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ASYMPTOTIC FORMULAS FOR q-REGULARLY VARYING SOLUTIONS OF HALF-LINEAR q-DIFFERENCE EQUATIONS

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ABSTRACT. This article studies the asymptotic behavior of positive solutions of the q-difference half-linear equation

$$D_q(p(t)\Phi(D_q(x(t)))) + r(t)\Phi(x(qt)) = 0, \quad t \in q^{\mathbb{N}_0} := \{q^n : n \in \mathbb{N}_0\},$$

where $q > 1$, $\Phi(x) = |x|^\alpha \operatorname{sgn} x$, $\alpha > 0$, $p : q^{\mathbb{N}_0} \to (0, \infty)$, $r : q^{\mathbb{N}_0} \to \mathbb{R}$, in the framework of q-regular variation. In particular, if r is eventually of one sign, p and $|r|$ are q-regularly varying functions such that $t^{\alpha+1}r(t)/p(t) \to 0$, as $t \to \infty$, we obtain asymptotic formulas for the q-regularly varying solutions. Moreover, when $p(t) \equiv 1$ and r is an eventually positive or eventually negative function, we obtain an asymptotic formula of a q-slowly varying solution. Using generalized regularly varying sequences, we apply these results to the half-linear difference equation case. At the end, we illustrate the obtained results with examples.

1. INTRODUCTION

This article studies the asymptotic behavior of q-regularly varying solutions of the half-linear q-difference equation

$$D_q(p(t)\Phi(D_q(x(t)))) + r(t)\Phi(x(qt)) = 0, \tag{1.1}$$

on the lattice $q^{\mathbb{N}_0} = \{q^n : n \in \mathbb{N}_0\}$, where $q > 1$, $\Phi(x) = |x|^\alpha \operatorname{sgn} x$, and $\alpha > 0$. To this end we use Karamata's theory which is a powerful tool in the study of regularly varying functions and the asymptotic properties of differential and difference equations.

The study of half-linear differential equations in the framework of regular variation started with papers [4, 5]. Namely determining necessary and sufficient conditions for the existence of regularly varying solutions in the case $p(t) \equiv 1$, and generalized regularly varying functions in the case p is positive, continuous function on $[a, \infty)$, for some $a \in \mathbb{R}$, with $r : [a, \infty) \to \mathbb{R}$ being continuous function, of the equation

$$(p(t)\Phi(x'(t)))' + r(t)\Phi(x(t)) = 0, \quad t \in [a, \infty). \tag{1.2}$$

For recent papers investigating asymptotic behavior of positive solutions of (1.2) see [13, 14, 15]. Once the existence of regularly varying solutions is proved, main

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investigation becomes the asymptotic behavior of these solutions. Results concerning asymptotic behavior of regularly varying solutions of half-linear differential equation (1.2) can be found in [8, 9, 13].

The results obtained in both the continuous and discrete case suggested investigating q-difference equations in the framework of q-regular variation. The theory of q-regularly varying functions has been applied in the asymptotic analysis of q-difference linear equations (see [11, 17, 19]), half-linear equations with $p(t) \equiv 1$ (see [16, 18]) and nonlinear equations (see [7]). Results concerning asymptotic formulas of q-regularly varying solutions, as far as we know, exist only for a linear equation (see [11]).

In [3], necessary and sufficient conditions for the existence of q-regularly varying solutions of the half-linear q-difference equation (1.1), with p being a positive, q-regularly varying function and with no sign condition on r, have been given. We state here the theorem proved in [3], which will be very useful throughout the paper, since it provides the existence of q-regularly varying solutions of certain indices, whose asymptotic behavior will be examined. This result was proved by using the Karamata's theory of regular variation and Banach fixed point theorem. Note that $\mathcal{RV}_q(\rho)$ denotes the set of all q-regularly varying function of index ρ and the symbol $[a]_q = \frac{q^a-1}{q-1}$, $a \in \mathbb{R}$ will be used throughout the paper. In Section 2 we recall the definition and some basic properties of q-Karamata functions and introduce notation that will be used through this paper.

Theorem 1.1 ([3, Theorems 3.1, 3.2]). *Let $p \in \mathcal{RV}_q(\lambda)$, $\lambda \neq \alpha$. Then* (1.1) *has eventually positive solutions*

$$x \in \mathcal{RV}_q(\rho_1) \text{ and } y \in \mathcal{RV}_q(\rho_2),$$

where ρ_1 and ρ_2 are such that $\lambda_1 = \Phi([\rho_1]_q)$ and $\lambda_2 = \Phi([\rho_2]_q)$ are real and different roots of the equation

$$h_q(x) - x + \frac{c}{[\alpha]_q} = 0, \tag{1.3}$$

if and only if

$$\lim_{t\to\infty} \frac{q^\alpha t^{\alpha+1} r(t)}{p(qt)} = c \in \Big(-\infty, \Big|\Big[\frac{\alpha-\lambda}{\alpha+1}\Big]_q\Big|^{\alpha+1}\Big), \tag{1.4}$$

where $h_q : \big(\Phi\big(\frac{1}{1-q}\big), \infty\big) \to \mathbb{R}$ is defined by

$$h_q(x) = \frac{x}{1-q^{-\alpha}}\Big(1 - q^{-\lambda}\big(1 + (q-1)\Phi^{-1}(x)\big)^{-\alpha}\Big).$$

To continue in this direction, our next goal is to establish asymptotic formulas for q-regularly varying solutions. Throughout this paper we will consider two approaches for establishing asymptotic formulas of q-regularly varying solutions of (1.1) in the case $c = 0$. First, in Section 3, we will consider equation (1.1) under the assumptions that coefficient p is a q-regularly varying function, i.e., $p \in \mathcal{RV}_q(\lambda)$, $\lambda \neq \alpha$ and r is a function of eventually one sign such that $|r| \in \mathcal{RV}_q(\lambda-\alpha-1)$. Under those assumptions, using the Karamata's integration theorem and reciprocity principle, asymptotic formulas for the existing q-regularly varying solutions of (1.1) will be established. Later, in Section 4, we will consider the special case of equation (1.1) with $p(t) \equiv 1$,

$$D_q(\Phi(D_q(x(t)))) + r(t)\Phi(x(qt)) = 0, \quad t \in q^{\mathbb{N}_0}, \tag{1.5}$$

but without assumption that $|r|$ is q-regularly varying function. Namely, under the assumption that r is eventually positive or eventually negative function, using the Riccati technique and Banach fixed point theorem, an asymptotic formula of a q-slowly varying solution will be given.

Results considering asymptotic behavior of positive solutions of the half-linear q-difference equation, will also give results about asymptotic behavior of some of the positive solutions of the half-linear difference equation, in the framework of generalized regularly varying sequences with respect to $\tau : \mathbb{N}_0 \to q^{\mathbb{N}_0}$, $\tau(k) = q^k$, defined in [12]. These results, presented in Section 5, are also new for the half-linear difference equation

$$\Delta(a(n)\Phi(\Delta(x(n)))) + b(n)\Phi(x(n+1)) = 0, \quad n \in \mathbb{N}_0, \tag{1.6}$$

since, as far as we know, the only type of equation (1.6) that was studied in the framework of regular variation, was with $a(n) \equiv 1$ (see [10]). In Section 6, the obtained results will be illustrated through the examples.

2. Preliminaries and classification

For the sake of completeness, we recall the definition and some of the basic properties of q-regularly varying functions. For some of the basic concepts of q-calculus, see [19]. First of all, let us state the notation that will be used through this paper. As usual, the symbol $\sim$ denotes asymptotic equivalence of two functions:

$$f(t) \sim g(t), \text{ as } t \to \infty \Leftrightarrow \lim_{t\to\infty} \frac{f(t)}{g(t)} = 1.$$

The interval $[t_0, \infty)_q$ represents $[t_0, \infty) \cap q^{\mathbb{N}_0}$. Moreover, let

$$\mathcal{R}^+_{t_0} = \{\delta : [t_0, \infty)_q \to \mathbb{R} : t(q-1)\delta(t) + 1 > 0,\, t \geq t_0\}$$

and for $\delta \in \mathcal{R}^+_1$ let us introduce the q-exponential function

$$e_\delta(t,s) = \begin{cases} \prod_{u\in[s,t)_q}((q-1)u\delta(u)+1), & s < t; \\ 1, & s = t; \\ 1/\prod_{u\in[t,s)_q}((q-1)u\delta(u)+1), & s > t, \end{cases}$$

where $s, t \in q^{\mathbb{N}_0}$.

Řehák and Vítovec [19], defined q-regularly varying functions as follows.

Definition 2.1. A function $f : [a, \infty)_q \to (0, \infty)$ is said to be q-regularly varying of index ρ, $\rho \in \mathbb{R}$, if there exists a function $\alpha : [a, \infty)_q \to (0, \infty)$ satisfying

$$\lim_{t\to\infty} \frac{f(t)}{\alpha(t)} = c \quad \text{and} \quad \lim_{t\to\infty} \frac{tD_q\alpha(t)}{\alpha(t)} = [\rho]_q, \tag{2.1}$$

with c being a positive constant. If $\rho = 0$, then f is said to be q-slowly varying.

The totality of q-regularly varying functions of index ρ is denoted by $\mathcal{RV}_q(\rho)$, while the totality of q-slowly varying functions is denoted by $\mathcal{SV}_q$. For q-regularly varying functions defined as above, most of the properties of regular variation in the continuous and discrete case are preserved, but because of the structure of $q^{\mathbb{N}_0}$, there are much simpler characterizations than in the continuous or discrete case. Řehák and Vítovec established in [19] several characterizations of such functions, we are presenting here a few of them. See [19] for more details.

Theorem 2.2. (i) *(Simple characterization) For a positive function f, $f \in \mathcal{RV}_q(\rho)$ if and only if f satisfies*

$$\lim_{t\to\infty} \frac{f(qt)}{f(t)} = q^\rho.$$

Moreover, $f \in \mathcal{RV}_q(\rho)$ if and only if f satisfies just the later condition in (2.1).

(ii) *(Representation I) $f \in \mathcal{RV}_q(\rho)$ if and only if f has the representation*

$$f(t) = \varphi(t)e_\delta(t, 1),$$

where $\varphi : q^{\mathbb{N}_0} \to (0, \infty)$ tends to a positive constant, $\delta : q^{\mathbb{N}_0} \to \mathbb{R}$ satisfies $\lim_{t\to\infty} t\delta(t) = [\rho]_q$ and $\delta \in \mathcal{R}_1^+$. Without loss of generality, in particular in the only if part, the function φ can be replaced by a constant.

Further, q-regularly varying functions have the following properties, proved in [11, 19]. Let us emphasize that the Karamata's integration theorem will play a central role in establishing the main results of this paper.

Proposition 2.3. (i) *$f \in \mathcal{RV}_q(\rho), \rho \in \mathbb{R}$ if and only if $f(t) = t^\rho \ell(t)$, where $\ell \in \mathcal{SV}_q$.*

(ii) *Let $f \in \mathcal{RV}_q(\rho), \rho \in \mathbb{R}$ and $\gamma \in \mathbb{R}$. Then $f^\gamma \in \mathcal{RV}_q(\gamma\rho)$.*

(iii) *Let $f \in \mathcal{RV}_q(\rho_1)$ and $g \in \mathcal{RV}_q(\rho_2)$, $\rho_1, \rho_2 \in \mathbb{R}$. Then $fg \in \mathcal{RV}_q(\rho_1 + \rho_2)$.*

(iv) *If $f \in \mathcal{RV}_q(\rho)$, with $\rho \in \mathbb{R}, \rho \neq 0$, then $|D_q f| \in \mathcal{RV}_q(\rho - 1)$. For $\rho = 0$ the statement may fail, even for monotone f.*

(v) *If $f \in \mathcal{SV}_q$, then $D_q \ln f(t) \sim \frac{D_q f(t)}{f(t)}$ as $t \to \infty$.*

Theorem 2.4 (Karamata's integration theorem, direct half). *Let $\ell \in \mathcal{SV}_q$ and $a \in q^{\mathbb{N}_0}$.*

(i) *if $\alpha > -1$, then $\int_a^x t^\alpha \ell(t)\, d_q t \sim \frac{x^{\alpha+1}}{[\alpha+1]_q}\ell(x)$ as $x \to \infty$;*

(ii) *if $\alpha < -1$, then $\int_x^\infty t^\alpha \ell(t)\, d_q t \sim -\frac{x^{\alpha+1}}{[\alpha+1]_q}\ell(x)$ as $x \to \infty$;*

(iii) *if $\int_a^\infty \frac{\ell(t)}{t}\, d_q t = \infty$, then $L(x) = \int_a^x \frac{\ell(t)}{t}\, d_q t$ for $x \in [a, \infty)_q$ is a $\mathcal{SV}_q$ function and $\lim_{x\to\infty} \frac{L(x)}{\ell(x)} = \infty$;*

(iv) *if $\int_a^\infty \frac{\ell(t)}{t}\, d_q t < \infty$, then $L(x) = \int_x^\infty \frac{\ell(t)}{t}\, d_q t$ for $x \in [a, \infty)_q$ is a $\mathcal{SV}_q$ function and $\lim_{x\to\infty} \frac{L(x)}{\ell(x)} = \infty$.*

Before we start establishing asymptotic formulas, let us consider a classification of the non-oscillatory solutions of (1.1). Since x is a solution of (1.1) if and only if $-x$ is a solution of this equation, we restrict our attention only to eventually positive solutions of (1.1). Since the coefficient r is an eventually positive or eventually negative function, all of the eventually positive solutions can be divided into two classes:

$$\mathbb{M}^- = \{x \in \mathbb{M} : D_q x(t) < 0 \text{ for t large enough}\},$$
$$\mathbb{M}^+ = \{x \in \mathbb{M} : D_q x(t) > 0 \text{ for t large enough}\},$$

where $\mathbb{M}$ is the set of all eventually positive solutions of (1.1). Furthermore, these classes will be divided into subclasses which will give more precise information about the asymptotic behavior at infinity of positive solutions. The asymptotic behavior of positive solutions of (1.1) depends on the divergence of the integrals

$$I_p = \int^{\infty} \frac{1}{p(s)^{1/\alpha}}\, d_q s, \quad I_r = \int^{\infty} r(s)\, d_q s.$$

We list the subclasses of positive solutions of (1.1), which will be used throughout this paper:

$$\begin{gathered}
\mathbb{M}^+_{\infty,\infty} = \{x \in \mathbb{M}^+ : \lim_{t\to\infty} x(t) = \infty, \lim_{t\to\infty} x^{[1]}(t) = \infty\},\\
\mathbb{M}^+_{\infty,B} = \{x \in \mathbb{M}^+ : \lim_{t\to\infty} x(t) = \infty, \lim_{t\to\infty} x^{[1]}(t) = c \in (0,\infty)\},\\
\mathbb{M}^+_{B,\infty} = \{x \in \mathbb{M}^+ : \lim_{t\to\infty} x(t) = c \in (0,\infty), \lim_{t\to\infty} x^{[1]}(t) = \infty\},\\
\mathbb{M}^+_{\infty,0} = \{x \in \mathbb{M}^+ : \lim_{t\to\infty} x(t) = \infty, \lim_{t\to\infty} x^{[1]}(t) = 0\},\\
\mathbb{M}^+_{B,0} = \{x \in \mathbb{M}^+ : \lim_{t\to\infty} x(t) = c \in (0,\infty), \lim_{t\to\infty} x^{[1]}(t) = 0\},\\
\mathbb{M}^-_{B,0} = \{x \in \mathbb{M}^- : \lim_{t\to\infty} x(t) = c \in (0,\infty), \lim_{t\to\infty} x^{[1]}(t) = 0\},\\
\mathbb{M}^-_{0,B} = \{x \in \mathbb{M}^- : \lim_{t\to\infty} x(t) = 0, \lim_{t\to\infty} x^{[1]}(t) = c \in (-\infty,0)\},\\
\mathbb{M}^-_{0,\infty} = \{x \in \mathbb{M}^- : \lim_{t\to\infty} x(t) = 0, \lim_{t\to\infty} x^{[1]}(t) = -\infty\},\\
\mathbb{M}^-_{B,\infty} = \{x \in \mathbb{M}^- : \lim_{t\to\infty} x(t) = c \in (0,\infty), \lim_{t\to\infty} x^{[1]}(t) = -\infty\},\\
\mathbb{M}^-_{0,0} = \{x \in \mathbb{M}^- : \lim_{t\to\infty} x(t) = 0, \lim_{t\to\infty} x^{[1]}(t) = 0\},
\end{gathered} \tag{2.2}$$

where $x^{[1]}(t) = p(t)\Phi(D_q x(t)), t \in q^{\mathbb{N}_0}$. In the case of positive solutions of a difference equation, we will use the same notation, with $\mathbb{MZ}$ instead of $\mathbb{M}$. Moreover, let us introduce notations $\mathbb{M}_{RV}(\rho) = \mathbb{M} \cap \mathcal{RV}_q(\rho)$, $\mathbb{M}_{SV} = \mathbb{M} \cap \mathcal{SV}_q$ and when we consider solutions of difference equations, the notation $\mathbb{MZ}_{\mathcal{SV}^\tau_{\mathbb{Z}}} = \mathbb{MZ} \cap \mathcal{SV}^\tau_{\mathbb{Z}}$, $\mathbb{MZ}_{\mathcal{RV}^\tau_{\mathbb{Z}}}(\rho) = \mathbb{MZ} \cap \mathcal{RV}^\tau_{\mathbb{Z}}(\rho)$ will be used.

(I) r is eventually negative. In this case, any nontrivial solution of (1.1) is non-oscillatory, eventually strictly monotone and both classes $\mathbb{M}^+$ and $\mathbb{M}^-$ of solutions of (1.1) are nonempty. To prove this, let us transform equation (1.1) to difference equation (1.6), where

$$a(n) = \frac{p(\tau(n))}{((q-1)\tau(n))^\alpha} \text{ and } b(n) = (q-1)\tau(n)r(\tau(n)), \quad n \in \mathbb{N}_0. \tag{2.3}$$

An application of [2, Lemma 1] provides that all nontrivial solutions of (1.6) are non-oscillatory and eventually strictly monotone. Also, Cecchi et al. [2] proved that the classes $\mathbb{MZ}^+$ and $\mathbb{MZ}^-$ of (1.6) are nonempty. Since x is a solution of (1.6) if and only if $y = x \circ \tau^{-1}$ is a solution of (1.1), we conclude that all nontrivial solutions of (1.1) are non-oscillatory, eventually strictly monotone and the classes $\mathbb{M}^+$ and $\mathbb{M}^-$ of (1.1) are nonempty.

If we assume the integral I_p diverges, it can be easily shown that

$$\mathbb{M}^+ = \mathbb{M}^+_{\infty,\infty} \cup \mathbb{M}^+_{\infty,B} \quad \text{and} \quad \mathbb{M}^- = \mathbb{M}^-_{B,0} \cup \mathbb{M}^-_{0,0}.$$

Indeed, if x is a positive and increasing solution of (1.1) on $[t_0,\infty)_q$ for some $t_0 \in q^{\mathbb{N}_0}$, then $x^{[1]}$ is positive, increasing function, thus $x^{[1]}(t) \ge c_1 > 0$, $t \ge t_0$. This further implies $x(t) \ge x(t_0) + c_1^{1/\alpha} \int_{t_0}^t \frac{d_q s}{p(s)^{1/\alpha}}$, $t \ge t_0$ and since I_p diverges, we conclude $x(t) \to \infty$, $t \to \infty$. Similarly, if x is a decreasing solution of (1.1), then

$x^{[1]}(t) \to -c \le 0$ as $t \to \infty$. If we assume $x^{[1]}(t) \to -c < 0$, $t \to \infty$, analogously to the previous consideration, we obtain $x(t) \ge -c_2 \int_{t_0}^{t} \frac{d_q s}{p(s)^{1/\alpha}}$, $t \ge t_0$ for some $t_0 \in q^{\mathbb{N}_0}$ and $c_2 > 0$, which bearing in mind the divergence of I_p implies $x(t) \to -\infty$, $t \to \infty$, so we obtain a contradiction. Hence, it must be $x^{[1]}(t) \to 0$ as $t \to \infty$.

Let us discuss what happens with the asymptotic behavior of positive solutions of (1.1) in the case of divergence of the integral I_r. Then

$$\mathbb{M}^+ = \mathbb{M}^+_{B,\infty} \cup \mathbb{M}^+_{\infty,\infty} \quad \text{and} \quad \mathbb{M}^- = \mathbb{M}^-_{0,B} \cup \mathbb{M}^-_{0,0}.$$

Indeed, if x is an increasing solution on $[t_0, \infty)_q$, then $x(t) \ge x(t_0)$, $t \ge t_0$. Since

$$x^{[1]}(t) = x^{[1]}(t_0) - \int_{t_0}^{t} r(s) x(qs)^\alpha d_q s \ge -x(t_0)^\alpha \int_{t_0}^{t} r(s) d_q s \to \infty, \quad t \to \infty,$$

it follows that $\lim_{t\to\infty} x^{[1]}(t) = \infty$. Similarly, if x is a decreasing solution of (1.1), assumption $\lim_{t\to\infty} x(t) = c > 0$ leads to $\lim_{t\to\infty} x^{[1]}(t) = \infty$, which is a contradiction since $x^{[1]}(t) < 0$, for t large enough. Therefore, $\lim_{t\to\infty} x(t) = 0$.

(II) r is eventually positive. First, let us consider the asymptotic behavior of positive solutions under the assumption $I_p = \infty$. Under this assumption, all of the positive solutions of (1.1) are increasing. Indeed, if we suppose the existence of a decreasing solution x, then, since $x^{[1]}$ is negative and decreasing function, it satisfies $x^{[1]}(t) \le -c < 0$, $t \ge t_0$, for some $t_0 \in q^{\mathbb{N}_0}$. This further implies $x(t) \le x(t_0) - c^{1/\alpha} \int_{t_0}^{t} \frac{d_q s}{p(s)^{1/\alpha}} \to -\infty$, $t \to \infty$, so we obtain a contradiction. Furthermore, it can be easily checked that increasing solutions can be divided into three classes:

$$\mathbb{M}^+ = \mathbb{M}^+_{\infty,0} \cup \mathbb{M}^+_{\infty,B} \cup \mathbb{M}^+_{B,0}.$$

It is left to consider the case $I_r = \infty$. Under this assumption, the set $\mathbb{M}^+$ is empty, while the set of decreasing solutions can be divided into three classes,

$$\mathbb{M}^- = \mathbb{M}^-_{0,\infty} \cup \mathbb{M}^-_{0,B} \cup \mathbb{M}^-_{B,\infty}.$$

This can be checked by standard techniques used in previous paragraphs.

3. Asymptotic formulas for some classes of q-regularly varying solutions of (1.1)

In what follows, we suppose that the coefficients in (1.1) satisfy the following conditions: $p \in \mathcal{RV}_q(\lambda)$, $\lambda \ne \alpha$, r is eventually of one sign such that $|r| \in \mathcal{RV}_q(\lambda - \alpha - 1)$ and use expressions

$$p(t) = t^\lambda l_p(t), \quad r(t) = \operatorname{sgn}(r(t)) t^{\lambda-\alpha-1} l_r(t), \quad t \in q^{\mathbb{N}_0}, \tag{3.1}$$

where $l_p, l_r \in \mathcal{SV}_q$. We will consider separately cases $\lambda < \alpha$ and $\lambda > \alpha$ which provide the divergence of I_p and I_r, respectively. The case $\lambda = \alpha$ will be excluded from our consideration. In this case, in general, convergence or divergence of the integrals I_p and I_r cannot be determined.

In this section we establish asymptotic formulas for $\mathcal{SV}_q$ and $\mathcal{RV}_q(1-\frac{\lambda}{\alpha})$ solutions of (1.1). Notice that under the above-mentioned assumptions for the coefficient p, Theorem 1.1 states that these solutions exist if and only if condition (1.4) is satisfied for $c = 0$. Regarding (3.1), condition (1.4) is then equivalent to

$$\lim_{t\to\infty} \frac{l_r(t)}{l_p(t)} = 0. \tag{3.2}$$

Moreover, under these assumptions, in the case r is eventually negative, it is proved in [3] that all of the eventually positive solutions of (1.1) are q-regularly varying by using the theory of q-regular variation and reciprocity principle. In that manner, by establishing asymptotic formulas of q-regularly varying solutions, in this case, we are establishing asymptotic formulas of all of the eventually positive solutions of (1.1). Here we state mentioned result.

Theorem 3.1 ([3, Theorem 4.1])**.** *Let $p \in \mathcal{RV}_q(\lambda)$, $\lambda \neq \alpha$, r is eventually negative, such that $|r| \in \mathcal{RV}_q(\lambda - \alpha - 1)$ and (3.2) is satisfied.*

(i) *If $\lambda < \alpha$, then $\mathbb{M}^+ = \mathbb{M}_{RV}(1 - \frac{\lambda}{\alpha})$ and $\mathbb{M}^- = \mathbb{M}_{SV}$.*
(ii) *If $\lambda > \alpha$, then $\mathbb{M}^- = \mathbb{M}_{RV}(1 - \frac{\lambda}{\alpha})$ and $\mathbb{M}^+ = \mathbb{M}_{SV}$.*

In what follows, we will use the notation

$$\delta = \Phi^{-1}\Big(\frac{1}{[\lambda - \alpha]_q}\Big), \quad G(t) = \Phi^{-1}\Big(\frac{tr(t)}{p(t)}\Big), \quad t \in q^{\mathbb{N}_0}. \tag{3.3}$$

Next auxiliary lemma will be very useful in determining the asymptotic formula of a q-slowly varying solution of (1.1).

Lemma 3.2. *Let $p \in \mathcal{RV}_q(\lambda)$, $\lambda \neq \alpha$, r is eventually of one sign such that $|r| \in \mathcal{RV}_q(\lambda - \alpha - 1)$. If x is a q-slowly varying solution of (1.1), then*

$$D_q \ln x(t) = -(1 + o(1))\delta G(t), \quad t \to \infty. \tag{3.4}$$

Proof. Suppose that x is a $\mathcal{SV}_q$ solution of (1.1) defined on $[a, \infty)_q$, for some $a \in q^{\mathbb{N}_0}$. Then, condition (3.2) is satisfied. Without loss of generality, suppose r is of one sign on $[a, \infty)_q$.

(i) Let us first consider the case $r(t) < 0$, $t \geq a$ and $\lambda < \alpha$. Application of Theorem 3.1 implies that x is a decreasing solution, while assumption $\lambda < \alpha$ implies $I_p = \infty$, hence, it must be $\lim_{t\to\infty} x^{[1]}(t) = 0$. After integrating (1.1) on the interval $[t, \infty)_q$, we obtain

$$x^{[1]}(t) = \int_t^\infty r(s)x(qs)^\alpha \, d_q s = -\int_t^\infty s^{\lambda-\alpha-1} l_r(s)x(qs)^\alpha \, d_q s, \quad t \geq a, \tag{3.5}$$

using expression (3.1) for r. Application of the Karamata's integration theorem in (3.5) leads to

$$x^{[1]}(t) \sim \frac{-tr(t)x(t)^\alpha}{[\lambda - \alpha]_q}, \quad t \to \infty,$$

which gives

$$\frac{D_q x(t)}{x(t)} \sim \Phi^{-1}\Big(\frac{-tr(t)}{p(t)[\lambda - \alpha]_q}\Big), \quad t \to \infty.$$

Since $x \in \mathcal{SV}_q$, using Proposition 2.3 (v), we obtain

$$D_q \ln x(t) \sim \Phi^{-1}\Big(\frac{-tr(t)}{p(t)[\lambda - \alpha]_q}\Big), \quad t \to \infty,$$

which leads to the desired asymptotic formula (3.4).

If we suppose $\lambda > \alpha$, an application of Theorem 3.1 implies $x \in \mathbb{M}^+$. Since, in this case, the integral I_r diverges, $\lim_{t\to\infty} x^{[1]}(t) = \infty$ holds for the solution x. Integration of (1.1) on $[a, t]_q$ gives

$$x^{[1]}(t) = x^{[1]}(a) - \int_a^t r(s)x(qs)^\alpha d_q s = x^{[1]}(a) + \int_a^t s^{\lambda-\alpha-1} l_r(s)x(qs)^\alpha d_q s, \quad t \geq a.$$

Proceeding exactly as in the previous part, application of Proposition 2.3 (v) and Theorem 2.4 imply that in this case, x also satisfies (3.4).

(ii) Next, we assume $r(t) > 0$, $t \geq a$. Let us consider the case $\lambda < \alpha$. Since in this case $\mathbb{M}^- = \emptyset$, x must be an increasing solution satisfying $\lim_{t\to\infty} x^{[1]}(t) = 0$. So, after integrating (1.1) on the interval $[t,\infty)_q$ and applying the Karamata's integration theorem we have

$$x^{[1]}(t) = \int_t^\infty r(s)x(qs)^\alpha\, d_qs = \int_t^\infty s^{\lambda-\alpha-1}l_r(s)x(qs)^\alpha d_qs \sim \frac{-tr(t)x(t)^\alpha}{[\lambda-\alpha]_q}$$

as $t \to \infty$. The above asymptotic relation leads to

$$D_q \ln x(t) \sim \frac{D_q x(t)}{x(t)} \sim \Phi^{-1}\Big(\frac{-tr(t)x(t)^\alpha}{[\lambda-\alpha]_q}\Big) = -\delta G(t), \quad t\to\infty,$$

so we come to the desired conclusion. It is left to verify if (3.4) holds in the case $\lambda > \alpha$. Then, $x \in \mathbb{M}^-$ and $\lim_{t\to\infty} x^{[1]}(t) = \infty$, so we integrate (1.1) on the interval $[a,t]_q$ to obtain

$$x^{[1]}(t) = x^{[1]}(a) - \int_a^t s^{\lambda-\alpha-1}l_r(s)x(qs)^\alpha d_qs \sim \frac{-tr(t)x(t)^\alpha}{[\lambda-\alpha]_q}, \quad t\to\infty.$$

Proceeding exactly as in the previous part, when r is eventually negative and $\lambda < \alpha$, we obtain (3.4). $\square$

Next two theorems give asymptotic formulas for q-slowly varying solutions of (1.1). We consider separately the cases when r is eventually negative and eventually positive function.

Theorem 3.3. *Assume that $p \in \mathcal{RV}_q(\lambda)$, $\lambda \neq \alpha$, r is eventually negative such that $|r| \in \mathcal{RV}_q(\lambda-\alpha-1)$ and (3.2) is satisfied. Every q-slowly varying solution x of (1.1) satisfies:*

(i) *If $\int^\infty G(t)d_qt = \infty$, then*

$$x(t) = \exp\Big(-(1+o(1))\delta\int_a^t G(s)d_qs\Big), \quad t\to\infty, \tag{3.6}$$

for some $a \in q^{\mathbb{N}_0}$. Moreover, if $\lambda < \alpha$, then $\mathbb{M}^- = \mathbb{M}^-_{0,0} = \mathbb{M}_{SV}$, while if $\lambda > \alpha$, then $\mathbb{M}^+ = \mathbb{M}^+_{\infty,\infty} = \mathbb{M}_{SV}$.

(ii) *If $\int^\infty G(t)d_qt < \infty$, then*

$$x(t) = N\exp\Big((1+o(1))\delta\int_t^\infty G(s)d_qs\Big), \quad t\to\infty, \tag{3.7}$$

where $N = \lim_{t\to\infty} x(t) \in (0,\infty)$. Moreover, if $\lambda < \alpha$, then $\mathbb{M}^- = \mathbb{M}^-_{B,0} = \mathbb{M}_{SV}$, while if $\lambda > \alpha$, then $\mathbb{M}^+ = \mathbb{M}^+_{B,\infty} = \mathbb{M}_{SV}$. In addition,

$$\frac{l_r(t)^{1/\alpha}}{l_p(t)^{1/\alpha}(N-x(t))} = o(1), \quad t\to\infty. \tag{3.8}$$

Proof. Without loss of generality, suppose $r(t) < 0$ on $[a,\infty)_q$, for some $a \in q^{\mathbb{N}_0}$ and x is a q-slowly varying solution of (1.1) defined on $[a,\infty)_q$. Condition (3.2) ensures the existence of such solution. Also, conditions of Lemma 3.2 are satisfied, so this solution satisfies asymptotic formula (3.4). Moreover, conditions of Theorem 3.1 are also satisfied, so in the case $\lambda < \alpha$ this implies $\mathbb{M}^- = \mathbb{M}_{SV}$, while in the case $\lambda > \alpha$, $\mathbb{M}^+ = \mathbb{M}_{SV}$.

(i) Assume $\int^\infty G(t)d_qt=\infty$. Integrating (3.4) on $[a,t]_q$ we have

$$\ln x(t)=\ln x(a)-\delta\int_a^t(1+o(1))G(s)d_qs,\quad t\to\infty. \tag{3.9}$$

Using the q-L'Hôpital rule (see [1, Theorem 1.119]) it can be easily verified that

$$\ln x(a)-\delta\int_a^t(1+o(1))G(s)d_qs\sim-\delta\int_a^t G(s)d_qs,\quad t\to\infty,$$

which further, using (3.9), leads to asymptotic formula (3.6) for a solution x. In the case $\lambda<\alpha$, condition $\int^\infty G(t)d_qt=\infty$ implies $\lim_{t\to\infty}x(t)=0$. Moreover, since in this case $I_p=\infty$, it must be $\lim_{t\to\infty}x^{[1]}(t)=0$, thus we obtain $x\in\mathbb{M}^-_{0,0}$. Since x was an arbitrary $\mathcal{SV}_q$ solution, it follows $\mathbb{M}_{SV}\subseteq\mathbb{M}^-_{0,0}$. According to Theorem 3.1(i), we have $\mathbb{M}^-=\mathbb{M}^-_{0,0}=\mathbb{M}_{SV}$.

Similarly, if we consider the case $\lambda>\alpha$, the divergence of integral I_r implies $\lim_{t\to\infty}x^{[1]}(t)=\infty$, while condition $\int^\infty G(t)d_qt=\infty$ implies $\lim_{t\to\infty}x(t)=\infty$, thus $x\in\mathbb{M}^+_{\infty,\infty}$. Again, since Theorem 3.1(ii) implies that in this case all of the increasing solutions are q-slowly varying, it follows $\mathbb{M}^+=\mathbb{M}^+_{\infty,\infty}=\mathbb{M}_{SV}$.

(ii) Assume $\int^\infty G(t)d_qt<\infty$. Integrating (3.4) on $[t,\infty)_q$ we have

$$\ln x(t)-\ln N=\delta\int_t^\infty(1+o(1))G(s)d_qs,\quad t\to\infty, \tag{3.10}$$

where $N=\lim_{t\to\infty}x(t)$. Using the q-L'Hospital rule it can be easily verified that

$$\delta\int_t^\infty(1+o(1))G(s)d_qs\sim\delta\int_t^\infty G(s)d_qs,\quad t\to\infty,$$

which, using (3.10), gives

$$\ln x(t)=\ln N+\delta(1+o(1))\int_t^\infty G(s)d_qs,\quad t\to\infty,$$

which is equivalent to (3.7). In the case $\lambda<\alpha$, for every decreasing solution x $\lim_{t\to\infty}x^{[1]}(t)=0$ holds, thus we have $\mathbb{M}_{SV}\subseteq\mathbb{M}^-_{B,0}$. Application of Theorem 3.1(i) gives $\mathbb{M}^-=\mathbb{M}^-_{B,0}=\mathbb{M}_{SV}$.

To prove (3.8), we notice that for a $\mathcal{SV}_q$ solution x, we have

$$x(t)-N=\int_t^\infty\frac{L(s)}{s}d_qs,\quad t\ge a,$$

where

$$L(t)=-t\Phi^{-1}\Big(\frac{1}{p(t)}\int_t^\infty r(u)x(qu)^\alpha d_qu\Big),\quad t\ge a,$$

is a q-slowly varying function, according to Theorem 2.4. The Karamata's integration theorem also implies that $\lim_{t\to\infty}\frac{L(t)}{x(t)-N}=0$. Considering

$$L(t)\sim\delta tG(t)x(t)\sim-N\delta\Big(\frac{l_r(t)}{l_p(t)}\Big)^{1/\alpha},\quad t\to\infty,$$

we obtain (3.8). In the case $\lambda>\alpha$, similarly it can be verified that every $\mathcal{SV}_q$ solution x belongs to the class $\mathbb{M}^+_{B,\infty}$. Moreover, for a solution x, we have

$$N-x(t)=\int_t^\infty\Big(\frac{1}{p(s)}\Phi^{-1}\Big(x^{[1]}(a)-\int_a^s r(u)x(qu)^\alpha d_qu\Big)\Big)d_qs$$

$$\sim \int_t^\infty \Phi^{-1}\Big(\frac{1}{p(s)}\Big(-\int_a^s r(u)x(qu)^\alpha d_q u\Big)\Big) d_q s, \quad t\to\infty\,.$$

Proceeding exactly as in the previous case, we prove that (3.8) also holds. □

Theorem 3.4. *Assume that $p\in\mathcal{RV}_q(\lambda)$, $\lambda\neq\alpha$, r is eventually positive such that $r\in\mathcal{RV}_q(\lambda-\alpha-1)$ and* (3.2) *is satisfied. Then every q-slowly varying solution x of* (1.1) *satisfies:*

- (i) *If $\int^\infty G(t)d_q t=\infty$, then x satisfies* (3.6)*. Moreover, if $\lambda<\alpha$, then $\mathbb{M}_{SV}\subseteq\mathbb{M}^+_{\infty,0}$, while if $\lambda>\alpha$, then $\mathbb{M}_{SV}\subseteq\mathbb{M}^-_{0,\infty}$.*
- (ii) *If $\int^\infty G(t)d_q t<\infty$, then x satisfies* (3.7)*, where $N=\lim_{t\to\infty}x(t)\in(0,\infty)$. Moreover, if $\lambda<\alpha$, then $\mathbb{M}_{SV}=\mathbb{M}^+_{B,0}$, while if $\lambda>\alpha$, then $\mathbb{M}_{SV}=\mathbb{M}^-_{B,\infty}$. In addition,* (3.8) *is satisfied.*

Proof. Without loss of generality, suppose $r(t)>0$ on $[a,\infty)_q$, for some $a\in q^{\mathbb{N}_0}$ and x is a q-slowly varying solution of (1.1) defined on $[a,\infty)_q$. Condition (3.2) ensures the existence of such solution. Also, Lemma 3.2 claims that such solution x satisfies (3.4).

(i) Assume $\int^\infty G(t)d_q t=\infty$. Integrating (3.4) from a to t, we obtain (3.9) which leads to the desired asymptotic formula (3.6) for the solution x. Further, let us first consider the case $\lambda<\alpha$. In this case, since $I_p=\infty$, the solution x is an increasing function satisfying $\lim_{t\to\infty}x^{[1]}(t)=0$. Moreover, condition $\int^\infty G(t)d_q t=\infty$ implies that $\lim_{t\to\infty}x(t)=\infty$, so we obtain $\mathbb{M}_{SV}\subseteq\mathbb{M}^+_{\infty,0}$. Similarly, if $\lambda>\alpha$, every $\mathcal{SV}_q$ solution x is decreasing and satisfies $\lim_{t\to\infty}x^{[1]}(t)=\infty$. In this case, $\int^\infty G(t)d_q t=\infty$ implies $\lim_{t\to\infty}x(t)=0$, so we obtain $\mathbb{M}_{SV}\subseteq\mathbb{M}^-_{0,\infty}$.

(ii) Assume $\int^\infty G(t)d_q t<\infty$. Integrating (3.4) on $[t,\infty)_q$ we have (3.10) which is equivalent to (3.7). This implies that $\lim_{t\to\infty}x(t)=N\in(0,\infty)$. Thus, in the case $\lambda<\alpha$, since every $\mathcal{SV}_q$ solution x is an increasing function satisfying $\lim_{t\to\infty}x^{[1]}(t)=0$, we have conclusion $\mathbb{M}_{SV}=\mathbb{M}^+_{B,0}$, while in the case $\lambda>\alpha$ it can be noticed that $\mathbb{M}_{SV}=\mathbb{M}^-_{B,\infty}$. Proceeding similarly as in the proof of Theorem 3.3, we obtain (3.8). □

In following two theorems we will establish asymptotic formulas for $\mathcal{RV}_q(1-\frac{\lambda}{\alpha})$ solutions of (1.1) under certain conditions. To provide this we will use reciprocity principle, which is based on following. Let $p(t)\neq 0$, $r(t)\neq 0$, $t\in[a,\infty)_q$. Then, x is a solution of (1.1) defined on $[a,\infty)_q$ if and only if $u(t)=x^{[1]}(t)$ is a solution of the equation

$$D_q\Big(\Phi^{-1}\Big(\frac{1}{r(t)}\Big)\Phi^{-1}(D_q u(t))\Big)+q\Phi^{-1}\Big(\frac{1}{p(qt)}\Big)\Phi^{-1}(u(qt))=0. \tag{3.11}$$

Since we have stronger assumptions for the coefficients p and r of (1.1) throughout this section, let us check which conditions will be satisfied by the coefficients in (3.11), under those assumptions. Assume that $p\in\mathcal{RV}_q(\lambda)$, $\lambda\neq\alpha$, r is of one sign on $[a,\infty)_q$ such that $|r|\in\mathcal{RV}_q(\lambda-\alpha-1)$ and (3.2) holds. Denote by

$$\hat\alpha=\frac{1}{\alpha},\quad \hat\lambda=1-\frac{\lambda}{\alpha}+\frac{1}{\alpha},\quad \hat p(t)=\frac{1}{|r(t)|^{1/\alpha}},\quad \hat r(t)=\operatorname{sgn}(r(t))\frac{q}{p(qt)^{1/\alpha}},\quad t\in q^{\mathbb{N}_0}.$$

Then equation (3.11) is equivalent to the equation

$$D_q\left(\hat p(t)\Phi_{\hat\alpha}(D_q u(t))\right)+\hat r(t)\Phi_{\hat\alpha}(u(qt))=0, \tag{3.12}$$

where $\Phi_{\hat{\alpha}}(x) = \operatorname{sgn}(x)|x|^{\hat{\alpha}}$, $x \in \mathbb{R}$. Notice that, $\hat{p} \in \mathcal{RV}_q(\hat{\lambda})$, $\hat{\lambda} \neq \hat{\alpha}$ and $|\hat{r}| \in \mathcal{RV}_q(\hat{\lambda}-\hat{\alpha}-1)$. Moreover, if $l_{\hat{p}}$ and $l_{\hat{r}}$ denote q-slowly varying parts of the functions $\hat{p}$ and $\hat{r}$, we have

$$\frac{l_{\hat{r}}(t)}{l_{\hat{p}}(t)} = q\frac{l_r(t)^{1/\alpha}}{l_p(qt)^{1/\alpha}} \sim q\Big(\frac{l_r(t)}{l_p(t)}\Big)^{1/\alpha}, \quad t \to \infty.$$

Thus, condition (3.2) implies $\lim_{t\to\infty} \frac{l_{\hat{r}}(t)}{l_{\hat{p}}(t)} = 0$. We will use the notation

$$\hat{G}(t) = \frac{t^\alpha r(t)}{p(t)} \quad \text{and} \quad \hat{\delta} = -\Phi\Big(\frac{1}{[\frac{\lambda}{\alpha}-1]_q}\Big).$$

Notice that $\hat{\delta}\hat{G}(t) \sim \Phi\big(\frac{1}{[\hat{\lambda}-\hat{\alpha}]_q}\big)\Phi\big(\frac{t\hat{r}(t)}{\hat{p}(t)}\big)$ as $t \to \infty$. Moreover, the notation for the classes of positive solutions of (3.12) will be analogue to the one in (2.2) with $\hat{\mathbb{M}}$ instead of $\mathbb{M}$.

Theorem 3.5. *Assume that $p \in \mathcal{RV}_q(\lambda)$, $\lambda \neq \alpha$, r is eventually negative such that $|r| \in \mathcal{RV}_q(\lambda-\alpha-1)$ and* (3.2) *holds. Every solution x of* (1.1)*, such that $x \in \mathcal{RV}_q(1-\frac{\lambda}{\alpha})$ satisfies*

(i) *If $\int^\infty \hat{G}(t)d_qt = \infty$, then*
$$x(t) = \frac{t}{p(t)^{1/\alpha}} \exp\Big(-(1+o(1))\frac{\hat{\delta}}{\alpha}\int_a^t \hat{G}(s)d_qs\Big), \quad t \to \infty, \tag{3.13}$$
for some $a \in q^{\mathbb{N}_0}$. Moreover, if $\lambda < \alpha$, then $\mathbb{M}^+ = \mathbb{M}^+_{\infty,\infty} = \mathbb{M}_{RV}(1-\frac{\lambda}{\alpha})$, while if $\lambda > \alpha$, then $\mathbb{M}^- = \mathbb{M}^-_{0,0} = \mathbb{M}_{RV}(1-\frac{\lambda}{\alpha})$.

(ii) *If $\int^\infty \hat{G}(t)d_qt < \infty$, then in the case $\lambda < \alpha$,*
$$x(t) = A + \int_a^t \Big(\frac{\hat{N}}{p(s)}\Big)^{1/\alpha} \exp\Big((1+o(1))\frac{\hat{\delta}}{\alpha}\int_s^\infty \hat{G}(u)d_qu\Big)d_qs, \quad t \to \infty, \tag{3.14}$$
for some $a \in q^{\mathbb{N}_0}$, $A \in (0,\infty)$ and $\mathbb{M}^+ = \mathbb{M}^+_{\infty,B} = \mathbb{M}_{RV}(1-\frac{\lambda}{\alpha})$, while in the case $\lambda > \alpha$,
$$x(t) = \int_t^\infty \Big(\frac{\hat{N}}{p(s)}\Big)^{1/\alpha} \exp\Big((1+o(1))\frac{\hat{\delta}}{\alpha}\int_s^\infty \hat{G}(u)d_qu\Big)d_qs \quad t \to \infty, \tag{3.15}$$
and $\mathbb{M}^- = \mathbb{M}^-_{0,B} = \mathbb{M}_{RV}(1-\frac{\lambda}{\alpha})$, where $\hat{N} = \lim_{t\to\infty}|x^{[1]}(t)|$. In addition,
$$\frac{l_r(t)}{l_p(t)(N-|x^{[1]}(t)|)} = o(1), \quad t \to \infty. \tag{3.16}$$

Proof. Let x be an arbitrary $\mathcal{RV}_q(1-\frac{\lambda}{\alpha})$ solution of (1.1), defined on $[a,\infty)_q$ and let r be negative on $[a,\infty)_q$, for some $a \in q^{\mathbb{N}_0}$. Condition (3.2) ensures the existence of such solution. Then, $u = |x^{[1]}|$ is a q-slowly varying solution of (3.12). Moreover, note that $u \in \hat{\mathbb{M}}^\pm_{u,v} \Leftrightarrow x \in \mathbb{M}^\pm_{v,u}$, for $u,v \in \{0,B,\infty\}$.

(i) Assume $\int^\infty \hat{G}(t)d_qt = \infty$. From the above observations, we can see that conditions of Theorem 3.3(i) are satisfied, so it can be applied to the $\mathcal{SV}_q$ solution u of (3.12) and this leads to the asymptotic formula

$$u(t) = \exp\Big(-(1+o(1))\hat{\delta}\int_a^t \hat{G}(s)d_qs\Big), \quad t \to \infty.$$

Consequently, the solution x of (1.1) is satisfies

$$|D_q x(t)| = \frac{1}{p(t)^{1/\alpha}} \exp\Big(-(1+o(1))\frac{\hat{\delta}}{\alpha}\int_a^t \hat{G}(s)d_q s\Big), \quad t\to\infty. \tag{3.17}$$

Let us consider the case $\lambda < \alpha$, that is $\hat{\lambda} > \hat{\alpha}$. In this case, Theorem 3.3(i) also implies that the solution u of (3.12) belongs to the class $\hat{\mathbb{M}}^+_{\infty,\infty}$ of positive solutions of (3.12). For equation (3.12) $\hat{\mathbb{M}}^+ = \hat{\mathbb{M}}^+_{\infty,\infty} = \hat{\mathbb{M}}_{SV}$ holds, as well. This further implies that solution x of (1.1) belongs to the class $\mathbb{M}^+_{\infty,\infty}$ and the classes of this equation satisfy $\mathbb{M}^+ = \mathbb{M}^+_{\infty,\infty} = \mathbb{M}_{RV}(1-\frac{\lambda}{\alpha})$. Integrating asymptotic relation (3.17) from a to t we obtain

$$\begin{aligned} x(t) &= x(a) + \int_a^t \frac{1}{p(s)^{1/\alpha}} \exp\Big(-(1+o(1))\frac{\hat{\delta}}{\alpha}\int_a^s \hat{G}(u)d_q u\Big) \\ &\sim \frac{t}{[1-\frac{\lambda}{\alpha}]_q}\frac{1}{p(t)^{1/\alpha}} \exp\Big(-(1+o(1))\frac{\hat{\delta}}{\alpha}\int_a^t \hat{G}(s)d_q s\Big) \\ &= \frac{t}{p(t)^{1/\alpha}} \exp\Big(-(1+o(1))\frac{\hat{\delta}}{\alpha}\int_a^t \hat{G}(s)d_q s\Big), \quad t\to\infty, \end{aligned}$$

by applying the Karamata's integration theorem, since u is a $\mathcal{SV}_q$ function. This further implies that x satisfies formula (3.13). Similarly, we obtain that in the case $\lambda > \alpha$ for equation (1.1), $\mathbb{M}^- = \mathbb{M}^-_{0,0} = \mathbb{M}_{RV}(1-\frac{\lambda}{\alpha})$ holds. Integration of the asymptotic relation (3.17) on $[t,\infty)_q$ leads to the desired asymptotic formula (3.13) for x.

(ii) Assume $\int^\infty \hat{G}(t)d_q t < \infty$. Then, an application of Theorem 3.3(ii) to the $\mathcal{SV}_q$ solution u of (3.11) leads to the asymptotic formula

$$u(t) = \hat{N}\exp\Big((1+o(1))\hat{\delta}\int_t^\infty \hat{G}(s)d_q s\Big), \quad t\to\infty,$$

where $\lim_{t\to\infty} u(t) = \hat{N}$. This implies that for the solution x of (1.1) satisfies

$$|D_q x(t)| = \Big(\frac{\hat{N}}{p(t)}\Big)^{1/\alpha} \exp\Big((1+o(1))\frac{\hat{\delta}}{\alpha}\int_t^\infty \hat{G}(s)d_q s\Big), \quad t\to\infty. \tag{3.18}$$

Moreover, Theorem 3.3(ii) implies $\frac{l_{\hat{r}}(t)^\alpha}{l_{\hat{p}}(t)^\alpha (N-u)} = o(1)$ as $t\to\infty$; hence (3.16) satisfied.

To obtain the asymptotic formula for x, let us first consider the case $\lambda < \alpha$, that is $\hat{\lambda} > \hat{\alpha}$. Under this assumption, for the positive solutions of (3.12), $\hat{\mathbb{M}}^+ = \hat{\mathbb{M}}^+_{B,\infty} = \hat{\mathbb{M}}_{SV}$ holds. This implies that for (1.1), $\mathbb{M}^+ = \mathbb{M}^+_{\infty,B} = \mathbb{M}_{RV}(1-\frac{\lambda}{\alpha})$. Integrating (3.18) on $[a,t]_q$ we obtain that x satisfies asymptotic formula (3.14). Similarly, if $\lambda > \alpha$, we obtain $\mathbb{M}^- = \mathbb{M}^-_{0,B} = \mathbb{M}_{RV}(1-\frac{\lambda}{\alpha})$, while integrating (3.18) on $[t,\infty)_q$ implies the asymptotic formula (3.15) for the solution x. □

Theorem 3.6. *Assume that $p \in \mathcal{RV}_q(\lambda)$, $\lambda \neq \alpha$, r is eventually positive such that $r \in \mathcal{RV}_q(\lambda-\alpha-1)$ and (3.2) holds. Every solution x of (1.1) such that $x \in \mathcal{RV}_q(1-\frac{\lambda}{\alpha})$ satisfies*

(i) *If $\int^\infty \hat{G}(t)d_q t = \infty$, then (3.13) holds. Moreover, if $\lambda < \alpha$, then $\mathbb{M}_{RV}(1-\frac{\lambda}{\alpha}) \subseteq \mathbb{M}^+_{\infty,0}$, while if $\lambda > \alpha$, then $\mathbb{M}_{RV}(1-\frac{\lambda}{\alpha}) \subseteq \mathbb{M}^-_{0,\infty}$.*

(ii) *If* $\int^{\infty}\hat{G}(t)d_qt<\infty$, *then in the case* $\lambda<\alpha$ (3.14) *holds and* $\mathbb{M}^{+}_{\infty,B}=\mathbb{M}_{RV}(1-\frac{\lambda}{\alpha})$, *while in the case* $\lambda>\alpha$ (3.15) *holds and* $\mathbb{M}^{-}_{0,B}=\mathbb{M}_{RV}(1-\frac{\lambda}{\alpha})$. *In addition,* (3.16) *is satisfied.*

Proof. Applying Theorem 3.4 to the $\mathcal{SV}_q$ solution $u=|x^{[1]}|$ of (3.12), as in the previous theorem, leads to desired results. Moreover, $u\in\hat{\mathbb{M}}^{\pm}_{u,v}\Leftrightarrow x\in\mathbb{M}^{\mp}_{v,u}$, where $u,v\in\{0,B,\infty\}$. ☐

Remark 3.7. When taking formally the limit as $q\to 1+$ in Theorems 3.3 and 3.5, the obtained results coincide with the corresponding results in the continuous case (see [14, Theorems 4.1, 5.1]). Under the analogue assumptions of Theorems 3.4 and 3.6 in the continuous case, the asymptotic formulas of regularly varying solutions of (1.2), with r being eventually positive, as far as we know, were not considered in the existing literature. So, letting $q\to 1+$ in Theorems 3.4 and 3.6 predicts corresponding results in the continuous case.

4. Asymptotic formula of a q-slowly varying solution of (1.5)

In this section, we consider equation (1.5) and assume that r is eventually positive or eventually negative function. Moreover, a condition stronger than $\lim_{t\to\infty}t^{\alpha+1}r(t)=0$, which ensures the existence of a $\mathcal{SV}_q$ solution, will be assumed for the coefficient r. We will use the notation

$$Q(t)=t^{\alpha}\int_t^{\infty}r(s)d_qs,\ t\in q^{\mathbb{N}_0}. \tag{4.1}$$

Let us recall that the condition $\lim_{t\to\infty}t^{\alpha+1}r(t)=0$ is equivalent to $\lim_{t\to\infty}Q(t)=0$ as $t\to\infty$. As shown in [18, Lemma 6], this is the consequence of the fact, specific just for q-calculus, that the existence of the finite limit $\lim_{t\to\infty}t^{\alpha}\int_t^{\infty}f(s)d_qs$ is equivalent to the existence of the finite limit $\lim_{t\to\infty}t^{\alpha+1}f(t)$, where $f:q^{\mathbb{N}_0}\to\mathbb{R}$ and $\alpha>0$. More precisely, $\lim_{t\to\infty}t^{\alpha}\int_t^{\infty}f(s)d_qs=A\in\mathbb{R}$ if and only if $\lim_{t\to\infty}t^{\alpha+1}f(t)=-[-\alpha]_qA\in\mathbb{R}$.

The next theorem shows the existence of a q-slowly varying solution affected with the decaying property of the function Q, while the second theorem establishes the asymptotic formula of the such solution.

Theorem 4.1. *Let r be eventually of one sign. Suppose that there exists a decreasing function $\phi:q^{\mathbb{N}_0}\to(0,+\infty)$ which tends to 0 as $t\to\infty$ and satisfies $|Q(t)|\le\phi(t)$ for t large enough. Then,* (1.5) *possesses a q-slowly varying solution x on $[t_0,\infty)_q$, for some $t_0\in q^{\mathbb{N}_0}$, expressed in the form*

$$x(t)=e_{\eta}(t,t_0),\quad t\ge t_0, \tag{4.2}$$

where $\eta(t)=\Phi^{-1}\Big(\frac{v(t)+Q(t)}{t^{\alpha}}\Big)$, $t\ge t_0$ *and* $v(t)=O\big(\phi(t)^{1+\frac{1}{\alpha}}\big)$, $t\to\infty$.

Proof. We will seek for a solution x of (1.5) expressed in the form (4.2), for some $t_0\in q^{\mathbb{N}_0}$. Function x expressed in the form (4.2) is a q-slowly varying function on $[t_0,\infty)_q$ if and only if

$$\lim_{t\to\infty}\Phi^{-1}\left(v(t)+Q(t)\right)=0 \tag{4.3}$$

and $\eta\in\mathcal{R}^{+}_{t_0}$, according to Theorem 2.2(ii). Moreover, such x is the positive solution of (1.5) defined on $[t_0,\infty)_q$ if and only if $w(t)=\frac{v(t)+Q(t)}{t^{\alpha}}$, $t\ge t_0$ is a solution of the Riccati q-difference equation on $[t_0,\infty)_q$,

$$D_q w(t) + r(t) + \frac{w(t)}{(q-1)t}\Big(1 - \frac{1}{\big(\Phi^{-1}(w(t))(q-1)t+1\big)^{\alpha}}\Big) = 0. \tag{4.4}$$

Let us note that $\Phi^{-1}(w(t))(q-1)t+1 = \frac{x(qt)}{x(t)}$, $t \ge t_0$. Furthermore, w is a solution of (4.4) on $[t_0,\infty)_q$ if and only if v is a solution of the equation

$$D_q\Big(\frac{v(t)}{t^{\alpha}}\Big) + \frac{v(t)+Q(t)}{(q-1)t^{\alpha+1}}\Big(1 - \frac{1}{(\Phi^{-1}(v(t)+Q(t))(q-1)+1)^{\alpha}}\Big) = 0, \tag{4.5}$$

defined on $[t_0,\infty)_q$. Condition (4.3), since $Q(t) \to 0$, $t \to \infty$ is equivalent to $v(t) \to 0$ as $t \to \infty$, hence integrating (4.5) on $[t,\infty)_q$ gives the integral equation

$$v(t) = t^{\alpha}\int_t^{\infty} \frac{v(s)+Q(s)}{(q-1)s^{\alpha+1}}\Big(1 - \frac{1}{\big(\Phi^{-1}(v(s)+Q(s))(q-1)+1\big)^{\alpha}}\Big)\, d_q s, \tag{4.6}$$

for $t \ge t_0$. Finally, finding a q-slowly varying solution x of (1.5) in the form (4.2) is equivalent to finding a solution v of the integral equation (4.6) satisfying $v(t) \to 0$ as $t \to \infty$ and $\eta \in \mathcal{R}^{+}_{t_0}$. To show the existence of such a solution, we will use the Banach fixed point theorem.

Let us choose $t_0 \in q^{\mathbb{N}_0}$ such that the following 3 conditions

$$\Phi^{-1}(Q(t))(q-1)+1 > 0, \tag{4.7}$$

$$\frac{-(q-1)(\alpha+1)\Phi^{-1}(Q(t))}{(\Phi^{-1}(Q(t))(q-1)+1)^{\alpha+1}} \le -\frac{1}{2}[-\alpha]_q, \tag{4.8}$$

$$\phi(t)^{1/\alpha} \le \min\Big\{\frac{2^{-1/\alpha}}{q-1}, -\frac{[-\alpha]_q 2^{-1-\alpha-\frac{1}{\alpha}}}{\alpha}, -\frac{[-\alpha]_q 2^{-1-\alpha-\frac{1}{\alpha}}}{(\alpha+1)(q-1)}\Big\} \tag{4.9}$$

are satisfied on $[t_0,\infty)_q$ and Q is of a constant sign on $[t_0,\infty)_q$. This is possible, since ϕ and Q tend to zero as $t \to \infty$.

Consider the Banach space $\mathcal{X}$ of bounded functions $f : [t_0,\infty)_q \to \mathbb{R}$ converging to zero at infinity, endowed with the supremum norm and let us denote by

$$\Omega = \{v \in \mathcal{X} :\ 0 \le v(t) \le \phi(t),\, t \ge t_0\}.$$

The operator $\mathcal{F} : \Omega \to \mathcal{X}$, defined as

$$(\mathcal{F}v)(t) = t^{\alpha}\int_t^{\infty} \frac{v(s)+Q(s)}{(q-1)s^{\alpha+1}}\Big(1 - \frac{1}{\big(\Phi^{-1}(v(s)+Q(s))(q-1)+1\big)^{\alpha}}\Big)\, d_q s,$$

for $v \in \Omega$, has following properties:

(i) Operator $\mathcal{F}$ maps Ω into itself. Let $v \in \Omega$. One can see that

$$\operatorname{sgn}\big(v(t)+Q(t)\big) = \operatorname{sgn}\Big(1 - \frac{1}{(\Phi^{-1}(v(t)+Q(t))(q-1)+1)^{\alpha}}\Big), \quad t \ge t_0,$$

which implies $(\mathcal{F}v)(t) \ge 0$ for $t \ge t_0$. On the other hand,

$$\begin{aligned}(\mathcal{F}v)(t) &\le t^{\alpha}\int_t^{\infty} \frac{2\phi(s)}{(q-1)s^{\alpha+1}}\Big(1 - \frac{1}{\big(\Phi^{-1}(2\phi(s))(q-1)+1\big)^{\alpha}}\Big)\, d_q s \\ &\le t^{\alpha}\int_t^{\infty} \frac{2\phi(s)}{(q-1)s^{\alpha+1}}\Big(\Big(\Phi^{-1}(2\phi(s))(q-1)+1\Big)^{\alpha} - 1\Big)\, d_q s, \quad t \ge t_0.\end{aligned}$$

Using the Lagrange mean value theorem and (4.9), we obtain

$$\begin{aligned}\big(\Phi^{-1}(2\phi(s))(q-1)+1\big)^{\alpha}-1 &= \alpha\left(\theta\Phi^{-1}(2\phi(s))(q-1)+1\right)^{\alpha-1}\Phi^{-1}(2\phi(s))(q-1)\\ &\le \alpha\left(\Phi^{-1}(2\phi(s))(q-1)+1\right)^{\alpha}\Phi^{-1}(2\phi(s))(q-1)\\ &\le \alpha 2^{\alpha}\Phi^{-1}(2\phi(s))(q-1), \quad \forall s\ge t_0,\end{aligned}$$

for some $0<\theta<1$. Therefore, using the monotonicity of the function ϕ, we obtain

$$(\mathcal{F}v)(t)\le \alpha 2^{\alpha}(2\phi(t))^{1+\frac{1}{\alpha}}t^{\alpha}\int_t^{\infty}\frac{d_q s}{s^{\alpha+1}}=\frac{-\alpha 2^{\alpha}}{[-\alpha]_q}(2\phi(t))^{1+\frac{1}{\alpha}}, \quad t\ge t_0.$$

Using (4.9), we finally obtain $(\mathcal{F}v)(t)\le\phi(t)$, for $t\ge t_0$, which provides that $\mathcal{F}$ maps Ω into itself.

(ii) Operator $\mathcal{F}$ is a contraction mapping. Let $v,w\in\Omega$ and observe that

$$\begin{aligned}&(\mathcal{F}v)(t)-(\mathcal{F}w)(t)\\ &=t^{\alpha}\int_t^{\infty}\frac{1}{(q-1)s^{\alpha+1}}\Big(H(v(s)+Q(s))-H(w(s)+Q(s))\Big)d_q s,\end{aligned}\tag{4.10}$$

for $t\ge t_0$, where $H(x)=x\big(1-\frac{1}{(\Phi^{-1}(x)(q-1)+1)^{\alpha}}\big)$ for $x\in\mathbb{R}$. Using the Lagrange mean value theorem, we obtain

$$H(v(s)+Q(s))-H(w(s)+Q(s))=H'(\xi(s))(v(s)-w(s)),$$

for some $\min\{v(s)+Q(s),w(s)+Q(s)\}\le\xi(s)\le\max\{v(s)+Q(s),w(s)+Q(s)\}$, $s\ge t_0$. Note that $Q(s)\le\xi(s)\le 2\phi(s)$ for $s\ge t_0$. So, if Q is positive on $[t_0,\infty)_q$, ξ is also positive on this interval, while in the case when Q is negative on $[t_0,\infty)_q$, ξ can take both, positive and negative values on this interval. We will prove that

$$|H'(\xi(s))|\le-\frac{1}{2}[-\alpha]_q, s\ge t_0.\tag{4.11}$$

Indeed, in the case $\xi(s)>0$, for some $s\ge t_0$, using the Lagrange mean value theorem and (4.9), we obtain

$$\begin{aligned}|H'(\xi(s))|=H'(\xi(s))&=\frac{(\Phi^{-1}(\xi(s))(q-1)+1)^{\alpha+1}-1}{(\Phi^{-1}(\xi(s))(q-1)+1)^{\alpha+1}}\\ &\le(\Phi^{-1}(\xi(s))(q-1)+1)^{\alpha+1}-1\\ &=(\alpha+1)(\theta\Phi^{-1}(\xi(s))(q-1)+1)^{\alpha}\Phi^{-1}(\xi(s))(q-1)\\ &\le(\alpha+1)2^{\alpha}(2\phi(s))^{1/\alpha}(q-1)\\ &\le-\frac{1}{2}[-\alpha]_q,\end{aligned}$$

for some $0<\theta<1$. On the other hand, if $\xi(s)<0$, for some $s\ge t_0$, similarly to the previous case, using (4.8), we obtain

$$\begin{aligned}|H'(\xi(s))|=-H'(\xi(s))&=\frac{1-(\Phi^{-1}(\xi(s))(q-1)+1)^{\alpha+1}}{(\Phi^{-1}(\xi(s))(q-1)+1)^{\alpha+1}}\\ &\le\frac{1-(\Phi^{-1}(Q(s))(q-1)+1)^{\alpha+1}}{(\Phi^{-1}(Q(s))(q-1)+1)^{\alpha+1}}\\ &\le\frac{-(\alpha+1)(\theta\Phi^{-1}(Q(s))(q-1)+1)^{\alpha}\Phi^{-1}(Q(s))(q-1)}{(\Phi^{-1}(Q(s))(q-1)+1)^{\alpha+1}}\end{aligned}$$

$$\begin{aligned}&\le \frac{-(\alpha+1)\Phi^{-1}(Q(s))(q-1)}{(\Phi^{-1}(Q(s))(q-1)+1)^{\alpha+1}}\\&\le -\frac{1}{2}[-\alpha]_q,\end{aligned}$$

for some $0<\theta<1$, so (4.11) is satisfied. According to (4.10) and (4.11),

$$\begin{aligned}|(\mathcal{F}v)(t)-(\mathcal{F}w)(t)| &\le -\frac{1}{2}[-\alpha]_q t^{\alpha}\int_t^{\infty}\frac{1}{(q-1)s^{\alpha+1}}|v(s)-w(s)|d_qs\\&\le \frac{1}{2}||v-w||,\quad t\ge t_0,\end{aligned}$$

which leads to the conclusion that $\mathcal{F}$ is a contraction mapping.

Thus, all the hypotheses of the Banach fixed point theorem are fulfilled, implying the existence of a fixed point $v\in\Omega$ of $\mathcal{F}$ satisfying (4.6). Moreover, x defined by (4.2), with such v, satisfying $v(t)=O\big(\phi(t)^{1+\frac{1}{\alpha}}\big)$ as $t\to\infty$, is the desired solution. □

Theorem 4.2. *Let r be eventually of one sign. Suppose that there exists a decreasing function $\phi: q^{\mathbb{N}_0}\to(0,+\infty)$ which tends to 0 as $t\to\infty$ and satisfies*

$$|Q(t)|\sim\phi(t),\quad t\to\infty. \tag{4.12}$$

(i) *If $\int^{\infty}\frac{\Phi^{-1}(Q(t))}{t}\,d_qt=\infty$, then (1.5) possesses a q-slowly varying solution x defined on $[t_0,\infty)_q$, for some $t_0\in q^{\mathbb{N}_0}$, such that*

$$x(t)=\exp\Big((1+o(1))\int_{t_0}^{t}\frac{\Phi^{-1}(Q(s))}{s}\,d_qs\Big),\quad t\to\infty. \tag{4.13}$$

Moreover, if r is eventually negative, then $x\in\mathbb{M}^{-}_{0,0}$, while if r is eventually positive, $x\in\mathbb{M}^{+}_{\infty,0}$.

(ii) *If $\int^{\infty}\frac{\Phi^{-1}(Q(t))}{t}\,d_qt<\infty$, then (1.5) possesses a q-slowly varying solution x defined on $[t_0,\infty)_q$, for some $t_0\in q^{\mathbb{N}_0}$, such that*

$$x(t)=N\exp\Big(-(1+o(1))\int_{t}^{\infty}\frac{\Phi^{-1}(Q(s))}{s}\,d_qs\Big),\quad t\to\infty, \tag{4.14}$$

where $N=\lim_{t\to\infty}x(t)$. Moreover, if r is eventually negative, then $x\in\mathbb{M}^{-}_{B,0}$, while if r is eventually positive, $x\in\mathbb{M}^{+}_{B,0}$.

Proof. Condition (4.12) implies $|Q(t)|\le(k+1)\phi(t)$ for t large enough and $k>0$, so the conditions of Theorem 4.1 are satisfied for $\phi(t)$ replaced with $(k+1)\phi(t)$. Therefore, there exists q-slowly varying solution x of (1.5) in the form (4.2), defined on $[t_0,\infty)_q$, for some $t_0\in q^{\mathbb{N}_0}$, where $v(t)=O(\phi(t)^{1+\frac{1}{\alpha}})$, $t\to\infty$. This solution satisfies

$$\frac{D_qx(t)}{x(t)}=\frac{\Phi^{-1}(v(t)+Q(t))}{t},\quad t\ge t_0.$$

Since Q satisfies condition (4.12), we obtain

$$\frac{D_qx(t)}{x(t)}=\frac{\Phi^{-1}\big(O(\phi(t)^{1+\frac{1}{\alpha}})+Q(t)\big)}{t}\sim\frac{\Phi^{-1}(Q(t))}{t},\quad t\to\infty. \tag{4.15}$$

An application of Proposition 2.3(v) then yields

$$D_q\ln x(t)\sim\frac{\Phi^{-1}(Q(t))}{t},\quad t\to\infty. \tag{4.16}$$

(i) Suppose $\int^\infty \frac{\Phi^{-1}(Q(t))}{t}\, d_q t = \infty$. Integrating (4.16) from t_0 to t, we obtain

$$\ln x(t) - \ln x(t_0) \sim \int_{t_0}^{t} \frac{\Phi^{-1}(Q(s))}{s}\, d_q s, \quad t \to \infty. \tag{4.17}$$

The divergence of the integral on the right-hand side of the above asymptotic relation, when $t \to \infty$, implies that the function on the left-hand side of (4.17) also tends to ∞ when $t \to \infty$. If r is eventually negative, (4.17) implies $x(t) \to 0$, $t \to \infty$, while (4.15) implies $x \in \mathbb{M}^-$. According to the classification, we conclude $x \in \mathbb{M}^-_{0,0}$. Similarly, if r is eventually positive, (4.17) implies $x(t) \to \infty$ as $t \to \infty$ and (4.15) implies $x \in \mathbb{M}^+$, so it must be $x \in \mathbb{M}^+_{\infty,0}$. Asymptotic relation (4.17) further implies

$$\ln x(t) \sim \int_{t_0}^{t} \frac{\Phi^{-1}(Q(s))}{s}\, d_q s, \quad t \to \infty,$$

which leads to the conclusion

$$\ln x(t) = (1 + o(1)) \int_{t_0}^{t} \frac{\Phi^{-1}(Q(s))}{s}\, d_q s, \quad t \to \infty$$

and hence we obtain the desired asymptotic expression (4.13) for the nontrivial q-slowly varying solution x.

(ii) Suppose $\int^\infty \frac{\Phi^{-1}(Q(t))}{t}\, d_q t < \infty$. Integrating (4.16) from t to ∞, we obtain

$$\ln N - \ln x(t) \sim \int_{t}^{\infty} \frac{\Phi^{-1}(Q(s))}{s}\, d_q s, \quad t \to \infty, \tag{4.18}$$

where $N = \lim_{t\to\infty} x(t)$, which further yields asymptotic formula (4.14) for the solution x. Similarly to the previous case, we come to the conclusion that in the case when r is eventually negative x belongs to $\mathbb{M}^-_{B,0}$, while if r is eventually positive, $x \in \mathbb{M}^+_{B,0}$. $\square$

Remark 4.3. Let us compare the obtained asymptotic formulas for a q-slowly varying solution of (1.5), where r is eventually of one sign such that $|r| \in \mathcal{RV}_q(-\alpha - 1)$. Then the Karamata's integration theorem implies $Q(t) \sim \frac{t^{\alpha+1} r(t)}{-[-\alpha]_q}$, as $t \to \infty$, which further implies $\frac{\Phi^{-1}(Q(t))}{t} \sim -\delta G(t)$, as $t \to \infty$, according to the notation in (3.3) and (4.1). Thus, in this case, asymptotic formulas (4.13) and (4.14) are equivalent to the asymptotic formulas (3.6) and (3.7), respectively, as expected.

Remark 4.4. When taking formally the limit as $q \to 1+$ in Theorem 4.2(i), the obtained results agree with the corresponding results in the continuous case (see [9, Theorem 3.1]). To be more precise, [9, Theorem 3.1] requires stronger conditions on the functions Q and ϕ, but it gives a more precise asymptotic formula. On the other hand, since the case $\int^\infty \frac{\Phi^{-1}(Q(t))}{t}\, dt < \infty$ is not considered in [9], letting $q \to 1+$ in Theorem 4.2(ii) predicts corresponding results in the continuous case.

5. Half-linear difference equations in the framework of discrete regular variation with respect to τ

Řehák [12] introduced a new class of regularly varying sequences with respect to τ, where $\tau : \mathbb{N}_0 \to q^{\mathbb{N}_0}$, $q > 1$, $\tau(k) = q^k$.

Definition 5.1. Let x be a positive sequence. It is said that x is a regularly varying sequence with respect to τ and written $x \in \mathcal{RV}^\tau_{\mathbb{Z}}(\rho)$, if and only if $x \circ \tau^{-1} \in \mathcal{RV}_q(\rho)$.

The following 4 statements are equivalent (see [12]):

(i) $x \in \mathcal{RV}_{\mathbb{Z}}^{\tau}(\rho)$;

(ii) $\lim_{k\to\infty} \frac{\triangle x(k)}{x(k)(q-1)} = [\rho]_q$;

(iii) $\lim_{k\to\infty} \frac{x(k+1)}{x(k)} = q^{\rho}$;

(iv) $x(k) = Cq^{k\rho} \exp\{\sum_{j=1}^{k-1} \Psi(j)\}$, $\Psi(j) \to 0$, $j \to \infty$ and $C \in (0, +\infty)$.

Since a half-linear q-difference equation can be transformed into a half-linear difference equation, our main results can be applied to obtain some new results in the discrete case. Indeed, if $\tau : \mathbb{N}_0 \to q^{\mathbb{N}_0}$, $\tau(k) = q^k$, it can be easily shown that $x : \mathbb{N}_0 \to \mathbb{R}$ is a solution of difference equation (1.6), which coefficients satisfy (2.3), if and only if $y = x \circ \tau^{-1}$ is a solution of q-difference equation (1.1).

Therefore, with the assumption that the sequences $a = \{a(n)\}_{n\in\mathbb{N}_0}$ and $|b| = \{|b(n)|\}_{n\in\mathbb{N}_0}$ are regularly varying sequences with respect to τ of a certain regularity index and b is eventually of one sign, obtained results can be applied to the half-linear difference equation (1.6), giving the asymptotic formulas of $\mathcal{SV}_{\mathbb{Z}}^{\tau}$ and $\mathcal{RV}_{\mathbb{Z}}^{\tau}(1-\frac{\lambda}{\alpha})$ solutions of this equation. To prove the following results it is enough to conclude that the assumptions $a \in \mathcal{RV}_{\mathbb{Z}}^{\tau}(\rho)$ and $p \in \mathcal{RV}_q(\rho + \alpha)$ are equivalent, as well as the assumptions $|b| \in \mathcal{RV}_{\mathbb{Z}}^{\tau}(\rho)$ and $|r| \in \mathcal{RV}_q(\rho - 1)$. Let us use expressions

$$a(n) = n^{\rho} l_a(n), \quad |b(n)| = n^{\rho} l_b(n), \quad n \in \mathbb{N}_0,$$

where $l_a, l_b \in \mathcal{SV}_{\mathbb{Z}}^{\tau}$. Next we present corollaries of Theorems 3.3–3.6.

Corollary 5.2. *Assume $a \in \mathcal{RV}_{\mathbb{Z}}^{\tau}(\rho)$, $\rho \neq 0$, b is eventually negative such that $|b| \in \mathcal{RV}_{\mathbb{Z}}^{\tau}(\rho)$ and*

$$\lim_{n\to\infty} \frac{b(n)}{a(n)} = 0. \tag{5.1}$$

Every solution $x \in \mathcal{SV}_{\mathbb{Z}}^{\tau}$ of (1.6) satisfies:

(i) *If $\sum_{n=1}^{\infty} \Phi^{-1}\Big(\frac{b(n)}{a(n)}\Big) = \infty$, then*

$$x(n) = \exp\Big((1 + o(1)) \frac{1}{\Phi^{-1}(1 - q^{\rho})} \sum_{k=n_0}^{n-1} \Phi^{-1}\Big(\frac{b(k)}{a(k)}\Big)\Big), \quad n \to \infty, \tag{5.2}$$

for some $n_0 \in \mathbb{N}$. Moreover, if $\rho < 0$, then $\mathbb{MZ}^- = \mathbb{MZ}_{0,0}^- = \mathbb{MZ}_{\mathcal{SV}_{\mathbb{Z}}^{\tau}}$, while if $\rho > 0$, then $\mathbb{MZ}^+ = \mathbb{MZ}_{\infty,\infty}^+ = \mathbb{MZ}_{\mathcal{SV}_{\mathbb{Z}}^{\tau}}$.

(ii) *If $\sum_{n=1}^{\infty} \Phi^{-1}\big(\frac{b(n)}{a(n)}\big) < \infty$, then*

$$x(n) = N \exp\Big((1 + o(1)) \frac{1}{\Phi^{-1}(q^{\rho} - 1)} \sum_{k=n}^{\infty} \Phi^{-1}\Big(\frac{b(k)}{a(k)}\Big)\Big), \quad n \to \infty, \tag{5.3}$$

where $N = \lim_{n\to\infty} x(n) \in (0, \infty)$. Moreover, if $\rho < 0$, then $\mathbb{MZ}^- = \mathbb{MZ}_{B,0}^- = \mathbb{MZ}_{\mathcal{SV}_{\mathbb{Z}}^{\tau}}$, while if $\rho > 0$, then $\mathbb{MZ}^+ = \mathbb{MZ}_{B,\infty}^+ = \mathbb{MZ}_{\mathcal{SV}_{\mathbb{Z}}^{\tau}}$. In addition,

$$\frac{l_b(n)^{1/\alpha}}{l_a(n)^{1/\alpha}(N - x(n))} = o(1), \quad n \to \infty. \tag{5.4}$$

Corollary 5.3. *Assume $a \in \mathcal{RV}_{\mathbb{Z}}^{\tau}(\rho)$, $\rho \neq 0$, b is eventually positive such that $|b| \in \mathcal{RV}_{\mathbb{Z}}^{\tau}(\rho)$ and (5.1) is satisfied. Every solution $x \in \mathcal{SV}_{\mathbb{Z}}^{\tau}$ of (1.6) satisfies:*

(i) *If $\sum_{n=1}^{\infty} \Phi^{-1}\big(\frac{b(n)}{a(n)}\big) = \infty$, then x satisfies (5.2). Moreover, if $\rho < 0$, then $\mathbb{MZ}_{\mathcal{SV}_{\mathbb{Z}}^{\tau}} \subseteq \mathbb{MZ}_{\infty,0}^+$, while if $\rho > 0$, then $\mathbb{MZ}_{\mathcal{SV}_{\mathbb{Z}}^{\tau}} \subseteq \mathbb{MZ}_{0,\infty}^-$.*

(ii) $\sum_{n=1}^{\infty} \Phi^{-1}\big(\frac{b(n)}{a(n)}\big) < \infty$, *then* x *satisfies* (5.3), *where* $N = \lim_{n\to\infty} x(n) \in (0,\infty)$. *Moreover, if* $\rho < 0$, *then* $\mathbb{MZ}_{\mathcal{SV}_{\mathbb{Z}}^{\tau}} = \mathbb{MZ}_{B,0}^{+}$, *while if* $\rho > 0$, *then* $\mathbb{MZ}_{\mathcal{SV}_{\mathbb{Z}}^{\tau}} = \mathbb{MZ}_{B,\infty}^{-}$. *In addition,* (5.4) *holds.*

Corollary 5.4. *Assume* $a \in \mathcal{RV}_{\mathbb{Z}}^{\tau}(\rho)$, $\rho \neq 0$, b *is eventually negative such that* $|b| \in \mathcal{RV}_{\mathbb{Z}}^{\tau}(\rho)$ *and* (5.1) *is satisfied. Every solution* $x \in \mathcal{RV}_{\mathbb{Z}}^{\tau}\big(-\frac{\rho}{\alpha}\big)$ *of* (1.6) *satisfies:*

(i) *If* $\sum_{n=1}^{\infty} \frac{b(n)}{q^n a(n)} = \infty$, *then*

$$x(n) = \frac{1}{a(n)^{1/\alpha}} \exp\Big((1+o(1)) \frac{1}{\alpha\Phi(q^{\rho/\alpha}-1)} \sum_{k=n_0}^{n-1} \frac{b(k)}{a(k)}\Big), \quad n \to \infty, \tag{5.5}$$

for some $n_0 \in \mathbb{N}_0$. *Moreover, if* $\rho < 0$, *then* $\mathbb{MZ}^{+} = \mathbb{MZ}_{\infty,\infty}^{+} = \mathbb{MZ}_{\mathcal{RV}_{\mathbb{Z}}^{\tau}}\big(-\frac{\rho}{\alpha}\big)$, *while if* $\rho > 0$, *then* $\mathbb{MZ}^{-} = \mathbb{MZ}_{0,0}^{-} = \mathbb{MZ}_{\mathcal{RV}_{\mathbb{Z}}^{\tau}}\big(-\frac{\rho}{\alpha}\big)$.

(ii) *If* $\sum_{n=1}^{\infty} \frac{b(n)}{q^n a(n)} < \infty$, *then in the case* $\rho < 0$,

$$x(n) = A + \sum_{k=n_0}^{n-1} \Big(\frac{\hat{N}}{a(k)}\Big)^{1/\alpha} \exp\Big((1+o(1)) \frac{1}{\alpha\Phi(1-q^{\rho/\alpha})} \sum_{j=k}^{\infty} \frac{b(j)}{a(j)}\Big), \tag{5.6}$$

as $n \to \infty$, *for some* $A \in \mathbb{N}_0$, *and* $\mathbb{MZ}^{+} = \mathbb{MZ}_{\infty,B}^{+} = \mathbb{MZ}_{\mathcal{RV}_{\mathbb{Z}}^{\tau}}\big(-\frac{\rho}{\alpha}\big)$. *While in the case* $\rho > 0$,

$$x(n) = \sum_{k=n}^{\infty} \Big(\frac{\hat{N}}{a(k)}\Big)^{1/\alpha} \exp\Big((1+o(1)) \frac{1}{\alpha\Phi(1-q^{\rho/\alpha})} \sum_{j=k}^{\infty} \frac{b(j)}{a(j)}\Big), \tag{5.7}$$

as $n \to \infty$, *and* $\mathbb{MZ}^{-} = \mathbb{MZ}_{0,B}^{-} = \mathbb{MZ}_{\mathcal{RV}_{\mathbb{Z}}^{\tau}}\big(-\frac{\rho}{\alpha}\big)$, *where* $\hat{N} = \lim_{t\to\infty} |x^{[1]}(n)|$. *In addition,*

$$\frac{l_b(n)}{l_a(t)(N - x^{[1]}(n))} = o(1), \quad t \to \infty. \tag{5.8}$$

Corollary 5.5. *Assume* $a \in \mathcal{RV}_{\mathbb{Z}}^{\tau}(\rho)$, $\rho \neq 0$, b *is eventually positive such that* $|b| \in \mathcal{RV}_{\mathbb{Z}}^{\tau}(\rho)$ *and* (5.1) *is satisfied. Every solution* $x \in \mathcal{RV}_{\mathbb{Z}}^{\tau}\big(-\frac{\rho}{\alpha}\big)$ *of* (1.6) *satisfies:*

(i) *If* $\sum_{n=1}^{\infty} \frac{b(n)}{q^n a(n)} = \infty$, *then* (5.5) *holds. Moreover, if* $\rho < 0$, *then* $\mathbb{MZ}_{\mathcal{RV}_{\mathbb{Z}}^{\tau}}\big(-\frac{\rho}{\alpha}\big) \subseteq \mathbb{MZ}_{\infty,0}^{+}$; *while if* $\rho > 0$, *then* $\mathbb{MZ}_{\mathcal{RV}_{\mathbb{Z}}^{\tau}}\big(-\frac{\rho}{\alpha}\big) \subseteq \mathbb{MZ}_{0,\infty}^{-}$.

(ii) *If* $\sum_{n=1}^{\infty} \frac{b(n)}{q^n a(n)} < \infty$, *then in the case* $\rho < 0$, (5.6) *holds and* $\mathbb{MZ}_{\infty,B}^{+} = \mathbb{MZ}_{\mathcal{RV}_{\mathbb{Z}}^{\tau}}\big(-\frac{\rho}{\alpha}\big)$; *while in the case* $\rho > 0$, (5.7) *holds and* $\mathbb{MZ}_{0,B}^{-} = \mathbb{MZ}_{\mathcal{RV}_{\mathbb{Z}}^{\tau}}\big(-\frac{\rho}{\alpha}\big)$. *In addition,* (5.8) *is satisfied.*

The next corollary of Theorem 4.2 gives the asymptotic formula of a slowly varying solution with respect to τ of equation (1.6), with $a(n) = \frac{1}{((q-1)q^n)^{\alpha}}$, $n \in \mathbb{N}_0$ and b being an arbitrary sequence eventually positive or eventually negative.

Corollary 5.6. *Let* $a = \big\{\frac{1}{((q-1)q^n)^{\alpha}}\big\}_{n\in\mathbb{N}_0}$ *and* b *be eventually of one sign. Suppose that there exists a decreasing sequence* $\{\varphi(n)\}_{n\in\mathbb{N}_0}$ *which tends to* 0 *as* $n \to \infty$ *and satisfies*

$$\Big|q^{n\alpha} \sum_{k=n}^{\infty} b(k)\Big| \sim \varphi(n), \quad n \to \infty.$$

(i) *If* $\sum_{n=1}^{\infty} q^n\Phi^{-1}\big(\sum_{k=n}^{\infty} b(k)\big) = \infty$, *then* (1.6) *possesses a slowly varying solution* $\{x(n)\}_{n\ge n_0}$ *with respect to* τ, *for some* $n_0 \in \mathbb{N}$, *such that*

$$x(n) = \exp\Big((1+o(1))(q-1)\sum_{k=n_0}^{n-1} q^k\Phi^{-1}\Big(\sum_{j=k}^{\infty} b(j)\Big)\Big), \quad n\to\infty.$$

Moreover, if b *is eventually negative, then* $x \in \mathbb{MZ}^{-}_{0,0}$, *while if* b *is eventually positive,* $x \in \mathbb{MZ}^{+}_{\infty,0}$.

(ii) *If* $\sum_{n=1}^{\infty} q^n\Phi^{-1}\big(\sum_{k=n}^{\infty} b(k)\big) < \infty$, *then* (1.6) *possesses a slowly varying solution* $\{x(n)\}_{n\ge n_0}$ *with respect to* τ, *for some* $n_0 \in \mathbb{N}$, *such that*

$$x(n) = N\exp\Big(-(1+o(1))(q-1)\sum_{k=n}^{\infty} q^k\Phi^{-1}\Big(\sum_{j=k}^{\infty} b(j)\Big)\Big), \quad n\to\infty,$$

where $N = \lim_{n\to\infty} x(n)$. *Moreover, if* b *is eventually negative, then* $x \in \mathbb{MZ}^{-}_{B,0}$, *while if* b *is eventually positive,* $x \in \mathbb{MZ}^{+}_{B,0}$.

6. Examples

The first example illustrates Theorems 3.3, 3.4 and 4.2 simultaneously, while the second example illustrates Theorems 3.3–3.6.

Example 6.1. Consider the half-linear q-difference equation (1.5) with

$$r(t) = \frac{\varphi(t)}{t^{\alpha+1}(\ln t)^{\alpha\theta}},$$

on $[q,\infty)_q$, for some $\alpha > 0$, $\theta > 0, \theta \neq 1$ and φ being an arbitrary function such that $\varphi(t) \to c \in \mathbb{R}\setminus\{0\}$, $t\to\infty$. First of all, let us remark that this equation possesses a q-slowly varying solution, since assumptions imply $t^{\alpha+1}r(t) \to 0$, $t\to\infty$. Further, let us verify if the conditions of Theorem 3.3 for eventually negative φ and the conditions of Theorem 3.4 for eventually positive φ, are satisfied. Note that

$$r(t) \sim \frac{c}{t^{\alpha+1}(\ln t)^{\alpha\theta}}, \quad t\to\infty,$$

so we conclude r is eventually of one sign and $r \in \mathcal{RV}_q(-\alpha-1)$. Definition of the q-integral for δ and G defined in (3.3), $a = q^{n_0}$, $t = q^n$, $n_0, n \in \mathbb{N}_0$, in the case $\theta \in (0,1)$ implies

$$\begin{aligned}
\delta\int_a^t G(s)d_qs &\sim \Phi^{-1}\Big(\frac{c}{[-\alpha_q]}\Big)(q-1)\sum_{s\in[a,t)_q}\frac{1}{(\ln s)^{\theta}}\\
&= \Phi^{-1}\Big(\frac{c}{[-\alpha_q]}\Big)\frac{(q-1)}{(\ln q)^{\theta}}\sum_{k=n_0}^{n-1}\frac{1}{k^{\theta}}\\
&\sim \Phi^{-1}\Big(\frac{c}{[-\alpha_q]}\Big)\frac{(q-1)}{(\ln q)^{\theta}}\frac{n^{1-\theta}}{1-\theta}\\
&= \Phi^{-1}\Big(\frac{c}{[-\alpha_q]}\Big)\frac{q-1}{(1-\theta)\ln q}\frac{1}{(\ln t)^{\theta-1}} \to \infty, \quad t\to\infty.
\end{aligned}$$

Theorem 3.3(i) can be applied to any $\mathcal{SV}_q$ solution of (1.5) if φ is eventually negative, or Theorem 3.4(i) if φ is eventually positive, which leads to the asymptotic formula of the such solution

$$x(t) = \exp\Big((1+o(1))\Phi^{-1}\Big(\frac{c}{[-\alpha_q]}\Big)\frac{q-1}{(\theta-1)\ln q}\frac{1}{(\ln t)^{\theta-1}}\Big), \quad t\to\infty. \tag{6.1}$$

Furthermore, if $\theta>1$, then

$$\delta\int_t^\infty G(s)d_qs \sim \Phi^{-1}\Big(\frac{c}{[-\alpha_q]}\Big)\frac{q-1}{(\theta-1)\ln q}\frac{1}{(\ln t)^{\theta-1}} \to 0, \quad t\to\infty.$$

An application of Theorem 3.3(ii) for eventually negative r, or Theorem 3.4(ii) for eventually positive r implies that every $\mathcal{SV}_q$ solution of (1.5) satisfies

$$x(t) = N\exp\Big((1+o(1))\Phi^{-1}\Big(\frac{c}{[-\alpha_q]}\Big)\frac{q-1}{(\theta-1)\ln q}\frac{1}{(\ln t)^{\theta-1}}\Big), \quad t\to\infty, \tag{6.2}$$

where $N=\lim_{t\to\infty}x(t)$.

To verify that the conditions of Theorem 4.2 are satisfied, let

$$\phi(t) = \frac{|c|}{-[-\alpha]_q}\frac{1}{(\ln t)^{\alpha\theta}}, \quad t\in[q,\infty)_q.$$

Using the Karamata's integration theorem, we obtain

$$Q(t)\sim t^\alpha\int_t^\infty \frac{c}{t^{\alpha+1}(\ln t)^{\alpha\theta}}d_qs \sim \frac{c}{-[-\alpha]_q}\frac{1}{(\ln t)^{\alpha\theta}} \sim \operatorname{sgn}(c)\phi(t), \quad t\to\infty,$$

which implies that condition (4.12) is satisfied. Thus, Theorem 4.2(i) shows the existence of a q-slowly varying solution with the same asymptotic formula (6.1), in the case $\theta\in(0,1)$, while in the case $\theta>1$, Theorem 4.2(ii) claims the existence of a q-slowly varying solution with asymptotic formula (6.2), as we have already noticed in Remark 4.3.

Example 6.2. Consider the q-difference half-linear equation

$$D_q(t^\lambda(\ln t)^{\theta_1}\varphi_1(t)\Phi(D_qx(t))) + t^{\lambda-\alpha-1}(\ln t)^{\theta_2}\varphi_2(t)\Phi(x(qt)) = 0, \tag{6.3}$$

on $[q,\infty)_q$, where $\alpha>0$, $\lambda\neq\alpha$, $\theta_1,\theta_2\in\mathbb{R}$, $\theta_1>\theta_2$, $\varphi_1(t)\to c_1>0$ and $\varphi_2(t)\to c_2\neq 0$, as $t\to\infty$. Such equation possesses $\mathcal{SV}_q$ and $\mathcal{RV}_q(1-\frac{\lambda}{\alpha})$ solutions. Indeed, since

$$\lim_{t\to\infty}\frac{t^{\alpha+1}r(t)}{p(t)} = \lim_{t\to\infty}\frac{(\ln t)^{\theta_2}\varphi_2(t)}{(\ln t)^{\theta_1}\varphi_1(t)} = 0,$$

Theorem 1.1 leads to the such conclusion. To obtain the asymptotic formulas for these solutions, notice that

$$G(t)\sim\Phi^{-1}\Big(\frac{c_2}{c_1}\Big)\frac{(\ln t)^{\theta_2-\theta_1}}{t} \quad\text{and}\quad \hat G(t)\sim\frac{c_2(\ln t)^{\theta_2-\theta_1}}{c_1t}, \quad t\to\infty.$$

For the sake of simplicity, we use the notation

$$I(a,t,-\delta,G) = \exp\Big(-\delta(1+o(1))\int_a^t G(s)d_qs\Big).$$

Similarly as in Example 6.1 we obtain that above defined I in formulas (3.6) and (3.7) satisfies

$$\begin{aligned} I(a,t,-\delta,G) &= \exp\Big(-(1+o(1))\Phi^{-1}\Big(\frac{c_2}{c_1}\Big)\frac{(q-1)\delta}{(\frac{\theta_2-\theta_1}{\alpha}+1)\ln q}(\ln t)^{\frac{\theta_2-\theta_1}{\alpha}+1}\Big) \\ &= I(t,\infty,\delta,G), \quad t\to\infty, \end{aligned}$$

where the left-hand relation holds if $\theta_2 - \theta_1 > -\alpha$, while the second relation holds if $\theta_2 - \theta_1 < -\alpha$. Analogously, above defined I in formulas (3.13) and (3.14) i.e. (3.15) satisfies

$$\begin{aligned} I(a, t, -\frac{\hat{\hat{\delta}}}{\alpha}, \hat{G}) &= \exp\Big(-(1+o(1)) \frac{(q-1)\hat{\hat{\delta}} c_2}{\alpha c_1(\theta_2 - \theta_1 + 1)\ln q} (\ln t)^{\theta_2 - \theta_1 + 1} \Big) \\ &= I(t, \infty, \frac{\hat{\hat{\delta}}}{\alpha}, \hat{G}), \quad t \to \infty, \end{aligned}$$

where the first relation holds for $\theta_2 - \theta_1 > -1$ and the second one for $\theta_2 - \theta_1 < -1$.

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