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MULTIPLE SOLUTIONS FOR INHOMOGENEOUS NONLINEAR ELLIPTIC PROBLEMS ARISING IN ASTROPHYISCS

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Abstract. Using variational methods we prove the existence and multiplicity of solutions for some nonlinear inhomogeneous elliptic problems on a bounded domain in $\mathbb{R}^n$, with $n \geq 2$ and a smooth boundary, and when the domain is $\mathbb{R}^n_+$.

1. Introduction

In this paper we study the boundary-value problem

$$\begin{gathered} -\Delta u + c(x)u = \lambda f(u) \quad \text{in } \Omega \\ u = h(x) \quad \text{on } \partial\Omega \end{gathered} \tag{1.1}$$

when Ω is a bounded domain in $\mathbb{R}^n$, with $n \geq 2$ and smooth boundary $\partial\Omega$, and when the domain is $\mathbb{R}^n_+ := \mathbb{R}^{n-1} \times \mathbb{R}_+$ with $\mathbb{R}_+ = \{y \in \mathbb{R} : y > 0\}$. The function $f :]-\infty, +\infty[\to \mathbb{R}$ is assumed to satisfy the following conditions:

- (f1) There exists $s_0 > 0$ such that $f(s) > 0$ for all $s \in]0, s_0[$.
- (f2) $f(s) = 0$ for $s \leq 0$ or $s \geq s_0$.
- (f3) $f(s) \leq as^\sigma$, a is a positive constant and $1 < \sigma < \frac{n+2}{n-2}$ if $n > 2$ or $\sigma > 1$ if $n = 2$.
- (f4) There exists $l > 0$ such that $|f(s_1) - f(s_2)| \leq l|s_1 - s_2|$, for all s_1, $s_2 \in \mathbb{R}$.

The function h is a non-negative bounded, smooth, $h \neq 0$, $\min h < s_0$ and $c \geq 0$, and $c \in L^\infty(\Omega) \bigcap C(\overline{\Omega})$.

Note that problem (1.1) is equivalent to

$$\begin{gathered} -\Delta \omega + c(x)\omega = \lambda f(\omega + \tau) \quad \text{in } \Omega \\ \omega = 0 \quad \text{on } \partial\Omega\,, \end{gathered} \tag{1.2}$$

where $\omega = u - \tau$ and τ is a solution of

$$\begin{gathered} -\Delta \tau + c(x)\tau = 0 \quad \text{in } \Omega \\ \tau = h(x) \quad \text{on } \partial\Omega\,. \end{gathered} \tag{1.3}$$

We will study (1.2) instead of (1.1). In section 2 using variational techniques we will find an interval $\Lambda \subset \mathbb{R}_+$ such that for all $\lambda \in \Lambda$ there exist at least three

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positive solutions of (1.2), for $\|\tau\|_{L^{\sigma+1}(\Omega)}$ small enough. This result is better than the one obtained by Calahorrano and Dobarro in [4].

In section 3, we will study the problem (1.2) for $\inf c(x) > 0$ and Ω big enough, by this we mean that there exists $x_0 \in \Omega$ such that the Euclidean ball with center x_0 and radius R is contained in Ω, with R large enough. In this case, we will eliminate the restrictions on τ, obtaining similar results.

Problem (1.1) is a generalization of an astrophysical gravity-free model of solar flares in the half plane $\mathbb{R}^2_+$, given in [7], [8] and [9], namely:

$$\begin{gathered} -\Delta u = \lambda f(u) \quad \mathbb{R}^2_+ \\ u(x,0) = h(x) \quad \forall x \in \mathbb{R} \end{gathered} \tag{1.4}$$

besides the above mentioned conditions for f and h, the authors are interested in finding a positive range of $\lambda' s$ in which there is multiplicity of solutions for (1.4), see [7, 8, 9] for a detail description.

In section 4, a related problem is reviewed

$$\begin{gathered} -\Delta\omega + c(x)\omega = \lambda f(\omega+\tau) \quad \text{in } \mathbb{R}^n_+ \\ \omega(x,0) = 0 \quad \forall x \in \mathbb{R}^{n-1} \end{gathered} \tag{1.5}$$

and we prove the existence of solutions of (1.5) as limit of a special family of solutions of

$$\begin{gathered} -\Delta\omega + c(x)\omega = \lambda f(\omega+\tau) \quad \text{in } D_R \\ \omega = 0 \quad \text{on } \partial D_R \end{gathered} \tag{1.6}$$

where

$$D_R = \{(x_1,\dots,x_n) \in \mathbb{R}^n_+ : \sum_{i=1}^n x_i^2 < R^2\}$$

and R is large enough. Besides these solutions are absolute minima of the natural associated functional for small $\lambda' s$ and local but not global minima for large $\lambda' s$.

2. Variational Method

Similarly to section 1, let τ be the solution of

$$\begin{gathered} -\Delta\tau + c(x)\tau = 0 \quad \text{in } \Omega \\ \tau = h(x) \quad \text{on } \partial\Omega\,. \end{gathered} \tag{2.1}$$

Problem (1.1) is equivalent to

$$\begin{gathered} -\Delta\omega + c(x)\omega = \lambda f(\omega+\tau) \quad \text{in } \Omega \\ \omega = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{2.2}$$

where $\omega = u - \tau$. Therefore, we are studying (2.2) instead of (1.1).

Since $f \ge 0$, then any solution of (2.2) is positive by the maximum principle, furthermore $\omega = 0$ is solution of (2.2) if and only if $\lambda = 0$. On the other hand τ achieves its maximum and minimum on the boundary, i.e. $\inf_{\partial\Omega}\tau \le \tau(x) \le \sup_{\partial\Omega}\tau$.

Let $H_0^1(\Omega)$ be the usual Sobolev space, with $\|u\|^2 = \int_\Omega |\nabla u|^2 dx$. We define for all $\lambda \ge 0$ and for all non-negative function τ such that $\|\tau\|_{L^{\sigma+1}(\Omega)} \equiv \Gamma < \infty$ the C^1 functional, [2], $\Phi_{\lambda,\tau} : H_0^1(\Omega) \to R$,

$$\Phi_{\lambda,\tau}(u) = \frac{1}{2}\int_\Omega [c(x)u^2 + |\nabla u|^2]dx - \lambda\int_\Omega F(u+\tau)dx$$

where, $F(s) = \int_0^s f(t)dt$.

If $u \in H_0^1(\Omega)$, $\Phi'_{\lambda,\tau}(u) = 0$ (Φ' is the gradient of Φ) then u is a weak and, by regularity strong solution of (2.2).

Since f is bounded, it is easy to prove that $\Phi_{\lambda,\tau}$ is coercive and verifies the Palais-Smale condition for all λ non negative (using methods like in the case c=0, [11]). Then $\Phi_{\lambda,\tau}$ attains its global infimum on a function $u_{\lambda,\tau} \in H_0^1(\Omega)$ for all λ non negative.

Theorem 2.1. *Let us assume (f1)–(f4). For all $\Gamma > 0$ small enough there exists an interval $]\underline{\lambda}, \overline{\lambda}(\Gamma)[$ with $\underline{\lambda} > 0$ such that for all $\lambda \in]\underline{\lambda}, \overline{\lambda}(\Gamma)[$ the problem* (2.2) *has at least three positive solutions. Moreover $\overline{\lambda}(\Gamma) \to +\infty$ as $\Gamma \to 0$.*

To prove Theorem 2.1, we will use arguments as those in [4], for which the following lemmas are necessary.

Lemma 2.2. *There exists $\omega_0 \geq 0$, $\omega_0 \neq 0$ and $\underline{\lambda} > 0$ such that for all $\lambda > \underline{\lambda}$ and for all $\tau \geq 0$, $\Phi_{\lambda,\tau}(\omega_0) < 0$*

Proof. . Let $B_r(x_0)$ denote an euclidean ball with center at x_0 and radius r. Let $x_0 \in \Omega$ and $R > 0$ such that $B_R(x_0) \subset \Omega$. Then for all $0 < \delta < R$, $B_\rho(x_0) \subset B_R(x_0)$, where $\rho = R - \delta$. Now, we define

$$\omega_{\delta,R}(x) = \begin{cases} s_0 & \text{if } |x - x_0| \leq \rho \\ \frac{s_0}{\delta}(R - |x - x_0|) & \text{if } \rho \leq |x - x_0| \leq R \\ 0 & \text{if } |x - x_0| \geq R \end{cases}$$

So, using the Hölder and Poincaré inequalities

$$\begin{aligned}
\Phi_{\lambda,\tau}(\omega_{\delta,R}) &= \frac{1}{2}\|\omega_{\delta,R}\|^2 + \frac{1}{2}\int_\Omega c(x)(\omega_{\delta,R})^2 dx - \lambda \int_\Omega F(\omega_{\delta,R} + \tau)dx \\
&\leq \frac{1}{2}\|\omega_{\delta,R}\|^2 + \frac{\|c\|_{L^\infty}}{2}\int_{B_R(x_0)} (\omega_{\delta,R})^2 dx - \lambda \int_{B_\rho(x_0)} F(s_0 + \tau)dx \\
&\leq \frac{1}{2}\|\omega_{\delta,R}\|^2 + \frac{\|c\|_{L^\infty}}{2}\Big(\frac{|B_R(x_0)|}{\omega_n}\Big)^{\frac{2}{n}} \|\omega_{\delta,R}\|^2 - \lambda F(s_0)\int_{B_\rho(x_0)} dx \\
&= \frac{s_0^2(1 + \|c\|_{L^\infty} R^2)}{2\delta^2}\int_{B_R(x_0) - B_\rho(x_0)} dx - \lambda F(s_0)\int_{B_\rho(x_0)} dx \\
&= \frac{s_0^2(1 + \|c\|_{L^\infty} R^2)(R^n - (R-\delta)^n)\omega_n}{2\delta^2} - \lambda F(s_0)(R-\delta)^n\omega_n
\end{aligned}$$

where ω_n denotes the volume of the unit ball in R^n. Let

$$\underline{\lambda}(\delta) \equiv \frac{s_0^2(1 + \|c\|_{L^\infty} R^2)(R^n - (R-\delta)^n)}{2F(s_0)\delta^2(R-\delta)^n}$$

If $\delta = tR$, $0 < t < 1$, results in

$$\underline{\lambda}(\delta) = \frac{s_0^2(1 + \|c\|_{L^\infty} R^2)}{2F(s_0)R^2}\Big(\frac{1 - (1-t)^n}{t^2(1-t)^n}\Big).$$

then $\Phi_{\lambda,\tau}(\omega_{\delta,R}) < 0$ for all $\lambda > \underline{\lambda}(\delta) > 0$, and for all $\tau \geq 0$. Let

$$\psi(t) \equiv \frac{1 - (1-t)^n}{t^2(1-t)^n}$$

and let $t_1 \in]0,1[$ such that $\psi(t_1) = \min_{]0,1[} \psi(t)$. If $\delta_1 = t_1 R$, $\omega_o = \omega_{\delta_1,R}$ and $\underline{\lambda} = \underline{\lambda}(\delta_1)$, then there results

$$\Phi_{\lambda,\tau}(\omega_0) < 0 \quad \forall\lambda > \underline{\lambda} > 0 \quad \textit{and} \quad \forall\tau \geq 0$$

Moreover,

$$\|\omega_0\| = s_0\Big(\omega_n\Big)^{1/2} R^{\frac{n-2}{2}} \Big(\frac{1-(1-t_1)^n}{t_1^2}\Big)^{1/2}$$

□

Lemma 2.3. *There exists a constant K=K(a,σ,Ω) such that for all $\lambda < \overline{\lambda}(\Gamma)$ and $\|u\| = \Gamma$, $\Phi_{\lambda,\tau}(u) > 0$ where $\overline{\lambda} \equiv K\Gamma^{1-\sigma}$.*

Proof. From (f3),

$$\int_\Omega F(u+\tau)dx = \int_\Omega \int_0^{u+\tau} f(t)dt\,dx \leq \int_\Omega \frac{a(u+\tau)^{\sigma+1}}{\sigma+1}\,dx$$

then, using the Sobolev immersion and Poincaré inequalities

$$\begin{aligned}
\Phi_{\lambda,\tau}(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{2}\int_\Omega c(x)u^2dx - \lambda\int_\Omega F(u+\tau)dx \\
&\geq \frac{1}{2}\|u\|^2 - \lambda\int_\Omega \frac{a(u+\tau)^{\sigma+1}}{\sigma+1}dx \\
&\geq \frac{1}{2}\|u\|^2 - \lambda\Big(\frac{a}{\sigma+1}\Big)(\|u\|_{L^{\sigma+1}(\Omega)} + \|\tau\|_{L^{\sigma+1}(\Omega)})^{\sigma+1} \\
&\geq \frac{1}{2}\|u\|^2 - \lambda\Big(\frac{a}{\sigma+1}\Big)(C(\Omega)\|u\| + \Gamma)^{\sigma+1},
\end{aligned}$$

where $C(\Omega)$ is a constant depending on Ω. Setting

$$K = \frac{\sigma+1}{2a(C(\Omega)+1)^{\sigma+1}}$$

it follows that for all $\lambda < \overline{\lambda}(\Gamma) \equiv K\Gamma^{1-\sigma}$, $\Phi_{\lambda,\tau}(u) > 0$. □

Remark 2.4. (i) Since $\overline{\lambda}(\Gamma) = K\Gamma^{1-\sigma}$ it follows $\overline{\lambda} \to +\infty$ as $\Gamma \to 0$.
(ii) $\Phi_{\lambda,\tau}(0)$ and $\Phi'_{\lambda,\tau}(0)(v)$ are negative for all $\lambda > 0$ and $v \geq 0$, $v \neq 0$.

Lemma 2.5. *For all $0 < \lambda < \overline{\lambda}(\Gamma)$ there exists $\overline{u} \in H_0^1(\Omega)$ with $\|\overline{u}\| < \Gamma$ such that $\Phi_{\lambda,\tau}(\overline{u}) < 0$ and $\Phi'_{\lambda,\tau}(\overline{u}) = 0$.*

Proof. Using Lemma 2.3 we prove that $\Phi_{\lambda,\tau}(u) > 0$, for $0 < \lambda < \overline{\lambda}(\Gamma)$ and u such that $\|u\| = \Gamma$. Moreover $\Phi_{\lambda,\tau}(0) < 0$ y $\Phi'_{\lambda,\tau}(0)(v) \neq 0$. Keeping in mind that the solution of

$$\begin{gathered}
\frac{d\alpha}{dt} = W(\alpha(t)) \\
\alpha(0) = 0
\end{gathered}$$

where $W = -V$, V pseudo-gradient vector field for $\Phi_{\lambda,\tau}$ in the set of regular points of $\Phi_{\lambda,\tau}$, with $0 < \lambda < \overline{\lambda}$.

Since $\Phi_{\lambda,\tau}$ verifies the Palais-Smale condition and is bounded from below, using [10, Theorem 5.4] we have that

(1) $\alpha : [0,+\infty[\to H_0^1(\Omega)$ is continuous.
(2) $\Phi_{\lambda,\tau}(\alpha(t))$ is strictly decreasing.

(3) $\alpha(t) \to \overline{u}$ as $t \to +\infty$, $\Phi'_{\lambda,\tau}(\overline{u}) = 0$.

then, $\overline{u}$ satisfies the required conditions. $\square$

Proof of Theorem 2.1. Let ω_0 and $\underline{\lambda}$ be defined in Lemma 2.2. Using Lemma 2.3 for $\Gamma < \|\omega_0\|$, there exists $\overline{\lambda}(\Gamma) > 0$ such that $\Phi_{\lambda,\tau}(u) > 0$ for all $\lambda < \overline{\lambda}$ and $\|u\| = \Gamma$. But since $\underline{\lambda}$ is independent of Γ, using Remark 2.4 $\underline{\lambda} < \overline{\lambda}(\Gamma)$ for Γ small enough.

Now we claim that for Γ small enough there exists $\widehat{u} \in H_0^1(\Omega)$, $\|\widehat{u}\| > \Gamma$ such that for all $\underline{\lambda} < \lambda < \overline{\lambda}(\Gamma)$ $\Phi_{\lambda,\tau}(\widehat{u}) < 0$ and $\Phi'_{\lambda,\tau}(\widehat{u}) = 0$. Indeed, we remember that for all $\underline{\lambda} < \lambda < \overline{\lambda}(\Gamma)$ lemmas 3 and 2 are verified. Keeping in mind that the solution of

$$\frac{d\beta}{dt} = W(\beta(t))$$
$$\beta(0) = \omega_0$$

Using similar arguments as those in Lemma 2.5 we find the critical point $\widehat{u}$ with $\|\widehat{u}\| > \Gamma$. Let

$$c \equiv \inf_{\delta\in\Theta} \sup_{u\in\delta} \Phi_{\lambda,\tau}(u)$$

where Θ is the set paths

$$\Theta = \{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = \overline{u}, \gamma(1) = \omega_0\}$$

we are able to apply the Mountain Pass Theorem of Ambrosetti-Rabinowitz [3]. Then c is achieved in $H_0^1(\Omega)$ at a function $\widetilde{u}$. Finally using Lemma 2.5 we prove Theorem 2.1. $\square$

Remark 2.6. (i) If we define $\mu \in R_-$,

$$\mu \equiv \min_{0\le t\le\Gamma} \frac{1}{2}t^2 - \lambda\frac{a}{\sigma+1}(C(\Omega)t+\Gamma)^{\sigma+1}$$

it is easy to prove

$$\Phi_{\lambda,\tau}(\widehat{u}) < \mu \le \Phi_{\lambda,\tau}(\overline{u}) < 0 < \Phi_{\lambda,\tau}(\widetilde{u})$$

(ii) Unlike [7], [8], [9] and [4], where the size of $\|\tau\|_{L^\infty(\Omega)}$ is relevant, in our approach the condition $\Gamma \equiv \|\tau\|_{L^{\sigma+1}(\Omega)}$ small is of primary importance. Note, that Γ small does not say anything about $\|\tau\|_{L^\infty(\Omega)}$.

3. Ω big enough

Now we study problem (2.2) for $\inf c(x) > 0$ and $\Omega \subset \mathbb{R}^n$ $(n \ge 3)$ big enough. By big enough we mean that there exists $x_0 \in \Omega$ such that the euclidean ball with center x_0 and radius R is contained in Ω, with R large enough.

Let $W_0^{1,2}(\Omega)$ be the usual Sobolev space, with $\|u\|^2_{W_0^{1,2}(\Omega)} = \int_\Omega [u^2 + |\nabla u|^2]dx$ and $\Gamma \equiv \|\tau\|_{L^2(\Omega)}$. If $\inf c(x) > 0$, then

$$\|u\|^2_{W_0^{1,2}(\Omega)} \le \frac{1}{m}\int_\Omega [c(x)u^2 + |\nabla u|^2]dx \tag{3.1}$$

where $m \equiv \min\{\inf c(x), 1\}$.

As was seen in section 2 we find an interval $\Lambda' \subset \mathbb{R}_+$ such that for all $\lambda \in \Lambda'$ there exists at least three positive solutions of (2.2) and we eliminate the restrictions on τ . Consequently we obtain:

Theorem 3.1. *Let us assume (f1)–(f4). For all $\Gamma > 0$ and R large enough there exists an interval $]\underline{\lambda}(R), \overline{\lambda}[$ with $\underline{\lambda}(R) > 0$ such that for all $\lambda \in]\underline{\lambda}, \overline{\lambda}[$ the problem (2.2) has at least three positive solutions.*

To prove this theorem, we need to redefine $\underline{\lambda}$ and $\overline{\lambda}$. Therefore, let

$$\omega_{\delta,R}(x) = \begin{cases} \dfrac{s_0}{\delta^{1/4}} & \text{if } |x - x_0| \le \rho \\ \dfrac{s_0}{\delta^{5/4}}(R - |x - x_0|) & \text{if } \rho \le |x - x_0| \le R \\ 0 & \text{if } |x - x_0| \ge R \end{cases}$$

If we define $\omega_o = \omega_{\delta_1,R}$ where $\delta_1 = t_1 R$ and $t_1 \in]0,1[$ such that $\psi(t_1) = \min_{]0,1[} \psi(t)$, $\psi(t) \equiv \frac{1-(1-t)^n}{t^{\frac{5}{2}}(1-t)^n}$; then with a similar development to Lemma 2.2, we obtain

$$\Phi_{\lambda,\tau}(\omega_0) < 0 \quad \forall \lambda > \underline{\lambda} > 0 \quad \textit{and} \quad \forall \tau \ge 0$$

where

$$\underline{\lambda}(R) = \frac{s_0^2(1 + \|c\|_{L^\infty} R^2)}{2F(\tau(x_0))R^{\frac{5}{2}}} \Big(\frac{1 - (1 - t_1)^n}{t_1^{\frac{5}{2}}(1 - t_1)^n} \Big).$$

On the other hand, using the modification, to $n \ge 3$

$$\|\nabla \omega_0\|_{L^2(\Omega)} = s_0 \Big(\omega_n \Big)^{1/2} R^{\frac{2n-5}{4}} \Big(\frac{1 - (1 - t_1)^n}{t_1^{\frac{5}{2}}} \Big)^{1/2} \to \infty \tag{3.2}$$

as $R \to \infty$. Since

$$0 \le \lim_{s \to 0^+} \frac{2F(s)}{s^2} \le \lim_{s \to 0^+} \frac{f(s)}{s} = 0$$

for (f3) and since F is bounded, we define

$$\frac{b}{2} \equiv \sup_{s > 0} \frac{F(s)}{s^2} < +\infty \tag{3.3}$$

Lemma 3.2. *For all $\lambda < \overline{\lambda}$ and $\|u\|_{W_0^{1,2}(\Omega)} = \Gamma$, $\Phi_{\lambda,\tau}(u) > 0$.*

Proof. Using (3.1) and (3.3)

$$\begin{aligned} \Phi_{\lambda,\tau}(u) &= \frac{1}{2} \int_\Omega [c(x)u^2 + |\nabla u|^2] dx - \lambda \int_\Omega F(u + \tau) dx \\ &\ge \frac{m}{2} \|u\|^2_{W_0^{1,2}(\Omega)} - \frac{\lambda b}{2} \int_\Omega (u + \tau)^2 dx \\ &\ge \frac{m}{2} \|u\|^2_{W_0^{1,2}(\Omega)} - \frac{\lambda b}{2} (\|u\|_{L^2(\Omega)} + \|\tau\|_{L^2(\Omega)})^2 \\ &> \frac{m}{2} \|u\|^2_{W_0^{1,2}(\Omega)} - \frac{\lambda b}{2} (\|u\|_{W_0^{1,2}(\Omega)} + \|\tau\|_{L^2(\Omega)})^2 \end{aligned}$$

So, when we define $\overline{\lambda} \equiv m/4b$, then for all $\lambda < \overline{\lambda}$, $\Phi_{\lambda,\tau}(u) > 0$. □

Proof of Theorem 3.1. Let ω_0 and $\underline{\lambda}(R)$ be as above, using Lemma 3.2 there exists $\overline{\lambda} > 0$ such that $\Phi_{\lambda,\tau}(u) > 0$ for all $\lambda < \overline{\lambda}$ and $\|u\|_{W_0^{1,2}(\Omega)} = \Gamma$. From the $\underline{\lambda}$, $\overline{\lambda}$ definition and (3.2) to R large enough $\underline{\lambda} < \overline{\lambda}$ and $\|\omega_0\|_{W_0^{1,2}(\Omega)} > \Gamma$. Finally using a similar development to Theorem 2.1, Theorem 3.1 is proven. □

Remark 3.3. For $n = 2$ Theorem 3.1 is false.

4. The problem in R^n_+

Let $W^{1,2}_0(\mathbb{R}^n_+)$ and $V^{1,2}_{c,0}(\mathbb{R}^n_+)$ be the completion of $C^\infty_0(\mathbb{R}^n_+)$ in $(\|.\|^2_2+\|\nabla(.)\|^2_2)^{1/2}$ and $(\|c.\|^2_2 + \|\nabla(.)\|^2_2)^{1/2}$ respectively, where $\|.\|_2$ is the usual L^2 norm for the respective domain. If $\inf c(x) > 0$, then by (3.1),

$$W^{1,2}_0(\mathbb{R}^n_+) \sim V^{1,2}_{c,0}(\mathbb{R}^n_+)$$

We define for all $\lambda \geq 0$ and for all non-negative function τ such that $\|\tau\|_{L^{\sigma+1}(\mathbb{R}^n_+)} < \infty$, the functional $\Phi_{\lambda,\tau,\infty} : W^{1,2}_0(\mathbb{R}^n_+) \to \mathbb{R}$

$$\Phi_{\lambda,\tau,\infty}(u) = \frac{1}{2}\int_{\mathbb{R}^n_+}[c(x)u^2 + |\nabla u|^2]dx - \lambda\int_{\mathbb{R}^n_+} F(u+\tau)dx$$

where $F(s) = \int_0^t f(t)dt$.

The function $\Phi_{\lambda,\tau,\infty}$ is well-defined; even more if $u \in W^{1,2}_0(\mathbb{R}^n_+)$, using (f3) and Sobolev immersion we obtain

$$\begin{aligned}
0 \leq \int_{\mathbb{R}^n_+} F(u+\tau) &\leq \frac{a}{\sigma+1}\int_{\mathbb{R}^n_+}(u+\tau)^{\sigma+1}\\
&\leq \frac{a}{\sigma+1}(\|u\|_{L^{\sigma+1}(\mathbb{R}^n_+)} + \|\tau\|_{L^{\sigma+1}(\mathbb{R}^n_+)})^{\sigma+1}\\
&\leq \frac{a}{\sigma+1}(C_s\|u\|_{W^{1,2}_0(\mathbb{R}^n_+)} + \|\tau\|_{L^{\sigma+1}(\mathbb{R}^n_+)})^{\sigma+1}
\end{aligned}$$

where C_s is the usual Sobolev immersion constant. Then using (3.1)

$$\Phi_{\lambda,\tau,\infty}(u) \geq \frac{m}{2}\|u\|^2_{W^{1,2}_0(\mathbb{R}^n_+)} - \lambda\frac{a}{\sigma+1}(C_s\|u\|_{W^{1,2}_0(\mathbb{R}^n_+)} + \|\tau\|_{L^{\sigma+1}(\mathbb{R}^n_+)})^{\sigma+1} \tag{4.1}$$

It is easy to verify that $\Phi_{\lambda,\tau,\infty}$ is a C^1 functional, so if $u \in W^{1,2}_0(\mathbb{R}^n_+)$ is a critical point of $\Phi_{\lambda,\tau,\infty}$ then u is a weak solution and by regularity, so classical solution of (1.5).

Proposition 4.1. *(i) Let m be as above then for all $\lambda < \frac{m}{b}$, $\Phi_{\lambda,\tau,\infty}$ is coercive and bounded from below.*
(ii) For all $\lambda < \frac{\inf c(x)}{l}$, (1.5) has at most one solution in $W^{1,2}_0(\mathbb{R}^n_+)$.

Proof. (i) Using (3.1) and (3.3)

$$\begin{aligned}
\Phi_{\lambda,\tau,\infty}(u) &\geq \frac{m}{2}\|u\|^2_{W^{1,2}_0(\mathbb{R}^n_+)} - \frac{\lambda b}{2}\int_{\mathbb{R}^n_+}(u+\tau)^2\\
&> \frac{m}{2}\|u\|^2_{W^{1,2}_0(\mathbb{R}^n_+)} - \frac{\lambda b}{2}(\|u\|_{W^{1,2}_0(\mathbb{R}^n_+)} + \|\tau\|_{L^2(\mathbb{R}^n_+)})^2\\
&= \Big(\frac{m-\lambda b}{2}\Big)\|u\|^2_{W^{1,2}_0(\mathbb{R}^n_+)} - \lambda b\|u\|_{W^{1,2}_0(\mathbb{R}^n_+)}\|\tau\|_{L^2(\mathbb{R}^n_+)} - \frac{\lambda b}{2}\|\tau\|^2_{L^2(\mathbb{R}^n_+)}
\end{aligned}$$

so, (i) is proven.
(ii) The uniqueness is proved as in [1]. Indeed: if u_1 and u_2 are two solutions of (1.5) then,

$$\inf c(x)\int_{\mathbb{R}^n_+}(u_1-u_2)^2dx \leq \int_{\mathbb{R}^n_+}[c(x)(u_1-u_2)^2+|\nabla(u_1-u_2)|^2]dx \leq \lambda l\int_{\mathbb{R}^n_+}(u_1-u_2)^2dx$$

$\square$

Now we consider problem (1.6) and we define $\Phi_{\lambda,\tau,R} : W_0^{1,2}(D_R) \to \mathbb{R}$ in the same way that $\Phi_{\lambda,\tau,\infty}$. It can be verified that, if $R' \geq R$, then

$$W_0^{1,2}(D_R) \subset W_0^{1,2}(D_{R'}) \subset W_0^{1,2}(\mathbb{R}^n_+)$$

in addition for all $u \in W_0^{1,2}(D_R)$, $\Phi_{\lambda,\tau,\infty}(u) \leq \Phi_{\lambda,\tau,R'}(u) \leq \Phi_{\lambda,\tau,R}(u)$, more precisely

$$\Phi_{\lambda,\tau,R'}(u) = \Phi_{\lambda,\tau,R}(u) - \lambda \int_{D_{R'}-D_R} F(\tau)dx \tag{4.2}$$

Remark 4.2. There exists a positive constant $C = C(a, \sigma, C_s, m)$ such that for all $\lambda < \overline{\overline{\lambda}}(\|\tau\|_{L^{\sigma+1}(\mathbb{R}^n_+)})$ and for all u: $\|u\|_{W_0^{1,2}(\mathbb{R}^n_+)} = \|\tau\|_{L^{\sigma+1}(\mathbb{R}^n_+)}$, $\Phi_{\lambda,\tau,\infty}(u) > 0$, where $\overline{\overline{\lambda}}(\|\tau\|_{L^{\sigma+1}(\mathbb{R}^n_+)}) \equiv C\|\tau\|^{1-\sigma}_{L^{\sigma+1}(\mathbb{R}^n_+)}$. In fact, applying (4.1) and taking

$$C \equiv \frac{(\sigma+1)m}{2a}[C_s+1]^{-\sigma-1}$$

the result is obvious. Furthermore for (4.2)

$$\Phi_{\lambda,\tau,R}(u) > 0 \quad \forall u \in W_0^{1,2}(D_R) \quad \|u\|_{W_0^{1,2}(D_R)} = \|\tau\|_{L^{\sigma+1}(\mathbb{R}^n_+)}$$

then as in Lemma 2.5, for $\lambda < \overline{\overline{\lambda}}$ there exists $\overline{u}_R \in W_0^{1,2}(D_R)$ with $\|\overline{u}_R\|_{W_0^{1,2}(D_R)} < \|\tau\|_{L^{\sigma+1}(\mathbb{R}^n_+)}$ such that $\Phi_{\lambda,\tau,R}(\overline{u}_R) < 0$ and $\Phi'_{\lambda,\tau,R}(\overline{u}_R) = 0$.

Now we will prove a sufficient condition to approximate solutions of (1.5) with solutions of (1.6) with R large enough.

Lemma 4.3. *Let f and τ be as above and $\lambda \in R_+$. Suppose $(R_n)_n$ is a sequence $\mathbb{R}_+$ such that $R_n \to +\infty$ and $(u_n)_n$ is a sequence of positive solutions of* (1.6) *with R_n instead of R, such that for all n, $u_n \in W_0^{1,2}(D_{R_n})$ and $(u_n)_n$ is bounded in $W_0^{1,2}(\mathbb{R}^n_+)$, i.e. there exists $\Gamma' > 0$ such that for all n, $\|u_n\|_{L^2(D_{R_n})} + \|\nabla u_n\|_{L^2(D_{R_n})} < \Gamma'$. Then, there exists a subsequence (called again $(u_n)_n$)) and a function $u \in W_0^{1,2}(\mathbb{R}^n_+)$ such that $u_n \to u$ weakly in $W_0^{1,2}(\mathbb{R}^n_+)$ and u is a classical solution of* (1.5).

Proof. Using the Calderón-Zygmund inequality for all n [6, theorems 9.9 and 9.11], $u_n \in W_0^{1,2}(D_{R_n}) \bigcap H^{2,p}(D_{R_n})$. ($H^{2,p}(D_{R_n})$ denotes the usual Sobolev space $W^{2,p}(D_{R_n})$). Fixed $R' > 0$, for any $\Omega' \subset\subset D_{R'}$,

$$\|u_n\|_{H^{2,p}(\Omega')} \leq C(\|u_n\|_{L^p(D_{R'})} + \|\lambda f(u_n+\tau)\|_{L^p(D_{R'})})$$

for all n such that $R_n > R'$. The constant C depends on $D_{R'}$, n, p and Ω'. Since (u_n) is bounded in $W_0^{1,2}(\mathbb{R}^n_+)$, using Sobolev immersion and Poincaré inequality

$$\|u_n\|_{H^{2,p}(\Omega')} \leq C(C_1\Gamma' + \lambda \sup f |D_{R'}|^{\frac{1}{p}})$$

for p such that

$$1 < p < \frac{2n}{n-2} \quad \text{if } n \geq 3$$
$$1 < p \quad \text{if } n = 2$$

and for all n such that $R_n > R'$. From this and the Sobolev embedding theorem for Ω', there exists a subsequence $(u_n)_n$ such that if n=2,3 $u_n \to u$ in $C^{1,\alpha}(\overline{\Omega'})$ and if $n \geq 4$ and $1 < p < \min\big(\frac{n}{2}, \frac{2n}{n-2}\big)$ is fixed, $u_n \to u$ in $L^q(\Omega')$, $1 \leq q < \frac{np}{n-2p}$. Since Ω' is an arbitrary and relatively compact such that $\Omega' \subset\subset D_{R_n}$ and $R_n \to +\infty$, we obtain that the above convergence are in $C^{1,\alpha}_{\rm loc}(\mathbb{R}^n_+)$ and $L^q_{\rm loc}(\mathbb{R}^n_+)$, respectively. In particular

$$u_n \to u \quad \text{in } L^1_{\rm loc}(\mathbb{R}^n_+) \tag{4.3}$$

On the other hand, since $(u_n)_n$ is bounded in $W^{1,2}_0(\mathbb{R}^n_+)$, and reflexivity

$$u_n \to u \quad \text{weakly in } W^{1,2}_0(\mathbb{R}^n_+) \tag{4.4}$$

then using Sobolev immersion

$$u_n \to u \quad \text{weakly in} \quad L^p(\mathbb{R}^n_+) \tag{4.5}$$

where

$$2 \le p < \frac{2n}{n-2} \quad \text{if } n \ge 3$$
$$2 \le p \quad \text{if } n = 2$$

By (4.4), if we prove that for all $v \in C^\infty_0(\mathbb{R}^n_+)$

$$\int_{\mathbb{R}^n_+} f(u_n+\tau)v dx \to \int_{\mathbb{R}^n_+} f(u+\tau)v dx$$

our lemma will follow. Based on this and for fixed $v \in C^\infty_0(\mathbb{R}^n_+)$, we consider the function

$$w = \frac{f(u+\tau)}{u+\tau} v$$

It is easy to see that $w \in L^{p'}(\mathbb{R}^n_+)$, where p' is such that $\frac{1}{p} + \frac{1}{p'} = 1$. Now

$$\begin{aligned} &\int_{\mathbb{R}^n_+} f(u_n+\tau)v dx \\ &= \int_{\mathbb{R}^n_+} \Big[f(u_n+\tau) - (u_n+\tau)\frac{f(u+\tau)}{u+\tau}\Big] v dx + \int_{\mathbb{R}^n_+} (u_n+\tau) w\, dx \end{aligned} \tag{4.6}$$

By (4.5), the last term of the right hand side of (4.6) tends to $\int_{\mathbb{R}^n_+} f(u+\tau)v$. On the other hand, by (f4)

$$\Big|\int_{\mathbb{R}^n_+} \Big[f(u_n+\tau) - (u_n+\tau)\frac{f(u+\tau)}{u+\tau}\Big] v dx\Big| \le 2l \int_{\mathrm{supp}(v)} |u-u_n||v| dx \tag{4.7}$$

so by (4.3), the first term of the second member in (4.6) tends to 0. □

Theorem 4.4. *Let Γ, f, τ and $\overline{\overline{\lambda}}$ be as above. Then, for all λ, $0 < \lambda < \overline{\overline{\lambda}}$ the local minima $\overline{u}_R$ of $\Phi_{\lambda,\tau,R}$ obtained in Remark 4.2, approximate the local minima of $\Phi_{\lambda,\tau,\infty}$ on the ball B_Γ of center 0 and radius Γ in $W^{1,2}_0(\mathbb{R}^n_+)$. As consequence $\nu_\infty \equiv \inf_{B_\Gamma} \Phi_{\lambda,\tau,\infty}$, is a minimum and by Proposition 4.1 it is the unique, if λ is small enough (i.e. $0 < \lambda < \frac{\inf c(x)}{l}$).*

Proof. Using the Lemma 4.3, we only need to prove that $\Phi_{\lambda,\tau,R}(\overline{u}_R) \to \nu_\infty$ as $R \to \infty$. Because of this we consider $(u_R)_R$ in $C^\infty_0(\mathbb{R}^n_+)$ such that $u_R \in W^{1,2}_0(D_R)$ and $\Phi_{\lambda,\tau,\infty}(u_R) \to \nu_\infty$ as $R \to \infty$. Then

$$\nu_\infty \le \Phi_{\lambda,\tau,R}(\overline{u}_R) \le \Phi_{\lambda,\tau,R}(u_R) = \Phi_{\lambda,\tau,\infty}(u_R) - \lambda \int_{\mathbb{R}^n_+ - D_R} F(\tau) dx$$

by (4.2), $\lambda \int_{\mathbb{R}^n_+ - D_R} F(\tau)dx \to 0$ as $R \to \infty$. □

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