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POSITIVE SOLUTIONS FOR A CLASS OF SINGULAR BOUNDARY-VALUE PROBLEMS

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Abstract. Using regularization and the sub-super solutions method, this note shows the existence of positive solutions for singular differential equation subject to four-point boundary conditions.

1. Introduction

This note concerns the existence of positive solutions to the boundary-value problem (BVP)

$$y'' = -\frac{\beta}{t}y' + \frac{\gamma}{y}|y'|^2 - f(t,y), \quad 0 < t < 1, \tag{1.1}$$

$$y(0) = y(1) = 0, \tag{1.2}$$

$$y'(0) = y'(1) = 0, \tag{1.3}$$

where $\beta > 0, \gamma > \beta + 1$ are constants, and f satisfies

(H1) $f(t,y) \in C^1([0,1] \times [0,\infty), [c_0,\infty))$ for sufficiently small $c_0 > 0$, and f is non-increasing with respect to y.

Equation (1.1) with the nonlinear right-hand side independent of y' has been discussed extensively in the literature; see for example [1, 7] and the references therein. Because of its background in applied mathematics and physics, problem (1.1) with right-hand side depending on y' has attracted the attention of many authors; see for instance [6, 8] and their references.

Guo et al. [6] studied the existence of positive solutions for the singular boundary-value problem with nonlinear boundary conditions

$$y'' + q(t)f(t,y,y') = 0, \quad 0 < t < 1,$$
$$y(0) = 0, \quad \theta(y'(1)) + y(1) = 0\,,$$

where $f(t,y,y') \geq 0$ is singular at $y = 0$. They use a nonlinear alternative of Leray-Schauder type and Urysohn's lemma.

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This work is motivated by [4] where the authors studied the problem

$$y'' + \frac{N-1}{t}y' - \frac{\gamma}{y}|y'|^2 + 1 = 0, \quad 0 < t < 1,$$
$$y(1) = 0, \quad y'(0) = 0.$$

There N is a positive integer, and the problem corresponds to $\beta = N-1$, $f \equiv 1$ in (1.1). Applying ordinary differential equation techniques, they obtained a decreasing positive solution which, subsequently, was used in [5] to study some properties of solutions for a class of degenerate parabolic equations (see [3] for further information).

In this note, we study problem (1.1) under boundary conditions that are mote complicated than those in [4]. By using a regularization method and constructing sub- and supersolutions, we obtain an existence result.

A function $y \in C^2(0,1) \cap C[0,1]$ is called a solution for (1.1) if it is positive in $(0,1)$ and satisfies (1.1) pointwise.

The main result of this note is as follows.

Theorem 1.1. *Under assumption (H1), the boundary-value problem* (1.1)–(1.3) *admits at least one solution.*

Since we need to calculate the derivatives of f, we assume that $f \in C^1([0,1] \times [0,\infty), [c_0,\infty))$. However, if $f \in C([0,1] \times [0,\infty), [c_0,\infty))$, Theorem 1.1 remains valid.

2. Proof of Theorem 1.1

Since problem (1.1) is singular at point $t = 0$, or $y(t) = 0$, we need to regularize it. Precisely, we discuss positive solutions of the regularized problem

$$-y'' - \frac{\beta}{t+\varepsilon}y' + \frac{\gamma}{|y|+\varepsilon^2}|y'|^2 - f(t,y) = 0, \quad 0 < t < 1, \tag{2.1}$$

subject to the boundary conditions (1.2), where $\varepsilon \in (0,1]$.

Denote $Ay = -y''$ and

$$b_\varepsilon(t,\xi,\eta) = \frac{\beta}{t+\varepsilon}\eta - \frac{\gamma}{|\xi|+\varepsilon^2}|\eta|^2 + f(t,\xi).$$

Note that $b_\varepsilon(\cdot,\xi,\eta) \in C^\mu[0,1]$ uniformly for (ξ,η) in bounded subsets of $\mathbb{R} \times \mathbb{R}$ for some $\mu \in (0,1]$, $\partial b_\varepsilon/\partial\xi, \partial b_\varepsilon/\partial\eta$ exist and are continuous on $[0,1] \times \mathbb{R}^2$. Moreover, there exists some positive constant C dependent of ε^{-1}, σ such that

$$|b_\varepsilon(t,\xi,\eta)| \le C(1+|\eta|^2)$$

for every $\sigma \ge 0$ and $(t,\xi,\eta) \in [0,1] \times [-\sigma,\sigma] \times \mathbb{R}$.

A function y is called a subsolution for BVP (2.1) (1.2) if $y \in C^{2+\mu}[0,1]$ and

$$Ay \le b_\varepsilon(\cdot,y,y') \quad \text{in } [0,1],$$
$$y(0) \le 0, \quad y(1) \le 0.$$

Supersolutions are defined by reversing the above inequality signs. We call y a solution for (2.1) (1.2), if y is a subsolution and a supersolution of (2.1) (1.2).

Let $v(t) = \frac{1}{2}t - \frac{1}{2}t^2$, it is easy to see that v is a nonnegative solution for problem

$$-v'' = 1, \quad 0 < t < 1,$$
$$v(0) = v(1) = 0.$$

Lemma 2.1. *Let $\underline{y} = C_1 v^2$, $y_{1\varepsilon} = C_2(t+\varepsilon)^2$, $y_{2\varepsilon} = C_2(1+\varepsilon-t)^2$, $\overline{y}_\varepsilon = \min\{y_{1\varepsilon}, y_{2\varepsilon}\}$, then* (2.1) (1.2) *admits at least one solution $y_\varepsilon \in [\underline{y}, \overline{y}_\varepsilon]$. Here C_1 and $C_2 \geq 1$ are some positive constants.*

Proof. By [2, Theorem 1.1], it suffices to prove $\underline{y}(\overline{y})$ is a subsolution (supersolution) for (2.1) (1.2). Hence we need to prove $A\underline{y} \leq b_\varepsilon(t, \underline{y}, \underline{y}')$, $Ay_{i\varepsilon} \geq b_\varepsilon(t, y_{i\varepsilon}, y'_{i\varepsilon})$ $(i = 1, 2)$.

From $0 \leq v(t) \leq t$ and $\underline{y}' = 2C_1 vv'$, $\underline{y}'' = 2C_1|v'|^2 - 2C_1 v$, we have

$$\begin{aligned} A\underline{y} - b_\varepsilon(t, \underline{y}, \underline{y}') &= 2C_1[v - \frac{\beta}{t+\varepsilon} vv' + |v'|^2(2\gamma \frac{C_1 v^2}{C_1 v^2 + \varepsilon^2} - 1)] - f(t, \underline{y}) \\ &\leq 2C_1[v + \beta|v'| + (2\gamma+1)|v'|^2] - f(t, \underline{y}). \end{aligned}$$

Since $f(t,\xi) \geq c_0 > 0$, we can choose

$$C_1 \leq \min\big\{\frac{c_0}{2\max_{[0,1]}[v + \beta|v'| + (2\gamma+1)|v'|^2]}, 1/2\big\},$$

hence

$$A\underline{y} \leq b_\varepsilon(t, \underline{y}, \underline{y}'), \quad 0 < t < 1.$$

Since $C_2(t+\varepsilon)^2 \geq \varepsilon^2$, it is easy to calculate that

$$\begin{aligned} Ay_{1\varepsilon} - b_\varepsilon(t, y_{1\varepsilon}, y'_{1\varepsilon}) &= 2C_2[\gamma \frac{2C_2(t+\varepsilon)^2}{C_2(t+\varepsilon)^2 + \varepsilon^2} - \beta - 1] - f(t, y_{1\varepsilon}), \\ &\geq 2C_2(\gamma - \beta - 1) - f(t, y_{1\varepsilon}). \end{aligned}$$

Choosing

$$C_2 \geq \max\{\frac{1}{2(\gamma-\beta-1)} \max_{[0,1]} f(t, \underline{y}(t)), 1\},$$

we see that $y_{1\varepsilon} \geq \underline{y}$ in $[0,1]$. It follows from (H1) that

$$Ay_{1\varepsilon} \geq b_\varepsilon(t, y_{1\varepsilon}, y'_{1\varepsilon}), \quad 0 < t < 1,$$

as asserted. The other inequality can be proved similarly. The proof is complete. □

Lemma 2.2. *For any $\tau \in (0,1)$, there exists a positive constant C_τ independent of ε such that*

$$|y'_\varepsilon| \leq C_\tau, \quad |y''_\varepsilon| \leq C_\tau, \quad \tau \leq t \leq 1 - \tau. \tag{2.2}$$

Proof. From Lemma 2.1, BVP (2.1) (1.2) admits a solution $y_\varepsilon \in C^{2+\mu}[0,1]$ which satisfies (2.1) (1.2) pointwise, hence it is also a solution of

$$[(t+\varepsilon)^\beta y'_\varepsilon]' = \frac{\gamma(t+\varepsilon)^\beta}{y_\varepsilon + \varepsilon^2}|y'_\varepsilon|^2 - (t+\varepsilon)^\beta f(t, y_\varepsilon).$$

Since $\gamma > 0$, from (H1) and Lemma 2.1 we obtain

$$[(t+\varepsilon)^\beta y'_\varepsilon]' \geq -(t+\varepsilon)^\beta f(t, y_\varepsilon) \geq -2^\beta \max_{[0,1]} f(t, \underline{y}(t)) := -M.$$

Therefore,

$$[(t+\varepsilon)^\beta y'_\varepsilon + Mt]' \geq 0, \quad 0 < t < 1,$$

which implies that the function $\varphi(t) := (t+\varepsilon)^\beta y'_\varepsilon + Mt$ is non-decreasing on $[0,1]$.

Since $y_\varepsilon \geq 0$ for all $t \in [0,1]$ and $y_\varepsilon(0) = y_\varepsilon(1) = 0$, we have

$$y_\varepsilon'(0) = \lim_{t\to 0^+} \frac{y_\varepsilon(t)}{t} \geq 0,$$
$$y_\varepsilon'(1) = \lim_{t\to 1^-} \frac{y_\varepsilon(t)}{t-1} \leq 0.$$

From which, it follows that

$$0 \leq \varphi(0) \leq \varphi(t) \leq \varphi(1) \leq M, \quad t \in [0,1],$$

which implies

$$|(t+\varepsilon)^\beta y_\varepsilon'(t)| \leq M. \tag{2.3}$$

Hence for any $\tau \in (0,1)$ there exists a positive constant C_τ independent of ε such that

$$|y_\varepsilon'| \leq C_\tau, \quad \tau \leq t \leq 1.$$

Multiplying (2.1) by $(t+\varepsilon)^{2\beta+1}$, from (2.3) (H1) and Lemma 2.1 it follows

$$\begin{aligned}
&|(t+\varepsilon)^{2\beta+1} y_\varepsilon''| \\
=&\Big|\gamma \frac{(t+\varepsilon)}{y_\varepsilon+\varepsilon^2}[(t+\varepsilon)^\beta y_\varepsilon']^2 - (t+\varepsilon)^{2\beta+1} f(t,y_\varepsilon) - (2\beta+1)(t+\varepsilon)^\beta((t+\varepsilon)^\beta y_\varepsilon')\Big| \\
\leq& C\Big(1 + \frac{t+\varepsilon}{\underline{y}+\varepsilon^2} + f(t,\underline{y})\Big),
\end{aligned}$$

where C is independent of ε. The second conclusion follows easily from the above inequality. $\square$

Now we complete the proof of Theorem 1.1. Differentiating formally (2.1) with respect to t, from (H1) and Lemma 2.1 we obtain

$$\begin{aligned}
|y_\varepsilon'''| &= \Big|\frac{\beta}{t+\varepsilon}\Big(\frac{y_\varepsilon'}{t+\varepsilon} - y_\varepsilon''\Big) + \gamma\frac{2(y_\varepsilon+\varepsilon^2)y_\varepsilon' y_\varepsilon'' - y_\varepsilon'|y_\varepsilon'|^2}{(y_\varepsilon+\varepsilon^2)^2} - f_t'(t,y_\varepsilon) - f_y'(t,y_\varepsilon)y_\varepsilon'(t)\Big| \\
&\leq \frac{\beta}{t+\varepsilon}\Big(\frac{|y_\varepsilon'|}{t+\varepsilon} + |y_\varepsilon''|\Big) + \gamma\Big[\frac{2|y_\varepsilon'||y_\varepsilon''|}{\underline{y}+\varepsilon^2} + \frac{|y_\varepsilon'|^3}{(\underline{y}+\varepsilon^2)^2}\Big] \\
&\quad + \max_{t\in[0,1], y\in[a,b]} |f_t'(t,y)| + |y_\varepsilon'| \cdot \max_{t\in[0,1], y\in[a,b]} |f_y'(t,y)|,
\end{aligned}$$

where $a = \min_{t\in[0,1]} \underline{y}(t)$, $b = \max_{t\in[0,1]} \overline{y}_\varepsilon(t)|_{\varepsilon=1}$. From (2.2) one infers that for any $\tau \in (0,1)$ there exists a positive constant C_τ independent of ε such that

$$|y_\varepsilon'''| \leq C_\tau, \quad \tau \leq t \leq 1-\tau.$$

This implies that

$$\|y_\varepsilon\|_{C^{2,1}[\tau,1-\tau]} \leq C_\tau.$$

Using Arzelá-Ascoli theorem and diagonal sequential process, we obtain that there exists a subsequence $\{y_{\varepsilon_n}\}$ of $\{y_\varepsilon\}$ and a function $y \in C^2(0,1)$ such that

$$y_{\varepsilon_n} \to y, \quad \text{uniformly in } C^2[\tau, 1-\tau],$$

as $\varepsilon_n \to 0$. By Lemma 2.1, we obtain

$$C_1 t^2(1-t)^2 \leq y(t) \leq C_2 t^2, \quad t \in [0,1],$$
$$C_1 t^2(1-t)^2 \leq y(t) \leq C_2 (1-t)^2, \quad t \in [0,1].$$

From this, it is not difficult to show that $y'(0) = y'(1) = 0$ and $y \in C[0,1]$. Clearly, y solves BVP (1.1)-(1.3), hence Theorem 1.1 is proved.

Example. Consider boundary-value problem

$$\begin{gathered} y'' + \frac{N-1}{t}y' - \frac{N+1}{y}|y'|^2 + t^2 + e^{-y} + 1 = 0, \quad 0 < t < 1, \\ y(0) = y(1) = y'(0) = y'(1) = 0. \end{gathered} \tag{2.4}$$

Let $N \geq 1$, $\beta = N-1$, $\gamma = N+1$, $f(t,y) = t^2 + e^{-y} + 1$, $c_0 = 1$. Clearly, all assumptions of Theorem 1.1 are satisfied. Hence the problem (2.4) has at least one positive solution $y \in C^2(0,1) \cap C[0,1]$. But the theorems in [6, 8] are not applicable to this example.

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References

[1] R. P. Agarwal, D. O'Regan, V. Lakshmikantham, S. Leela; *Nonresonant singular boundary-value problems with sign changing nonlinearities*, Applied Mathematics and Computation, **167**(2005), 1236-1248.

[2] Herbert Amann; *Existence and multiplicity theorems for semi-linear elliptic boundary-value problems*, Math. Z., 1976, **150**: 281-295.

[3] G. I. Barenblatt, M. Bertsch, A. E. Chertock, V. M. Prostokishin; *Self-similar intermediate asymptotic for a degenerate parabolic filtration-absorption equation*, Proc. Nat. Acad. Sci. (USA), 2000, **18**: 9844-9848.

[4] M. Bertsch, M. Ughi; *Positivity properties of viscosity solutions of a degenerate parabolic equation*, Nonlinear Anal. TMA, 1990, **14**: 571-592.

[5] M. Bertsch, R. D. Passo, M. Ughi; *Discontinuous viscosity solutions of a degenerate parabolic equation*, Trans. Amer. Math. Soc., 1990, **320**: 779-798.

[6] Guo Yanping, Shan Wenrui, Ge Weigao; *Positive solutions of singular ordinary differential equations with nonlinear boundary conditions*, Applied Mathematics Letters, **18**(2005), 1-9.

[7] Johnny Henderson, Haiyan Wang; *Positive Solutions for Nonlinear Eigenvalue Problems*, J. Math. Appl. Anal., **208**(1997), 252-259.

[8] A. Tineo; *Existence theorems for a singular two point Dirichlet problem*, Nonlinear Analysis, **19**(1992) 323-333.

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