^2 - \lambda \int_{\mathbb{R}^N} g(x)|u|^{1-\gamma}dx - \int_{\mathbb{R}^N} \phi_u |u|^{2^*-1}dx = 0\Big\},$$
and split $\mathcal{N}_\lambda$ as follows:
$$\begin{aligned} \mathcal{N}_\lambda^+ &= \{u \in \mathcal{N}_\lambda : \psi(u) > 0\}, \\ \mathcal{N}_\lambda^0 &= \{u \in \mathcal{N}_\lambda : \psi(u) = 0\}, \\ \mathcal{N}_\lambda^- &= \{u \in \mathcal{N}_\lambda : \psi(u) < 0\}, \end{aligned}$$
where
$$\psi(u) = 2\|u\|^2 - \lambda(1-\gamma)\int_{\mathbb{R}^N} g(x)|u|^{1-\gamma}dx - 2(2^*-1)\int_{\mathbb{R}^N} \phi_u |u|^{2^*-1}dx.$$
Before proving our Theorem 1.1, we recall the following lemma (see [3]).

Lemma 2.1. *For every $u \in D^{1,2}(\mathbb{R}^N)$, there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^N)$ solution of*
$$-\Delta \phi = |u|^{2^*-1}, \ \text{in} \mathbb{R}^N .$$
Also

(1) $\phi_u \geq 0$ *for* $x \in \mathbb{R}^N$.
(2) *For each* $t \neq 0$, $\phi_{tu} = t^{2^*-1}\phi_u$.
(3)
$$\int_\Omega \phi_u |u|^{2^*-1} dx = \int_\Omega |\nabla \phi_u|^2 dx \leq S^{-2^*} \|u\|^{2(2^*-1)}.$$
(4) *Assume that* $u_n \rightharpoonup u$ *in* $D^{1,2}(\mathbb{R}^N)$, *then*
$$\int_{\mathbb{R}^N} \phi_{u_n} |u_n|^{2^*-1} dx - \int_{\mathbb{R}^N} \phi_{u_n - u} |u_n - u|^{2^*-1} dx = \int_{\mathbb{R}^N} \phi_u |u|^{2^*-1} dx + o_n(1).$$

Set
$$\Lambda_0 = |g|^{-1}_{\frac{2^*}{2^*+\gamma-1}} \frac{2(2^*-2) S^{\frac{1-\gamma}{2}}}{2 \cdot 2^* + \gamma - 3} \Big[\frac{(1+\gamma) S^{2^*}}{(2 \cdot 2^* + \gamma - 3)}\Big]^{\frac{1+\gamma}{2(2^*-2)}}.$$

Lemma 2.2. *Assume $\lambda \in (0, \Lambda_0)$. Then (1) $\mathcal{N}_\lambda^\pm \neq \emptyset$ and (2) $\mathcal{N}_\lambda^0 = \{0\}$.*

Proof. (i) For each $u \in D^{1,2}(\mathbb{R}^N) \backslash \{0\}$, we have
$$\begin{aligned}
&t[\frac{d}{dt} I_\lambda(tu)] \\
&= t^2 \|u\|^2 - \lambda t^{1-\gamma} \int_{\mathbb{R}^N} g(x)|u|^{1-\gamma} dx - t^{2(2^*-1)} \int_{\mathbb{R}^N} \phi_u |u|^{2^*-1} dx \\
&= t^{1-\gamma} \Big[t^{1+\gamma}\|u\|^2 - t^{2\cdot 2^* + \gamma - 3} \int_{\mathbb{R}^N} \phi_u |u|^{2^*-1} dx - \lambda \int_{\mathbb{R}^N} g(x)|u|^{1-\gamma} dx\Big].
\end{aligned}$$
Set
$$\Gamma(t) = t^{1+\gamma}\|u\|^2 - t^{2\cdot 2^* + \gamma - 3} \int_{\mathbb{R}^N} \phi_u |u|^{2^*-1} dx, \ t \geq 0.$$
We see that $\Gamma(0) = 0$ and $\lim_{t\to\infty} \Gamma(t) = -\infty$. Then Γ achieves its maximum at
$$t_{\max} = \Big[\frac{(1+\gamma)\|u\|^2}{(2\cdot 2^* + \gamma - 3) \int_{\mathbb{R}^N} \phi_u |u|^{2^*-1} dx}\Big]^{\frac{1}{2(2^*-2)}},$$
and so,
$$\Gamma(t_{\max}) = \frac{2(2^*-2)\|u\|^2}{2\cdot 2^* + \gamma - 3} \Big[\frac{(1+\gamma)\|u\|^2}{(2\cdot 2^* + \gamma - 3) \int_{\mathbb{R}^N} \phi_u |u|^{2^*-1} dx}\Big]^{\frac{1+\gamma}{2(2^*-2)}}.$$
Consequently,
$$\begin{aligned}
&\Gamma(t_{\max}) - \lambda \int_{\mathbb{R}^N} g(x)|u|^{1-\gamma} dx \\
&= \frac{2(2^*-2)\|u\|^2}{2\cdot 2^* + \gamma - 3} \Big[\frac{(1+\gamma)\|u\|^2}{(2\cdot 2^* + \gamma - 3) \int_{\mathbb{R}^N} \phi_u |u|^{2^*-1} dx}\Big]^{\frac{1+\gamma}{2(2^*-2)}} - \lambda \int_{\mathbb{R}^N} g(x)|u|^{1-\gamma} dx \\
&\geq \frac{2(2^*-2)\|u\|^2}{2\cdot 2^* + \gamma - 3} \Big[\frac{(1+\gamma) S^{2^*} \|u\|^2}{(2\cdot 2^* + \gamma - 3) \|u\|^{2(2^*-1)}}\Big]^{\frac{1+\gamma}{2(2^*-2)}} - \lambda |g|_{\frac{2^*}{2^*+\gamma-1}} S^{-\frac{1-\gamma}{2}} \|u\|^{1-\gamma} \\
&= \Big\{\frac{2(2^*-2)}{2\cdot 2^* + \gamma - 3} \Big[\frac{(1+\gamma) S^{2^*}}{(2\cdot 2^* + \gamma - 3)}\Big]^{\frac{1+\gamma}{2(2^*-2)}} - \lambda |g|_{\frac{2^*}{2^*+\gamma-1}} S^{-\frac{1-\gamma}{2}}\Big\} \|u\|^{1-\gamma} > 0,
\end{aligned} \tag{2.1}$$

the last inequality holds provided $0 < \lambda < \Lambda_0$. Consequently, there exactly exist two points $0 < t_u^+ < t_{\max} < t_u^-$ such that

$$\Gamma(t_u^+) = \Gamma(t_u^-) = \lambda \int_{\mathbb{R}^N} g(x)|u|^{1-\gamma}dx, \Gamma'(t_u^+) > 0 > \Gamma'(t_u^-),$$

which imply that $t_u^+u \in \mathcal{N}_\lambda^+$, $t_u^-u \in \mathcal{N}_\lambda^-$. That is, $\mathcal{N}_\lambda^\pm \neq \emptyset$.

(ii) We prove (ii) by contradiction, suppose that there exists $u_0 \neq 0$ such that $u_0 \in \mathcal{N}_\lambda^0$, similar to (2.1), it holds that

$$\begin{aligned}
0 &< \Big\{\frac{2(2^*-2)}{2\cdot 2^*+\gamma-3}\Big[\frac{(1+\gamma)S^{2^*}}{(2\cdot 2^*+\gamma-3)}\Big]^{\frac{1+\gamma}{2(2^*-2)}} - \lambda|g|_{\frac{2^*}{2^*+\gamma-1}}S^{-\frac{1-\gamma}{2}}\Big\}\|u_0\|^{1-\gamma} \\
&\leq \frac{2(2^*-2)\|u_0\|^2}{2\cdot 2^*+\gamma-3}\Big[\frac{(1+\gamma)\|u_0\|^2}{(2\cdot 2^*+\gamma-3)\int_{\mathbb{R}^N}\phi_{u_0}|u_0|^{2^*-1}dx}\Big]^{\frac{1+\gamma}{2(2^*-2)}} \\
&\quad - \lambda\int_{\mathbb{R}^N} g(x)|u_0|^{1-\gamma}dx = 0,
\end{aligned}$$

this is a contradiction, thereby $\mathcal{N}_\lambda^0 = \{0\}$ for $\lambda \in (0, \Lambda_0)$. The proof is complete. □

Lemma 2.3. *The functional I_λ is coercive and bounded below on $\mathcal{N}_\lambda$.*

Proof. Suppose $u \in \mathcal{N}_\lambda$, then by Sobolev inequality,

$$\begin{aligned}
I_\lambda(u) &= \frac{1}{2}\|u\|^2 - \frac{\lambda}{1-\gamma}\int_{\mathbb{R}^N} g(x)|u|^{1-\gamma}dx - \frac{1}{2(2^*-1)}\int_{\mathbb{R}^N}\phi_u|u|^{2^*-1}dx \\
&= \frac{2^*-2}{2(2^*-1)}\|u\|^2 - \lambda\Big[\frac{1}{1-\gamma} - \frac{1}{2(2^*-1)}\Big]\int_{\mathbb{R}^N} g(x)|u|^{1-\gamma}dx \\
&\geq \frac{2}{N+2}\|u\|^2 - \lambda\Big[\frac{1}{1-\gamma} - \frac{1}{2(2^*-1)}\Big]|g|_{\frac{2^*}{2^*+\gamma-1}}S^{-\frac{1-\gamma}{2}}\|u\|^{1-\gamma},
\end{aligned}$$

as $0 < \gamma < 1$, it follows that I_λ is coercive and bounded below on $\mathcal{N}_\lambda$. □

We remark that by Lemma 2.2 we have $\mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^- \cup \mathcal{N}_\lambda^0$ for all $\lambda \in (0, \Lambda_0)$. Moreover, we know that $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$ are non-empty and by Lemma 2.3 we may define

$$\alpha_\lambda = \inf_{u\in\mathcal{N}_\lambda} I_\lambda(u), \quad \alpha_\lambda^+ = \inf_{u\in\mathcal{N}_\lambda^+} I_\lambda(u), \quad \alpha_\lambda^- = \inf_{u\in\mathcal{N}_\lambda^-} I_\lambda(u).$$

Lemma 2.4. $\alpha_\lambda \leq \alpha_\lambda^+ < 0$.

Proof. Assume $u \in \mathcal{N}_\lambda^+$. Then

$$\int_{\mathbb{R}^N}\phi_u|u|^{2^*-1}dx < \frac{1+\gamma}{2\cdot 2^*+\gamma-3}\|u\|^2,$$

so that

$$\begin{aligned}
I_\lambda(u) &= \frac{1}{2}\|u\|^2 - \frac{\lambda}{1-\gamma}\int_{\mathbb{R}^N} g(x)|u|^{1-\gamma}dx - \frac{1}{2(2^*-1)}\int_{\mathbb{R}^N}\phi_u|u|^{2^*-1}dx \\
&= \Big(\frac{1}{2} - \frac{1}{1-\gamma}\Big)\|u\|^2 + \Big(\frac{1}{1-\gamma} - \frac{1}{2(2^*-1)}\Big)\int_{\mathbb{R}^N}\phi_u|u|^{2^*-1}dx \\
&< \Big[\Big(\frac{1}{2} - \frac{1}{1-\gamma}\Big) + \Big(\frac{1}{1-\gamma} - \frac{1}{2(2^*-1)}\Big)\frac{1+\gamma}{2\cdot 2^*+\gamma-3}\Big]\|u\|^2 \\
&= \Big[-\frac{1+\gamma}{2(1-\gamma)} + \frac{1+\gamma}{2(2^*-1)(1-\gamma)}\Big]\|u\|^2
\end{aligned}$$

$$= -\frac{(2^*-2)(1+\gamma)}{2(2^*-1)(1-\gamma)}\|u\|^2 < 0.$$

By the definitions of α_λ and α_λ^+, one obtains $\alpha_\lambda \le \alpha_\lambda^+ < 0$. □

Lemma 2.5. *For $u \in \mathcal{N}_\lambda$ (respectively $\mathcal{N}_\lambda^-$), there exist $\varepsilon > 0$ and a continuous function $f = f(w) > 0, w \in D^{1,2}(\mathbb{R}^N)$, $\|w\| < \varepsilon$ satisfying*

$$f(0) = 1, \quad f(w)(u+w) \in \mathcal{N}_\lambda \quad (\textit{respectively } \mathcal{N}_\lambda^-),$$

for all $w \in D^{1,2}(\mathbb{R}^N)$, $\|w\| < \varepsilon$.

Proof. For $u \in \mathcal{N}_\lambda$, define $F : \mathbb{R} \times D^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ by

$$\begin{aligned} F(t,w) = t^2\|u+w\|^2 - t^{2(2^*-1)} \int_{\mathbb{R}^N} \phi_{u+w}|u+w|^{2^*-1}dx \\ - \lambda t^{1-\gamma} \int_{\mathbb{R}^N} g(x)|u+w|^{1-\gamma}dx. \end{aligned}$$

Since $u \in \mathcal{N}_\lambda$, it is easily obtained that $F(1,0) = 0$ and

$$F_t(1,0) = 2\|u\|^2 - 2(2^*-1)\int_{\mathbb{R}^N} \phi_u|u|^{2^*-1}dx - \lambda(1-\gamma)\int_{\mathbb{R}^N} g(x)|u|^{1-\gamma}dx.$$

As $u \neq 0$, by Lemma 2.2, we know that $F_t(1,0) \neq 0$. Thus, we can apply the implicit function theorem at the point $(0,1)$, and obtain $\varepsilon > 0$ and a continuous function $f : B(0,\varepsilon) \subset D^{1,2}(\mathbb{R}^N) \to \mathbb{R}^+$ satisfying

$$f(0) = 1, \quad f(w) > 0, \quad f(w)(u+w) \in \mathcal{N}_\lambda,$$

for all $w \in D^{1,2}(\mathbb{R}^N)$ with $\|w\| < \varepsilon$.

The case $u \in \mathcal{N}_\lambda^-$ can be obtained in the same way. The proof is complete. □

Lemma 2.6. *If $\{u_n\} \subset \mathcal{N}_\lambda$ is a minimizing sequence of I_λ, for each $\phi \in D^{1,2}(\mathbb{R}^N)$, it holds*

$$-\frac{|f_n'(0)|\|u_n\| + \|\varphi\|}{n} \le \langle J'(u_n), \varphi\rangle \le \frac{|f_n'(0)|\|u_n\| + \|\varphi\|}{n}, \tag{2.2}$$

where

$$\langle J'(u_n), \varphi\rangle = \int_{\mathbb{R}^N} (\nabla u_n, \nabla\varphi)dx - \int_{\mathbb{R}^N} \phi_{u_n} u_n^{2^*-2}\varphi dx - \lambda\int_{\mathbb{R}^N} g(x)u_n^{-\gamma}\varphi dx.$$

Proof. According to Lemma 2.3, I_λ is coercive on $\mathcal{N}_\lambda$. Applying Ekeland's variational principle, there exists a minimizing sequence $\{u_n\} \subset \mathcal{N}_\lambda$ of I_λ such that

$$I_\lambda(u_n) < \alpha_\lambda + \frac{1}{n}, \quad I_\lambda(v) - I_\lambda(u_n) \ge -\frac{1}{n}\|v-u_n\|, \quad \forall v \in \mathcal{N}_\lambda. \tag{2.3}$$

Based on $I_\lambda(|u_n|) = I_\lambda(u_n)$, we may assume that $u_n \ge 0$ in $\mathbb{R}^N$, and there exist a subsequence (by denoted itself) and u_* in $D^{1,2}(\mathbb{R}^N)$ such that

$$\begin{gathered} u_n \rightharpoonup u_* \quad \text{weakly in } D^{1,2}(\mathbb{R}^N), \\ u_n(x) \to u_*(x) \quad \text{a.e. in } \mathbb{R}^N. \end{gathered}$$

Let $t > 0$ small enough, $\varphi \in D^{1,2}(\mathbb{R}^N)$, we set $u = u_n, w = t\varphi \in D^{1,2}(\mathbb{R}^N)$ in Lemma 2.5, then we have $f_n(t) = f_n(t\varphi)$ with $f_n(0) = 1, f_n(t)(u_n + t\varphi) \in \mathcal{N}_\lambda$. Note that

$$\|u_n\|^2 - \lambda\int_{\mathbb{R}^N} g(x)|u_n|^{1-\gamma}dx - \int_{\mathbb{R}^N} \phi_{u_n}|u_n|^{2^*-1}dx = 0. \tag{2.4}$$

From (2.3), it follows that

$$\begin{aligned} \frac{1}{n}\left[|f_n(t)-1|\cdot\|u_n\|+tf_n(t)\|\varphi\|\right] &\geq \frac{1}{n}\|f_n(t)(u_n+t\varphi)-u_n\| \\ &\geq I_\lambda(u_n)-I_\lambda[f_n(t)(u_n+t\varphi)], \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} &I_\lambda(u_n)-I_\lambda[f_n(t)(u_n+t\varphi)] \\ &=\frac{1-f_n^2(t)}{2}\|u_n\|^2+\frac{f_n^{2(2^*-1)}(t)-1}{2(2^*-1)}\int_{\mathbb{R}^N}\phi_{(u_n+t\varphi)}|u_n+t\varphi|^{2^*-1}dx \\ &\quad+\lambda\frac{f_n^{1-\gamma}(t)-1}{1-\gamma}\int_{\mathbb{R}^N}g(x)|u_n+t\varphi|^{1-\gamma}dx+\frac{f_n^2(t)}{2}(\|u_n\|^2-\|u_n+t\varphi\|^2) \\ &\quad+\frac{1}{2(2^*-1)}\int_{\mathbb{R}^N}[\phi_{(u_n+t\varphi)}|u_n+t\varphi|^{2^*-1}-\phi_{u_n}|u_n|^{2^*-1}]dx \\ &\quad+\frac{\lambda}{1-\gamma}\int_{\mathbb{R}^N}g(x)((u_n+t\varphi)^{1-\gamma}-u_n^{1-\gamma})dx. \end{aligned}$$

Combined with (2.4) and (2.5), dividing by t and letting $t\to 0$, we obtain

$$\begin{aligned} &\frac{|f_n'(0)|\|u_n\|+\|\varphi\|}{n} \\ &\geq -f_n'(0)\Big\{\|u_n\|^2-\lambda\int_{\mathbb{R}^N}g(x)u_n^{1-\gamma}dx-\int_{\mathbb{R}^N}\phi_{u_n}|u_n|^{2^*-1}dx\Big\} \\ &\quad-\int_{\mathbb{R}^N}(\nabla u_n,\nabla\varphi)dx+\int_{\mathbb{R}^N}\phi_{u_n}|u_n|^{2^*-3}u\varphi dx+\lambda\int_{\mathbb{R}^N}g(x)u_n^{-\gamma}\varphi dx \\ &=-\int_{\mathbb{R}^N}(\nabla u_n,\nabla\varphi)dx+\int_{\mathbb{R}^N}\phi_{u_n}|u_n|^{2^*-3}u\varphi dx+\lambda\int_{\mathbb{R}^N}g(x)u_n^{-\gamma}\varphi dx, \end{aligned}$$

so, we obtain that for $\varphi\in D^{1,2}(\mathbb{R}^N)$, $\varphi\geq 0$, it holds

$$\begin{aligned} &\int_{\mathbb{R}^N}(\nabla u_n,\nabla\varphi)dx-\int_{\mathbb{R}^N}[\phi_{u_n}|u_n|^{2^*-3}u+\lambda g(x)u_n^{-\gamma}]\varphi dx \\ &\geq-\frac{|f_n'(0)|\|u_n\|+\|\varphi\|}{n}. \end{aligned} \tag{2.6}$$

Since the above inequality holds for $-\varphi$, it follows that

$$\begin{aligned} &\int_{\mathbb{R}^N}(\nabla u_n,\nabla\varphi)dx-\int_{\mathbb{R}^N}[\phi_{u_n}|u_n|^{2^*-3}u+\lambda g(x)u_n^{-\gamma}]\varphi dx \\ &\leq\frac{|f_n'(0)|\|u_n\|+\|\varphi\|}{n}. \end{aligned}$$

Set

$$\langle J'(u),\varphi\rangle=\int_{\mathbb{R}^N}(\nabla u,\nabla\varphi)dx-\int_{\mathbb{R}^N}\phi_{u_n}|u_n|^{2^*-3}u\varphi dx-\lambda\int_{\mathbb{R}^N}g(x)u_n^{-\gamma}\varphi dx,$$

consequently (2.2) holds. As in [31], we can prove that $\{f_n'(0)\}$ is bounded for all n. The proof is complete. □

Lemma 2.7. *Supposes $\{v_n\}\subset\mathcal{N}_\lambda^-$ is a minimizing sequence for I_λ with*

$$\alpha_\lambda^-<\frac{2}{N+2}S^{\frac{N}{2}}-D\lambda^{\frac{2}{1+\gamma}}\quad\textit{where}\quad D=D(N,\gamma,S,|g|_{\frac{2^*}{2^*+\gamma-1}}).$$

Then there exists $v_* \in D^{1,2}(\mathbb{R}^N)$ such that $v_n \to v_*$ in $D^{1,2}(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} \phi_{v_n} |v_n|^{2^*-1} dx \to \int_{\mathbb{R}^N} \phi_{v_*} |v_*|^{2^*-1} dx$$

as $n \to \infty$.

Proof. Let $\{v_n\} \subset \mathcal{N}_\lambda^-$ be a minimizing sequence for I_λ, similarly to the proof of Lemma 2.6, one obtains

$$-\frac{|f_n'(0)| \|v_n\| + \|\varphi\|}{n} \le \langle J'(v_n), \varphi \rangle \le \frac{|f_n'(0)| \|v_n\| + \|\varphi\|}{n}. \tag{2.7}$$

Since $\{v_n\}$ is bounded in $D^{1,2}(\mathbb{R}^N)$, there exist a subsequence, still denoted by itself, and a function $v_* \in D^{1,2}(\mathbb{R}^N)$ such that

$$\begin{gathered} v_n \rightharpoonup v_*, \quad \text{weakly in } D^{1,2}(\mathbb{R}^N), \\ v_n(x) \to v_*(x), \quad \text{a.e. in } \mathbb{R}^N \end{gathered}$$

as $n \to \infty$. We firstly claim that

$$\int_{\mathbb{R}^N} g(x) v_n^{1-\gamma} dx \to \int_{\mathbb{R}^N} g(x) v_*^{1-\gamma} dx.$$

In fact, by Hölder's inequality and the boundedness of $\{v_n\}$, it holds that

$$\begin{aligned} &\Big| \int_{|x|>m} g(x)[v_n^{1-\gamma} - v_*^{1-\gamma}] dx \Big| \\ &\le \int_{|x|>m} g(x) |v_n^{1-\gamma} - v_*^{1-\gamma}| dx \\ &\le \int_{|x|>m} g(x) (|v_n|^{1-\gamma} + |v_*|^{1-\gamma}) dx \\ &= \int_{|x|>m} g(x) |v_n|^{1-\gamma} dx + \int_{|x|>m} g(x) |v_*|^{1-\gamma} dx \\ &\le \Big(\int_{|x|>m} g(x)^{\frac{2^*}{2^*+\gamma-1}} dx \Big)^{\frac{2^*+\gamma-1}{2^*}} |v_n|_{2^*}^{1-\gamma} + \Big(\int_{|x|>m} g(x)^{\frac{2^*}{2^*+\gamma-1}} dx \Big)^{\frac{2^*+\gamma-1}{2^*}} |v_*|_{2^*}^{1-\gamma} \\ &\le C \Big(\int_{|x|>m} g(x)^{\frac{2^*}{2^*+\gamma-1}} dx \Big)^{\frac{2^*+\gamma-1}{2^*}} \\ &\to 0, \quad \text{as } m \to \infty, \end{aligned}$$

which implies that for any $\varepsilon > 0$, there exists $N_1 > 0$ such that

$$\Big| \int_{|x|>m} g(x)[v_n^{1-\gamma} - v_*^{1-\gamma}] dx \Big| < \frac{\varepsilon}{2}, \quad \text{for each } m > N_1.$$

Let $\mathcal{M} = \{x \in \mathbb{R}^N : |x| \le N_1 + 1\}$. Note that $\{v_n\}$ is bounded in $D^{1,2}(\mathbb{R}^N)$, then $\big(\int_{|x| \le N_1+1} v_n^{2^*} dx \big)^{\frac{1-\gamma}{2^*}} \le M'$ for some $M' > 0$. Moreover, from absolute continuity of the Lebesgue integral, for every $\varepsilon > 0$, there exists $\delta' > 0$ such that for each $E \subset \mathcal{M}$ with meas $E < \delta'$, it holds

$$\int_E g(x)^{\frac{2^*}{2^*+\gamma-1}} dx < \Big(\frac{\varepsilon}{M'} \Big)^{\frac{2^*}{2^*+\gamma-1}}.$$

Consequently,

$$\int_E g(x)v_n^{1-\gamma}dx \le \Big(\int_E g(x)^{\frac{2^*}{2^*+\gamma-1}}dx\Big)^{\frac{2^*+\gamma-1}{2^*}}\Big(\int_E |v_n|^{2^*}dx\Big)^{\frac{1-\gamma}{2^*}} < \varepsilon.$$

Hence $\{\int_{|x|\le N_1+1} g(x)v_n^{1-\gamma}dx, n \in N^+\}$ is equi-absolutely-continuous. It follows easily from Vitali Convergence Theorem that

$$\int_{|x|\le N_1+1} g(x)v_n^{1-\gamma}dx \to \int_{|x|\le N_1+1} g(x)v_*^{1-\gamma}dx, \quad \text{as } n\to\infty.$$

That is, there exists $N_2>0$ such that

$$\Big|\int_{|x|\le N_1+1} g(x)[v_n^{1-\gamma}-v_*^{1-\gamma}]dx\Big| < \frac{\varepsilon}{2}, \quad \text{for each } n>N_2.$$

Therefore, from the above inequalities, it follows that

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^N} g(x)v_n^{1-\gamma}dx - \int_{\mathbb{R}^N} g(x)v_*^{1-\gamma}dx\Big| \\
&= \Big|\int_{|x|\le N_1+1} g(x)[v_n^{1-\gamma}-v_*^{1-\gamma}]dx + \int_{|x|>N_1+1} h(x)[v_n^{1-\gamma}-v_*^{1-\gamma}]dx\Big| \\
&\le \Big|\int_{|x|\le N_1+1} g(x)[v_n^{1-\gamma}-v_*^{1-\gamma}]dx\Big| + \Big|\int_{|x|>N_1+1} g(x)[v_n^{1-\gamma}-v_*^{1-\gamma}]dx\Big| < \varepsilon
\end{aligned}$$

for $n>N_2$, which implies

$$\int_{\mathbb{R}^N} g(x)v_n^{1-\gamma}dx \to \int_{\mathbb{R}^N} g(x)v_*^{1-\gamma}dx, \quad \text{as } n\to\infty.$$

Now, set $w_n = v_n - v_*$, then $\|w_n\|\to 0$. Otherwise, there exists a subsequence (still denoted by w_n) such that

$$\lim_{n\to\infty}\|w_n\| = l > 0.$$

From (2.7), letting $n\to\infty$, for every $\varphi\in D^{1,2}(\mathbb{R}^N)$, it follows

$$\int_{\mathbb{R}^N}(\nabla v_*,\nabla\varphi)dx - \lambda\int_{\mathbb{R}^N} g(x)v_*^{-\gamma}\varphi dx - \int_{\mathbb{R}^N}\phi_{v_*}v_*^{2^*-2}\varphi dx = 0. \tag{2.8}$$

Taking the test function $\varphi = v_*$ in (2.8), it follows that

$$\|v_*\|^2 - \lambda\int_{\mathbb{R}^N} g(x)v_*^{1-\gamma}dx - \int_{\mathbb{R}^N}\phi_{v_*}v_*^{2^*-1}dx = 0. \tag{2.9}$$

Putting $\varphi = v_n$ in (2.7), by the Brézis-Lieb's lemma (see [6]) and Lemma 2.1, it follows that

$$\begin{aligned}
&\|w_n\|^2 + \|v_*\|^2 - \int_{\mathbb{R}^N}[\phi_{w_n}|w_n|^{2^*-1} + \phi_{v_*}|v_*|^{2^*-1}]dx \\
&- \lambda\int_{\mathbb{R}^N} g(x)v_*^{1-\gamma}dx = o(1).
\end{aligned} \tag{2.10}$$

It follows from (2.9) and (2.10) that

$$\|w_n\|^2 - \int_{\mathbb{R}^N}\phi_{w_n}|w_n|^{2^*-1}dx = o(1). \tag{2.11}$$

Note that $\int_{\mathbb{R}^N}\phi_{w_n}|w_n|^{2^*-1}dx \le S^{-2^*}\|w_n\|^{2(2^*-1)}$; then

$$l \ge S^{\frac{2^*}{2(2^*-2)}}, \quad l>0.$$

On the one hand, from (2.9), by the Young inequality,

$$
\begin{aligned}
I_\lambda(v_*) &= \frac{1}{2}\|v_*\|^2 - \frac{1}{2(2^*-1)}\int_{\mathbb{R}^N}\phi_{v_*}v_*^{2^*-1}dx - \frac{\lambda}{1-\gamma}\int_{\mathbb{R}^N} g(x)v_*^{1-\gamma}dx\\
&= \frac{2^*-2}{2(2^*-1)}\|v_*\|^2 - \lambda\big[\frac{1}{1-\gamma} - \frac{1}{2(2^*-1)}\big]\int_{\mathbb{R}^N} g(x)v_*^{1-\gamma}dx\\
&\geq \frac{2}{N+2}\|v_*\|^2 - \lambda\big[\frac{1}{1-\gamma} - \frac{1}{2(2^*-1)}\big]|g|_{\frac{2^*}{2^*+\gamma-1}}S^{-\frac{1-\gamma}{2}}\|v_*\|^{1-\gamma}\\
&\geq -D\lambda^{\frac{2}{1+\gamma}},
\end{aligned}
$$

where $D = D(N,\gamma,S,|g|_{\frac{2^*}{2^*+\gamma-1}}) > 0$ is a constant (independent of λ).

On the other hand, from (2.11),

$$
\begin{aligned}
I_\lambda(v_*) &= I_\lambda(v_n) - \frac{1}{2}\|w_n\|^2 + \frac{1}{2(2^*-1)}\int_{\mathbb{R}^N}\phi_{w_n}|w_n|^{2^*-1}dx + o(1)\\
&= I_\lambda(v_n) - \frac{2^*-2}{2(2^*-1)}\|w_n\|^2 + o(1)\\
&\leq \alpha_\lambda^- - \frac{2}{N+2}l^2\\
&< \frac{2}{N+2}S^{\frac{N}{2}} - D\lambda^{\frac{2}{1+\gamma}} - \frac{2}{N+2}S^{\frac{N}{2}}\\
&= -D\lambda^{\frac{2}{1+\gamma}}.
\end{aligned}
$$

This is a contradiction. Therefore, $l = 0$, it implies that $v_n \to v_*$ in $D^{1,2}(\mathbb{R}^N)$. Note that

$$
\begin{aligned}
0 &\leq \int_{\mathbb{R}^N}\phi_{v_n}v_n^{2^*-1}dx - \int_{\mathbb{R}^N}\phi_{v_*}v_*^{2^*-1}dx\\
&= \int_{\mathbb{R}^N}\phi_{w_n}w_n^{2^*-1}dx + o(1)\\
&\leq S^{-2^*}\|w_n\|^{2(2^*-1)} + o(1) \to 0,
\end{aligned}
$$

which implies that $\int_{\mathbb{R}^N}\phi_{v_n}v_n^{2^*-1}dx \to \int_{\mathbb{R}^N}\phi_{v_*}v_*^{2^*-1}dx$ as $n \to \infty$. The proof is complete. $\square$

Theorem 2.8. *Under the assumptions of Theorem 1.1, system* (1.1) *has a positive ground state solution* $(u_\lambda, \phi_{u_\lambda}) \in D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ *with* $I_\lambda(u_\lambda) < 0$.

Proof. There exists a constant $\delta > 0$ such that $\frac{2}{N+2}S^{\frac{N}{2}} - D\lambda^{\frac{2}{1+\gamma}} > 0$ for $\lambda < \delta$. Set $\Lambda_1 = \min\{\Lambda_0, \delta\}$, then Lemmas 2.1–2.7 hold for all $0 < \lambda < \Lambda_1$. Therefore, there exist a bounded minimizing sequence $\{u_n\} \subset \mathcal{N}_\lambda$ of I_λ and $u_\lambda \in D^{1,2}(\mathbb{R}^N)$ such that

$$
\begin{gathered}
u_n \rightharpoonup u_\lambda, \quad \text{weakly in } D^{1,2}(\mathbb{R}^N),\\
u_n(x) \to u_\lambda(x), \quad \text{a.e. in } \mathbb{R}^N,
\end{gathered}
$$

as $n \to \infty$. Now we will prove that u_λ is a positive ground state solution of system (1.1).

Indeed, by Lemmas 2.4–2.7, we can deduce that

$$
\lim_{n\to\infty}\int_{\mathbb{R}^N}\phi_{u_n}u_n^{2^*-1}dx = \int_{\mathbb{R}^N}\phi_{u_\lambda}u_\lambda^{2^*-1}dx.
$$

By Fatou's lemma,

$$\begin{aligned} 0 &= \lim_{n\to\infty} \Big\{ \|u_n\|^2 - \lambda \int_{\mathbb{R}^N} g(x) u_n^{1-\gamma} dx - \int_{\mathbb{R}^N} \phi_{u_n} u_n^{2^*-1} dx \Big\} \\ &\geq \|u_\lambda\|^2 - \lambda \int_{\mathbb{R}^N} g(x) u_\lambda^{1-\gamma} dx - \int_{\mathbb{R}^N} \phi_{u_\lambda} u_\lambda^{2^*-1} dx. \end{aligned} \tag{2.12}$$

Letting $n \to \infty$ in (2.2) and using the Fatou's lemma again, for each $\varphi \in D^{1,2}(\mathbb{R}^N)$, $\varphi \geq 0$, it holds

$$\int_{\mathbb{R}^N} (\nabla u_\lambda, \nabla \varphi) dx - \lambda \int_{\mathbb{R}^N} g(x) u_\lambda^{-\gamma} \varphi dx - \int_{\mathbb{R}^N} \phi_{u_\lambda} u_\lambda^{2^*-2} \varphi dx \geq 0. \tag{2.13}$$

Now, for any $v \in D^{1,2}(\mathbb{R}^N)$, we set $\Psi = (u_\lambda + \varepsilon v)^+$, it follows from (2.12) and (2.13) that

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}^N} [(\nabla u_\lambda, \nabla \Psi) - \phi_{u_\lambda} u_\lambda^{2^*-2} \Psi - \lambda g(x) u_\lambda^{-\gamma} \Psi] dx \\ &= \int_{\{u_\lambda + \varepsilon v > 0\}} \big[(\nabla u_\lambda, \nabla(u_\lambda + \varepsilon v)) - \phi_{u_\lambda} u_\lambda^{2^*-2} (u_\lambda + \varepsilon v) \\ &\quad - \lambda g(x) u_\lambda^{-\gamma} (u_\lambda + \varepsilon v) \big] dx \\ &= \Big(\int_{\mathbb{R}^N} - \int_{\{u_\lambda + \varepsilon v \leq 0\}} \Big) [(\nabla u_\lambda, \nabla(u_\lambda + \varepsilon v) \\ &\quad - \phi_{u_\lambda} u_\lambda^{2^*-2} (u_\lambda + \varepsilon v) - \lambda g(x) u_\lambda^{-\gamma} (u_\lambda + \varepsilon v)] dx \\ &\leq \|u_\lambda\|^2 - \int_{\mathbb{R}^N} \phi_{u_\lambda} u_\lambda^{2^*-1} dx - \lambda \int_{\mathbb{R}^N} g(x) u_\lambda^{1-\gamma} dx \\ &\quad + \varepsilon \int_{\mathbb{R}^N} \Big[(\nabla u_\lambda, \nabla v) - \phi_{u_\lambda} u_\lambda^{2^*-2} v - \lambda g(x) u_\lambda^{-\gamma} v \Big] dx \\ &\quad - \int_{\{u_\lambda + \varepsilon v \leq 0\}} (\nabla u_\lambda, \nabla(u_\lambda + \varepsilon v)) dx \\ &\quad + \int_{\{u_\lambda + \varepsilon v \leq 0\}} \Big[\phi_{u_\lambda} u_\lambda^{2^*-2} (u_\lambda + \varepsilon v) + \lambda g(x) u_\lambda^{-\gamma} (u_\lambda + \varepsilon v) \Big] dx \\ &\leq \varepsilon \int_{\mathbb{R}^N} \Big[(\nabla u_\lambda, \nabla v) - \phi_{u_\lambda} u_\lambda^{2^*-2} v - \lambda g(x) u_\lambda^{-\gamma} v \Big] dx \\ &\quad - \varepsilon \int_{\{u_\lambda + \varepsilon v \leq 0\}} (\nabla u_\lambda, \nabla v) dx. \end{aligned} \tag{2.14}$$

Since $\nabla u_\lambda = 0$ for a.e. $x \in \mathbb{R}^3$ with $u_\lambda(x) = 0$ and

$$\operatorname{meas}\{x | u_\lambda(x) + \varepsilon v(x) < 0, u_\lambda(x) > 0\} \to 0 \quad \text{as } \varepsilon \to 0,$$

then, we have

$$\Big| \int_{\{u_\lambda + \varepsilon v < 0\}} (\nabla u_\lambda, \nabla v) dx \Big| = \int_{\{u_\lambda + \varepsilon v < 0, u_\lambda > 0\}} (\nabla u_\lambda, \nabla v) dx \to 0 \quad \text{as } \varepsilon \to 0.$$

Therefore, dividing by ε and letting $\varepsilon \to 0$ in (2.14), one gets

$$\int_{\mathbb{R}^N} (\nabla u_\lambda, \nabla v) dx - \int_{\mathbb{R}^N} \phi_{u_\lambda} u_\lambda^{2^*-2} v dx - \lambda \int_{\mathbb{R}^N} g(x) u_\lambda^{-\gamma} v dx \geq 0.$$

As v is arbitrarily, consequently, u_λ is a nonzero negative solution of system (1.1). Note that $u_\lambda \in \mathcal{N}_\lambda$ and $\alpha_\lambda < 0$ (by Lemma 2.4), then

$$\begin{aligned}\Big[\frac{1}{1-\gamma} - \frac{1}{2(2^*-1)}\Big]\lambda \int_{\mathbb{R}^N} g(x)u_\lambda^{1-\gamma}dx &= \frac{2}{N+2}\|u_\lambda\|^2 - I_\lambda(u_\lambda)\\ &\geq \frac{2}{N+2}\|u_\lambda\|^2 - \alpha_\lambda > 0,\end{aligned}$$

which implies that $u_\lambda \not\equiv 0$. Note that $u_\lambda \geq 0$ in $\mathbb{R}^N$. By standard arguments as in DiBenedetto [11] and Tolksdorf [27], we have that $u_\lambda \in L^\infty(\mathbb{R}^N)$ and $u_\lambda \in C^{1,\alpha}_{loc}(\mathbb{R}^N)$ with $0 < \alpha < 1$. Furthermore, by Harnack's inequality (see Trudinger [28]), $u_\lambda > 0$ for any $x \in \mathbb{R}^N$. Furthermore, we have

$$\alpha_\lambda = \lim_{n\to\infty} I_\lambda(u_n) = I_\lambda(u_\lambda). \tag{2.15}$$

Next, we claim that $u_\lambda \in \mathcal{N}_\lambda^+$. On the contrary, assume that $u_\lambda \in \mathcal{N}_\lambda^-$ ($\mathcal{N}_\lambda^0 = \{0\}$ for $\lambda \in (0, \Lambda_0)$), then by Lemma 2.2, there exist positive numbers $t^+ < t_{\max} < t^- = 1$ such that $t^+u_\lambda \in \mathcal{N}_\lambda^+$, $t^-u_\lambda \in \mathcal{N}_\lambda^-$ and

$$\alpha_\lambda < I_\lambda(t^+u_\lambda) < I_\lambda(t^-u_\lambda) = I_\lambda(u_\lambda) = \alpha_\lambda,$$

this is a contradiction. Hence, $u_\lambda \in \mathcal{N}_\lambda^+$. By the definition of α_λ^+, we have $\alpha_\lambda^+ \leq I_\lambda(u_\lambda)$. It follows from Lemma 2.4 and (2.15) that

$$I_\lambda(u_\lambda) = \alpha_\lambda^+ = \alpha_\lambda < 0.$$

From the above analysis, we obtain that u_λ is a positive ground state solution of system (1.1). The proof is complete. □

3. Existence of the second positive solution of system (1.1)

From [26], For $x \in \mathbb{R}^N$, it is well known that the function

$$\Phi(x) = \frac{(\frac{N}{N-2})^{\frac{N-2}{4}}}{(1+|x|^2)^{\frac{N-2}{2}}}$$

solves

$$\begin{gathered}-\Delta u = u^{2^*-1}\ x \in \mathbb{R}^N,\\ \|\Phi\|^2 = \int_{\mathbb{R}^N} \Phi^{2^*}dx = S^{\frac{N}{2}}.\end{gathered}$$

Lemma 3.1. *There exists $\Lambda_3 > 0$ such that for each $\lambda \in (0, \Lambda_3)$, it holds*

$$\sup_{t\geq 0} I_\lambda(t\Phi) < \frac{2}{N+2}S^{\frac{N}{2}} - D\lambda^{\frac{2}{1+\gamma}}. \tag{3.1}$$

Proof. We are going to give an estimate of the value of I_λ. Observe that, multiplying the second equation of system (1.1) by $|u|$ and integrating, one has

$$|u|_{2^*}^{2^*} = \int_{\mathbb{R}^N} \nabla\phi_u|\nabla|u||dx \leq \frac{1}{2}|\nabla\phi_u|_2^2 + \frac{1}{2}|\nabla|u||_2^2.$$

So, if we introduce the new functional $J_\lambda : D^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ defined in the following way

$$J_\lambda(u) = \frac{N}{N+2}\|u\|^2 - \frac{1}{2^*-1}\int_{\mathbb{R}^N} |u|^{2^*}dx - \frac{\lambda}{1-\gamma}\int_{\mathbb{R}^N} g(x)|u|^{1-\gamma}dx.$$

Then, we have $I_\lambda(u) \le J_\lambda(u)$ for any $u \in D^{1,2}(\mathbb{R}^N)$. For $t \ge 0$, set

$$\begin{aligned} h(t) &= \frac{Nt^2}{N+2}\|\Phi\|^2 - \frac{t^{2^*}}{2^*-1}\int_{\mathbb{R}^N}\Phi^{2^*}dx \\ &= \frac{Nt^2}{N+2}S^{\frac{N}{2}} - \frac{t^{2^*}}{2^*-1}S^{\frac{N}{2}}. \end{aligned}$$

Then

$$\sup_{t\ge 0} h(t) = \sup_{t\ge 0}\Big\{\frac{Nt^2}{N+2}S^{\frac{N}{2}} - \frac{t^{2^*}}{2^*-1}S^{\frac{N}{2}}\Big\} = \frac{2}{N+2}S^{\frac{N}{2}}.$$

When $\lambda \in (0,\delta)$, we have $\frac{2}{N+2}S^{\frac{N}{2}} - D\lambda^{\frac{2}{1+\gamma}} > 0$, which implies that there exists $t_0 > 0$ small such that

$$\sup_{0\le t\le t_0} I_\lambda(t\Phi) < \frac{2}{N+2}S^{\frac{N}{2}} - D\lambda^{\frac{2}{1+\gamma}} \quad \text{for each } \lambda \in (0,\delta).$$

We next consider the case where $t > t_0$. Since $\frac{2}{1+\gamma} > 1$, there exists $\Lambda_2 > 0$ such that

$$-\lambda\frac{t_0^{1-\gamma}}{1-\gamma}\int_{\mathbb{R}^N} g(x)\Phi^{1-\gamma}dx < -D\lambda^{\frac{2}{1+\gamma}} \quad \text{for each } \lambda \in (0,\Lambda_2).$$

Then, for each $\lambda \in (0,\Lambda_2)$, it follows

$$\begin{aligned} \sup_{t\ge t_0} I_\lambda(t\Phi) &\le \frac{2}{N+2}S^{\frac{N}{2}} - \lambda\frac{t^{1-\gamma}}{1-\gamma}\int_{\mathbb{R}^N} g(x)\Phi^{1-\gamma}dx \\ &\le \frac{2}{N+2}S^{\frac{N}{2}} - \lambda\frac{t_0^{1-\gamma}}{1-\gamma}\int_{\mathbb{R}^N} g(x)\Phi^{1-\gamma}dx \\ &< \frac{2}{N+2}S^{\frac{N}{2}} - D\lambda^{\frac{2}{1+\gamma}}. \end{aligned}$$

Set $\Lambda_3 = \min\{\delta,\Lambda_2\}$. From the above information, it holds that

$$\sup_{t\ge 0} I_\lambda(t\Phi) < \frac{2}{N+2}S^{\frac{N}{2}} - D\lambda^{\frac{2}{1+\gamma}} \quad \text{for each } \lambda \in (0,\Lambda_3).$$

Therefore, (3.1) holds true when $\lambda < \Lambda_3$. The proof is complete. □

Theorem 3.2. *There exists $\lambda_* > 0$ such that problem (1.1) has a positive solution v_λ with $v_\lambda \in \mathcal{N}_\lambda^-$ for each $0 < \lambda < \lambda_*$.*

Proof. Let $\lambda_* = \min\{\Lambda_0,\Lambda_3\}$. Since I_λ is also coercive on $\mathcal{N}_\lambda^-$, we apply the Ekeland's variational principle to the minimization problem $\alpha_\lambda^- = \inf_{v\in\mathcal{N}_\lambda^-} I_\lambda(v)$, there exists a minimizing sequence $\{v_n\} \subset \mathcal{N}_\lambda^-$ of I_λ with the following properties:

(i) $I_\lambda(v_n) < \alpha_\lambda^- + \frac{1}{n}$;
(ii) $I_\lambda(u) \ge I_\lambda(v_n) - \frac{1}{n}\|u - v_n\|$ for all $u \in \mathcal{N}_\lambda^-$.

Since $\{v_n\}$ is bounded in $D^{1,2}(\mathbb{R}^N)$, up to a subsequence if necessary, there exists $v_\lambda \in D^{1,2}(\mathbb{R}^N)$ such that

$$\begin{gathered} v_n \rightharpoonup v_\lambda, \quad \text{weakly in } D^{1,2}(\mathbb{R}^N), \\ v_n(x) \to v_\lambda(x), \quad \text{a.e. in} \mathbb{R}^N \text{ as } n \to \infty. \end{gathered}$$

Using Lemmas 2.5–2.7 and Lemma 3.1, similarly, we can get that v_λ is a non-negative solution of system (1.1).

Now, we prove that $v_\lambda > 0$ in $\mathbb{R}^N$. Since $v_n \in \mathcal{N}_\lambda^-$, it holds

$$\begin{aligned}(1+\gamma)\|v_n\|^2 &< (2\cdot 2^* + \gamma - 3)\int_{\mathbb{R}^N} \phi_{v_n} v_n^{2^*-1}dx \\ &< (2\cdot 2^* + \gamma - 3) S^{-2^*}\|v_n\|^{2(2^*-1)},\end{aligned}$$

so that

$$\|v_n\| > \Big(\frac{(1+\gamma)S^{2^*}}{2\cdot 2^* + \gamma - 3}\Big)^{\frac{1}{2(2^*-2)}} \quad \forall v_n \in \mathcal{N}_\lambda^-,$$

which implies that $v_\lambda \not\equiv 0$. Similarly, by Harnacks inequality, we also obtain $v_\lambda > 0$ for any $x \in \mathbb{R}^N$.

Next, we prove that $v_\lambda \in \mathcal{N}_\lambda^-$, it suffices to prove that $\mathcal{N}_\lambda^-$ is closed.

Indeed, by Lemma 2.7 and Lemma 3.1, for $\{v_n\} \subset \mathcal{N}_\lambda^-$, it holds

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} \phi_{v_n} v_n^{2^*-1}dx = \int_{\mathbb{R}^N} \phi_{v_\lambda} v_\lambda^{2^*-1}dx.$$

By the definition of $\mathcal{N}_\lambda^-$, it holds that

$$2\|v_n\|^2 - (2^*-1)\int_{\mathbb{R}^N} \phi_{v_n} v_n^{2^*-1}dx - \lambda(1-\gamma)\int_{\mathbb{R}^N} g(x) v_n^{1-\gamma}dx < 0,$$

thus

$$2\|v_\lambda\|^2 - (2^*-1)\int_{\mathbb{R}^N} \phi_{v_\lambda} v_\lambda^{2^*-1}dx - \lambda(1-\gamma)\int_{\mathbb{R}^N} g(x) v_\lambda^{1-\gamma}dx \le 0,$$

which implies that $v_\lambda \in \mathcal{N}_\lambda^0 \cup \mathcal{N}_\lambda^-$. If $\mathcal{N}_\lambda^-$ is not closed, then we have $v_\lambda \in \mathcal{N}_\lambda^0$, by Lemma 2.2, it follows that $v_\lambda = 0$, this contradicts $v_\lambda > 0$. Consequently, $v_\lambda \in \mathcal{N}_\lambda^-$. Note that, $\mathcal{N}_\lambda^+ \cap \mathcal{N}_\lambda^- = \emptyset$, then u_λ and v_λ are different positive solutions of (1.1). This completes the proof. □

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