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STABILIZATION OF THE CRITICAL NONLINEAR KLEIN-GORDON EQUATION WITH VARIABLE COEFFICIENTS ON $\mathbb{R}^3$

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Abstract. We prove the exponential stability of the defocusing critical semilinear wave equation with variable coefficients and locally distributed damping on $\mathbb{R}^3$. The construction of the variable coefficients is almost equivalent to the geometric control condition. We develop the traditional Morawetz estimates and the compactness-uniqueness arguments for the semilinear wave equation to prove the unique continuation result. The observability inequality and the exponential stability are obtained subsequently.

1. Introduction

In this article, we consider the defocusing critical nonlinear Klein-Gordon equation

$$
\begin{gathered}
u_{tt} - \operatorname{div} A(x)\nabla u + a(x)u_t + u + u^5 = 0, \quad (x,t) \in \mathbb{R}^3 \times (0,+\infty), \\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^3,
\end{gathered} \tag{1.1}
$$

where $A(x) = \{a_{ij}(x)\}_{ij=1}^3$ is a positive definite matrix such that

$$
a_{ij}(x) \in W^{2,\infty}(\mathbb{R}^3) \quad \text{for } i,j = 1,2,3.
$$

Let the damping term $a(x)$ be a real nonnegative function of class $W^{2,\infty}(\mathbb{R}^3)$, and let the initial data $(u_0, u_1) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$.

The Klein-Gordon equation is the basic equation in relativistic quantum mechanics and in quantum field theory. It is a special relativistic form of the Schrödinger equation, which describes particles with zero spin. The studying of Klein-Gordon equations with nonlinear perturbation is essential in both physics and mathematics. In particular, the stability of the semilinear Klein-Gordon equations have attracted much attention, but the critical case hard to study. See for instance [25] and references therein.

1.1. Notation and statement of the problem.

Let O be the origin in the space $\mathbb{R}^3$, and $r(x) = |x|$ be the Euclidean norm of $x \in \mathbb{R}^3$. Let $\langle \cdot, \cdot \rangle$, div, ∇, Δ, and $I_3 = (\delta_{ij})_{3\times 3}$ be the standard inner product, divergence operator, gradient operator, Laplace operator, and the unit matrix in $\mathbb{R}^3$, respectively.

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We define

$$g = A^{-1}(x) = G(x) \quad \text{for } x \in \mathbb{R}^3$$

as a Riemannian metric on $\mathbb{R}^3$. Thus, we consider $(\mathbb{R}^3, g)$ as a Riemannian manifold and

$$\langle X, Y\rangle_g = \langle A^{-1}(x)X, Y\rangle, \quad |X|_g^2 = \langle X, X\rangle_g, \quad X, Y \in \mathbb{R}^3_x,\ x \in \mathbb{R}^3, \tag{1.2}$$

where $\mathbb{R}^3_x$ is the tangential space at $x \in \mathbb{R}^3$. We assume that there exist positive constants m_1, m_2 such that

$$m_1|X|^2 \le \langle A(x)X, X\rangle = |X|_g^2 \le m_2|X|^2 \quad \text{for } X \in \mathbb{R}^3_x,\ x \in \mathbb{R}^3. \tag{1.3}$$

Let D be the Levi-Civita connection in the metric g, and H be a vector field. We will use many times that $H(u) = \langle H, \nabla_g u\rangle_g$. The covariant differential DH of the vector field H is a tensor field of rank 2, and

$$DH(X, Y)(x) = \langle D_Y H, X\rangle_g(x) \quad X, Y \in \mathbb{R}^3_x,\ x \in \mathbb{R}^3. \tag{1.4}$$

For a given $y > 0$, we define the ball

$$B(y) = \{x \in \mathbb{R}^3 : |x| \le y\}. \tag{1.5}$$

We also set div_g, ∇_g, and Δ_g the divergence operator, e gradient operator, and Laplace-Beltrami operator in the metric g, respectively.

We define the energy functional associated with (1.1) as

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^3} (u_t^2 + |\nabla_g u|_g^2 + u^2)dx + \frac{1}{6}\int_{\mathbb{R}^3} u^6 dx. \tag{1.6}$$

In this article we consider mainly the exponential decay of $E(t)$.

We say that a subdomain $\omega \subset \Omega$ satisfies the Geometric Control Condition (GCC) if each unit geodesic initiated from Ω enters ω before a finite time T. In particular, if ω is the boundary Γ of Ω, then the Geometric Control Condition states that each geodesic initiated from Ω mush hit the boundary Γ in time less than T.

1.2. **Previous results.** There is a large number of results for the wave equations with either locally distributed damping or suitable boundary dissipation. For stability results of linear wave equations in compact domains, we refer to [15, 26, 27, 33, 34, 43]. For the linear wave equations in non-compact domains, we refer to [2, 3, 27, 30, 31, 42].

A lot of contributions to the stability analysis of the nonlinear wave equations arose subsequently. We mention that [8] concerned the wave equation on compact surfaces with nonlinear locally distributed damping, described by

$$u_{tt} - \Delta_g u + a(x)h(u_t) = 0.$$

The authors obtained the stability result that $E(t) \le S(t)(t/T_0 - 1)$ for fixed $T_0 > 0$ under some assumptions on the function h and on the compact domain, where $S(t)$ vanishes as t tends to infinity. Moreover, the energy decays exponentially with respect to the initial energy if the feedback h is linear. Later, in [1, 8] the authors studied the well-posedness and sharp uniform decay rates of the energy related to the Klein-Gordon equation. This is done subject to a nonlinear and locally distributed damping, posed in a complete and noncompact n dimensional Riemannian manifold (M, g) without boundary, $u_{tt} - \Delta u + u + a(x)h(u_t) = 0$. They obtained the exponential stability result under some suitable assumptions on h, a and the geometric conditions of (M, g).

For the long time behavior of the nonlinear wave equations in compact spaces, we refer to [7, 9, 20, 22, 24, 44, 46]. For the nonlinear wave equations in noncompact spaces, we refer to [3, 10, 20, 28, 29, 38, 39, 40, 45, 48]. We note that most of the noncompact spaces concerned in literature are either the whole spaces $\mathbb{R}^n$ or domains outside a convex obstacle. The geometric control condition (GCC) is always used as a necessary assumption to get the stability results.

For the energy subcritical semilinear wave equations, we point out that [20] studied the exponential stability of the semilinear wave equation with a damping effective in a zone satisfying the geometric control condition only. The nonlinearity is assumed to be subcritical, defocusing, and analytic. The new contribution compared to previous results, is their proof of a unique continuation result in large time for some undamped equation. For the stabilization of the subcritical semilinear wave equations, we refer to [3, 7, 10, 48] and references therein.

We know that the global well-posedness and the stability results related to the subcritical nonlinear wave-type equations are easier than the critical ones. In [25], exponential stability of the critical semilinear Klein-Gordon equation was proved on a 3-D compact manifold with small initial data. They posed a geometric assumption slightly stronger than the classical GCC. The smallness of the initial data in the norm $L^2 \times H^{-1}$ was assumed in order to avoid the missing unique continuation theorem:

$u = 0$ is the unique strong solution in the energy space of

$$\begin{gathered} u_{tt} - \Delta u + u + |u|^4 u = 0 \quad \text{in } M \times (0,T), \\ u_t = 0 \quad \text{in } \omega \times (0,T), \end{gathered} \tag{1.7}$$

where ω is an open subset of M satisfies the GCC in a given time $T_0 > 0$. For general case, we do not clearly know how to eliminate the smallness of the initial data. In this article, because of the complexity of the critical case, the unique continuation property of the energy critical semilinear wave equations is difficult to obtain. Comparing to the previous results, a stronger assumption on $A(x)$ is assumed to obtain a unique continuation result (see Assumption and Proposition 3.1). Therefore, the exponential stability can be achieved by developing the traditional Morawetz estimates and the compactness-uniqueness arguments for the semilinear equation.

1.3. **Main assumptions and main result.**

We use the following assumption in this article:

(A1) There exists a constant $0 < \delta \le 1$ such that

$$\langle \big((1-\delta)A(x) - \frac{r}{2}\frac{\partial A(x)}{\partial r}\big)X, X\rangle \ge 0 \quad \text{for } X \in \mathbb{R}^3_x,\ x \in \mathbb{R}^3. \tag{1.8}$$

We will give an example that satisfies (1.8) and will show the relationship between (A1) and the Geometric Control Condition in the appendix.

Condition (1.8) below which seems strong, is used to guarantee the classical Morawetz mutiplier $H = x = r\frac{\partial}{\partial r}$ works in the metric g. That is, we have

$$DH(X,X) \ge \delta |X|_g^2.$$

More precisely, such a technical assumption is helpful to obtain the the following unique continuation result:

$u = 0$ is the only solution to

$$\begin{aligned} u_{tt} - \operatorname{div} A(x)\nabla u + u + u^5 = 0, \quad (x,t) \in \mathbb{R}^3 \times (0,T), \\ u_t = 0, \quad (x,t) \in (\mathbb{R}^3 \backslash B(R_0)) \times (0,T). \end{aligned} \tag{1.9}$$

Generally, the unique continuation property for the critical semilinear wave equations is still an open problem, even in compact spaces.

For the global well-posedness of (1.1), we assume that

(A2) System (1.1) admits a unique solution such that

$$u \in C^1(0,\infty; L^2(\mathbb{R}^3)) \cap C(0,\infty; H^1(\mathbb{R}^3)).$$

Remark 1.1. The global existence results for critical wave equations are complex. Fortunately, the powerful Strichartz estimate, as a space-time estimate, offers us an effective tool to handle the critical case. In general, for lower regularity initial data $(u_0, u_1) \in H^1 \times L^2$, we have

$$u(t) \in C([0,+\infty); H^1) \cap L_t^5 L_x^{10}([0,T) \times \mathbb{R}^3), \quad \text{for all } T < +\infty.$$

Here we list some more references on this topic. For Cauchy problem, global existence of C^2-solutions in dimension $n = 3$ was first obtained by Rauch [32], assuming the initial energy to be small. Later, the global existence results have improved in many subsequent papers: [4, 13, 14, 16, 21, 36, 37]. Now, the global well-posedness of the energy critical defocusing wave equations are classical. We refer to [47] for the critical wave equations with variable coefficients on $\mathbb{R}^3$, and to [25] for the critical Klein-Gordon equations on 3-D compact Riemannian manifolds.

The main result in this article reads as follows.

Theorem 1.2. *Suppose that* (A1), (A2) *hold. Let $E(0) \le E_0$ and*

$$a(x) \ge a_0, \quad x \in \mathbb{R}^3 \backslash B(R_0), \tag{1.10}$$

where E_0, a_0, R_0 are positive constants. Then there exist positive constants C_1, C_2, which are dependent on $E(0)$, such that

$$E(t) \le C_1 e^{-C_2 t} E(0), \quad \forall t > 0. \tag{1.11}$$

2. Multiplier identities and key lemmas

Here we establish several geometric multiplier identities, which are useful for the unique continuation results.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. Let $\nu(x)$ be the unit normal vector of $\partial\Omega$, pointing outside on Ω. Suppose that $u(x,t)$ is a solution of the equation*

$$u_{tt} - \operatorname{div} A(x)\nabla u + a(x)u_t + u + u^5 = 0, \quad (x,t) \in \Omega \times (0,+\infty). \tag{2.1}$$

Let $\mathcal{H}$ be a C^1 vector field defined on $\mathbb{R}^3$. Then

$$\begin{aligned} &\int_0^T \int_{\partial\Omega} \langle \nabla_g u, \nu\rangle \mathcal{H}(u) d\Gamma dt + \frac{1}{2}\int_0^T \int_{\partial\Omega} (u_t^2 - |\nabla_g u|_g^2 - u^2 - \frac{1}{3}u^6)\mathcal{H}\cdot\nu d\Gamma dt \\ &= \int_\Omega u_t \mathcal{H}(u)\,dx\Big|_0^T + \int_0^T\int_\Omega D\mathcal{H}(\nabla_g u, \nabla_g u)\,dx\,dt + \int_0^T\int_\Omega a(x)u_t\mathcal{H}(u)\,dx\,dt \\ &\quad + \frac{1}{2}\int_0^T\int_\Omega (u_t^2 - |\nabla_g u|_g^2 - u^2 - \frac{1}{3}u^6)\operatorname{div}\mathcal{H}\,dx\,dt. \end{aligned} \tag{2.2}$$

Moreover, if we assume that $P \in C^2(\mathbb{R}^3)$, then

$$\begin{aligned} &\int_0^T \int_\Omega (u_t^2 - |\nabla_g u|_g^2 - u^2 - u^6) P \,dx\,dt \\ &= \int_\Omega Puu_t \,dx\Big|_0^T + \frac{1}{2}\int_0^T \int_{\partial\Omega} u^2 \langle \nabla_g P, \nu\rangle d\Gamma dt - \int_0^T \int_{\partial\Omega} Pu\langle \nabla_g u, \nu\rangle d\Gamma dt \\ &\quad - \frac{1}{2}\int_0^T \int_\Omega u^2 (\operatorname{div} A(x)\nabla P)\,dx\,dt + \frac{1}{2}\int_\Omega a(x) P u^2\,dx\Big|_0^T. \end{aligned} \tag{2.3}$$

Proof. Note that

$$\begin{aligned} \nabla_g u(\mathcal{H}(u)) &= \nabla_g u\langle \nabla_g u, \mathcal{H}\rangle_g = D^2 u(\mathcal{H}, \nabla_g u) + D\mathcal{H}(\nabla_g u, \nabla_g u) \\ &= D^2 u(\nabla_g u, \mathcal{H}) + D\mathcal{H}(\nabla_g u, \nabla_g u) \\ &= \frac{1}{2}\mathcal{H}(|\nabla_g u|_g^2) + D\mathcal{H}(\nabla_g u, \nabla_g u) \\ &= D\mathcal{H}(\nabla_g u, \nabla_g u) + \frac{1}{2}\operatorname{div}(|\nabla_g u|_g^2 \mathcal{H}) - \frac{1}{2}|\nabla_g u|_g^2 \operatorname{div}\mathcal{H}. \end{aligned} \tag{2.4}$$

Hence, we have

$$\begin{aligned} (\operatorname{div} A(x)\nabla u)\mathcal{H}(u) &= \operatorname{div}(\mathcal{H}(u)\nabla_g u) - \nabla_g u(\mathcal{H}(u)) \\ &= \operatorname{div}(\mathcal{H}(u)\nabla_g u) - D\mathcal{H}(\nabla_g u, \nabla_g u) - \frac{1}{2}\operatorname{div}(|\nabla_g u|_g^2 \mathcal{H}) \\ &\quad + \frac{1}{2}|\nabla_g u|_g^2 \operatorname{div}\mathcal{H}. \end{aligned} \tag{2.5}$$

We multiply the wave equation (2.1) by $\mathcal{H}(u)$ and integrate over $\Omega\times(0,T)$ to obtain

$$\begin{aligned} &(\operatorname{div} A(x)\nabla u)(\mathcal{H}(u)) \\ &= (u_{tt} + a(x)u_t + u + u^5)\mathcal{H}(u) \\ &= (u_t\mathcal{H}(u))_t - \frac{1}{2}\mathcal{H}(u_t^2) + \frac{1}{2}\mathcal{H}(u^2) + \frac{1}{6}\mathcal{H}(u^6) + a(x)u_t\mathcal{H}(u) \\ &= (u_t\mathcal{H}(u))_t - \frac{1}{2}\operatorname{div}(u_t^2\mathcal{H}) + \frac{1}{2}u_t^2\operatorname{div}\mathcal{H} + \frac{1}{2}\operatorname{div}(u^2\mathcal{H}) \\ &\quad - \frac{1}{2}u^2\operatorname{div}\mathcal{H} + \frac{1}{6}\operatorname{div}(u^6\mathcal{H}) - \frac{1}{6}u^6\operatorname{div}\mathcal{H} + a(x)u_t\mathcal{H}(u). \end{aligned} \tag{2.6}$$

From this and (2.5), the equality (2.2) follows from Green's formula.

Similarly, we multiply the wave equation (2.1) by Pu and integrate over $\Omega\times(0,T)$. Note that

$$\begin{aligned} 0 &= (u_{tt} - \operatorname{div} A(x)\nabla u + a(x)u_t + u + u^5)Pu \\ &= (u_t Pu)_t - Pu_t^2 - \operatorname{div}(Pu\nabla_g u) + P|\nabla_g u|_g^2 \\ &\quad + \frac{1}{2}\nabla_g P(u^2) + Pu^2 + Pu^6 + Pa(x)uu_t \\ &= (u_t Pu)_t - Pu_t^2 - \operatorname{div}(Pu\nabla_g u) + P|\nabla_g u|_g^2 \\ &\quad + \frac{1}{2}\operatorname{div}(u^2\nabla_g P) - \frac{1}{2}u^2\operatorname{div} A(x)\nabla P + Pu^2 + Pu^6 + \frac{1}{2}(Pa(x)u^2)_t\,. \end{aligned} \tag{2.7}$$

Then equality (2.3) follows from Green's formula. $\square$

Lemma 2.2. *Let $u(x,t)$ be a solution of* (1.1). *Then*

$$E(t)\Big|_0^T = -\int_0^T \int_{\mathbb{R}^3} a(x) u_t^2 \, dx \, dt, \tag{2.8}$$

which implies $E(t)$ is decreasing.

Proof. Multiply the first equation in (1.1) by u_t and integrate over $\mathbb{R}^3 \times (0,T)$, the equality (2.8) holds immediately. □

3. Unique continuation

In this section, we prove two unique continuation results, which are crucial for the compactness-uniqueness arguments.

Lemma 3.1. *There exists a constant $C > 0$ such that*

$$\int_{\mathbb{R}^3} \frac{w^2}{r^2} dx \le C \int_{\mathbb{R}^3} |\nabla w|^2 \, dx \tag{3.1}$$

for all $w \in H^1(\mathbb{R}^3)$.

Proof. Note that

$$\operatorname{div}\Big(\frac{w^2}{r}\frac{\partial}{\partial r}\Big) = w^2 \operatorname{div}\Big(\frac{1}{r}\frac{\partial}{\partial r}\Big) + \frac{2}{r} w w_r = \frac{1}{r^2} w^2 + \frac{2}{r} w w_r. \tag{3.2}$$

Integrating (3.2) over $\mathbb{R}^3$ yields

$$\int_{\mathbb{R}^3} \frac{1}{r^2} w^2 dx = -\int_{\mathbb{R}^3} \frac{2}{r} w w_r dx, \tag{3.3}$$

which implies (3.1). □

Lemma 3.2. *Let E_0 be a positive constant. Assume that $E(0) \le E_0$ and*

$$f(u) = u_t^2 + u^2 + |\nabla_g u|_g^2 + u^6.$$

Then

$$\liminf_{y\to\infty} \int_{|x|=y} r f(u) d\Gamma = 0. \tag{3.4}$$

Proof. Suppose that (3.4) is not true. Then there exist positive constants M and β such that

$$\int_{|x|=y} f(u) d\Gamma \ge \frac{\beta}{y}, \quad y \ge M. \tag{3.5}$$

Note that

$$\begin{aligned}
\int_{\mathbb{R}^3} f(u) dx &= \int_0^\infty \int_{|x|=y} f(u) d\Gamma dy \\
&= \Big(\int_0^M + \int_M^\infty\Big) \int_{|x|=y} f(u) d\Gamma dy \\
&\ge \int_0^M \int_{|x|=y} f(u) d\Gamma dy + \int_M^\infty \frac{\beta}{y} dy = +\infty,
\end{aligned} \tag{3.6}$$

which contradicts

$$\int_{\mathbb{R}^3} (u_t^2 + u^2 + |\nabla_g u|_g^2 + u^6) dx \le 6E_0 < +\infty. \tag{3.7}$$

□

Proposition 3.3. *Let* (A1), (A2) *hold and let $R_0 > 0$ be the constant given in Theorem 1.2. Then there exists a constant $T_0 > 0$ such that for any $T > T_0$, the only solution $(u, u_t) \in C([0,T], H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3))$ to the system*

$$\begin{gathered} u_{tt} - \operatorname{div} A(x)\nabla u + u + u^5 = 0, \quad (x,t) \in \mathbb{R}^3 \times (0,T), \\ u_t = 0, \quad (x,t) \in (\mathbb{R}^3 \backslash B(R_0)) \times (0,T), \end{gathered} \tag{3.8}$$

is $u \equiv 0$.

Proof. Letting $a(x) \equiv 0$, it follows from (2.8) that

$$E(t) = E(0), \quad \forall\, t \geq 0. \tag{3.9}$$

Let $\phi \in C^\infty(\mathbb{R}^3)$ be a nonnegative cut-off function such that

$$\phi = 1, \quad x \in \mathbb{R}^3 \backslash B(R_0 + 1) \quad \text{and} \quad \phi = 0, \quad x \in B(R_0). \tag{3.10}$$

Let $\Omega = B(y)$ with a radius $y > 0$, $\mathcal{H} = x$, and $P \in C^2(\mathbb{R}^3) \cap W^{1,\infty}(\mathbb{R}^3)$. Notice that $x = r\frac{\partial}{\partial r}$, it follows from (3.4) that

$$\begin{aligned} &\liminf_{y\to\infty} \int_{\partial B(y)} [\langle \nabla_g u, \nu\rangle \mathcal{H}(u) + \frac{1}{2}(u_t^2 - |\nabla_g u|_g^2 - u^2 - \frac{1}{3}u^6)\langle \mathcal{H}, \nu\rangle] d\Gamma \\ &\leq \liminf_{y\to\infty} \int_{|x|=y} r(u_t^2 + |\nabla_g u|_g^2 + u^2 + u^6) d\Gamma = 0, \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} &\liminf_{y\to\infty} \int_{\partial B(y)} [\frac{1}{2}u^2 \langle \nabla_g P, \nu\rangle - Pu\langle \nabla_g u, \nu\rangle] d\Gamma \\ &\leq ||P||_{W^{1,\infty}(\mathbb{R}^3)} \liminf_{y\to\infty} \int_{|x|=y} [u^2 + (\frac{1}{r}u^2 + r|\nabla_g u|_g^2)] d\Gamma = 0. \end{aligned} \tag{3.12}$$

Let $P = \phi$, $a(x) \equiv 0$ and $\Omega = B(y)$ in (2.3). Let $y \to +\infty$, it follows from (2.3) and (3.12) that

$$\int_0^T \int_{\mathbb{R}^3} (|\nabla_g u|_g^2 + u^2 + u^6) P \,dx\,dt \leq CE(0) + C \int_0^T \int_{B(R_0+1)} u^2 \,dx\,dt. \tag{3.13}$$

From (3.1), we have

$$\int_0^T \int_{B(R_0+1)} u^2 \,dx\,dt \leq C(R_0) \int_0^T \int_{\mathbb{R}^3} |\nabla_g u|_g^2 \,dx\,dt. \tag{3.14}$$

Thus, we have

$$\int_0^T \int_{\mathbb{R}^3} u^2 \,dx\,dt \leq CE(0) + C(R_0) \int_0^T \int_{\mathbb{R}^3} |\nabla_g u|_g^2 \,dx\,dt. \tag{3.15}$$

Let $\mathcal{H}=x$, $a(x)\equiv 0$ and $\Omega=B(y)$ in (2.2). Let $y\to+\infty$, it follows from (2.2), (5.6), and (3.11) that

$$\begin{aligned}
0&=\int_{\mathbb{R}^3} u_t\mathcal{H}(u)\,dx\Big|_0^T+\int_0^T\int_{\mathbb{R}^3} D\mathcal{H}(\nabla_g u,\nabla_g u)\,dx\,dt+\int_0^T\int_{\mathbb{R}^3} a(x)u_t\mathcal{H}(u)\,dx\,dt\\
&\quad+\frac{3}{2}\int_0^T\int_{\mathbb{R}^3}(u_t^2-|\nabla_g u|_g^2-u^2-\frac{1}{3}u^6)\,dx\,dt\\
&\geq\int_{\mathbb{R}^3} u_t\mathcal{H}(u)\,dx\Big|_0^T+\delta\int_0^T\int_{\mathbb{R}^3}|\nabla_g u|_g^2\,dx\,dt+\int_0^T\int_{\mathbb{R}^3} a(x)u_t\mathcal{H}(u)\,dx\,dt\\
&\quad+\frac{3}{2}\int_0^T\int_{\mathbb{R}^3}(u_t^2-|\nabla_g u|_g^2-u^2-u^6)\,dx\,dt+\int_0^T\int_{\mathbb{R}^3}u^6\,dx\,dt.
\end{aligned}\tag{3.16}$$

Again let $\Omega=B(y)$ and $a(x)=0$ in (2.3). Combining (2.3) with (3.16) and letting $y\to+\infty$, we obtain

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^3}\big[(\frac{3}{2}-P)u_t^2+(P-\frac{3}{2}+\delta)|\nabla_g u|_g^2+(P-\frac{3}{2})u^2+(P-\frac{1}{2})u^6\big]\,dx\,dt\\
&\leq-\int_{\mathbb{R}^3}[Puu_t+u_t\mathcal{H}(u)]\,dx\Big|_0^T+\frac{1}{2}\int_0^T\int_{\mathbb{R}^3}u^2\operatorname{div}(A(x)\nabla P)\,dx\,dt.
\end{aligned}\tag{3.17}$$

We denote

$$\delta_c=\frac{\delta}{1+C(R_0)}<1,\tag{3.18}$$

where $C(R_0)$ is given by (3.15).

Taking $P=\frac{3-\delta_c}{2}$, $a(x)\equiv 0$ in (3.17), we have

$$\int_0^T\int_{\mathbb{R}^3}\big[\frac{1}{2}\delta_c u_t^2+\delta_1|\nabla_g u|_g^2-\frac{1}{2}\delta_c u^2+\delta_2 u^6\big]\,dx\,dt\leq CE(0),\tag{3.19}$$

where $\delta_1=\delta-\frac{1}{2}\delta_c=\frac{\delta(1+2C(R_0))}{2(1+C(R_0))}$ and $\delta_2=1-\frac{1}{2}\delta_c=\frac{2(1+C(R_0))-\delta}{2(1+C(R_0))}>0$. On the other hand, by (3.15), for $\delta_0>0$, we have

$$\frac{\delta(1+\delta_0)}{2(1+C(R_0))}\int_0^T\int_{\mathbb{R}^3}u^2\,dx\,dt\leq CE(0)+\frac{\delta(1+\delta_0)C(R_0)}{2(1+C(R_0))}\int_0^T\int_{\mathbb{R}^3}|\nabla_g u|_g^2\,dx\,dt.$$

Taking $\delta_0=1$, we have

$$\frac{\delta(1+\delta_0)}{2(1+C(R_0))}-\frac{1}{2}\delta_c=\frac{1}{2}\delta_c\quad\text{and}\quad\delta_1-\frac{\delta(1+\delta_0)C(R_0)}{2(1+C(R_0))}=\frac{1}{2}\delta_c.\tag{3.20}$$

Thus, with (3.17)-(3.20), we conclude that

$$\int_0^T E(t)dt\leq CE(0),\tag{3.21}$$

which implies $(T-C)E(0)\leq 0$. Therefore, the assertion (3.8) holds and the proof is complete. □

The following proposition has a similar proof the one above.

Proposition 3.4. *Let* (A1), (A2) *hold and let $R_0 > 0$ be the constant given in Theorem 1.2. Then there exists a constant $T_0 > 0$ such that for any $T > T_0$, the only solution $(u, u_t) \in C([0,T], H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3))$ to the system*

$$\begin{gathered} u_{tt} - \operatorname{div} A(x)\nabla u + u = 0, \quad (x,t) \in \mathbb{R}^3 \times (0,T), \\ u_t = 0, \quad (x,t) \in (\mathbb{R}^3 \backslash B(R_0)) \times (0,T), \end{gathered} \tag{3.22}$$

is $u \equiv 0$.

4. Proofs of the main theorem

Lemma 4.1. *Let* (A1), (A2) *hold, and $u(x,t)$ solve system* (1.1)*. then*

$$E(0) \le C \int_0^T \int_{\mathbb{R}^3} a(x) u_t^2 \, dx \, dt + C \int_0^T \int_{B(R_0)} u^2 \, dx \, dt \tag{4.1}$$

holds for sufficiently large T.

Proof. Recall that $a(x) \ge a_0$ for $x \in \mathbb{R}^3 \backslash B(R_0)$, then there exists a small constant $\varepsilon_0 > 0$ such that

$$a(x) \ge \frac{a_0}{2}, \quad x \in \mathbb{R}^3 \backslash B(R_0 - 2\varepsilon_0). \tag{4.2}$$

Let $b(z)$ be a smooth nonnegative function on $[0, +\infty)$ satisfying

$$b(z) = 1, \quad 0 \le z \le R_0 - \varepsilon_0, \quad b(z) = 0, \quad z \ge R_0. \tag{4.3}$$

Let $H(x)$ be a vector field on $B(R_0)$ satisfying

$$H(x) = b(r)x, \quad x \in B(R_0).$$

It follows from (5.6) that

$$\begin{gathered} DH(X,X) \ge \delta |X|_g^2 \quad \text{for } X \in \mathbb{R}^3_x, \ x \in B(R_0 - \varepsilon_0), \\ \operatorname{div} H = 3 \quad \text{for } x \in B(R_0 - \varepsilon_0). \end{gathered} \tag{4.4}$$

Let $\mathcal{H} = H$ and $\Omega = B(R_0)$ in (2.2). Then

$$\begin{aligned} 0 \ge & \int_\Omega u_t H(u) \, dx \Big|_0^T + \delta \int_0^T \int_{B(R_0-\varepsilon_0)} |\nabla_g u|_g^2 \, dx \, dt \\ & - C \int_0^T \int_{B(R_0) \backslash B(R_0 - \varepsilon_0)} |\nabla_g u|_g^2 \, dx \, dt + \int_0^T \int_{B(R_0)} a(x) u_t H(u) \, dx \, dt \\ & + \frac{1}{2} \int_0^T \int_{B(R_0)} (u_t^2 - |\nabla_g u|_g^2 - u^2 - \frac{1}{3} u^6) \operatorname{div} H \, dx \, dt \\ = & \int_\Omega u_t H(u) \, dx \Big|_0^T + \delta \int_0^T \int_{B(R_0-\varepsilon_0)} |\nabla_g u|_g^2 \, dx \, dt \\ & - C \int_0^T \int_{B(R_0) \backslash B(R_0 - \varepsilon_0)} |\nabla_g u|_g^2 \, dx \, dt + \int_0^T \int_{B(R_0)} a(x) u_t H(u) \, dx \, dt \\ & + \int_0^T \int_{B(R_0)} [\frac{1}{3} u^6 + \frac{1}{2} (u_t^2 - |\nabla_g u|_g^2 - u^2 - u^6)] \operatorname{div} H \, dx \, dt. \end{aligned} \tag{4.5}$$

Let $P = (\operatorname{div} H - b(r)\delta)/2$ and $\Omega = B(R_0)$ in (2.3). Substituting (2.3) into (4.5), we obtain

$$\begin{aligned}
&\int_{B(R_0)} u_t(H(u) + Pu)\,dx\Big|_0^T - \frac{1}{2}\int_0^T\int_{B(R_0)} u^2(\operatorname{div} A(x)\nabla P)\,dx\,dt \\
&+ \frac{1}{2}\int_{B(R_0)} a(x)Pu^2\,dx\Big|_0^T + \int_0^T\int_{B(R_0)} a(x)u_tH(u)\,dx\,dt \\
&+ \frac{\delta}{2}\int_0^T\int_{B(R_0-\varepsilon_0)} (u_t^2 + |\nabla_g u|_g^2 + u^2 + u^6)dx \\
&\le C\int_0^T\int_{B(R_0)\setminus B(R_0-\varepsilon_0)} (u^2 + |\nabla_g u|_g^2 + u^6)\,dx\,dt \\
&+ C\int_0^T\int_{B(R_0-\varepsilon_0)} u^2\,dx\,dt + C\int_0^T\int_{B(R_0)} a(x)u_t^2\,dx\,dt.
\end{aligned} \tag{4.6}$$

Therefore,

$$\begin{aligned}
&\int_0^T\int_{B(R_0-\varepsilon_0)} \Big(u_t^2 + |\nabla_g u|_g^2 + u^2 + \frac{1}{3}u^6\Big)\,dx\,dt \\
&\le C(E(0) + E(T)) + \int_0^T\int_{B(R_0)} a(x)(C_\epsilon u_t^2 + \epsilon|\nabla_g u|_g^2)\,dx\,dt \\
&+ C\int_0^T\int_{B(R_0)\setminus B(R_0-\varepsilon_0)} (u^2 + |\nabla_g u|_g^2 + u^6)\,dx\,dt \\
&+ C\int_0^T\int_{B(R_0-\varepsilon_0)} u^2\,dx\,dt + C\int_0^T\int_{B(R_0)} a(x)u_t^2\,dx\,dt.
\end{aligned} \tag{4.7}$$

Taking ϵ sufficiently small, we have

$$\begin{aligned}
&\int_0^T\int_{B(R_0-\varepsilon_0)} \Big(u_t^2 + |\nabla_g u|_g^2 + u^2 + \frac{1}{3}u^6\Big)\,dx\,dt \\
&\le C(E(0) + E(T)) + C\int_0^T\int_{B(R_0)} a(x)u_t^2\,dx\,dt + C\int_0^T\int_{B(R_0-\varepsilon_0)} u^2\,dx\,dt \\
&+ C\int_0^T\int_{B(R_0)\setminus B(R_0-\varepsilon_0)} (u^2 + |\nabla_g u|_g^2 + u^6)\,dx\,dt.
\end{aligned} \tag{4.8}$$

Therefore,

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^3} \Big(u_t^2 + |\nabla_g u|_g^2 + u^2 + \frac{1}{3}u^6\Big)\,dx\,dt \\
&\le C(E(0) + E(T)) + C\int_0^T\int_{\mathbb{R}^3} a(x)u_t^2\,dx\,dt \\
&+ C\int_0^T\int_{\mathbb{R}^3\setminus B(R_0-\varepsilon_0)} (u^2 + |\nabla_g u|_g^2 + u^6)\,dx\,dt + C\int_0^T\int_{B(R_0-\varepsilon_0)} u^2\,dx\,dt.
\end{aligned} \tag{4.9}$$

Let $w(z)$ be a smooth nonnegative function on $[0, +\infty)$ satisfying

$$w(z) = 0, \quad 0 \le z \le R_0 - 2\varepsilon_0 \quad \text{and} \quad w(z) = 1, \quad z \ge R_0 - \varepsilon_0.$$

Let $P = w(r)$ and $\Omega = B(y)$ in (2.3). Let $y \to +\infty$, it follows from (2.3) and Lemma 3.2 that

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^3} (u_t^2 - |\nabla_g u|_g^2 - u^2 - u^6) P \, dx \, dt \\ &= (u_t, uP)\Big|_0^T - \frac{1}{2} \int_0^T \int_{\mathbb{R}^3} u^2 \operatorname{div} A(x) \nabla P \, dx \, dt + \frac{1}{2} \int_{\mathbb{R}^3} a(x) P u^2 \, dx \Big|_0^T. \end{aligned} \tag{4.10}$$

From (4.2), we obtain

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^3} (|\nabla_g u|_g^2 + u^2 + u^6) P \, dx \, dt \\ &\le C(E(0) + E(T)) + C \int_0^T \int_{\mathbb{R}^3} a(x) u_t^2 \, dx \, dt \\ &\quad + C \int_0^T \int_{B(R_0 - \varepsilon_0) \setminus B(R_0 - 2\varepsilon_0)} u^2 \, dx \, dt. \end{aligned} \tag{4.11}$$

Substituting (4.11) into (4.9) yields

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^3} (u_t^2 + u^2 + |\nabla_g u|_g^2 + \frac{1}{3} u^6) \, dx \, dt \\ &\le C(E(0) + E(T)) + C \int_0^T \int_{\mathbb{R}^3} a(x) u_t^2 \, dx \, dt + C \int_0^T \int_{B(R_0 - \varepsilon_0)} u^2 \, dx \, dt. \end{aligned} \tag{4.12}$$

With (2.8), we deduce that

$$CE(T) = CE(0) - C \int_0^T \int_{\mathbb{R}^3} a(x) u_t^2 \, dx \, dt, \tag{4.13}$$

and

$$\begin{aligned} 4CE(0) &= \int_0^{4C} E(t) dt - \int_0^{4C} (E(t) - E(0)) dt \\ &\le \int_0^{4C} E(t) dt + 4C \int_0^{4C} \int_{\mathbb{R}^3} a(x) u_t^2 \, dx \, dt. \end{aligned} \tag{4.14}$$

Inserting (4.13) and (4.14) into (4.12), taking $T > 4C$, we have

$$E(0) \le C \int_0^T \int_{\mathbb{R}^3} a(x) u_t^2 \, dx \, dt + C \int_0^T \int_{B(R_0 - \varepsilon_0)} u^2 \, dx \, dt. \tag{4.15}$$

The proof is complete. $\square$

Lemma 4.2 (Observability inequality)**.** *Let* (A1), (A2) *hold. Let* $u(x,t)$ *solve system* (1.1). *Then for any* $E(0) \le E_0 < \infty$,

$$E(0) \le C(E_0, T) \int_0^T \int_{\mathbb{R}^3} a(x) u_t^2 \, dx \, dt, \tag{4.16}$$

for sufficiently large T.

Proof. We apply the compactness-uniqueness arguments to prove the conclusion. It follows from (4.1) that

$$E(0) \le C \int_0^T \int_{\mathbb{R}^3} a(x) u_t^2 \, dx \, dt + C \int_0^T \int_{B(R_0)} u^2 \, dx \, dt. \tag{4.17}$$

By contradiction. Suppose that estimate (4.16) does not hold, then there exists a sequence $\{u_k\}_{k=1}^{\infty}$ such that

$$E_k(0) \le E_0, \tag{4.18}$$

where

$$E_k(t) = \frac{1}{2}\int_{\mathbb{R}^3}(u_{kt}^2 + u_k^2 + |\nabla_g u_k|_g^2)dx + \frac{1}{6}\int_{\mathbb{R}^3} u_k^6 dx,$$

and

$$\int_0^T \int_{B(R_0)} u_k^2 \, dx \, dt \ge k \int_0^T \int_{\mathbb{R}^3} a(x) u_{kt}^2 \, dx \, dt. \tag{4.19}$$

From (2.8), we have

$$E_k(t) \le E_0, \quad 0 \le t \le T, \tag{4.20}$$

and

$$\int_0^T E_k(t) dt \le T E_0.$$

Therefore, there exists $\hat{u}$ and a subset of $\{u_k\}_{k=1}^{\infty}$, still denoted by $\{u_k\}_{k=1}^{\infty}$, such that

$$u_k \to \hat{u} \quad \text{weakly in } H^1(\mathbb{R}^3 \times (0,T)), \tag{4.21}$$

$$u_k \to \hat{u} \quad \text{strongly in } L^2(B(R_0) \times (0,T)), \tag{4.22}$$

Case a:

$$\int_0^T \int_{B(R_0)} \hat{u}^2 \, dx \, dt > 0. \tag{4.23}$$

Note that $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ and $L^6(\mathbb{R}^3)$ is the dual space of $L^{6/5}(\mathbb{R}^3)$. It follows from (4.20) that

$$\{u_k^5\} \quad \text{is bounded in } L^\infty([0,T], L^{6/5}(\mathbb{R}^3)). \tag{4.24}$$

Then

$$\{u_k^5\} \quad \text{is bounded in } L^{6/5}(\mathbb{R}^3 \times (0,T)), \tag{4.25}$$

which implies

$$u_k^5 \to \hat{u}^5 \quad \text{weakly in } L^{6/5}(\mathbb{R}^3 \times (0,T)). \tag{4.26}$$

It follows from (4.19) that

$$a(x)\hat{u}_t = 0, \quad (x,t) \in \mathbb{R}^3 \times (0,T).$$

Therefore, with (4.21) and (4.26), we obtain

$$\begin{gathered} \hat{u}_{tt} - \operatorname{div} A(x)\nabla \hat{u} + \hat{u} + \hat{u}^5 = 0, \quad (x,t) \in \mathbb{R}^3 \times (0,T), \\ \hat{u}_t = 0, \quad (x,t) \in (\mathbb{R}^3 \backslash B(R_0)) \times (0,T). \end{gathered} \tag{4.27}$$

It follows from Proposition 3.3 that

$$\hat{u}(x,t) \equiv 0, \quad (x,t) \in \mathbb{R}^3 \times (0,T), \tag{4.28}$$

which contradicts (4.23).

Case b:

$$\hat{u}(x,t) \equiv 0 \qquad (x,t) \in B(R_0) \times (0,T). \tag{4.29}$$

We denote

$$v_k = u_k/\sqrt{c_k} \quad \text{for } k \ge 1, \tag{4.30}$$

where

$$c_k = \int_0^T \int_{B(R_0)} u_k^2 \, dx \, dt. \tag{4.31}$$

Then v_k satisfies

$$v_{ktt} - \operatorname{div} A(x)\nabla v_k + a(x)v_{kt} + v_k + u_k^4 v_k = 0, \quad (x,t) \in \mathbb{R}^3 \times (0,T), \tag{4.32}$$

$$\int_0^T \int_{B(R_0)} v_k^2 \, dx \, dt = 1. \tag{4.33}$$

It follows from (4.19) that

$$1 \geq k \int_0^T \int_{\mathbb{R}^3} a(x) v_{kt}^2 \, dx \, dt. \tag{4.34}$$

From this and (4.17), we have

$$\widehat{E}_k(0) \leq 1 + \frac{1}{k} \leq 2, \tag{4.35}$$

where

$$\widehat{E}_k(t) = \frac{1}{2} \int_{\mathbb{R}^3} (v_{kt}^2 + v_k^2 + |\nabla_g v_k|_g^2) dx + \frac{1}{6} \int_{\mathbb{R}^3} u_k^4 v_k^2 dx.$$

Hence, there exists a $\hat{v}$ and a subsequence of $\{v_k\}_{k=1}^{\infty}$, still denoted by $\{v_k\}_{k=1}^{\infty}$, such that

$$\begin{gathered} v_k \to \hat{v} \quad \text{weakly in } H^1(\mathbb{R}^3 \times (0,T)), \\ v_k \to \hat{v} \quad \text{strongly in } L^2(B(R_0) \times (0,T)). \end{gathered} \tag{4.36}$$

Collecting (2.8), (4.30), and (4.35), we obtain

$$\widehat{E}_k(t) \leq \widehat{E}_k(0) \leq 2, \quad \forall 0 \leq t \leq T. \tag{4.37}$$

Notice that $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$. Therefore $\{v_k\}$ are bounded in $L^\infty([0,T], L^6(\mathbb{R}^3))$. Hence, we have

$$\int_0^T \int_{\mathbb{R}^3} \left| u_k^4 v_k \right|^{6/5} dx \, dt = c_k^{12/5} \int_0^T \int_{\mathbb{R}^3} v_k^6 \, dx \, dt \leq c_k^{12/5} C(T). \tag{4.38}$$

We combine (4.29) with (4.31) to obtain

$$\lim_{k \to +\infty} \int_0^T \int_{\mathbb{R}^3} \left| u_k^4 v_k \right|^{6/5} dx \, dt = 0. \tag{4.39}$$

By (4.34) and (4.36), we have

$$a(x)\hat{v}_t = 0, \quad (x,t) \in \mathbb{R}^3 \times (0,T).$$

Therefore, from (4.32) and (4.39) it follows that

$$\begin{gathered} \hat{v}_{tt} - \operatorname{div} A(x)\nabla \hat{v} + \hat{v} = 0, \quad (x,t) \in \mathbb{R}^3 \times (0,T), \\ \hat{v}_t = 0, \quad (x,t) \in (\mathbb{R}^3 \backslash B(R_0)) \times (0,T). \end{gathered} \tag{4.40}$$

The following holds by Proposition 3.4,

$$\hat{v} \equiv 0, \quad (x,t) \in \mathbb{R}^3 \times (0,T). \tag{4.41}$$

Then it follows from (4.33) that

$$\int_0^T \int_{B(R_0)} \hat{v}^2 \, dx \, dt = 1, \tag{4.42}$$

which contradicts (4.41). The proof is complete. $\square$

Proof of Theorem 1.2. From (2.8) and (4.16), we obtain

$$E(0) \le C(E_0, T)(E(0) - E(T)).$$

Then

$$E(T) \le \frac{C(E_0, T) - 1}{C(E_0, T)} E(0),$$

which implies $E(t)$ is of exponential decay. □

5. Appendix: Comments on assumption (A1)

As an example, (A1) is satisfied by the function $A(x) = \mathrm{diag}\{\alpha_1(x), \alpha_2(x), \alpha_3(x)\}$, where $\alpha_i(x)$ are all smooth positive functions on $\mathbb{R}^3$, for $1 \le i \le 3$. Assume that, for $1 \le i \le 3$,

$$0 < m_1 \le \alpha_i(x) \le m_2 < +\infty, \quad x \in \mathbb{R}^3, \tag{5.1}$$

$$(1-\delta)\alpha_i(x) - \frac{r(x)}{2} \frac{\partial \alpha_i(x)}{\partial r} \ge 0, \quad x \in \mathbb{R}^3. \tag{5.2}$$

Then

$$m_1 |X|^2 \le \langle A(x)X, X\rangle \le m_2 |X|^2, \quad X \in \mathbb{R}^3_x,\ x \in \mathbb{R}^3, \tag{5.3}$$

$$\big\langle \big((1-\delta)A(x) - \frac{r(x)}{2} \frac{\partial A(x)}{\partial r}\big) X, X\big\rangle \ge 0, \quad X \in \mathbb{R}^3_x,\ x \in \mathbb{R}^3. \tag{5.4}$$

It is easy to see that the standard unit matrix, $I_3 = (\delta_{ij})_{1\le i,j\le 3}$, satisfies (1.8). Another example satisfying (5.2) is $\alpha_i(x) = e^{-r^2}$.

In the following, let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a smooth boundary $\partial\Omega$, we will show the relationship between (A1) and the geometric control condition (GCC). The proof is similar to the one for [31].

Proposition 5.1. *Let $H(x) = x$. Then*

$$DH(X, X) = \big\langle \big(G(x) + \frac{r(x)}{2} \frac{\partial G(x)}{\partial r}\big) X, X\big\rangle, \quad X \in \mathbb{R}^3_x,\ x \in \mathbb{R}^3. \tag{5.5}$$

Proof. Let $x \in \mathbb{R}^3$, $X = \sum_{i=1}^3 X_i \frac{\partial}{\partial x_i} \in \mathbb{R}^3_x$. Note that

$$H(x) = \sum_{i=1}^3 x_i \frac{\partial}{\partial x_i}.$$

Then

$$\begin{aligned}
DH(X, X) &= \sum_{i,j,k=1}^3 \big\langle D_{\frac{\partial}{\partial x_i}} \big(x_k \frac{\partial}{\partial x_k}\big), \frac{\partial}{\partial x_j}\big\rangle_g X_i X_j \\
&= \sum_{i,j=1}^3 g_{ij} X_i X_j + \sum_{i,j,k=1}^3 x_k \big\langle D_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_j}\big\rangle_g X_i X_j \\
&= |X|_g^2 + \sum_{i,j,k=1}^3 x_k \big\langle D_{\frac{\partial}{\partial x_k}} \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\big\rangle_g X_i X_j \\
&= |X|_g^2 + \sum_{i,j,k=1}^3 \frac{x_k}{2} \frac{\partial g_{ij}}{\partial x_k} X_i X_j = \big\langle \big(G(x) + \frac{r(x)}{2} \frac{\partial G(x)}{\partial r}\big) X, X\big\rangle. \quad \square
\end{aligned}$$

Proposition 5.2. *Let* (A1) *hold and let $H(x) = x$. Then*

$$DH(X,X) \geq \delta |X|_g^2, \quad X \in \mathbb{R}_x^3,\ x \in \mathbb{R}^3. \tag{5.6}$$

Proof. Let $x \in \mathbb{R}^3$, $X, Y \in \mathbb{R}_x^3$ and $Y = G(x)X$. We deduce that

$$\begin{aligned} 0 &\leq Y^T\big((1-\delta)A(x) - \frac{r}{2}\frac{\partial A(x)}{\partial r}\big)Y \\ &= \langle G(x)\big((1-\delta)A(x) - \frac{r}{2}\frac{\partial A(x)}{\partial r}\big)G(x)X, X\rangle \\ &= \big\langle\big((1-\delta)G(x) + \frac{r}{2}\frac{\partial(G(x))}{\partial r}\big)X, X\big\rangle. \end{aligned} \tag{5.7}$$

Inequality (5.6) follows from (5.5). □

Proposition 5.3. *Let* (A1) *hold. Then, for any $x \in \Omega$ and any unit-speed geodesic $\gamma(t)$ starting from x, if $\gamma(t) \in \Omega$ for $0 \leq t \leq t_0$, then*

$$t_0 \leq \frac{2}{\delta}\sup\{|H|_g(x) : x \in \overline{\Omega}\}.$$

Proof. Note that $|\gamma'(t)|_g = 1$ and $D_{\gamma'(t)}\gamma'(t) = 0$. From (5.6), we deduce that

$$\langle H, \gamma'(t)\rangle_g\big|_0^{t_0} = \int_0^{t_0} \gamma'(t)\langle H, \gamma'(t)\rangle_g dt = \int_0^{t_0} DH(\gamma'(t), \gamma'(t))dt \geq \delta t_0. \tag{5.8}$$

The proof is complete. □

Let $S(r)$ be the sphere in $\mathbb{R}^3$ with a radius r. Then

$$\langle X, \frac{\partial}{\partial r}\rangle = 0, \quad \text{for } X \in S(r)_x,\ x \in \mathbb{R}^3 \backslash O,$$

where $S(r)_x$ is the tangential space of $S(r)$ at x. The following lemma shows that GCC may not hold if $A(x)$ satisfies (5.9) and (5.10) below.

Proposition 5.4. *Assume that*

$$A(x)\frac{\partial}{\partial r} = \frac{\partial}{\partial r}, \quad x \in \mathbb{R}^3, \tag{5.9}$$

$$\langle (A(x) - \frac{r}{2}\frac{\partial A(x)}{\partial r})X, X\rangle = 0 \quad \text{for } X \in S(R_1)_x, \quad |x| = R_1. \tag{5.10}$$

where R_1 is a positive constant. Then, for any $x \in S(R_1)$ and any unit-speed geodesic $\gamma(t)$ starting from x with $\gamma'(0) \in S(R_1)_x$, we have

$$\gamma(t) \in S(R_1), \quad \forall\ t \geq 0.$$

Proof. Note that

$$G(x)\frac{\partial}{\partial r} = \frac{\partial}{\partial r}, \quad x \in \mathbb{R}^3.$$

Therefore,

$$D(r\frac{\partial}{\partial r}) = D(rDr) = Dr \otimes Dr + rD^2r.$$

By a proof similar to the one of Proposition 5.2, we obtain

$$D(rDr)(X,X) = 0, \quad X \in S(R_1)_x,\ |x| = R_1.$$

Then

$$D^2r(X,X) = 0, \quad X \in S(R_1)_x, \quad |x| = R_1.$$

Let $\widehat{g}$ be a Riemannian metric induced by g in $S(R_1)$ and $\widehat{D}$ be the associated Levi-Civita connection. Let $\widehat{\gamma}(t)$ be a unit-speed geodesic of $(S(R_1), \widehat{g})$ starting from $x \in S(R_1)$, then

$$\langle \widehat{\gamma}'(t), \frac{\partial}{\partial r} \rangle_g = 0, \quad \widehat{D}_{\widehat{\gamma}'(t)} \widehat{\gamma}'(t) = 0, \quad \forall t \geq 0.$$

Therefore,

$$\begin{aligned} D_{\widehat{\gamma}'(t)} \widehat{\gamma}'(t) &= \widehat{D}_{\widehat{\gamma}'(t)} \widehat{\gamma}'(t) + \langle D_{\widehat{\gamma}'(t)} \widehat{\gamma}'(t), \frac{\partial}{\partial r} \rangle_g \frac{\partial}{\partial r} \\ &= \widehat{D}_{\widehat{\gamma}'(t)} \widehat{\gamma}'(t) - D^2 r(\widehat{\gamma}'(t), \widehat{\gamma}'(t)) \frac{\partial}{\partial r} = 0, \end{aligned} \tag{5.11}$$

which implies $\widehat{\gamma}(t)$ is also a geodesic of $(\mathbb{R}^3, g)$. Then

$$\gamma(t) = \widehat{\gamma}(t) \in S(R_1), \quad \forall t \geq 0,$$

for unit-speed geodesic $\gamma(t)$ of $(\mathbb{R}^3, g)$ satisfying $\gamma(0) = \widehat{\gamma}(0)$ and $\gamma'(0) = \widehat{\gamma}'(0)$. The proof is complete. $\square$

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