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Solutions to elliptic systems of Hamiltonian type in $\mathbb{R}^N$ *

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Abstract

The paper proves existence of a solution for elliptic systems of Hamiltonian type on $\mathbb{R}^N$ by a variational method. We use the Benci-Rabinowitz technique, which cannot be applied here directly for lack of compactness. However, a concentration compactness technique allows us to construct a finite-dimensional pseudogradient that restores the Benci-Rabinowitz method to power also for problems on unbounded domains.

1 Introduction

The present paper deals with a variational elliptic problem of Hamiltonian type, i.e., with a functional that has a saddle-point geometry where both positive and negative subspaces of the quadratic form are infinte-dimensional. The Benci-Rabinowitz approach to such functionals requires the functional to be a sum of a quadratic form and a weakly continuous term (we refer the reader to the elaborate exposition in [1]). To assure linking of infinite-dimensional spheres, and thus existence of a critical sequence, they restrict the class of deformations to flows of vector fields which are sums of a field, roughly speaking, with radial direction, and a field that over every bounded set has a finite-dimensional span. We remark that inifinte-dimensional spheres do not link even when the deformations are restricted to rotations and parallel translations ([7]).

We construct Benci-Rabinowitz deformations without requiring compactness for the perturbation of the quadratic form, using instead the concentrated compactness on $\mathbb{R}^N$. The construction is isolated into a separate lemma (Lemma 2.2). Section 2 of the paper deals with the application to elliptic system of a Hamiltonian type (cf. [2] and references therein for the case of bounded domains), while leaving the proof of Lemma 2.2 to Section 3. The application serves merely as an example (and follows several steps from [1, 2] and similar work) to justify the construction of Section 3, which can be used in further variational problems where lack of compactness complicates construction of deformations that preserve linking.

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2 A semilinear elliptic system

We shall study existence of a nonzero solution to the problem

$$\begin{aligned} -\Delta u + au &= \gamma v + F_u(u,v) \\ -\Delta v + bv &= -\gamma u - F_v(u,v) \qquad (2.1) \\ u, v &\in W^{1,2}(\mathbb{R}^N) \setminus \{0\}, N \geq 3\,. \qquad (2.2) \end{aligned}$$

We will use the notation $2^* = 2N/(N-2)$ for the critical exponent. We make the following assumptions:

$$a, b > 0, \quad \gamma \neq 0, \quad F \in C^1(\mathbb{R}^2); \qquad (2.3)$$

$$\begin{aligned} F(u,0) &\leq C|u|^q, \quad q > 2, \qquad (2.4) \\ F_v(u,v) &\leq C(|v| + |v|^r)(1+u^2), \quad C > 0, 2 < r < 2^*; \end{aligned}$$

$$|F_u(u,v)| + |F_v(u,v)| \leq C(|u| + |v| + |u|^{p-1} + |v|^{p-1}), \qquad (2.5)$$

where $C > 0$ and $p \in (2, 2^*)$;

$$\begin{aligned} F_u(u,v)u + F_v(u,v)v \geq \sigma F(u,v) \geq 0, \quad \sigma > 2; \qquad (2.6) \\ F_u(u,v)u - F_v(u,v)v \leq CF(u,v), \quad C > 0\,. \qquad (2.7) \end{aligned}$$

An example of a function satisfying all these conditions for $N = 3$ is $F(u,v) = u^4 + 2v^4 - u^2v^2$.

We denote now as H the space $W^{1,2}(\mathbb{R}^N \to \mathbb{R}^2)$ of 2-component Sobolev functions with the norm

$$\|(u,v)\|^2 = \|u\|_a^2 + \|v\|_b^2 = \int (|\nabla u|^2 + au^2) dx + \int (|\nabla v|^2 + bv^2)\, dx,$$

Scalar products will be denoted as $\langle x, y\rangle$ for points in H, and $\langle u, \varphi\rangle_a$ or $\langle v, \varphi\rangle_b$ for the u- (resp. the v-) components of vectors in H. An open ball on H of radius R centered at w will be denoted as $B(w, R)$.

Solutions of (2.1) are critical points for the following C^1- functional on H:

$$G(u,v) = \int_{\mathbb{R}^N} (\frac{1}{2}|\nabla u|^2 - \frac{1}{2}|\nabla v|^2 + \frac{1}{2}au^2 - \frac{1}{2}bv^2 - \gamma uv - F(u,v))\, dx\,.$$

It should be noted that under Assumption (2.5), the derivative G' is not only continuous, but also weak-to weak continuous on H, that is

$$x_k \overset{w}{\to} x \Rightarrow G'(x_k) \overset{w}{\to} G'(x).$$

The main result of this section is

Theorem 2.1 *Under assumptions (2.3)-(2.7), the system (2.1) has a nonzero solution.*

The crucial technical statement needed for the proof of this theorem is the following lemma.

Lemma 2.2 *Assume (2.3) and (2.5). Let $\kappa > 0$. If the set*

$$\begin{aligned}\Omega(\eta,\kappa) \quad = \quad & \big\{(u,v)\in H:\ |\langle G'(u,v),(u,0)\rangle| \le \eta\|u\|_a^2 \text{ and} \\ & (|\langle G'(u,v),(0,v)\rangle| \le \eta\|v\|_b^2 \text{ or } \|v\|_b \le \eta, |G(u,v)-\kappa| \le \eta)\big\}\end{aligned} \tag{2.8}$$

is bounded for some $\eta > 0$, and $G'(u,v) \neq 0$ whenever $|G(u,v)-\kappa| \le \eta$, then there exists a finite-dimensional subspace W of H, bounded Lipschitz functions $\varphi,\psi : H \to \mathbb{R}$ and a Lipschitz map $z : H \to W$ support in $\Omega(\eta,\kappa)$, such that the map

$$Z(u,v) := (\varphi(u,v)u, \psi(u,v)v) + z(u,v)$$

satisfies the following relations

$$\begin{gathered}|G(u,v)-\kappa| \ge \eta \Rightarrow Z(u,v) = 0 \\ (u,v)\in H \Rightarrow \langle G'(u,v), Z(u,v)\rangle \ge 0, \\ |G(u,v)-\kappa| \le \eta/2 \Rightarrow \langle G'(u,v), Z(u,v)\rangle \ge 1\,.\end{gathered}$$

The proof of this lemma is left for Section 3 and it does not refer to any of the statements in this section.

Lemma 2.3 *Assume (2.4)-(2.7). Then there exists an $\eta > 0$ such that the set $\Omega(\eta,\kappa)$ is bounded.*

Proof. Let us rewrite (2.8). If $(u,v) \in \Omega(\eta,\kappa)$, then

$$-\eta\|u\|_a^2 \le \|u\|_a^2 - \textstyle\int \gamma uv - \int F_u(u,v)u \le \eta\|u\|_a^2, \tag{2.9}$$

$$-\eta\|v\|_b^2 \le \|v\|_b^2 + \textstyle\int \gamma uv + \int F_v(u,v)v \le \eta\|v\|_b^2, \text{ or} \tag{2.10}$$

$$\|v\|_b \le \eta, \tag{2.11}$$

$$\kappa - \eta \le \tfrac{1}{2}\|u\|_a^2 - \tfrac{1}{2}\|v\|_b^2 - \textstyle\int \gamma uv - \int F(u,v) \le \kappa + \eta. \tag{2.12}$$

First, assume (2.10). Let us multiply (2.12) by σ from (2.6), subtract (2.9) and add (2.10). We will have

$$\begin{aligned}&(\sigma/2-1)(\|u\|_a^2 - \|v\|_b^2 - 2\int \gamma uv) - \sigma\int F(u,v) + \int (F_u(u,v)u + F_v(u,v)v) \\ &\le \quad \eta(\|u\|_a^2 + \|v\|_b^2) + \sigma(\kappa+\eta).\end{aligned}$$

which yields, due to (2.6),

$$(\sigma/2-1)(\|u\|_a^2 - \|v\|_b^2 - 2\int \gamma uv) \le \eta(\|u\|_a^2 + \|v\|_b^2) + \sigma(\kappa+\eta).) \tag{2.13}$$

By (2.12),

$$\int F(u,v) \le \frac{1}{2}\|u\|_a^2 - \frac{1}{2}\|v\|_b^2 - \int \gamma uv - \kappa + \eta. \tag{2.14}$$

If we add now (2.9) and (2.10), (2.11) then, using (2.7) we obtain

$$\begin{aligned} \|u\|_a^2 + \|v\|_b^2 &\leq \int (F_u(u,v)u - F_v(u,v)v) + \eta(\|u\|_a^2 + \|v\|_b^2) \\ &\leq C\int F(u,v) + \eta(\|u\|_a^2 + \|v\|_b^2). \end{aligned}$$

We now combine this inequality with (2.14) and (2.13) to obtain

$$\begin{aligned} \|u\|_a^2 + \|v\|_b^2 &\leq C(\frac{1}{2}\|u\|_a^2 - \frac{1}{2}\|v\|_b^2 - \int \gamma uv - \kappa + \eta) + \eta(\|u\|_a^2 + \|v\|_b^2) \\ & \frac{C\eta}{\sigma/2-1}(\|u\|_a^2 + \|v\|_b^2) + \frac{C\sigma}{\sigma/2-1}(\kappa+\eta) \\ & +C(\eta-\kappa) + \eta(\|u\|_a^2 + \|v\|_b^2). \end{aligned}$$

This implies

$$(1 - \frac{C\eta}{\sigma/2-1} - \eta)\|(u,v)\|^2 \leq C'.$$

Therefore, if η is sufficiently small and (2.10) is assumed, the norm of (u,v) on $\Omega(\eta,\kappa)$ is bounded.

Now assume (2.11). From (2.12) follows

$$\frac{1}{2}\|u\|_a^2 - \int F(u,v) \leq \kappa + \eta + C\gamma\eta\|u\| + \frac{1}{2}\eta^2,$$

and therefore

$$\frac{\sigma}{2}\|u\|_a^2 - \sigma\int F(u,v) \leq \sigma\kappa + C\eta + C\eta\|u\|^2. \tag{2.15}$$

¿From (2.9) we derive

$$-C\eta\|u\|_a^2 - C\eta \leq \|u\|_a^2 - \int F_u(u,v)u. \tag{2.16}$$

Subtracting (2.16) from (2.15) we get

$$(\frac{\sigma}{2}-1)\|u\|_a^2 + \int (F_u(u,v)u - \sigma F(u,v)) \leq C\kappa + 2C\eta + 2C\|u\|_a^2,$$

so that applying (2.6) and (2.4) we get

$$\begin{aligned} (\frac{\sigma}{2} - 1 - 2C\eta)\|u\|_a^2 &\leq C + \int (\sigma F(u,v) - F_u(u,v)u) \\ &= C + \int (\sigma F(u,v) - F_u(u,v)u - F_v(u,v)v) + \int F_v(u,v)v \\ &\leq C + \int F_v(u,v)v \leq C + C\eta\|u\|_a^2, \end{aligned}$$

which in turn implies that $\|(u,v)\|$ is bounded. □

We will check now the geometric conditions for the critical point argument.

Lemma 2.4 *There exist $\rho > 0$, $R > 0$ and $u_0 \in W^{1,2}(\mathbb{R}^N)$ such that*

$$\inf G(A) > 0 \text{ and } \sup G(B) = 0,$$

where $A = \{(u,0) \in H : \|u\|_a = \rho)\}$ and

$$B = [0,R]u_0 \times \{v : (0,v) \in H : \|v\|_b = R\} \bigcup \{0, Ru_0\} \times \{(0,v) \in H : \|v\|_b \le R\}.$$

Proof. To estimate the functional G on A, we use (2.4),

$$G(u,0) \ge \frac{1}{2}\|u\|_a^2 - C\int |u|^q \ge \frac{1}{2}\|u\|_a^2 - C\|u\|_a^q = 1/2\rho^2 - C\rho^q,$$

which is a positive quantity for a certain ρ, which form now on will be fixed.

To estimate G on B, we will consider it as a union of three subsets:

$$\begin{aligned} B_1 &= \{(tu_0, v) : 0 \le t \le R, \|v\|_b = R\}, \\ B_2 &= \{(0,v) : \|v\|_b \le R\}, \text{ and} \\ B_3 &= \{(Ru_0, v) : \|v\|_b \le R\}. \end{aligned}$$

The functional G is non-positive on B_2 due to (2.6). On B_1, one can use (2.6) to get the estimate

$$\begin{aligned} G(tu_0, v) &\le -\frac{1}{2}R^2 + \frac{1}{2}R^2\|u_0\|_a^2 - t\gamma \int u_0 v\, dx \\ &\le -\frac{1}{2}R^2(1 - \|u_0\|_a^2 - C\gamma\|u_0\|_a) \le 0 \end{aligned}$$

when $\epsilon := \|u_0\|_a$ is sufficiently small. Finally, on B_3, using the first inequality of (2.6), we have

$$\begin{aligned} G(Ru_0, v) &\le \frac{1}{2}R^2\epsilon^2 - R\gamma \int u_0 v dx - CR^\sigma \int |u_0|^\sigma \\ &\le \frac{1}{2}R^2\epsilon^2 + CR^2\epsilon - CR^\sigma\epsilon^\sigma \\ &\le 0 \end{aligned}$$

for R sufficiently large. $\square$

Let H_U, H_V be the subspaces of H consisting of vectors of the form $(u,0)$ and $(0,v)$ respectively.

Definition 2.5 *We shall say that a map $S \in C([0,1] \times H; H)$ is almost radial if there is a neighborhood of the origin where $S(t,\cdot)$ is the identity function for all t, the subspaces H_U and H_V admit an orthogonal decomposition into spaces Y_U, W_U and Y_V, W_V respectively, $\dim W_U + \dim W_V < \infty$ and there are locally Lipschitz and uniformly bounded maps $\alpha, \beta : [0,1] \times H \to \mathbb{R} \setminus \{0\}$ such that*

$$S(t,u,v) - (\alpha(t,u,v)u, \beta(t,u,v)v) \in W := W_U \oplus W_V.$$

Lemma 2.6 *If A is as in Lemma 2.4,*

$$B_0 = \{(u,v) \in H : u \in [0,R]u_0, \|v\|_b \leq R\}$$

and S is an almost radial map such that $S(t,u,v) = (u,v)$ for all $(u,v) \in B := \partial B_0$, then for any $t \in [0,1]$,

$$S(t,B_0) \bigcap A \neq \emptyset. \tag{2.17}$$

Proof. For every $t \in [0,1]$ consider a map

$$\Phi_t : B_0 \to H \times \mathbb{R}, \Phi_t(x) = (P_V S(t,x), \|S(t,x)\|),$$

where P_V is the orthogonal projection $P_V(u,v) = (0,v)$. Then a point $x \in B$ contributes to the intersection set (2.17) if and only if

$$\Phi_t(x) = (0,\rho). \tag{2.18}$$

Without loss of generality we assume that $u_0 \in W$. Since the map S is almost radial, $S(t,\theta u_0,v) = (0,\beta(t,\theta u_0,v)v)$ modulo W. Therefore (2.18) will be satisfied if one sets the components of v in the complement of W to zero, namely, $P_{V\ominus W}(0,v) = 0$, and satisfies (2.18) restricted to W and to the relative interior of B, namely,

$$(P_{W\cap V} S(t,\theta u_0,v), \|S(t,x)\|) = (0_{V\cap W}, \rho),$$

with $\theta \in (0,R), andv \in V \cap W, \|v\|_b < R$. In other words, the intersection set (2.17) will be nonempty if the set $\Psi_t^{-1}(0,\rho)$ is nonempty, where, identifying points $(0,v)$ as v,

$$\Psi_t(v,\theta) = (P_V P_W S(t,\theta u_0,v), \|S(t,x)\|),$$

where $\Psi : (B(0,R) \cap W_V) \times (0,R) \to W_V \times mathbbR$. For the sake of convenience we will identify now points $(0,v)$ of W_V as v. For $t = 0$, the map has the form $\Psi_0(v,\theta) = (v, (\theta^2\|u_0\|_a^2 + \|v\|_b^2)^{\frac{1}{2}})$ and the pre-image of $(0_{W_V},\rho)$ consists of one point $(0_{W_V}, \rho^{-\frac{1}{2}}\|u_0\|_a^{-1})$ at which Ψ_0 has a surjective derivative, so that the Brower degree of Ψ_0 at the intersection value, $d(\Psi_0, (B(0,R) \cap W_V) \times (0,R), (0,\rho))$ equals 1 up to a sign. Note that $S(t,\cdot)$ is identity on the boundary B of B_0. This immediately implies that $\Psi_t \neq (0,\rho)$ on the boundary of its domain. Consequently, the Brouwer degree is preserved and the intersection is nonempty for all t. $\square$

Completion of the proof of Theorem 2.1 is now standard. We assume that $G'(u,v) \neq 0$, unless $u = v = 0$. Let

$$\begin{aligned} M \quad = \quad & \big\{ S \in C([0,1] \times H \to H) : S(t,\cdot) \text{ is almost radial and} \\ & \text{equal to the identity near } B \text{ for all } t \in [0,1] \big\}, \end{aligned}$$

and let

$$\kappa := \inf_{\Phi\in M} \sup_{(u,v)\in B_0} G(\Phi(1,u,v)).$$

Note that, by Lemma 2.6, for any $\Phi \in M$ and every t,

$$\sup_{(u,v)\in B_0} G(\Phi(1,u,v)) \geq \inf G(A) > 0$$

and therefore $\kappa > 0$. The conditions of Lemma 2.2 are now satisfied, due to Lemma 2.3.

Let Z be as in Lemma 2.2. Then the equation

$$\frac{dx(t)}{dt} = -Z(x(t)),\ x(0) = (u,v)$$

has a unique solution for all initial data and values of $t \in mathbbR$, and the map $S : (t,u,v) \to x(t)$ is almost radial. By Lemma 2.4, with η sufficiently small, $Z = 0$ on the set B.

Let Φ_η be such that $G(\Phi_\eta(u,v)) \leq \kappa + \eta/2$ for all $(u,v) \in B_0$. Then due to Lemma 2.2, using the standard deformation argument (eg [6]) one has

$$G(S(t,\Phi_\eta(u,v))) \leq \kappa - \eta/2, (u,v) \in B_0.$$

for t sufficiently large. However, by Lemma 2.6, since composition of almost radial maps is an almost radial map, $\kappa \leq \kappa - \eta/2$, a contradiction.

3 The almost radial pseudogradient

In this section we prove Lemma 2.2. We will use the terminology of [5], saying that a sequence $u_k \in W^{1,2}(\mathbb{R}^N)$ converges weakly with concentration to a point u, $u_k \overset{cw}{\to} u$ if for any sequence of shifts $\alpha_k \in \mathbb{R}^N$, $(u_k - u)(\cdot + \alpha_k) \overset{w}{\to} 0$. As an immediate corollary of Lemma 6 from [3] (see also Lemma I1 from [4]), $u_k \overset{cw}{\to} u$ implies for $N \geq 3$ that $u_k \overset{L^p}{\to} u$ with $p \in (2, 2^*)$. Indeed, even if all components of u_k are subject to same shifts, we reduce the problem to the scalar case by using test functions $(\varphi, 0, \dots, 0), (0, \varphi, \dots, 0), \dots, (0, \dots, 0, \varphi)$

Definition 3.1 *The following set will be called an extended weak limit set of a sequence* $\{u_k\} \subset W^{1,2}(\mathbb{R}^N)$

$$\mathrm{wLim}(u_k) = \{u \in W^{1,2}(\mathbb{R}^N) :\ \exists \alpha_j \in \mathbb{R}^N, k_j \in \mathbb{N}, u_{k_j}(\cdot + \alpha_j) \overset{w}{\to} u\}.$$

Proposition 3.2 *The extended weak limit set of every bounded sequence* $\{u_k\} \subset W^{1,2}(\mathbb{R}^N)$ *contains 0.*

Proof. Let $\alpha_j \in \mathbb{R}^N, |\alpha_j| \to \infty$. Let $v_n, n \in \mathbb{N}$, be a basis on $W^{1,2}(\mathbb{R}^N)$. Then, obviously, there exists a sequence $j_k^1 \in \mathbb{N}$ such that

$$|(u_k(\cdot + \alpha_j), v_1)| \leq 2^{-k} \text{ for all } j \geq j_k^1.$$

Similarly, there is a sequence $j_k^2 \geq j_k^1$ such that

$$|(u_k(\cdot + \alpha_j), v_2)| \leq 2^{-k} \text{ for all } j \geq j_k^2.$$

Selecting further subsequences in a similar way, we get on the nth step

$$|(u_k(\cdot+\alpha_j), v_m)| \le 2^{-k} \text{ for all } m \le n, j \ge j_k^n.$$

Then

$$|(u_k(\cdot+\alpha_{j_k^k}), v_m)| \le 2^{-k} \text{ for all } m \le k.$$

Therefore, $u_k(\cdot+\alpha_{j_k^k}) \xrightarrow{w} 0$. □

Naturally, the statements and the definitions above extend immediately to the space $H = W^{1,2}(\mathbb{R}^N \to \mathbb{R}^2)$.

Proof of Lemma 2.2. For the sake of convenience we will abbreviate the set $\Omega(\eta,\kappa)$ defined in (2.8) as Ω.

1.) We start with an observation that if $(u_k, v_k) \in \Omega$, then

$$\mathrm{wLim}\{(u_k, v_k)\} \setminus \{0\} \ne \emptyset.$$

If it were otherwise, then $(u_k, v_k) \to 0$ in $L^p, 2 < p < 2^*$. Thus by (2.8), $|\langle G'(u_k, v_k), (u_k, 0)\rangle| \le \eta\|u_k\|_a^2$ implies $u_k \to 0$. Then $\limsup G(u_k, v_k) \le \limsup(-\frac{1}{2}\|v_k\|_b^2 - \int F(u_k, v_k)) \le 0$, which contradicts the condition

$$G(u, v) \ge \kappa - \eta$$

in (2.8), when η is small. This observation allows us to introduce a map r from sequences on Ω to H, assigning to every sequence $(u_k, v_k) \in \Omega$ a point $r(\{(u_k, v_k)\}) \in wLim\{(u_k, v_k)\} \setminus \{0\}$. Of course, the map is not expected to be continuous in any sense. We will use this map to introduce a *pseudoclosure* of Ω:

$$\Omega^+ = \Omega \bigcup \{r(\{(u_k, v_k)\}), (u_k, v_k) \in \Omega\}.$$

Obviously, $0 \notin \Omega^+$, so that G' does not vanish on Ω^+. Therefore the set Ω^+ can be covered by open sets

$$\mathcal{O}_w^1 := \{(u, v) \in H : \langle G'(u, v), w\rangle > \delta_w\}, w \in C_0^\infty(\mathbb{R}^N, \mathbb{R}^2) \tag{3.1}$$

with appropriate $\delta_w > 0$. We will use instead a covering by larger sets that contain correspondent $\mathcal{O}_w^1$:

$$\mathcal{O}_w := \{(u, v) \in H : \sup_{\alpha \in \mathbb{R}^N \times \mathbb{R}^N} \langle G'(u, v), w(\cdot+\alpha)\rangle > \delta_w\}$$

with the same $\delta_w > 0$ as above.

2.) We claim that Ω can be covered by finitely many sets $\mathcal{O}_w$. Since H is separable, we assume without loss of generality that the covering by $\mathcal{O}_w$ is countable. Let now

$$\Omega_m := \Omega \setminus \cup_{k=1}^m \mathcal{O}_{w_k}.$$

If $\Omega_m \neq \emptyset$ for every m, then one can select a sequence $(u_m, v_m) \in \Omega_m$. Since the point $r(\{(u_m, v_m)\}) \in \Omega^+$, it belongs to one of the sets $\mathcal{O}$, say, $\mathcal{O}_{w_\mu}$ and there is an $\alpha_\mu \in \mathbb{R}^N$ such that

$$\langle G'(r(\{(u_m, v_m)\})), w_\mu(\cdot + \alpha_\mu)\rangle > \delta_{w_\mu}$$

Since G' is weak-to-weak continuous, there is a sequence of translations $\alpha_m \in \mathbb{R}^N$ such that for a renamed subsequence of m, $(u_m, v_m)(\cdot + \alpha_m) \overset{w}{\to} r(\{(u_m, v_m)\}$ and

$$\langle G'(u_m, v_m), w_\mu(\cdot + \alpha_m)\rangle > \delta_{w_\mu},$$

i.e. $(u_m, v_m) \in \mathcal{O}_{w_\mu}$. At the same time, we chose of (u_m, v_m) so that for all $m \geq \mu$, $(u_m, v_m) \notin \mathcal{O}_{w_\mu}$. The contradiction proves that there is a n such that the set Ω_n is empty, which by (4.3) implies that $\{\mathcal{O}_{w_m}, m = 1, \dots, n\}$ is a covering of Ω.

3.) This implies that the sets $\{\mathcal{O}^0(m, \alpha, \delta), m = 1, \dots, n,\ \alpha \in R^N\}$, defined as

$$\mathcal{O}^0(m, \alpha, \delta) := \{(u, v) \in H : \langle G'(u, v), w_m(\cdot + \alpha)\rangle > \delta\},$$

with $\delta = \min\{\delta_{w_m}, m = 1, .., n\}$ also cover Ω. Let $R > 0$ be such that $\Omega \subset \bar{B}(0, R-2)$ and let $\epsilon_R > 0$ be such that whenever $|\alpha - \beta| < \epsilon_R, m = 1, \dots n$,

$$\mathcal{O}^0(m, \alpha, \delta) \cap \bar{B}(0, R) \subset \mathcal{O}^0(m, \beta, \delta/2).$$

Let us show that $\epsilon_R > 0$ exists. Indeed, the magnitude of $\alpha - \beta$ may be defined by the requirement

$$\begin{gathered}\|G'(u, v)\| \|w_m(\cdot - \alpha) - w_m(\cdot - \beta)\| \leq \delta/2, \\ (u, v) \in \cup \mathcal{O}^0(m, \alpha, \delta/2) \cap \bar{B}(0, R),\ m = 1, \dots, n,\end{gathered}$$

which can be satisfied by a uniform bound on $\alpha - \beta$, since G' is bounded on bounded sets and $w_m \in C_0^\infty$ by assumption in (3.1). Then Ω is covered by $\mathcal{O}^0(m, \beta_j, \delta/2)$, $m = 1, \dots, n$, where β_j are, say, points of a cubic lattice in $\mathbb{R}^N$.

4.) We shall show now that multiplicity of the covering $\mathcal{O}^0(m, \beta_j, \delta/2)$ does not exceed a finite number M for any point in $\bar{B}(0, R)$. If it were not true, there would exist a sequence $(u_i, v_i) \in \bar{B}(0, R)$ such that with some lattice translations $\beta_{i,j}$,

$$\langle G'(u_i, v_i), w_1(\cdot - \beta_{i,j})\rangle > \delta/2, j = 1, 2, \dots j(i), j(i) \to \infty. \tag{3.2}$$

(The index 1 in w_1 is of course no offense to generality.) It is easy to see that (3.2) implies that $\|G'(u_i, v_i)\| \to \infty$, which contradicts the assumption $(u_i, v_i) \in \bar{B}(0, R)$.

We remark, that Ω remains covered by similar sets with some new lattice points β_j and with $\delta/2$ replaced by $\delta/4$, since the finite multiplicity argument was carried out for an arbitrary δ and any lattice $\{\beta_j\}$ with a sufficiently small step, and the covering remains finite on the whole $\bar{B}(0, R)$.

5.) Let now y_r be an orthonormal basis in H. let

$$\|x\|_w := \sum_r 2^{-r} \langle x, y_r\rangle^2$$

and $d_w(x,A) := \inf_{y\in A} \|x-y\|_w$. We define now

$$\chi_{ij}(x) = \frac{d_w(x, H\setminus \mathcal{O}^0(i,\beta_j,\delta/4))}{d_w(x,\mathcal{O}^0(i,\beta_j,\delta/2)) + d_w(x, H\setminus \mathcal{O}^0(i,\beta_j,\delta/4))}$$

and set

$$z_0(x) = \sum \chi_{ij}(x) w_i(\cdot - \beta_j). \tag{3.3}$$

Note that the sum in (3.3) is uniformly finite for all $x \in \bar{B}(0,R)$, since w_i have compact support by (3.1), they are finitely many and β_j is a lattice. Note also that the map (3.3), restricted to $\bar{B}(0,R)$, is bounded, Lipschitz, weakly continuous and

$$\langle G'(u,v), z_0(u,v)\rangle \geq \delta/2 \text{ for } (u,v)\in\Omega.$$

Then there is a finite-dimensional orthogonal projector $P : H \to H$, such that

$$\langle G'(u,v), Pz_0(u,v)\rangle \geq \delta/3 \text{ for } u(u,v)\in\Omega,$$

Let $\Sigma \equiv \Sigma(\eta) := \{(u,v)\in H : |G(u,v)-\kappa| \leq \eta\}$. We shall define now subsets of Σ where $(u,0)$ or $(v,0)$ is a pseudogradient. More precisely, we set

$$\begin{aligned}
\Sigma_u^+ &= \{(u,v)\in\Sigma : \langle G'(u,v),(u,0)\rangle > \eta\|u\|^2\},\\
\Sigma_u^- &= \{(u,v)\in\Sigma : \langle G'(u,v),(u,0)\rangle < -\eta\|u\|^2\},\\
\Sigma_v^+ &= \{(u,v)\in\Sigma : \langle G'(u,v),(0,v)\rangle > \eta^4\}, \text{ and}\\
\Sigma_v^- &= \{(u,v)\in\Sigma : \langle G'(u,v),(0,v)\rangle < -\eta^4\},
\end{aligned}$$

Clearly, these sets form a covering of $\Sigma\setminus\Omega$. Moreover, from (2.6) one can easily conclude that if $(u,v)\in\Sigma$, then $\|u\|$ is also bounded away from zero, so we can replace the right hand sides of the inequalities in (3.20i,ii) by constants. This implies that that the set $\Sigma(\eta/2)\setminus\Omega$ is covered by the union of

$$\Sigma_u^{1+} = \{(u,v)\in int\Sigma(2\eta/3) : \langle G'(u,v),(u,0)\rangle > \delta\}, \tag{3.4}$$
$$\Sigma_u^{1-}- = \{(u,v)\in int\Sigma(2\eta/3) : \langle G'(u,v),(u,0)\rangle < -\delta\}, \tag{3.5}$$
$$\Sigma_v^{1+} = \{(u,v)\in int\Sigma(2\eta/3) : \langle G'(u,v),(0,v)\rangle > \delta\}, \text{ and} \tag{3.6}$$
$$\Sigma_v^{1-} = \{(u,v)\in int\Sigma(2\eta/3) : \langle G'(u,v),(0,v)\rangle < -\delta\}, \tag{3.7}$$

with some $\delta > 0$. By selecting a partition of unity $\chi_u^\pm, \chi_v^\pm, \chi_\Omega$, subordinated to the sets (3.4)-(3.7) together with the interior of $\Omega(\eta,\kappa)$, we construct the a pseudogradient on the set $\Sigma(\eta/2)$ in the following form:

$$Z_0(u,v) := (\varphi(u,v)u, \psi(u,v)v) + \chi_\Omega P z_0(u,v),$$

where $\varphi = \lambda(\chi_u^+ - \chi_u^-)$, $\psi = \lambda(\chi_v^+ - \chi_v^-)$ and $\lambda > 0$ is sufficiently large. Let $\nu \in C^\infty(\mathbb{R}\to[0,1])$, $\nu(t) = 1$ for $t\in[-1,1]$, $\nu(t) = 0$ for $t \notin [-2,2]$. We leave to the reader to verify that the functional

$$Z(u,v) := \nu(6\eta^{-1}(G(u,v)-\kappa))Z_0(u,v)$$

satisfies the assertions of Lemma 2.2 with η reduced to $\eta/3$.

References

[1] Benci V., Capozzi A., Fortunato D., *Periodic solutions of Hamiltonian systems with superquadratic potential*, Annali Mat. Pura Applicata **4**, 1-46 (1986)

[2] De Figuejredo D.J., Felmer, P., *On superquadratic elliptic systems*, Trans. Amer. Math. Soc. **343**, 99-116 (1994)

[3] Lieb, E., *On the lowest eigenvalue of the Laplacian for the intersection of two domains*, Invent. Math. **74**, 441-448 (1983).

[4] Lions P. L., *The concentration-compactness principle in the calculus of variations. The locally compact case, part 2.* Ann. Inst. H. Poincare, Analyse Non Lineaire **1**, 223-283 (1984)

[5] Schindler I., Tintarev,K., *Abstract concentration compactness and elliptic equations on unbounded domains*, Proceedings of ASDE 98, University of Lisbon (to appear)

[6] Struwe M., *Variational Methods*, Springer 1990 .

[7] Tintarev K., *Isotopic linking and critical points of functionals*, J. Nonlin. Anal. **30**, 4145-4149 (1997)

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