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Existence results for second-order neutral functional differential and integrodifferential inclusions in Banach spaces *

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Abstract

In this paper, we investigate the existence of mild solutions on a compact interval to second order neutral functional differential and integrodifferential inclusions in Banach spaces. The results are obtained by using the theory of continuous cosine families and a fixed point theorem for condensing maps due to Martelli.

1 Introduction

In this paper we prove the existence of mild solutions, defined on a compact interval, for second-order neutral functional differential and integrodifferential inclusions in Banach spaces. In Section 3 we consider the second-order neutral functional differential inclusion

$$\begin{gathered}\frac{d}{dt}[y'(t) - g(t, y_t)] \in Ay(t) + F(t, y_t), \quad t \in J = [0, T], \\ y_0 = \phi, \quad y'(0) = x_0,\end{gathered} \tag{1.1}$$

where $J_0 = [-r, 0]$, $F : J \times C(J_0, E) \to 2^E$ is a bounded, closed, convex valued multivalued map, $g : J \times C(J_0, E) \to E$ is given function, $\phi \in C(J_0, E)$, $x_0 \in E$, and A is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in R\}$ in a real Banach space E with the norm $|\cdot|$.

For a continuous function y defined on the interval $J_1 = [-r, T]$ and $t \in J$, we denote by y_t the element of $C(J_0, E)$ defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in J_0.$$

Here $y_t(\cdot)$ represents the history of the state from time $t - r$, up to the present time t.

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In Section 4 we investigate the existence of mild solutions for second order neutral functional integrodifferential inclusion

$$\begin{gathered}\frac{d}{dt}[y'(t) - g(t, y_t)] \in Ay(t) + \int_0^t K(t,s)F(s, y_s)ds, \quad t \in J = [0, T], \\ y_0 = \phi, \quad y'(0) = x_0,\end{gathered} \tag{1.2}$$

where A, F, g, ϕ are as in the problem (1.1) and $K : D \to R$, $D = \{(t, s) \in J \times J : t \geq s\}$.

In many cases it is advantageous to treat the second order abstract differential equations directly rather than to convert them into first order systems. A useful tool for the study of abstract second order equations is the theory of strongly continuous cosine families. Here we use of the basic ideas from cosine family theory [17, 18].

Existence results for differential inclusions on compact intervals, are given in the papers of Avgerinos and Papageorgiou [1], Papageorgiou [15, 16], and Benchohra [3, 4] for differential inclusions on noncompact intervals.

This paper is motivated by the recent papers of Benchohra and Ntouyas [4, 5, 6] and Ntouyas [14]. In [4] second order functional differential inclusions are studied. In [5,6] functional differential and integrodifferential inclusions are studied. In [14] neutral functional integrodifferential equations was studied. Here we compose the above results and prove the existence of mild solutions for problems (1.1) and (1.2), relying on a fixed point theorem for condensing maps due to Martelli [13].

2 Preliminaries

In this section, we introduce notation, definitions, and preliminary facts from multivalued analysis which are used throughout this paper.

Let $C(J, E)$ be the Banach space of continuous functions from J into E with the norm

$$\|y\|_\infty := \sup\{|y(t)| : t \in J\}.$$

Let $B(E)$ denote the Banach space of bounded linear operators from E into E. A measurable function $y : J \to E$ is Bochner integrable if and only if $|y|$ is Lebesque integrable. (For properties of the Bochner integral see Yosida [19].)

Let $L^1(J, E)$ denotes the Banach space of continuous functions $y : J \to E$ which are Bochner integrable, with the norm

$$\|y\|_{L^1} = \int_0^T |y(t)|dt \quad \text{for all } y \in L^1(J, E).$$

Let $(X, \|\cdot\|)$ be a Banach space. A multivalued map $G : X \to 2^X$ is convex (closed) valued, if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(D) = \bigcup_{x \in D} G(x)$ is bounded in X, for any bounded set D of X, i.e.,

$$\sup_{x \in D}\{\sup\{\|y\| : y \in G(x)\}\} < \infty.$$

A map G is called upper semicontinuous on X if, for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X and if for each open set V of X containing $G(x_0)$, there exists an open neighborhood A of x_0 such that $G(A) \subseteq V$.

A map G is said to be completely continuous if $G(D)$ is relatively compact for every bounded subset $D \subseteq X$. If the multivalued map G is completely continuous with nonempty compact values, then G is upper semicontinuous if and only if G has a closed graph, i.e., for $x_n \to x_*$, $y_n \to y_*$, with $y_n \in Gx_n$ we have $y_* \in Gx_*$. The map G has a fixed point if there is $x \in X$ such that $x \in Gx$.

In the following, $BCC(X)$ denotes the set of all nonempty bounded closed and convex subsets of X. A multivalued map $G : J \to BCC(X)$ is said to be measurable if for each $x \in X$, the distance between x and $G(t)$ is a measurable function on J. For more details on multivalued maps, see the books of Deimling [7] and Hu and Papageorgiou [11].

An upper semicontinuous map $G : X \to 2^X$ is said to be condensing if, for any bounded subset $D \subseteq X$, with $\alpha(D) \neq 0$, we have

$$\alpha(G(D)) < \alpha(D),$$

where α denotes the Kuratowski measure of noncompactness. For properties of the Kuratowski measure, we refer to Banas and Goebel [2].

We remark that a completely continuous multivalued map is the easiest example of a condensing map.

We say that the family $\{C(t) : t \in R\}$ of operators in $B(E)$ is a strongly continuous cosine family if

(i) $C(0) = I$, is the identity operator in E

(ii) $C(t+s) + C(t-s) = 2C(t)C(s)$ for all $s, t \in R$

(iii) The map $t \to C(t)y$ is strongly continuous for each $y \in X$.

The strongly continuous sine family $\{S(t) : t \in R\}$, associated to the given strongly continuous cosine family $\{C(t) : t \in R\}$, is defined by

$$S(t)y = \int_0^t C(s)y\,ds, \quad y \in E,\ t \in R.$$

The infinitesimal generator $A : E \to E$ of a cosine family $\{C(t) : t \in R\}$ is defined by

$$Ay = \frac{d^2}{dt^2} C(t)y\Big|_{t=0}.$$

For more details on strongly continuous cosine and sine families, we refer the reader to the books of Goldstein [10] and to the papers of Fattorini [8, 9] and of Travis and Webb [17, 18].

The considerations of this paper are based on the following fixed point theorem.

Lemma 2.1 ([13]) *Let X be a Banach space and $N : X \to BCC(X)$ be a condensing map. If the set $\Omega := \{y \in X : \lambda y \in Ny,$ for some $\lambda > 1\}$ is bounded, then N has a fixed point.*

3 Second Order Neutral Differential Inclusions

In this section we give an existence result for the problem (1.1). Let us list the following hypotheses.

(H1) A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in R$, of bounded linear operators from E into itself.

(H2) $C(t)$, $t > 0$ is compact.

(H3) $F : J \times C(J_0, E) \to BCC(E)$; $(t, u) \to F(t, u)$ is measurable with respect to t for each $u \in C(J_0, E)$, upper semicontinuous with respect to u for each $t \in J$, and for each fixed $u \in C(J_0, E)$, the set

$$S_{F,u} = \{f \in L^1(J, E) : f(t) \in F(t, u) \text{ for a.e. } t \in J\}$$

is nonempty.

(H4) The function $g : J \times C(J_0, E) \to E$ is completely continuous and for any bounded set K in $C(J_1, E)$, the set $\{t \to g(t, y_t) : y \in K\}$ is equicontinuous in $C(J, E)$.

(H5) There exist constants c_1 and c_2 such that

$$|g(t, v)| \le c_1\|v\| + c_2, \quad t \in J,\ v \in C(J_0, E)$$

(H6) $\|F(t, u)\| := \sup\{|v| : v \in F(t, u)\} \le p(t)\Psi(\|u\|)$ for almost all $t \in J$ and $u \in C(J_0, E)$, where $p \in L^1(J, R_+)$ and $\Psi : R_+ \to (0, \infty)$ is continuous and increasing with

$$\int_0^T m(s)ds < \int_c^\infty \frac{ds}{s + \Psi(s)},$$

where $c = M\|\phi\| + MT[|x_0| + c_1\|\phi\| + 2c_2]$, $m(t) = \max\{Mc_1, MTp(t)\}$ and $M = \sup\{|C(t)| : t \in J\}$.

Remark (i) If $\dim E < \infty$, then for each $v \in C(J_0, E)$, $S_{F,u} \neq \phi$ (see Lasota and Opial [10]).
(ii) $S_{F,u}$ is nonempty if and only if the function $Y : J \to R$ defined by

$$Y(t) := \inf\{|v| : v \in F(t, u)\}$$

belongs to $L^1(J, R)$ (see Papageorgiou[15]).

In order to define the concept of mild solution for (1.1), by comparison with abstract Cauchy problem

$$\begin{gathered} y''(t) = Ay(t) + h(t) \\ y(0) = y_0, \quad y'(0) = y_1 \end{gathered}$$

whose properties are well known [17, 18], we associate problem (1.1) to the integral equation

$$y(t) = C(t)\phi(0) + S(t)[x_0 - g(0,\phi)] + \int_0^t C(t-s)g(s,y_s)ds + \int_0^t S(t-s)f(s)ds, \tag{3.1}$$

$t \in J$, where

$$f \in S_{F,y} = \{f \in L^1(J,E) : f(t) \in F(t,y_t) \text{ for a.e. } t \in J\}.$$

Definition A function $y : (-r,T) \to E$, $T > 0$ is called a mild solution of the problem (1.1) if $y(t) = \phi(t)$, $t \in [-r,0]$, and there exists a $v \in L^1(J,E)$ such that $v(t) \in F(t,y_t)$ a.e. on J, and the integral equation (3.1) is satisfied.

The following lemmas are crucial in the proof of our main theorem.

Lemma 3.1 ([12]) *Let I be a compact real interval, and let X be a Banach space. Let F be a multivalued map satisfying $(H3)$, and let Γ be a linear continuous mapping from $L^1(I,X)$ to $C(I,X)$. Then, the operator*

$$\Gamma \circ S_F : C(I,X) \to BCC(C(I,X)), \quad y \to (\Gamma \circ S_F)(y) = \Gamma(S_{F,y})$$

is a closed graph operator in $C(I,X) \times C(I,X)$.

Now, we are able to state and prove our main theorem.

Theorem 3.2 *Assume that Hypotheses (H1)-(H6) are satisfied. Then system* (1.1) *has at least one mild solution on J_1.*

Proof. Let $C := C(J_1,E)$ be the Banach space of continuous functions from J_1 into E endowed with the supremum norm

$$\|y\|_\infty := \sup\{|y(t)| : t \in J_1\}, \quad \text{for } y \in C.$$

Now we transform the problem into a fixed point problem. Consider the multivalued map, $N : C \to 2^C$ defined by Ny the set of functions $h \in C$ such that

$$h(t) = \begin{cases} \phi(t), & \text{if } t \in J_0 \\ C(t)\phi(0) + S(t)[x_0 - g(0,\phi)] & \\ + \displaystyle\int_0^t C(t-s)g(s,y_s)ds + \int_0^t S(t-s)f(s)ds, & \text{if } t \in J \end{cases}$$

where

$$f \in S_{F,y} = \{f \in L^1(J,E) : f(t) \in F(t,y_t) \text{ for a.e. } t \in J\}.$$

We remark that the fixed points of N are mild solutions to (1.1).

We shall show that N is completely continuous with bounded closed convex values and it is upper semicontinuous. The proof will be given in several steps.
Step 1. Ny is convex for each $y \in C$. Indeed, if h_1, h_2 belong to Ny, then there exist $f_1, f_2 \in S_{F,y}$ such that, for each $t \in J$ and $i = 1,2$, we have

$$h_i(t) = C(t)\phi(0) + S(t)[x_0 - g(0,\phi)] + \int_0^t C(t-s)g(s,y_s)ds + \int_0^t S(t-s)f_i(s)ds.$$

Let $0 \le \alpha \le 1$. Then, for each $t \in J$, we have

$$\begin{aligned}(\alpha h_1 + (1-\alpha)h_2)(t) &= C(t)\phi(0) + S(t)[x_0 - g(0,\phi)] + \int_0^t C(t-s)g(s,y_s)ds \\ &\quad + \int_0^t S(t-s)[\alpha f_1(s) + (1-\alpha)f_2(s)]\,ds.\end{aligned}$$

Since $S_{F,y}$ is convex (because F has convex values), then

$$\alpha h_1 + (1-\alpha)h_2 \in Ny.$$

Step 2. N maps bounded sets into bounded sets in C. Indeed, it is enough to show that there exists a positive constant ℓ such that, for each $h \in Ny$, $y \in B_q = \{y \in C : \|y\|_\infty \le q\}$, one has $\|h\|_\infty \le \ell$. If $h \in Ny$, then there exists $f \in S_{F,y}$ such that for each $t \in J$ we have

$$h(t) = C(t)\phi(0) + S(t)[x_0 - g(0,\phi)] + \int_0^t C(t-s)g(s,y_s)ds + \int_0^t S(t-s)f(s)ds.$$

By (H5) and (H6), we have that, for each $t \in J$,

$$\begin{aligned}|h(t)| &\le |C(t)\phi(0)| + |S(t)[x_0 - g(0,\phi)]| + \Big|\int_0^t C(t-s)g(s,y_s)ds\Big| \\ &\quad + \Big|\int_0^t S(t-s)f(s)ds\Big| \\ &\le M\|\phi\| + MT[|x_0| + c_1\|\phi\| + 2c_2] + Mc_1\int_0^t \|y_s\|ds \\ &\quad + MT \sup_{y\in[0,q]} \Psi(y)\Big(\int_0^t p(s)ds\Big)\end{aligned}$$

Then for each $h \in N(B_q)$ we have

$$\begin{aligned}\|h\|_\infty &\le M\|\phi\| + MT[|x_0| + c_1\|\phi\| + 2c_2] + Mc_1\int_0^T \|y_s\|ds \\ &\quad + MT \sup_{y\in[0,q]} \Psi(y)\Big(\int_0^T p(s)ds\Big) := \ell.\end{aligned}$$

Step 3. N maps bounded sets into equicontinuous sets of C. Let $t_1, t_2 \in J$, $0 < t_1 < t_2$, and let $B_q = \{y \in C : \|y\|_\infty \leq q\}$ be a bounded set of $C(J_1, E)$. For each $y \in B_q$ and $h \in Ny$, there exists $f \in S_{F,y}$ such that for $t \in J$,

$$h(t) = C(t)\phi(0) + S(t)[x_0 - g(0,\phi)] + \int_0^t C(t-s)g(s,y_s)ds + \int_0^t S(t-s)f(s)\,ds\,.$$

Thus,

$$\begin{aligned}
&|h(t_2) - h(t_1)|\\
&\leq \big|[C(t_2) - C(t_1)]\phi(0)\big| + \big|[S(t_2) - S(t_1)][x_0 - g(0,\phi)]\big|\\
&\quad +\big|\int_0^{t_2} [C(t_2 - s) - C(t_1 - s)]g(s,y_s)ds\big| + \big|\int_{t_1}^{t_2} C(t_1 - s)g(s,y_s)ds\big|\\
&\quad +\big|\int_0^{t_2} [S(t_2 - s) - S(t_1 - s)]f(s)ds\big| + \big|\int_{t_1}^{t_2} S(t_1 - s)f(s)ds\big|\\
&\leq |C(t_2) - C(t_1)|\|\phi\| + |S(t_2) - S(t_1)|[|x_0| + c_1\|\phi\| + c_2]\\
&\quad + \int_0^{t_2} |C(t_2 - s) - C(t_1 - s)|[c_1\|y_s\| + c_2]ds\\
&\quad + \int_{t_1}^{t_2} |C(t_1 - s)|[c_1\|y_s\| + c_2]ds\\
&\quad + \int_0^{t_2} |S(t_2 - s) - S(t_1 - s)|\|f(s)\|ds + \int_{t_1}^{t_2} |S(t_1 - s)|\|f(s)\|ds.
\end{aligned}$$

As $t_2 \to t_1$ the right-hand side of the above inequality tend to zero. The equicontinuities for the cases $t_1 < t_2 \leq 0$ and $t_1 \leq 0 \leq t_2$ are obvious. As a consequence of Step 2, Step 3, (H2) and (H4) together with the Ascoli-Arzela theorem, we can conclude that $N : C \to 2^C$ is a compact multivalued map, and therefore, a condensing map.

Step 4. N has a closed graph. Let $y_n \to y_*$, $h_n \in Ny_n$, and $h_n \to h_*$. We shall prove that $h_* \in Ny_*$. $h_n \in Ny_n$ means that there exists $f_n \in S_{F,y_n}$, such that for $t \in J$,

$$h_n(t) = C(t)\phi(0) + S(t)[x_0 - g(0,\phi)] + \int_0^t C(t-s)g(s,y_{ns})ds + \int_0^t S(t-s)f_n(s)ds\,.$$

We must prove that there exists $f_* \in S_{F,y_*}$ such that for $t \in J$,

$$h_*(t) = C(t)\phi(0) + S(t)[x_0 - g(0,\phi)] + \int_0^t C(t-s)g(s,y_{*s})ds + \int_0^t S(t-s)f_*(s)ds\,.$$

Clearly, we have that as $n \to \infty$,

$$\begin{aligned}
&\Big\|\Big(h_n - C(t)\phi(0) - S(t)[x_0 - g(0,\phi)] - \int_0^t C(t-s)g(s,y_{ns})ds\Big)\\
&-\Big(h_* - C(t)\phi(0) - S(t)[x_0 - g(0,\phi)] - \int_0^t C(t-s)g(s,y_{*s})ds\Big)\Big\|_\infty \to 0\,.
\end{aligned}$$

Consider the linear and continuous operator $\Gamma : L^1(J,E) \to C(J,E)$ defined as

$$f \to \Gamma(f)(t) = \int_0^t S(t-s)f(s)ds.$$

From Lemma 3.1, it follows that $\Gamma \circ S_F$ is a closed graph operator. Moreover, we have that

$$h_n(t) - C(t)\phi(0) - S(t)[x_0 - g(0,\phi)] - \int_0^t C(t-s)g(s,y_{ns})ds \in \Gamma(S_{F,y_n}).$$

Since $y_n \to y_*$, it follows from Lemma 3.1 that

$$h_*(t) - C(t)\phi(0) - S(t)[x_0 - g(0,\phi)] - \int_0^T C(t-s)g(s,y_{*s})ds = \int_0^t S(t-s)f_*(s)ds$$

for some $f_* \in S_{F,y_*}$. Therefore N is a completely continuous multivalued map, upper semicontinuous with convex closed values. In order to prove that N has a fixed point, we need one more step.

Step 5. The set

$$\Omega := \{y \in C : \lambda y \in Ny, \text{ for some } \lambda > 1\}$$

is bounded. Let $y \in \Omega$. Then $\lambda y \in Ny$ for some $\lambda > 1$. Thus, there exists $f \in S_{F,y}$ such that

$$\begin{aligned} y(t) &= \lambda^{-1}C(t)\phi(0) + \lambda^{-1}S(t)[x_0 - g(0,\phi)] + \lambda^{-1}\int_0^t C(t-s)g(s,y_s)ds \\ &\quad + \lambda^{-1}\int_0^t S(t-s)f(s)ds, \quad t \in J. \end{aligned}$$

This implies by (H5)-(H6) that for each $t \in J$, we have

$$\begin{aligned} |y(t)| &\le M\|\phi\| + MT[|x_0| + c_1\|\phi\| + 2c_2] \\ &\quad + Mc_1\int_0^t \|y_s\|ds + MT\int_0^t p(s)\Psi(\|y_s\|)ds. \end{aligned}$$

We consider the function

$$\mu(t) = \sup\{|y(s)| : -r \le s \le t\}, \quad t \in J.$$

Let $t^* \in [-r,t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in J$, by the previous inequality we have for $t \in J$,

$$\begin{aligned} \mu(t) &\le M\|\phi\| + MT[|x_0| + c_1\|\phi\| + 2c_2] + Mc_1\int_0^{t^*} \|y_s\|ds \\ &\quad + MT\int_0^{t^*} p(s)\Psi(\|y_s\|)ds \\ &\le M\|\phi\| + MT[|x_0| + c_1\|\phi\| + 2c_2] + Mc_1\int_0^t \mu(s)ds \\ &\quad + MT\int_0^t p(s)\Psi(\mu(s))ds. \end{aligned}$$

If $t^* \in J_0$, then $\mu(t) \le \|\phi\|$ and the previous inequality obviously holds. Let us denote the right-hand side of the above inequality as $v(t)$. Then, we have

$$\begin{gathered} c = v(0) = M\|\phi\| + MT[|x_0| + c_1\|\phi\| + 2c_2], \\ \mu(t) \le v(t), \quad t \in J, \\ v'(t) = Mc_1\mu(t) + MTp(t)\Psi(\mu(t)), \quad t \in J. \end{gathered}$$

Using the nondecreasing character of Ψ, we get

$$v'(t) \le Mc_1 v(t) + MTp(t)\Psi(v(t)) \le m(t)[v(t) + \Psi(v(t))], \quad t \in J.$$

This implies that for each $t \in J$ that

$$\int_{v(0)}^{v(t)} \frac{ds}{s + \Psi(s)} \le \int_0^T m(s)ds < \int_{v(0)}^{\infty} \frac{ds}{s + \Psi(s)}.$$

This inequality implies that there exists a constant L such that $v(t) \le L$, $t \in J$, and hence $\mu(t) \le L$, $t \in J$. Since for every $t \in J$, $\|y_t\| \le \mu(t)$, we have

$$\|y\|_\infty := \sup\{|y(t)| : -r \le t \le T\} \le L,$$

where L depends only on T and on the function p and Ψ. This shows that Ω is bounded.

Set $X := C$. As a consequence of Lemma 2.1, we deduce that N has a fixed point which is a mild solution of the system (1.1).

4 Second Order Neutral Integrodifferential Inclusions

In this section we consider the solvability of the problem (1.2). We need the following assumptions

(H7) For each $t \in J$, $K(t, s)$ is measurable on $[0, t]$ and

$$K(t) = \text{ess sup}\{|K(t, s)|, 0 \le s \le t\}$$

is bounded on J.

(H8) The map $t \to K_t$ is continuous from J to $L^\infty(J, R)$, here $K_t(s) = K(t, s)$.

(H9) $\|F(t, u)\| := \sup\{|v| : v \in F(t, u)\} \le p(t)\Psi(\|u\|)$ for almost all $t \in J$ and $u \in C(J_0, E)$, where $p \in L^1(J, R_+)$ and $\Psi : R_+ \to (0, \infty)$ is continuous and increasing with

$$\int_0^T m(s)ds < \int_c^\infty \frac{ds}{s + \Psi(s)},$$

where $c = M\|\phi\| + MT[|x_0| + c_1\|\phi\| + 2c_2]$, $m(t) = \max\{Mc_1, MT^2 \sup_{t\in J} K(t)p(t)\}$ and $M = \sup\{|C(t)| : t \in J\}$.

We define the mild solution for the problem (1.2) by the integral equation

$$\begin{aligned} y(t) =& C(t)\phi(0) + S(t)[x_0 - g(0,\phi)] + \int_0^t C(t-s)g(s,y_s)ds \\ &+ \int_0^t S(t-s)\int_0^s K(s,u)f(u)duds, \quad t \in J, \end{aligned} \tag{4.1}$$

where $f \in S_{F,y} = \{f \in L^1(J,E) : f(t) \in F(t,y_t) \text{ for a.e. } t \in J\}$.

Definition A function $y : (-r,T) \to E$, $T > 0$ is called a mild solution of the problem (1.2) if $y(t) = \phi(t)$, $t \in [-r,0]$, and there exists a $v \in L^1(J,E)$ such that $v(t) \in F(t,y_t)$ a.e. on J, and the integral equation (4.1) is satisfied.

Theorem 4.1 *Assume that hypotheses (H1)–(H5), (H7)–(H9) are satisfied. Then system* (1.2) *has at least one mild solution on* J_1.

Proof. Let $C := C(J_1,E)$ be the Banach space of continuous functions from J_1 into E endowed with the supremum norm

$$\|y\|_\infty := \sup\{|y(t)| : t \in J_1\}, \text{ for } y \in C.$$

We transform the problem into a fixed point problem. Consider the multivalued map, $Q : C \to 2^C$ defined by Qy, the set of functions $h \in C$ such that

$$h(t) = \begin{cases} \phi(t), & \text{if } t \in J_0 \\ C(t)\phi(0) + S(t)[x_0 - g(0,\phi)] + \int_0^t C(t-s)g(s,y_s)\,ds & \\ + \int_0^t S(t-s)\int_0^s K(s,u)f(u)\,du\,ds, & \text{if } t \in J, \end{cases}$$

where

$$f \in S_{F,y} = \{f \in L^1(J,E) : f(t) \in F(t,y_t) \text{ for a.e. } t \in J\}.$$

We remark that the fixed points of Q are mild solutions to (1.2).

As in Theorem 3.1 we can show that Q is completely continuous with bounded closed convex values and it is upper semicontinuous, and therefore a condensing map. We repeat only the Step 5, i.e. we show that the set

$$\Omega := \{y \in C : \lambda y \in Qy, \text{ for some } \lambda > 1\}$$

is bounded. Let $y \in \Omega$. Then $\lambda y \in Qy$ for some $\lambda > 1$. Thus, there exists $f \in S_{F,y}$ such that

$$\begin{aligned} y(t) &= \lambda^{-1}C(t)\phi(0) + \lambda^{-1}S(t)[x_0 - g(0,\phi)] + \lambda^{-1}\int_0^t C(t-s)g(s,y_s)ds \\ &\quad + \lambda^{-1}\int_0^t S(t-s)\int_0^s K(s,u)f(u)\,du\,ds, \quad t \in J. \end{aligned}$$

This implies by (H5)-(H6) that for each $t \in J$, we have

$$\begin{aligned} |y(t)| &\leq M\|\phi\| + MT[|x_0| + c_1\|\phi\| + 2c_2] \\ &\quad + Mc_1 \int_0^t \|y_s\| ds + MT^2 \sup_{t\in J} K(t) \int_0^t p(s)\Psi(\|y_s\|) ds. \end{aligned}$$

We consider the function

$$\mu(t) = \sup\{|y(s)| : -r \leq s \leq t\}, \quad t \in J.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in J$, by the previous inequality we have for $t \in J$,

$$\begin{aligned} \mu(t) &\leq M\|\phi\| + MT[|x_0| + c_1\|\phi\| + 2c_2] \\ &\quad + Mc_1 \int_0^{t^*} \|y_s\| ds + MT^2 \sup_{t\in J} K(t) \int_0^{t^*} p(s)\Psi(\|y_s\|) ds \\ &\leq M\|\phi\| + MT[|x_0| + c_1\|\phi\| + 2c_2] \\ &\quad + Mc_1 \int_0^t \mu(s) ds + MT^2 \sup_{t\in J} K(t) \int_0^t p(s)\Psi(\mu(s)) ds. \end{aligned}$$

If $t^* \in J_0$, then $\mu(t) \leq \|\phi\|$ and the previous inequality obviously holds.

Let us denote the right-hand side of the above inequality as $v(t)$. Then, we have

$$\begin{gathered} c = v(0) = M\|\phi\| + MT[|x_0| + c_1\|\phi\| + 2c_2], \\ \mu(t) \leq v(t), \quad t \in J, \\ v'(t) = Mc_1\mu(t) + MT^2 \sup_{t\in J} K(t)p(t)\Psi(\mu(t)), \quad t \in J. \end{gathered}$$

Using the nondecreasing character of Ψ, for $t \in J$,

$$v'(t) \leq Mc_1 v(t) + MT^2 \sup_{t\in J} K(t)p(t)\Psi(v(t)) \leq m(t)[v(t) + \Psi(v(t))].$$

This implies that for each $t \in J$,

$$\int_{v(0)}^{v(t)} \frac{ds}{s + \Psi(s)} \leq \int_0^T m(s) ds < \int_{v(0)}^{\infty} \frac{ds}{s + \Psi(s)}.$$

This inequality implies that there exists a constant L such that $v(t) \leq L$, $t \in J$, and hence $\mu(t) \leq L$, $t \in J$. Since for every $t \in J$, $\|y_t\| \leq \mu(t)$, we have

$$\|y\|_\infty := \sup\{|y(t)| : -r \leq t \leq T\} \leq L,$$

where L depends only on T and on the function p and Ψ. This shows that Ω is bounded.

Set $X := C$. As a consequence of Lemma 2.1, we deduce that Q has a fixed point and thus system (1.1) is controllable on J_1.

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