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A PRIORI BOUNDS AND EXISTENCE OF NON-REAL EIGENVALUES OF FOURTH-ORDER BOUNDARY VALUE PROBLEM WITH INDEFINITE WEIGHT FUNCTION

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Abstract. In this article, we give a priori bounds on the possible non-real eigenvalue of regular fourth-order boundary value problem with indefinite weight function and obtain a sufficient conditions for such problem to admit non-real eigenvalue.

1. Introduction

In this article we study non-real eigenvalues of differential equations with indefinite weights. The Sturm-Liouville problem with weighted functions is called right-definite if the weighted function do not change signs. Otherwise, the problem is called indefinite problem. The spectral theory of the right-definite problem with self-adjoint boundary conditions has been accomplished, but the spectral structure of indefinite problems, especially both right and left indefinite problem, i.e., indefinite problem, is quite different from and more complicated than that of right-definite problems. For example, there is neither upper nor lower bound for real eigenvalues of indefinite Sturm-Liouville boundary problems. What is more, the indefinite problem may have non-real eigenvalues. Such problems occur in certain physical models, particularly in transport theory and statistical physics. The indefinite nature of the problem was noticed by Haupt [7] and Richardson [12] at the beginning of the previous century. For a review of the early work in this direction, see [9].

In [10], the author considered the indefinite spectral problem

$$-y'' + qy = \lambda wy, \quad y(-1) = y(1) = 0, \quad y \in L^2_{|w|}[-1,1]$$

combined with conditions that q and w are real-valued functions satisfying

$$w(x) \neq 0 \text{ a.e. on } [-1,1], \quad q, w \in L^1[-1,1],$$

and $w(x)$ changes sign on $[-1,1]$. Here, $L^2_{|w|}[-1,1]$ is a Krein space, equipped with the indefinite inner product

$$[f,g] = \int_{-1}^{1} f(x)\overline{g(x)}w(x)dx, f, g \in L^2_{|w|}[-1,1].$$

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The indefinite problems has discrete, real eigenvalues, unbounded from both below and above, and may also admit non-real eigenvalues.

But most articles consider second-order differential equations; in this article, we consider the indefinite spectral problem

$$\begin{gathered}\tau y := y^{(4)} + qy = \lambda w y,\\ y(-1) = y(1) = y''(-1) = y''(1) = 0, \quad y \in L^2_{|w|}[-1,1]\end{gathered} \tag{1.1}$$

combined with conditions that q and w are real-valued functions satisfying

$$w(x) \neq 0 \text{ a.e. on } [-1,1], \quad q, w \in L^1[-1,1], \tag{1.2}$$

and $w(x)$ changes sign on $[-1,1]$. We will first obtain a priori bounds for possible non-real eigenvalues and then find sufficient conditions for the existence of non-real eigenvalues of (1.1).

2. A priori bounds of non-real eigenvalues

For the indefinite problem (1.1), let

$$\begin{gathered}\tau y := y^{(4)} + qy = \lambda |w| y,\\ y(-1) = y(1) = y''(-1) = y''(1) = 0, \quad y \in L^2_{|w|}[-1,1]\end{gathered} \tag{2.1}$$

be the corresponding right-definite problem.

Firstly, we consider the fourth-order differential equation

$$y^4 + qy = \lambda w y, \quad x \in [a,b] \tag{2.2}$$

combined with the boundary conditions

$$\begin{gathered}B_1 y := y(a)\cos(\theta_1) - y'''(a)\sin(\theta_1) = 0,\\ B_2 y := y(b)\cos(\theta_2) - y'''(b)\sin(\theta_2) = 0,\\ B_3 y := y'(a)\cos(\theta_3) - y''(a)\sin(\theta_3) = 0,\\ B_4 y := y'(b)\cos(\theta_4) - y''(b)\sin(\theta_4) = 0,\end{gathered} \tag{2.3}$$

where q and w satisfies (1.2) and $\theta_1, \theta_2, \theta_3, \theta_4 \in \mathbb{R}$. The corresponding right-definite problem is

$$y^4 + qy = \lambda |w| y \tag{2.4}$$

combined with (2.3).

Assumption: $\lambda = 0$ is not an eigenvalue of the boundary problems in questions.

Proposition 2.1. *If problem (2.2)-(2.3) has non-real eigenvalues, then problem (2.4)-(2.3) has at least one negative eigenvalue.*

Proof. Let $y = y(t,\lambda)$ be the corresponding non-real eigenfunction of the non-real eigenvalue λ. Multiplying (2.2) by $\overline{y}$ integrating over $[a,b]$, we find

$$\begin{aligned}&\cot(\theta_2)|y(b)|^2 - \cot(\theta_1)|y(a)|^2 + \cot(\theta_3)|y'(a)|^2\\ &-\cot(\theta_4)|y'(b)|^2 + \int_a^b [|y''|^2 + q|y|^2]dx\\ &= \lambda \int_a^b w|y|^2 dx\end{aligned}$$

Then the smallest eigenvalue v of (2.4) is given by the minimum of the left side of the equation. Let $y \in S_0$, where

$$\begin{aligned} S_0 = \big\{ & y \in L^2_{|w|}[a,b] | y, y', y'' \in AC_{\rm loc}[a,b], y^{(4)} + qy \in L^2_{|w|}[a,b], \\ & B_1 y = B_2 y = B_3 y = B_4 y = 0 \big\} \end{aligned}$$

Now the non-real eigenfunction y makes the left side of equation vanish. Moreover $y \in S_0$. Hence $v < 0$ since $v = 0$ is not an eigenvalue of (2.4), i.e., the problem has at least one negative eigenvalue. □

Proposition 2.2 ([8]). *If problem* (2.4)-(2.3) *has n negative eigenvalues, then problem* (2.2)-(2.3) *has at most $2n$ non-real eigenvalues.*

Denote by $\|\cdot\|_p$ the norm of the space $L^p[-1,1]$ and by $\|\cdot\|_C$ the maximum norm of $C[-1,1]$. If $xw(x) > 0$ a.e. on $[-1,1]$, we set

$$S_1(\varepsilon_1) = \{x \in [-1,1] : xw(x) < \varepsilon_1\}, \quad m_1(\varepsilon_1) = \operatorname{meas} S_1(\varepsilon_1). \tag{2.5}$$

If $w \in AC_{\rm loc}[-1,1]$, $w' \in L^2[-1,1]$, $w'' \in L^2[-1,1]$, we set

$$S_2(\varepsilon_2) = \{x \in [-1,1] : w^2(x) < \varepsilon_2\}, \quad m_2(\varepsilon_2) = \operatorname{meas} S_2(\varepsilon_2). \tag{2.6}$$

A value of x about which $w(x)$ changes its sign will be called a turning point. If $w(x)$ has only one turning point, we will obtain the following a priori bounds for possible non-real eigenvalues.

Theorem 2.3. *Suppose that λ is, if it exists, a non-real eigenvalue of* (1.1). *If $xw(x) > 0$ a.e. on $[-1,1]$, then*

$$|\operatorname{Re}\lambda| \le \frac{2\sqrt{2}\|\phi\|_C(1 + \sqrt{\|q_-\|_1}\|\phi\|_C + \sqrt{2}\|q_-\|_1\|\phi\|_C)}{\varepsilon_1},$$

$$|\operatorname{Im}\lambda| \le \frac{2\sqrt{2}\|\phi\|_C(1 + \sqrt{\|q_-\|_1}\|\phi\|_C)}{\varepsilon_1},$$

where $\varepsilon_1 > 0$ satisfies $(1 - m_1(\varepsilon_1)\|\phi\|_C^2) \ge \frac{1}{2}$ and $q_-(x) = -\min\{0, q(x)\}$.

Proof. Let λ be a non-real eigenvalue of (1.1) and ϕ is the corresponding eigenfunction with $\|\phi\|_2 = 1$, $\|\phi\|_C = \max\{|\phi|, |\phi'|, |\phi'''|\}$. Multiplying both sides of $\phi^{(4)} + q\phi = \lambda w\phi$ by $\overline{\phi}$ and integrating over the interval $[x,1]$ we have

$$-(\phi'''\overline{\phi})(x) + (\phi''\overline{\phi'})(x) + \int_x^1 |\phi''|^2 dx + \int_x^1 q|\phi|^2 dx = \lambda \int_x^1 w|\phi|^2 dx. \tag{2.7}$$

Separating the real and imaginary parts of both sides of (2.4) yields

$$\operatorname{Re}\lambda \int_x^1 w|\phi|^2 dx = \operatorname{Re}(-\phi'''\overline{\phi})(x) + \operatorname{Re}(\phi''\overline{\phi'})(x) + \int_x^1 |\phi''|^2 dx + \int_x^1 q|\phi|^2 dx, \tag{2.8}$$

$$\operatorname{Im}\lambda \int_x^1 w|\phi|^2 dx = \operatorname{Im}(-\phi'''\overline{\phi})(x) + \operatorname{Im}(\phi''\overline{\phi'})(x). \tag{2.9}$$

We will use (2.8) and (2.9) to estimate $\operatorname{Re}\lambda$ and $\operatorname{Im}\lambda$. To do this, let $x = -1$ in (2.9). From $\operatorname{Im}\lambda \ne 0$ and $\phi(-1) = 0$ and $\phi''(-1) = 0$, we have $\int_{-1}^1 w|\phi|^2 dx = 0$ and hence, by (2.8),

$$\int_{-1}^1 |\phi''|^2 dx + \int_{-1}^1 q|\phi|^2 dx = 0. \tag{2.10}$$

Let $\|q_-\|_1 = \int_{-1}^1 q_- dx$, then

$$\int_{-1}^1 |\phi''|^2 dx = -\int_{-1}^1 q|\phi|^2 dx \le \int_{-1}^1 q_-|\phi|^2 dx \le \|\phi\|_C^2 \|q_-\|_1$$

and $\int_{-1}^1 q_-|\phi|^2 dx \le \|\phi\|_C^2 \|q_-\|_1$, hence

$$\|\phi''\|_2^2 \le \|\phi\|_C^2 \|q_-\|_1, \quad \int_{-1}^1 q_-|\phi|^2 dx \le \|\phi\|_C^2 \|q_-\|_1. \tag{2.11}$$

Since $xw(x) > 0$ a.e. on $[-1,1]$, one can find $\varepsilon_1 > 0$ such that $(1 - m_1(\varepsilon_1)\|\phi\|_C^2) \ge \frac{1}{2}$, where $m_1(\varepsilon_1)$ is defined in (2.5). Using $\int_{-1}^1 w|\phi|^2 dx = 0$, from (2.11), we have

$$\begin{aligned} \int_{-1}^1 \int_x^1 w(t)|\phi(t)|^2 \, dt \, dx &= \int_{-1}^1 xw(x)|\phi(x)|^2 dx \\ &\ge \varepsilon_1 \Big(\int_{-1}^1 |\phi(x)|^2 dx - \int_{S_1(\varepsilon_1)} |\phi(x)|^2 dx \Big) \\ &\ge \varepsilon_1 (1 - m_1(\varepsilon_1)\|\phi\|_C^2) \ge \frac{\varepsilon_1}{2}. \end{aligned} \tag{2.12}$$

Set $q_+(x) = \max\{0, q(x)\}$, then $q = q_+ - q_-$ and $|q| = q_+ + q_- = q + 2q_-$. Repeatedly using (2.10), we have

$$\begin{aligned} \Big| \int_{-1}^1 \int_x^1 (|\phi''|^2 + q|\phi|^2) \, dt \, dx \Big| &= \Big| \int_{-1}^1 x(|\phi''|^2 + q|\phi|^2) dx \Big| \\ &\le \int_{-1}^1 (|\phi''|^2 + q|\phi|^2 + 2q_-|\phi|^2) dx \\ &\le 2 \int_{-1}^1 q_-|\phi|^2 dx \le 2\|\phi\|_C^2 \|q_-\|_1. \end{aligned}$$

Now, by (2.11) integrating (2.8) gives

$$\begin{aligned} &|\operatorname{Re}\lambda| \int_{-1}^1 \int_x^1 w(t)|\phi(t)|^2 \, dt \, dx \\ &= \Big| \int_{-1}^1 \operatorname{Re}(-\phi'''\overline{\phi})(x) dx + \int_{-1}^1 \operatorname{Re}(\phi''\overline{\phi'})(x) dx + \int_{-1}^1 \int_x^1 (|\phi''|^2 + q|\phi|^2) \, dt \, dx \Big| \\ &\le \sqrt{2}\|\phi\|_C (1 + \sqrt{\|q_-\|_1}\|\phi\|_C + \sqrt{2}\|q_-\|_1 \|\phi\|_C). \end{aligned}$$

Therefore, in view of (2.11), we conclude that

$$|\operatorname{Re}\lambda| \le \frac{2\sqrt{2}\|\phi\|_C (1 + \sqrt{\|q_-\|_1}\|\phi\|_C + \sqrt{2}\|q_-\|_1 \|\phi\|_C)}{\varepsilon_1}. \tag{2.13}$$

Moreover, integrating (2.9) and using (2.12) and (2.11), we have

$$\begin{aligned} \frac{\varepsilon_1}{2} |\operatorname{Im}\lambda| &\le |\operatorname{Im}\lambda| \int_{-1}^1 \int_x^1 w|\phi|^2 \, dt \, dx \\ &= \Big| \int_{-1}^1 (\operatorname{Im}(-\phi'''\overline{\phi})(x) + \operatorname{Im}(\phi''\overline{\phi'})(x)) \Big| \\ &\le \sqrt{2}\|\phi\|_C (1 + \sqrt{\|q_-\|_1}\|\phi\|_C). \end{aligned} \tag{2.14}$$

This completes the proof. $\square$

When $w(x)$ is allowed to have more turning points, we have the following result.

Theorem 2.4. *Suppose that λ is, if it exists, a non-real eigenvalue of* (1.1). *If $w \in AC_{\rm loc}[-1,1]$, $w', w'' \in L^2[-1,1]$, then*

$$|\operatorname{Re}\lambda| \le \frac{2\|\phi\|_C[2\|w\|_C\|q_-\|_1\|\phi\|_C + 2\sqrt{2}\|w'\|_2\sqrt{\|q_-\|_1}\|\phi\|_C + \sqrt{\|q_-\|_1}\|w''\|_2]}{\varepsilon_2},$$
$$|\operatorname{Im}\lambda| \le \frac{2\|\phi\|_C\sqrt{\|q_-\|_1}[2\sqrt{2}\|w'\|_2\|\phi\|_C + \|w''\|_2]}{\varepsilon_2},$$

where $\varepsilon_2 > 0$ satisfies $(1 - m_2(\varepsilon_2)\|\phi\|_C^2) \ge 1/2$.

Proof. Let λ be a non-real eigenvalue of (1.1) and ϕ the corresponding eigenfunction with $\|\phi\|_2 = 1$, $\|\phi\|_C = \max\{|\phi|, |\phi'|, |\phi'''|\}$. In this case we still can make use of (2.7), (2.8) and (2.9). From (2.9), since $\operatorname{Im}\lambda \neq 0$, we have $\int_{-1}^1 w|\phi|^2 dx = 0$. Thus, (2.10) and (2.11) holds, and particularly,

$$\begin{gathered} |\phi|^2 \le \|\phi\|_C^2, \quad \|\phi'\|_2^2 \le 2\|\phi\|_C^2, \\ \|\phi''\|_2^2 \le \|\phi\|_C^2\|q_-\|_1, \quad \int_{-1}^1 q_-|\phi|^2 dx \le \|\phi\|_C^2\|q_-\|_1. \end{gathered} \tag{2.15}$$

Multiplying both sides of $\phi^{(4)} + q\phi = \lambda w\phi$ by $w\overline{\phi}$ and integrating over the interval $[-1,1]$ we have

$$\begin{aligned} &\int_{-1}^1 w|\phi''|^2 dx + 2\int_{-1}^1 \phi'' w'\overline{\phi'} dx + \int_{-1}^1 \phi'' w''\overline{\phi} dx + \int_{-1}^1 wq|\phi|^2 dx \\ &= \lambda\int_{-1}^1 w^2|\phi|^2 dx. \end{aligned} \tag{2.16}$$

Separating the real and imaginary parts of both sides of (2.17) yields

$$\begin{aligned} \operatorname{Re}\lambda\int_{-1}^1 w^2|\phi|^2 dx &= \operatorname{Re}\Big(2\int_{-1}^1 \phi'' w'\overline{\phi'} dx\Big) + \operatorname{Re}\Big(\int_{-1}^1 \phi'' w''\overline{\phi} dx\Big) \\ &\quad + \int_{-1}^1 w|\phi''|^2 dx + \int_{-1}^1 wq|\phi|^2 dx, \end{aligned} \tag{2.17}$$

$$\operatorname{Im}\lambda\int_{-1}^1 w^2|\phi|^2 dx = \operatorname{Im}\Big(2\int_{-1}^1 \phi'' w'\overline{\phi'} dx\Big) + \operatorname{Im}\Big(\int_{-1}^1 \phi'' w''\overline{\phi} dx\Big). \tag{2.18}$$

Now, using (2.15) and $|q| = q_+ + q_- = q + 2q_-$ and $\int_{-1}^1 q|\phi|^2 dx = -\int_{-1}^1 |\phi''|^2 dx$, we obtain

$$\begin{aligned} \Big|\int_{-1}^1 w|\phi''|^2 dx\Big| &\le \|w\|_C\|\phi\|_C^2\|q_-\|_1, \\ \Big|\int_{-1}^1 wq|\phi|^2 dx\Big| &\le \|w\|_C\|\phi\|_C^2\|q_-\|_1, \\ \Big|\int_{-1}^1 \phi'' w'\overline{\phi'} dx\Big| &\le \Big(\int_{-1}^1 |\phi''|^2 dx\Big)^{1/2}\Big(\int_{-1}^1 |w'|^2 dx\Big)^{1/2}\Big(\int_{-1}^1 |\phi'|^2 dx\Big)^{1/2} \\ &\le \sqrt{2\|q_-\|_1}\|w'\|_2\|\phi\|_C^2, \\ \Big|\int_{-1}^1 \phi'' w''\overline{\phi} dx\Big| &\le \sqrt{\|q_-\|_1}\|w''\|_2\|\phi\|_C. \end{aligned} \tag{2.19}$$

Recall that $m_2(\varepsilon_2) = \operatorname{meas} S_2(\varepsilon_2)$ defined by (2.6) and $w^2(x) \geq \varepsilon_2$ on the set $\Omega(\varepsilon_2) := [-1,1]\backslash S_2(\varepsilon_2)$. Then $(1 - m_2(\varepsilon_2)\|\phi\|_C^2) \geq \frac{1}{2}$ yields

$$\begin{aligned} \int_{-1}^{1} w^2(x)|\phi(x)|^2 dx &\geq \varepsilon_2 \int_{\Omega(\varepsilon_2)} |\phi(x)|^2 dx \\ &= \varepsilon_2 \Big(\int_{-1}^{1} |\phi(x)|^2 dx - \int_{S_2(\varepsilon_2)} |\phi(x)|^2 dx \Big) \\ &\geq \varepsilon_2 (1 - m_2(\varepsilon_2)\|\phi\|_C^2) \geq \frac{\varepsilon_2}{2}, \end{aligned} \tag{2.20}$$

which, together with (2.17), (2.18) and (2.19), completes the proof. □

In the particular case when $q \geq 0$, by Theorems 2.3 and 2.4 we see that (1.1) has no any non-real eigenvalues, which is in accordance with the conclusion in Proposition 2.1 since (2.1) does not have any negative eigenvalues.

In what follows, we impose the symmetry conditions on q and w, namely,

$$q(x) = q(-x), \quad w(-x) = -w(x). \tag{2.21}$$

Under the conditions (1.2) and (2.21), it is easy to see that if $\lambda \in \mathbb{C}$ be a eigenvalue of (1.1) and ϕ the corresponding eigenfunction, then $-\overline{\lambda}$ is an eigenvalue of (1.1) with the eigenfunction $\overline{\phi(-x)}$. Thus, if $\lambda = i\alpha$ with $\alpha \in \mathbb{R}$, then $\overline{\phi(-x)} = c\phi(x)$ for some $c \neq 0$ since the geometric multiplicity is one. Then it follows that $|c| = 1$ from $\overline{\phi(0)} = c\phi(0)$, $\overline{\phi'(0)} = c\phi'(0)$, and $|\phi(0)| + |\phi'(0)| \neq 0$. To sum up, we have a lemma.

Lemma 2.5. *Let* (1.2) *and* (2.21) *hold. If* $\lambda \in \mathbb{C}$ *is an eigenvalue of* (1.1) *with an eigenfunction* ϕ*, then* $-\overline{\lambda}$ *is an eigenvalue of* (1.1) *with the eigenfunction* $\overline{\phi(-x)}$*. Particularly, if* $\lambda = i\alpha$ *with* $\alpha \in \mathbb{R}$ *and* $\alpha \neq 0$*, then* $\overline{\phi(-x)} = c\phi$ *for some* $c \in \mathbb{C}$ *with* $|c| = 1$*.*

In this case, more accurate a priori bounds on imaginary eigenvalues can be found if q is bounded below and w keeps away from zero.

Theorem 2.6. *Suppose that* (2.21) *holds and* $xw(x) > 0$ *a.e. on* $[-1, 1]$*. If, for some* $q_0 < 0$ *and* $w_0 > 0$*,*

$$q(x) \geq q_0, \quad |w(x)| \geq w_0, \quad \text{a.e. } x \in [-1, 1], \tag{2.22}$$

then for any possible pure imaginary eigenvalue λ *of* (1.1)*, we have*

$$|\operatorname{Im} \lambda| \leq \frac{8\sqrt{2}\|\phi\|_C^3(1 + \sqrt{-q_0})}{w_0}. \tag{2.23}$$

Proof. Let ϕ be an eigenfunction corresponding to $\lambda = i\alpha$ with $\|\phi\|_2 = 1$, $\|\phi\|_C = \max\{|\phi|, |\phi'|, |\phi'''|\}$. It follows from Lemma 2.5 that there exists an $\omega \in [0, 2\pi)$ such that $\overline{\phi(-x)} = e^{i\omega}\phi(x)$ and $-\overline{\phi'(-x)} = e^{i\omega}\phi'(x)$. So, $|\phi(x)|$ and $|\phi'(x)|$ are even functions. We see that (2.7)-(2.11) hold for this ϕ. And

$$|\phi(x)|^2 \leq (x+1)\int_{-1}^{x} |\phi'(t)|^2 dt \leq \int_{-1}^{0} |\phi'(t)|^2 dt = \frac{1}{2}\|\phi'\|_2^2, \quad x \in [-1, 0] \tag{2.24}$$

since $|\phi'(x)|$ is even. Actually, (2.24) is true for $x \in [-1, 1]$ since $|\phi(x)|$ is even.

Since $q(x) \geq q_0$, on $[-1,1]$, it follows from (2.10) and $\|\phi\|_2 = 1$, that $\|\phi''\|_2^2 = -\int_{-1}^1 q|\phi|^2 dx \leq -q_0$. Then integrating (2.9) produces

$$\begin{aligned} |\operatorname{Im}\lambda\| \int_{-1}^1 \int_x^1 w|\phi|^2 \, dt \, dx| &= |\int_{-1}^1 (\operatorname{Im}(-\phi'''\overline{\phi})(x) + \operatorname{Im}(\phi''\overline{\phi'})(x)) dx| \\ &\leq \sqrt{2}\|\phi\|_C (1 + \sqrt{-q_0}). \end{aligned} \tag{2.25}$$

Let $\delta = 1/(4\|\phi\|_C^2)$. By (2.24), we have

$$\begin{aligned} &|\int_{-1}^1 \int_x^1 w(t)|\phi(t)|^2 \, dt \, dx| \\ &= \int_{-1}^1 x w(x)|\phi(x)|^2 dx \geq w_0 \int_{-1}^1 |x||\phi(x)|^2 dx \\ &\geq w_0 \delta \int_{|x|\geq\delta} |\phi(x)|^2 dx \\ &\geq w_0 \delta \Big(\int_{-1}^1 |\phi(x)|^2 dx - \int_{-\delta}^{\delta} |\phi(x)|^2 dx \Big) \\ &\geq w_0 \delta (1 - 2\delta \|\phi\|_C^2) \geq \frac{w_0}{8\|\phi\|_C^2}. \end{aligned} \tag{2.26}$$

Now, (2.23) follows from (2.25) and (2.26). ☐

3. Existence of non-real eigenvalues

Although a priori estimate can be given in section 2 and the exact number of non-real eigenvalues are still difficult; there are recent studies by means of the operator theory in Krein spaces [6]. In this section we prove the existence of non-real eigenvalues.

Lemma 3.1 ([6]). *If $w_j \in L^1[-1,1]$ and $w_j(x) > 0$ a.e. on $[-1,1]$ for $j = 1,2$, then the two eigenvalue problems*

$$y^{(4)} + qy = \lambda w_j(x) y, \quad y(-1) = y(1) = y''(-1) = y''(1) = 0, \quad j = 1,2 \tag{3.1}$$

have the same number of negative eigenvalues.

Let K be the Krein space $L^2_{|w|}[-1,1]$, equipped with the indefinite inner product

$$[f,g] = \int_{-1}^1 f(x)\overline{g(x)} w(x) dx, \quad f,g \in L^2_{|w|}[-1,1], \tag{3.2}$$

and T be a self-adjoint operator in K with domain $D(T) = \{y \in L^2_{|w|}[-1,1] | y, y', y'' \in AC_{\rm loc}[-1,1], T \in L^2_{|w|}[-1,1]\}$. See [1, 3, 5]. We say that the operator T has k negative squares, $k \in N_0$, if there exists a k-dimensional subspace X of K in $D(T)$ such that $[Tf,f] < 0$ if $f \in X$and $f \neq 0$, but no $(k+1)$-dimensional subspace with this property.

Theorem 3.2. *Let* (2.21) *hold. If the eigenvalue problem*

$$y^{(4)} + qy = \lambda |w| y, \quad y(-1) = y(1) = y''(-1) = y''(1) = 0 \tag{3.3}$$

has one negative eigenvalue and the rest eigenvalues are all positive, then (1.1) *has exactly two purely imaginary eigenvalues.*

Proof. Let $A = \frac{1}{w}\tau$ and $B = \frac{1}{|w|}\tau$ be the operators associated with $y^{(4)} + qy = \lambda wy$ and $y^{(4)} + qy = \lambda|w|y$ with boundary conditions, respectively. Then B is self-adjoint with respect to the definite inner product

$$(f,g) = \int_{-1}^{1} f(x)\overline{g(x)}|w(x)|dx, f, g \in L^2_{|w|}[-1,1]$$

and A is self-adjoint with respect to the indefinite inner product (3.2).

It follows from Lemma 3.1 and the assumption in Theorem 3.2 that B has one negative eigenvalue and the rest are positive, and hence, A has exactly one negative square since $[Af,f] = (Bf,f)$ and 0 is a resolvent point of A. It is well known (see, [5, 6]) that this implies the existence of exactly one eigenvalue λ of (1.1) in $\mathbb{R}$ or the upper half-plane $\mathbb{C}^+$ and that if $\lambda \in \mathbb{R}$ with eigenfunction ϕ then $[Af,f] = \lambda(f,f) \le 0$. Let λ be such an eigenvalue with eigenfunction ϕ. If λ is real, then $-\lambda = -\overline{\lambda}$ is also an eigenvalue with the eigenfunction $\overline{\phi(-x)}$ by Lemma 2.5 and

$$-\lambda[\overline{\phi(-x)}, \overline{\phi(-x)}] = \lambda[\phi,\phi] \le 0$$

by the odd symmetry of w. Thus, we get that λ and $-\lambda$ are two such eigenvalues, which is a contradiction. Since $\lambda \in \mathbb{C}^+$ implies $-\overline{\lambda} \in \mathbb{C}^+$, we see that $\lambda = -\overline{\lambda}$, i.e., λ is purely imaginary. The proof is complete. □

For more details about non-real eigenvalue of second-order boundary value problems, please see [1, 2, 3, 4, 11, 13].

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References

[1] J. Behrndt, S. Chen, F. Philipp, J. Qi; *Estimates on the non-real eigenvalues of regular indefinite Sturm-Liouville problems*, Proc. Roy. Soc. Edinburgh Sect. A 144 (2014), 1113-1126.

[2] J. Behrndt, Q. Katatbeh, C. Trunk; *Non-real eigenvalues of singular indefinite Sturm-Liouville operators*, Proc. Amer. Math. Soc., 137 (2009), 3797-3806.

[3] J. Behrndt, Q. Katatbeh, C. Trunk; *Accumulation of complex eigenvalues of indefinite Sturm-Liouville operators*, J. Phys. A: Math. Theor. 41 (2008), 244003.

[4] J. Behrndt, F. Philipp, C. Trunk; *Bounds on the non-real spectrum of differential operators with indefinite weights*, arXiv: 1204.1112 (2012), Math.

[5] J. Behrndt, C. Trunk; *On the negative squares of indefinite Sturm-Liouville operators*, J. Differential Equations, 238 (2007), 491-519.

[6] B. C̈urgus, H. Langer; *A Krein space approach to symmetric ordinary differential operators with an indefinite weight functions*, J. Differential Equations, 79 (1989), 31-61.

[7] O. Haupt; *Über eine methode zum beweis von oszillations theoremen*, Math. Ann., 76 (1915), 67-104.

[8] A. B. Mingarelli; *Indefinite Sturm-Liouville problems. Ordinary and partial differential equations* (Dundee, 1982), pp. 519-528, Lecture Notes in Math., 964, Springer, Berlin-New York, 1982.

[9] A. B. Mingarelli; *A survey of the regular weighted Sturm-Liouville problem-The non-definite case*, arXiv:1106.6013v1 [math.CA] (2011).

[10] J. Qi, S. Chen; *A priori bounds and existence of non-real eigenvalues of indefinite Sturm-Liouville problems*, axXiv:1306.5517, J. Spectral Theory, 4 (2014), 53-63.

[11] J. Qi, B. Xie, S. Chen; *The upper and lower bounds on non-real eigenvalues of indefinite Sturm-Liouville problems*, Proc. Amer. Math. Soc. 144 (2016), 547-559.

[12] R. G. D. Richardson; *Theorems of oscillation for two linear differential equations of second order with two parameters*, Trans. Amer. Math. Soc., 13 (1912), 22-34.

[13] B. Xie, J. Qi; *Non-real eigenvalues of indefinite Sturm-Liouville problems*, J. Differential Equations, 255 (2013), 2291-2301.

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