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MULTIPLE POSITIVE SOLUTIONS FOR FOURTH-ORDER THREE-POINT p-LAPLACIAN BOUNDARY-VALUE PROBLEMS

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Abstract. In this paper, we study the three-point boundary-value problem for a fourth-order one-dimensional p-Laplacian differential equation

$$\big(\phi_p(u''(t))\big)'' + a(t)f\big(u(t)\big) = 0, \quad t \in (0,1),$$

subject to the nonlinear boundary conditions:

$$u(0) = \xi u(1), \quad u'(1) = \eta u'(0),$$
$$(\phi_p(u''(0))' = \alpha_1(\phi_p(u''(\delta))', \quad u''(1) = \sqrt[p-1]{\beta_1}\, u''(\delta),$$

where $\phi_p(s) = |s|^{p-2}s$, $p > 1$. Using the five functional fixed point theorem due to Avery, we obtain sufficient conditions for the existence of at least three positive solutions.

1. Introduction

This paper concerns the existence of three positive solutions for the fourth-order three-point boundary-value problem (BVP for short) consisting of the p-Laplacian differential equation

$$\big(\phi_p(u''(t))\big)'' + a(t)f(u(t)) = 0, \quad t \in (0,1), \tag{1.1}$$

with the nonlinear boundary conditions

$$\begin{gathered} u(0) = \xi u(1), \quad u'(1) = \eta u'(0), \\ (\phi_p(u''(0))' = \alpha_1(\phi_p(u''(\delta))', \quad u''(1) = \sqrt[p-1]{\beta_1}\, u''(\delta), \end{gathered} \tag{1.2}$$

where $f : R \to [0,+\infty)$ and $a : (0,1) \to [0,+\infty)$ are continuous functions, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, α_1, $\beta_1 \geq 0$, $\xi \neq 1$, $\eta \neq 1$ and $0 < \delta < 1$.

Two-point boundary-problems for differential equation are used to describe a number of physical, biological and chemical phenomena. For additional background and results, we refer the reader to the monograph by Agawarl, O'Regan and Wong [1] as well as to the recent contributions by [2, 9, 13, 14, 20].

Boundary-value problems for n-th order differential equation [15, 16, 22] and even-order can arise, especially for fourth-order equations, in applications, see [4, 5, 6, 7, 8] and references therein.

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Recently, three-point boundary-value problems of the differential equations were presented and studied by many authors, see [10, 11, 12, 21] and the references cite there. However, three-point BVP (1.1), (1.2) have not received as much attention in the literature as Lidstone condition BVP

$$\begin{gathered} u''''(t) = a(t)f(u(t)), \quad t \in (0,1), \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{gathered} \tag{1.3}$$

and the three-point BVP for the second-order differential equation

$$\begin{gathered} u''(t) + a(t)f(u(t)) = 0, \quad t \in (0,1), \\ u(0) = 0, \quad u(1) = \alpha u(\eta), \end{gathered} \tag{1.4}$$

that were extensively considered, in [13, 14, 20] and [21], respectively. The results of existence of positive solutions of BVP (1.1), (1.2) are relatively scarce.

Most recently, Liu and Ge studied two class of four-order four-point BVPs successively in [17, 18]. They proved that existence of at least two or three positive solutions. To the best of our knowledge, existence results of multiple positive solutions for fourth-order three-point BVP (1.1), (1.2) have not been found in literature. Motivated by the works in [17, 18], the purpose of this paper is to establish the existence of at least three positive solutions of (1.1), (1.2).

For the remainder of the paper, we assume that:

(i) $0 < \int_0^1 a(s)ds < \infty$;
(ii) q satisfies $\frac{1}{p} + \frac{1}{q} = 1$ and $\phi_q(z) = |z|^{q-2}z$.

2. Background and definitions

For the convenience of the reader, we provide some background material from the theory of cones in Banach spaces. We also state in this section a fixed point theorem by Avery.

Definition 2.1. Let X be a real Banach space. A nonempty closed set $P \subset X$ is said to be a cone provided that

(i) $x \in P$ and $\lambda \geq 0$ implies $\lambda x \in X$, and
(ii) $x \in P$ and $-x \in P$ implies $x = 0$.

Every cone $P \subset X$ induces an ordering in X given by $x \leq y$ if and only if $y - x \in P$.

Definition 2.2. The map ψ is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E provided that $\psi : P \to [0, \infty)$ is continuous and

$$\psi(tx + (1-t)y) \geq t\psi(x) + (1-t)\psi(y)$$

for all $x, y \in P$ and $0 \leq t \leq 1$. Similarly, we say the map β is a nonnegative continuous convex functional on a cone P of a real Banach space E provided that $\beta : P \to [0, \infty)$ is continuous and

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y)$$

for all $x, y \in P$ and $0 \leq t \leq 1$.

Let γ, β, θ be nonnegative, continuous, convex functionals on P and α, ψ be nonnegative, continuous, concave functionals on P. Then for nonnegative numbers h, a, b, d and c we define the following sets:

$$\begin{gathered}
P(\gamma, c) = \{x \in P : \gamma(x) < c\}, \\
P(\gamma, \alpha, a, c) = \{x \in P : a \leq \alpha(x),\ \gamma(x) \leq c\}, \\
Q(\gamma, \beta, d, c) = \{x \in P : \beta(x) \leq d,\ \gamma(x) \leq c\}, \\
P(\gamma, \theta, \alpha, a, b, c) = \{x \in P :\ a \leq \alpha(x),\ \theta(x) \leq b,\ \gamma(x) \leq c\}, \\
Q(\gamma, \beta, \psi, h, d, c) = \{x \in P : h \leq \psi(x),\ \beta(x) \leq d,\ \gamma(x) \leq c\}.
\end{gathered}$$

To prove our results, we need the following Five Functionals Fixed Point Theorem due to Avery [3] which is a generalization of the Leggett-Williams fixed point theorem.

Theorem 2.1. *Suppose X is a real Banach space and P is a cone of X, γ, β, θ are three nonnegative, continuous, convex functionals and α, ψ are nonnegative, continuous, concave functionals such that*

$$\alpha(x) \leq \beta(x), \quad ||x|| \leq M\gamma(x)$$

for $x \in \overline{P(\gamma, c)}$ and some positive numbers c, M. Again, assume that

$$T : \overline{P(\gamma, c)} \to \overline{P(\gamma, c)}$$

be a completely continuous operator and there are positive numbers h, d, a, b with $0 < d < a$ such that

- (i) *$\{x \in P(\gamma, \theta, \alpha, a, b, c) : \alpha(x) > a\} \neq \emptyset$ and $x \in P(\gamma, \theta, \alpha, a, b, c)$ implies $\alpha(Tx) > a$.*
- (ii) *$\{x \in Q(\gamma, \beta, \psi, h, d, c) : \beta(x) < d\} \neq \emptyset$ and $x \in Q(\gamma, \beta, \psi, h, d, c)$ implies $\beta(Tx) < d$.*
- (iii) *$x \in P(\gamma, \alpha, a, c)$ with $\theta(Tx) > b$ implies $\alpha(Tx) > a$.*
- (iv) *$x \in Q(\gamma, \beta, d, c)$ with $\psi(Tx) < h$ implies $\beta(Tx) < d$.*

Then T has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, c)}$ such thah

$$\beta(x_1) < d, \quad a < \alpha(x_2), \quad d < \beta(x_3), \quad \text{with} \quad \alpha(x_3) < a.$$

3. Related lemmas

Lemma 3.1 ([17])**.** *Suppose $f \in C(R, R)$, then the three-point BVP*

$$\begin{gathered}
-y'' = f(t), \quad t \in (0, 1) \\
y'(0) = \alpha_1 y'(\delta), \quad y(1) = \beta_1 y(\delta)
\end{gathered} \tag{3.1}$$

has a unique solution

$$y(t) = \int_0^1 g(t, s) f(s) ds, \quad t \in (0, 1),$$

where $M = (1-\alpha_1)(1-\beta_1) \neq 0$ *and*

$$g(t,s) = \frac{1}{M}\begin{cases} 1-\beta_1\delta - t + \beta_1 t, & \text{if } 0 \le s \le t < \delta < 1 \\ & \text{or } 0 \le s \le \delta \le t \le 1, \\ 1-\beta_1\delta + (1-\beta_1)(\alpha_1 s - s - \alpha_1 t), & \text{if } 0 \le t \le s \le \delta < 1, \\ 1-\alpha_1 - \beta_1 s + \alpha_1\beta_1 s - t & \\ +\alpha_1 t + \beta_1 t + \alpha_1\beta_1 t, & \text{if } 0 \le \delta \le s \le t \le 1, \\ (1-s)(t-\alpha_1), & \text{if } 0 < \delta \le t \le s \le 1 \\ & \text{or } 0 \le t < \delta \le s \le 1. \end{cases}$$

Lemma 3.2 ([19]). *Suppose* $f \in C(R,R)$, *then the two-point BVP*

$$\begin{gathered} -y'' = f(t), \qquad t \in (0,1) \\ y(0) = \xi y(1), \quad y'(1) = \eta y'(0) \end{gathered} \tag{3.2}$$

has a unique solution

$$y(t) = \int_0^1 h(t,s)f(s)ds, \quad t \in [0,1],$$

where $M_1 = (1-\xi)(1-\eta) \neq 0$ *and*

$$h(t,s) = \frac{1}{M_1}\begin{cases} s + \eta(t-s) + \xi\eta(1-t), & 0 \le s \le t \le 1, \\ t + \xi(s-t) + \xi\eta(1-s), & 0 \le t \le s \le 1. \end{cases}$$

Remark 3.3. It is easy to check that if $\alpha_1 < 1$, and $0 \le \beta_1 < 1$, then $g(t,s) \ge 0$ for $(t,s) \in [0,1]\times[0,1]$. If $\xi,\eta \ge 0$ and $M_1 = (1-\xi)(1-\eta) \ge 0$, then $h(t,s) \ge 0$ for $(t,s) \in [0,1]\times[0,1]$

If $u(t)$ is a solution of BVP (1.1) and (1.2). By Lemma 3.1 and (3.1), one has

$$\phi_p(u''(t)) = -\int_0^1 g(t,s)a(s)f(u(s))ds. \tag{3.3}$$

By Lemma 3.2 and (3.2), one obtains

$$u(t) = \int_0^1 h(t,s)\phi_q\Big(\int_0^1 g(s,\tau)a(\tau)f(u(\tau))d\tau\Big)ds. \tag{3.4}$$

Lemma 3.4 ([19]). *Suppose* $0 \le \xi,\eta < 1$, $0 < t_1 < t_2 < 1$ *and* $\delta \in (0,1)$. *Then, for* $s \in [0,1]$,

$$\frac{h(t_1,s)}{h(t_2,s)} \ge \frac{t_1}{t_2}, \tag{3.5}$$

$$\frac{h(1,s)}{h(\delta,s)} \le \frac{1}{\delta}. \tag{3.6}$$

Lemma 3.5 ([19]). *Suppose* $\xi,\eta > 1$, $0 < t_1 < t_2 < 1$ *and* $\delta \in (0,1)$. *Then, for* $s \in [0,1]$,

$$\frac{h(t_2,s)}{h(t_1,s)} \ge \frac{1-t_2}{1-t_1}, \tag{3.7}$$

$$\frac{h(0,s)}{h(\delta,s)} \le \frac{1}{1-\delta}. \tag{3.8}$$

4. Triple positive solutions to (1.1), (1.2)

Now let the Banach space $E = C[0,1]$ be endowed with the maximum norm $\|u\| = \max_{t\in[0,1]} |u(t)|$. Let the two cones $P_1, P_2 \subset X$ defined by

$$\begin{aligned} P_1 &= \{u \in X : u(t) \text{ is nonnegative, concave, nondecreasing on } (0,1)\}, \\ P_2 &= \{u \in X : u(t) \text{ is nonnegative, concave, nonincreasing on } (0,1)\}. \end{aligned}$$

Next, choose $t_1, t_2, t_3 \in (0,1)$ and $t_1 < t_2$. Define nonnegative, continuous, concave functions α, ψ and nonnegative, continuous, convex functions β, θ, γ on P_1 by

$$\begin{aligned} \gamma(u) &= \max_{t\in[0,t_3]} u(t) = u(t_3), \quad u \in P_1, \\ \psi(u) &= \min_{t\in[\delta,1]} u(t) = u(\delta), \quad u \in P_1, \\ \beta(u) &= \max_{t\in[\delta,1]} u(t) = u(1), \quad u \in P_1, \\ \alpha(u) &= \min_{t\in[t_1,t_2]} u(t) = u(t_1), \quad u \in P_1, \\ \theta(u) &= \max_{t\in[t_1,t_2]} u(t) = u(t_2), \quad u \in P_1. \end{aligned}$$

It is easy to verify that $\alpha(u) = u(t_1) \le u(1) = \beta(u)$ and $\|u\| = u(1) \le \frac{1}{t_3} u(t_3) = \frac{1}{t_3}\gamma(u)$ for $u \in P_1$.

Theorem 4.1. *Suppose $0 \le \xi, \eta < 1$, $\alpha_1 < 1$, $0 \le \beta_1 < 1$ and there exist numbers $0 < a < b < c$ such that $0 < a < b < \frac{t_2}{t_1} b \le c$ and $f(w)$ satisfies the following conditions:*

$$f(w) < \phi_p(\frac{a}{C}), \quad 0 \le w \le a, \tag{4.1}$$

$$f(w) > \phi_p(\frac{b}{B}), \quad b \le w \le \frac{t_2}{t_1} b, \tag{4.2}$$

$$f(w) \le \phi_p(\frac{c}{A}), \quad 0 \le w \le \frac{1}{t_3} c, \tag{4.3}$$

where

$$\begin{aligned} A &= \int_0^1 h(t_3,s)\phi_q\Big(\int_0^1 g(s,\tau)a(\tau)d\tau\Big)ds, \\ B &= \int_0^1 h(t_1,s)\phi_q\Big(\int_{t_1}^{t_2} g(s,\tau)a(\tau)d\tau\Big)ds, \\ C &= \int_0^1 h(1,s)\phi_q\Big(\int_0^1 g(s,\tau)a(\tau)d\tau\Big)ds. \end{aligned}$$

Then (1.1), (1.2) has at least three positive solutions $u_1, u_2, u_3 \in \overline{P_1(\gamma, c)}$ such that

$$u_1(t_1) > b, \quad u_2(1) < a, \quad u_3(t_1) < b \tag{4.4}$$

with $u_3(1) > a$, $u_i(\delta) \le c$ for $i = 1,2,3$.

Proof. We begin by defining the completely continuous operator $T : P_1 \to X$ by (3.4) as

$$(Tu)(t) = \int_0^1 h(t,s)\phi_q\Big(\int_0^1 g(s,\tau)a(\tau)f(u(\tau))d\tau\Big)ds$$

for $u \in P_1$. It is easy to prove that (1.1), (1.2) has a positive solution $u = u(t)$ if and only if the operator T has a fixed point on P_1.

Firstly, we prove $T : \overline{P_1(\gamma, c)} \subset \overline{P_1(\gamma, c)}$. For $u \in P_1$, by Remark 3.1, it is easy to check that $Tu \geq 0$. Moreover,

$$\begin{aligned}(Tu)'(t) = (1-\xi)\Big(\eta \int_0^t \phi_q\Big(\int_0^1 g(s,\tau)a(\tau)f(u(\tau))d\tau\Big)ds \\ + \int_t^1 \phi_q\Big(\int_0^1 g(s,\tau)a(\tau)f(u(\tau))d\tau\Big)ds\Big) \geq 0\end{aligned}$$

and

$$(Tu)''(t) = -\phi_q\Big(\int_0^1 g(t,s)a(s)f(u(s))ds\Big) \leq 0.$$

So, we have $TP_1 \subset P_1$.

For $u \in \overline{P_1(\gamma, c)}$, $0 \leq u(t) \leq \|u\| \leq \frac{1}{t_3}\gamma(u) \leq \frac{1}{t_3}c$. By (4.3), it follows that

$$\begin{aligned}\gamma(Tu) &= \max_{0 \leq t \leq t_3}(Tu)(t) = (Tu)(t_3) \\ &= \int_0^1 h(t_3,s)\phi_q\Big(\int_0^1 g(s,\tau)a(\tau)f(u(\tau))d\tau\Big)ds \\ &\leq \int_0^1 h(t_3,s)\phi_q\Big(\int_0^1 g(s,\tau)a(\tau)d\tau\ \phi_p(c/A)\Big)ds \\ &= \frac{c}{A}\int_0^1 h(t_3,s)\phi_q\Big(\int_0^1 g(s,\tau)a(\tau)d\tau\Big)ds \\ &= \frac{c}{A}A = c.\end{aligned}$$

Thus, $T : \overline{P_1(\gamma, c)} \subset \overline{P_1(\gamma, c)}$.

Secondly, by taking

$$\begin{aligned}u_1(t) &= b + \varepsilon_1 \quad \text{for } 0 < \varepsilon_1 < \frac{t_2}{t_1}b - b, \\ u_2(t) &= a - \varepsilon_2 \text{ for } 0 < \varepsilon_2 < a - \delta a,\end{aligned}$$

It is immediate that

$$\begin{aligned}u_1(t) &\in \{P(\gamma,\theta,\alpha,b,\frac{t_2}{t_1}b,c) :\ \alpha(u) > b\} \neq \emptyset, \\ u_2(t) &\in \{Q(\gamma,\beta,\psi,\delta a,a,c) :\ \beta(u) < a\} \neq \emptyset.\end{aligned}$$

In the following steps, we verify the remaining conditions of Theorem 2.1. Now the proof is divided into four steps.

Step 1: We prove that

$$u \in P(\gamma,\theta,\alpha,b,\frac{t_2}{t_1}b,c) \quad \text{implies} \quad \alpha(Tu) > b. \tag{4.5}$$

In fact, $u(t) \geq u(t_1) = \alpha(u) \geq b$ for $t_1 \leq t \leq t_2$, and $u(t) \leq u(t_2) = \theta(u) \leq \frac{t_2}{t_1} b$ for $t_1 \leq t \leq t_2$. Thus using (4.2), one gets

$$\begin{aligned}
\alpha(Tu) &= \min_{t_1 \leq t \leq t_2} (Tu)(t) = (Tu)(t_1) \\
&= \int_0^1 h(t_1, s)\phi_q\Big(\int_0^1 g(s,\tau)a(\tau)f(u(\tau))d\tau\Big)ds \\
&\geq \int_0^1 h(t_1, s)\phi_q\Big(\int_{t_1}^{t_2} g(s,\tau)a(\tau)f(u(\tau))d\tau\Big)ds \\
&> \int_0^1 h(t_1, s)\phi_q\Big(\int_{t_1}^{t_2} g(s,\tau)a(\tau)d\tau\ \phi_p(b/B)\Big)ds \\
&= \frac{b}{B}\int_0^1 h(t_1, s)\phi_q\Big(\int_{t_1}^{t_2} g(s,\tau)a(\tau)d\tau\Big)ds \\
&= \frac{b}{B}B = b.
\end{aligned}$$

Step 2: We show that

$$u \in Q(\gamma, \beta, \psi, \delta a, a, c) \quad \text{implies} \quad \beta(Tu) < a. \tag{4.6}$$

In fact, $0 \leq u(t) \leq u(1) = \beta(u) \leq a$ for $0 \leq t \leq 1$, Thus using (4.1), one arrives at

$$\begin{aligned}
\beta(Tu) &= \max_{\delta \leq t \leq 1} (Tu)(t) = (Tu)(1) \\
&= \int_0^1 h(1, s)\phi_q\Big(\int_0^1 g(s,\tau)a(\tau)f(u(\tau))d\tau\Big)ds \\
&< \int_0^1 h(1, s)\phi_q\Big(\int_0^1 g(s,\tau)a(\tau)d\tau\ \phi_p(a/C)\Big)ds \\
&= \frac{a}{C}\int_0^1 h(1, s)\phi_q\Big(\int_0^1 g(s,\tau)a(\tau)d\tau\Big)ds \\
&= \frac{a}{C}C = a.
\end{aligned}$$

Step 3: We verify that

$$u \in Q(\gamma, \beta, a, c) \quad \text{with} \quad \psi(Tu) < \delta a \quad \text{implies} \quad \beta(Tu) < a. \tag{4.7}$$

By Lemma 3.4,

$$\begin{aligned}
\beta(Tu) &= \max_{\delta \leq t \leq 1} (Tu)(t) = (Tu)(1) \\
&= \int_0^1 h(1, s)\phi_q\Big(\int_0^1 g(s,\tau)a(\tau)f(u(\tau))d\tau\Big)ds \\
&= \int_0^1 \frac{h(1,s)}{h(\delta,s)} h(\delta, s)\phi_q\Big(\int_0^1 g(s,\tau)a(\tau)f(u(\tau))d\tau\Big)ds \\
&\leq \frac{1}{\delta}(Tu)(\delta) = \frac{1}{\delta}\psi(Tu) < a.
\end{aligned}$$

Step 4: We prove that

$$u \in P(\gamma, \alpha, b, c) \quad \text{with} \quad \theta(Tu) > \frac{t_2}{t_1} b \quad \text{implies} \quad \alpha(Tu) > b. \tag{4.8}$$

By Lemma 3.4,

$$\begin{aligned}\alpha(Tu) &= \min_{t_1\le t\le t_2}(Tu)(t) = (Tu)(t_1)\\ &= \int_0^1 h(t_1,s)\phi_q\Big(\int_0^1 g(s,\tau)a(\tau)f(u(\tau))d\tau\Big)ds\\ &= \int_0^1 \frac{h(t_1,s)}{h(t_2,s)}h(t_2,s)\phi_q\Big(\int_0^1 g(s,\tau)a(\tau)f(u(\tau))d\tau\Big)ds\\ &\ge \frac{t_1}{t_2}(Tu)(t_2) = \frac{t_1}{t_2}\theta(Tu) > b.\end{aligned}$$

Therefore, the hypotheses of Theorem 2.1 are satisfied and there exist three positive solutions x_1, x_2, x_3 for BVP (1.1), (1.2) satisfying (4.4). □

Similarly, choose $t_1, t_2, t_3 \in (0,1)$ and $t_1 < t_2$. Define nonnegative, continuous, concave functions α, ψ and nonnegative, continuous, convex functions β, θ, γ on P_2 by

$$\begin{gathered}\gamma(u) = \max_{t\in[t_3,1]} u(t) = u(t_3), \quad x \in P_2,\\ \psi(x) = \min_{t\in[0,\delta]} u(t) = u(\delta), \quad u \in P_2,\\ \beta(u) = \max_{t\in[0,\delta]} u(t) = u(0), \quad u \in P_2,\\ \alpha(u) = \min_{t\in[t_1,t_2]} u(t) = u(t_2), \quad u \in P_2,\\ \theta(u) = \max_{t\in[t_1,t_2]} u(t) = u(t_1), \quad u \in P_2.\end{gathered}$$

It is easy to verify that $\alpha(u) = u(t_2) \le u(0) = \beta(u)$ and $\|u\| = u(0) \le \frac{1}{t_3}u(t_3) = \frac{1}{t_3}\gamma(u)$ for $u \in P_2$. So, we obtain the following result.

Theorem 4.2. *Suppose ξ, $\eta > 1$, $\alpha_1 < 1$, $0 \le \beta_1 < 1$ and there exist numbers $0 < a < b < c$ such that $0 < a < b < \frac{1-t_1}{1-t_2}b \le c$ and $f(w)$ satisfy the following conditions:*

$$f(w) < \phi_p\Big(\frac{a}{C}\Big), \quad 0 \le w \le a, \tag{4.9}$$

$$f(w) > \phi_p\Big(\frac{b}{B}\Big), \quad b \le w \le \frac{1-t_1}{1-t_2}b, \tag{4.10}$$

$$f(w) \le \phi_p\Big(\frac{c}{A}\Big), \quad 0 \le w \le \frac{1}{t_3}c, \tag{4.11}$$

where

$$\begin{aligned}A &= \int_0^1 h(t_3,s)\phi_q\Big(\int_0^1 g(s,\tau)a(\tau)d\tau\Big)ds,\\ B &= \int_0^1 h(t_2,s)\phi_q\Big(\int_{t_1}^{t_2} g(s,\tau)a(\tau)d\tau\Big)ds,\\ C &= \int_0^1 h(0,s)\phi_q\Big(\int_0^1 g(s,\tau)a(\tau)d\tau\Big)ds.\end{aligned}$$

Then (1.1), (1.2) *has at least three positive solutions $u_1, u_2, u_3 \in \overline{P_1(\gamma, c)}$ such that*

$$u_1(t_2) > b, \quad u_2(0) < a, \quad u_3(t_2) < b,$$

with $u_3(0) > a$ *and* $u_i(\delta) \leq c$ *for* $i = 1, 2, 3$.

Since the proof of the above theorem is similar to that of Lemma 3.5, we omit it.

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