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EXISTENCE OF POSITIVE SOLUTIONS FOR BOUNDARY-VALUE PROBLEMS FOR SINGULAR HIGHER-ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. We study the existence of positive solutions for the boundary-value problem of the singular higher-order functional differential equation

$$(Ly^{(n-2)})(t) + h(t)f(t, y_t) = 0, \quad \text{for } t \in [0,1],$$

$$y^{(i)}(0) = 0, \quad 0 \le i \le n-3,$$

$$\alpha y^{(n-2)}(t) - \beta y^{(n-1)}(t) = \eta(t), \quad \text{for } t \in [-\tau, 0],$$

$$\gamma y^{(n-2)}(t) + \delta y^{(n-1)}(t) = \xi(t), \quad \text{for } t \in [1, 1+a],$$

where $Ly := -(py')' + qy$, $p \in C([0,1],(0,+\infty))$, and $q \in C([0,1],[0,+\infty))$. Our main tool is the fixed point theorem on a cone.

1. Introduction

As pointed out in [5], boundary-value problems associated with functional differential equations arise from problems in physics, from variational problems in control theory, and from applied mathematics; see for example [6, 8]. Many authors have investigated the existence of solutions for boundary-value problems of functional differential equations; see [3, 9, 15, 18]. Recently an increasing interest in studying the existence of positive solutions for such problems has been observed. Among others publication, we refer to [1, 2, 10, 11, 13, 19].

In this paper, we investigate the existence of positive solutions for singular boundary-value problems (BVP) of an n-th order ($n \ge 3$) functional differential equation (FDE) of the form

$$(Ly^{(n-2)})(t) + h(t)f(t, y_t) = 0, \quad \text{for } t \in [0,1], \tag{1.1}$$

$$y^{(i)}(0) = 0, \quad 0 \le i \le n-3, \tag{1.2}$$

$$\alpha y^{(n-2)}(t) - \beta y^{(n-1)}(t) = \eta(t), \quad \text{for } t \in [-\tau, 0], \tag{1.3}$$

$$\gamma y^{(n-2)}(t) + \delta y^{(n-1)}(t) = \xi(t), \quad \text{for } t \in [1, 1+a], \tag{1.4}$$

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where $Ly := -(py')' + qy$, $p \in C([0,1],(0,+\infty))$, and $q \in C([0,1],[0,+\infty))$; $\alpha,\beta,\gamma,\delta \geq 0$, and $\alpha\delta+\alpha\gamma+\beta\gamma>0$; $\eta \in C([-\tau,0],\mathbb{R})$, $\xi \in C([1,b],\mathbb{R})$ $(b=1+a)$, and $\eta(0)=\xi(1)=0$; $h \in C((0,1),\mathbb{R})$ ($h(t)$ is allowed to have singularity at $t=0$ or 1); $f \in C([0,1]\times D,\mathbb{R})$, $D=C([-\tau,a],\mathbb{R})$, for every $t\in[0,1]$, $y_t \in D$ is defined by $y_t(\theta)=y(t+\theta), \theta\in[-\tau,a]$.

The study of higher-order functional differential equation has received also some attention; see for example [3, 10, 17]. Recently, Hong et al. [12] imposed conditions on $f(t,y^t)$ to yield at least one positive solution to (1.1)-(1.4) for the special case $h(t)\equiv 1$, $p(t)\equiv 1$, and $q(t)\equiv 0$. They applied the Krasnosel'skii fixed-point theorem.

The purpose of this paper is to establish the existence of positive solutions of the singular higher-order functional differential equation (1.1) with boundary conditions (1.2)-(1.4) under suitable conditions on f.

2. Preliminaries

To abbreviate our discussion, we assume the following hypotheses:

(H1) $G(t,s)$ is the Green's function of the differential equation
$$(Ly^{(n-2)})(t)=0, \quad 0<t<1$$
subject to the boundary condition (1.2)-(1.4) with $\tau=a=0$.

(H2) $g(t,s)$ is the Green's function of the differential equation
$$Ly(t)=0 \quad t\in(0,1)$$
subject to the boundary conditions
$$\alpha y(0)-\beta y'(0)=0, \quad \gamma y(1)+\delta y'(1)=0,$$
where α,β,γ and δ are as in (1.3) and (1.4).

(H3) $h\in C((0,1),[0,+\infty))$ and satisfies
$$0<\int_0^1 g(s,s)h(s)ds<+\infty.$$

(H4) $f\in C([0,1]\times D^+,\ [0,\infty))$, where $D^+=C([-\tau,a],[0,+\infty))$.

(H5) $\eta\in C([-\tau,0],[0,+\infty))$, $\xi\in C([1,1+a],[0,+\infty))$, and $\eta(0)=\xi(1)=0$.

It is easy to see that
$$\frac{\partial^{n-2}}{\partial t^{n-2}}G(t,s)=g(t,s), \quad t,s\in[0,1].$$
It is also well known that the Green's function $g(t,s)$ is
$$g(t,s)=\frac{1}{c}\begin{cases}\phi(s)\psi(t), & \text{if } 0\leq s\leq t\leq 1,\\ \phi(t)\psi(s), & \text{if } 0\leq t\leq s\leq 1,\end{cases}$$
where ϕ and ψ are solutions, respectively, of
$$L\phi=0, \quad \phi(0)=\beta, \quad \phi'(0)=\alpha, \tag{2.1}$$
$$L\psi=0, \quad \psi(1)=\delta, \quad \psi'(1)=-\gamma. \tag{2.2}$$
One can show that $c=-p(t)(\phi(t)\psi'(t)-\phi'(t)\psi(t))>0$ and $\phi'(t)>0$ on $(0,1]$ and $\psi'(t)<0$ on $[0,1)$. Clearly
$$g(t,s)\leq g(s,s), \quad 0\leq t,s\leq 1. \tag{2.3}$$

By (H3), there exists $t_0 \in (0,1)$ such that $h(t_0) > 0$. We may choose $\varepsilon \in (0,1/2)$ such that $t_0 \in (\varepsilon, 1-\varepsilon)$. Then for $\varepsilon \le t \le 1-\varepsilon$ we have $\phi(\varepsilon) \le \phi(t) \le \phi(1-\varepsilon)$ and $\psi(1-\varepsilon) \le \psi(t) \le \psi(\varepsilon)$. Also for $(t,s) \in [\varepsilon, 1-\varepsilon] \times (0,1)$

$$\frac{g(t,s)}{g(s,s)} \ge \min\big\{\frac{\psi(1-\varepsilon)}{\psi(s)}, \frac{\phi(\varepsilon)}{\phi(s)}\big\} \ge \min\big\{\frac{\psi(1-\varepsilon)}{\psi(0)}, \frac{\phi(\varepsilon)}{\phi(1)}\big\} := \sigma. \tag{2.4}$$

Let $E = C^{(n-2)}([-\tau, b]; \mathbb{R})$ with a norm $\|u\|_{[-\tau,b]} = \sup_{-\tau \le t \le b} |u^{(n-2)}(t)|$ for $u \in E$. Obviously, E is a Banach space. And let $C = C^{(n-2)}([-\tau, a], \mathbb{R})$ be a space with norm $\|\psi\|_{[-\tau,a]} = \sup_{-\tau \le t \le a} |\psi^{(n-2)}(x)|$ for $\psi \in C$. Let

$$C^+ = \{\psi \in C : \psi(x) \ge 0, x \in [-\tau, a]\}.$$

It is easy to see that C^+ is a subspace of C.

Define a cone $K \subset E$ as follows:

$$K = \{y \in E : y(t) \ge 0, \min_{t \in [\varepsilon, 1-\varepsilon]} y^{(n-2)}(t) \ge \overline{\sigma} \|y\|_{[-\tau,b]}\}, \tag{2.5}$$

where $\overline{\sigma} = \frac{1}{b} \min\{\varepsilon, \sigma\}$, σ is as in (2.4).

For each $\rho > 0$, we define $K_\rho = \{y \in K : \|y\|_{[-\tau,b]} < \rho\}$. Furthermore, we define a set Ω_ρ as follows:

$$\Omega_\rho = \big\{y \in K : \min_{\varepsilon \le t \le 1-\varepsilon} y^{(n-2)}(t) < \overline{\sigma}\rho\big\}.$$

Similar to the [14, Lemma 2.5], we have

Lemma 2.1. *Ω_ρ defined above has the following properties:*

(a) *Ω_ρ is open relative to K.*
(b) *$K_{\overline{\sigma}\rho} \subset \Omega_\rho \subset K_\rho$.*
(c) *$y \in \partial\Omega_\rho$ if and only if $\min_{\varepsilon \le t \le 1-\varepsilon} y^{(n-2)}(t) = \overline{\sigma}\rho$.*
(d) *If $y \in \partial\Omega_\rho$, then $\overline{\sigma}\rho \le y^{(n-2)}(t) \le \rho$ for $t \in [\varepsilon, 1-\varepsilon]$.*

To obtain the positive solutions of (1.1)-(1.4), the following fixed point theorem in cones will be fundamental.

Lemma 2.2. *Let K be a cone in a Banach space E. Let D be an open bounded subset of E with $D_K = D \cap K \ne \emptyset$ and $\overline{D}_K \ne K$. Assume that $A : \overline{D}_K \to K$ is a compact map such that $x \ne Ax$ for $x \in \partial D_K$. Then the following results hold.*

(1) *$\|Ax\| \le \|x\|$, $\quad x \in \partial D_K$, then $i_K(A, D_K) = 1$.*
(2) *If there exists $e \in K \backslash \{0\}$ such that $x \ne Ax + \lambda e$ for all $x \in \partial D_K$ and all $\lambda > 0$, then $i_K(A, D_K) = 0$.*
(3) *Let U be an open set in E such that $\overline{U} \subset D_K$. If $i_K(A, D_K) = 1$ and $i_K(A, U_K) = 0$, then A has a fixed point in $D_K \backslash \overline{U}_K$. The same results holds if $i_K(A, D_K) = 0$ and $i_K(A, U_K) = 1$.*

Suppose that $y(t)$ is a solution of (1.1)-(1.4), then it can be written as

$$y(t) = \begin{cases} y(-\tau; t), & -\tau \le t \le 0, \\ \int_0^1 G(t,s)h(s)f(s, y_s)ds, & 0 \le t \le 1, \\ y(b; t), & 1 \le t \le b, \end{cases}$$

where $y(-\tau; t)$ and $y(b; t)$ satisfy

$$y^{(n-2)}(-\tau; t) = \begin{cases} e^{\frac{\alpha}{\beta} t} \left(\frac{1}{\beta} \int_t^0 e^{-\frac{\alpha}{\beta} s} \eta(s) ds + y^{(n-2)}(0)\right), & t \in [-\tau, 0],\ \beta \ne 0, \\ \frac{1}{\alpha} \eta(t), & t \in [-\tau, 0],\ \beta = 0, \end{cases}$$

and

$$y^{(n-2)}(b;t)=\begin{cases} e^{-\frac{\gamma}{\delta}t}\Big(\frac{1}{\delta}\int_1^t e^{\frac{\gamma}{\delta}s}\xi(s)ds+e^{\frac{\gamma}{\delta}}y^{(n-2)}(1)\Big), & t\in[1,b],\ \delta\neq 0,\\ \frac{1}{\gamma}\xi(t), & t\in[1,b],\ \delta=0. \end{cases}$$

Throughout this paper, we assume that $u_0(t)$ is the solution of (1.1)-(1.4) with $f\equiv 0$, and $\|u_0\|_{[-\tau,b]}=: M_0$. Clearly, $u_0^{(n-2)}(t)$ can be expressed as follows:

$$u_0^{(n-2)}(t)=\begin{cases} u_0^{(n-2)}(-\tau;t), & -\tau\le t\le 0,\\ 0, & 0\le t\le 1,\\ u_0^{(n-2)}(b;t), & 1\le t\le b. \end{cases}$$

where

$$u_0^{(n-2)}(-\tau;t)=\begin{cases} \frac{1}{\beta}e^{\frac{\alpha}{\beta}t}\int_t^0 e^{-\frac{\alpha}{\beta}s}\eta(s)ds, & t\in[-\tau,0],\ \beta\neq 0,\\ \frac{1}{\alpha}\eta(t), & t\in[-\tau,0],\ \beta=0, \end{cases}$$

and

$$u_0^{(n-2)}(b;t)=\begin{cases} \frac{1}{\delta}e^{-\frac{\gamma}{\delta}t}\int_1^t e^{\frac{\gamma}{\delta}s}\xi(s)ds, & t\in[1,b],\ \delta\neq 0,\\ \frac{1}{\gamma}\xi(t), & t\in[1,b],\ \delta=0. \end{cases}$$

Let $y(t)$ be a solution of BVP (1.1)-(1.4) and $u(t)=y(t)-u_0(t)$. Noting that $u(t)\equiv y(t)$ for $0\le t\le 1$, we have

$$u^{(n-2)}(t)=\begin{cases} u^{(n-2)}(-\tau;t), & -\tau\le t\le 0,\\ \int_0^1 g(t,s)h(s)f(s,u_s+(u_0)_s)ds, & 0\le t\le 1,\\ u^{(n-2)}(b;t), & 1\le t\le b, \end{cases}$$

where

$$u^{(n-2)}(-\tau;t)=\begin{cases} e^{\frac{\alpha}{\beta}t}\int_0^1 g(0,s)h(s)f(s,u_s+(u_0)_s)ds, & t\in[-\tau,0],\ \beta\neq 0,\\ 0, & t\in[-\tau,0],\ \beta=0, \end{cases}$$

and

$$u^{(n-2)}(b;t)=\begin{cases} e^{-\frac{\gamma}{\delta}(t-1)}\int_0^1 g(1,s)h(s)f(s,u_s+(u_0)_s)ds, & t\in[1,b],\ \delta\neq 0,\\ 0, & t\in[1,b],\ \delta=0. \end{cases}$$

It is easy to see that $y(t)$ is a solution of BVP (1.1)-(1.4) if and only if $u(t)=y(t)-u_0(t)$ is a solution of the operator equation

$$u(t)=Au(t)\quad\text{for } t\in[-\tau,b]. \tag{2.6}$$

Here, operator $A:E\to E$ is defined by

$$Au(t):=\begin{cases} B_1u(t), & -\tau\le t\le 0,\\ \int_0^1 G(t,s)h(s)f(s,u_s+(u_0)_s)ds, & 0\le t\le 1,\\ B_2u(t), & 1\le t\le b, \end{cases}$$

where

$$B_1u(t):=\begin{cases} \big(\frac{\beta}{\alpha}\big)^{n-2}e^{\frac{\alpha}{\beta}t}\int_0^1 g(0,s)h(s)f(s,u_s+(u_0)_s)ds, & \beta\neq 0,\ \alpha\neq 0,\\ \frac{t^{n-2}}{(n-2)!}\int_0^1 g(0,s)h(s)f(s,u_s+(u_0)_s)ds, & \beta\neq 0,\ \alpha=0,\\ 0, & \beta=0 \end{cases}$$

for each $t \in [-\tau, 0]$, and

$$B_2u(t) := \begin{cases} \big(-\frac{\delta}{\gamma}\big)^{n-2} e^{-\frac{\gamma}{\delta}(t-1)} \int_0^1 g(1,s)h(s)f(s,u_s+(u_0)_s)ds, & \delta \neq 0,\ \gamma \neq 0, \\ \frac{t^{n-2}}{(n-2)!} \int_0^1 g(1,s)h(s)f(s,u_s+(u_0)_s)ds, & \delta \neq 0,\ \gamma = 0, \\ 0, & \delta = 0 \end{cases}$$

for any $t \in [1, b]$. Obviously,

$$(Au)^{(n-2)}(t) := \begin{cases} (B_1u)^{(n-2)}(t), & -\tau \le t \le 0, \\ \int_0^1 g(t,s)h(s)f(s,u_s+(u_0)_s)ds, & 0 \le t \le 1, \\ (B_2u)^{(n-2)}(t), & 1 \le t \le b, \end{cases}$$

where

$$(B_1u)^{(n-2)}(t) := \begin{cases} e^{\frac{\alpha}{\beta}t} \int_0^1 g(0,s)h(s)f(s,u_s+(u_0)_s)ds, & t \in [-\tau, 0],\ \beta \neq 0, \\ 0, & t \in [-\tau, 0],\ \beta = 0, \end{cases}$$

and

$$(B_2u)^{(n-2)}(t) := \begin{cases} e^{-\frac{\gamma}{\delta}(t-1)} \int_0^1 g(1,s)h(s)f(s,u_s+(u_0)_s)ds, & t \in [1, b],\ \delta \neq 0, \\ 0, & t \in [1, b],\ \delta = 0. \end{cases}$$

Lemma 2.3. *With the above notation, $A(K) \subset K$.*

Proof. By the assumptions of (H1)-(H5), it is easy to know that $Au \in E$ and $Au \ge 0$ for any $u \in K$. Moreover, it follows from

$$0 \le (Au)^{(n-2)}(t) \le (Au)^{(n-2)}(0) \quad \text{for } -\tau \le t \le 0$$
$$0 \le (Au)^{(n-2)}(t) \le (Au)^{(n-2)}(1) \quad \text{for } 1 \le t \le b$$

that $\|Au\|_{[-\tau,b]} = \|Au\|_{[0,1]}$. By (2.3) we have, for any $u \in K$ and $t \in [0, 1]$ that

$$\|Au\|_{[-\tau,b]} = \|Au\|_{[0,1]} \le \int_0^1 g(s,s)h(s)f(s,u_s+(u_0)_s)ds. \tag{2.7}$$

From (2.4), we get

$$\begin{aligned} \min_{\varepsilon \le t \le 1-\varepsilon} (Au)^{(n-2)}(t) &= \min_{\varepsilon \le t \le 1-\varepsilon} \int_0^1 g(t,s)h(s)f(s,u_s+(u_0)_s)ds \\ &\ge \sigma \int_0^1 g(s,s)h(s)f(s,u_s+(u_0)_s)ds \\ &\ge \overline{\sigma} \int_0^1 g(s,s)h(s)f(s,u_s+(u_0)_s)ds. \end{aligned} \tag{2.8}$$

In view of (2.7) and (2.8), we obtain

$$\min_{\varepsilon \le t \le 1-\varepsilon} (Au)^{(n-2)}(t) \ge \overline{\sigma} \|Au\|_{[-\tau,b]}, \quad u \in K,$$

which implies $A(K) \subset K$. □

Let

$$C^+_{[k,r]} = \{\varphi \in C^+ : \ k \le \|\varphi\|_{[-\tau,a]} \le r\},$$
$$C^+_{[k,\infty)} = \{\varphi \in C^+ : \ k \le \|\varphi\|_{[-\tau,a]} < \infty\},$$

where $0 \le k < r$.

Lemma 2.4. *$A : K \to K$ is completely continuous.*

Proof. We apply a truncation technique (cf. [16]). We define the function h_m for $m \geq 2$, by

$$h_m(t) = \begin{cases} \min\left\{h(t), h(\frac{1}{m})\right\}, & 0 < t \leq \frac{1}{m}, \\ h(t), & \frac{1}{m} < t < 1 - \frac{1}{m}, \\ \min\left\{h(t), h(\frac{m-1}{m})\right\}, & \frac{m-1}{m} \leq t < 1. \end{cases}$$

It is clear that $h_m(t)$ is nonnegative and continuous on $[0,1]$. We define the operator A_m by

$$A_m u(t) := \begin{cases} B_{1m}u(t), & -\tau \leq t \leq 0, \\ \int_0^1 G(t,s)h_m(s)f(s, u_s + (u_0)_s)ds, & 0 \leq t \leq 1, \\ B_{2m}u(t), & 1 \leq t \leq b, \end{cases}$$

where

$$B_{1m}u(t) := \begin{cases} \left(\frac{\beta}{\alpha}\right)^{n-2} e^{\frac{\alpha}{\beta}t} \int_0^1 g(0,s)h_m(s)f(s, u_s + (u_0)_s)ds, & \beta \neq 0,\ \alpha \neq 0, \\ \frac{t^{n-2}}{(n-2)!} \int_0^1 g(0,s)h_m(s)f(s, u_s + (u_0)_s)ds, & \beta \neq 0,\ \alpha = 0, \\ 0, & \beta = 0 \end{cases}$$

for each $t \in [-\tau, 0]$, and

$$B_{2m}u(t) := \begin{cases} \left(-\frac{\delta}{\gamma}\right)^{n-2} e^{-\frac{\gamma}{\delta}(t-1)} \int_0^1 g(1,s)h_m(s)f(s, u_s + (u_0)_s)ds, & \delta \neq 0,\ \gamma \neq 0, \\ \frac{t^{n-2}}{(n-2)!} \int_0^1 g(1,s)h_m(s)f(s, u_s + (u_0)_s)ds, & \delta \neq 0,\ \gamma = 0, \\ 0, & \delta = 0 \end{cases}$$

for any $t \in [1,b]$. By Lemma 2.3, it is easy to check that $A_m : K \to K$. And, A_m is continuous, the proof is similar to that of [11, Theorem 2.1].

Next let $B \subset K$ be a bounded subset of K, and $M_1 > 0$ be a constant such that $\|u\|_{[-\tau,b]} \leq M_1$ for $u \in B$. Noting that if $x_t \in C = C^{n-2}([-\tau,a], \mathbb{R})$, then $x_t^{(n-2)} \in C([-\tau,a], \mathbb{R})$, and $x_t^{(n-2)}(\theta) = x^{(n-2)}(t+\theta)$, $\theta \in [-\theta, a]$. Thus

$$\begin{aligned} \|u_s + (u_0)_s\|_{[-\tau,a]} &= \sup_{-\tau \leq \theta \leq a} |(u^s + u_0^s)^{(n-2)}(\theta)| \\ &\leq \sup_{-\tau \leq \theta \leq a} |(u^{(n-2)}(s+\theta)| + \sup_{-\tau \leq \theta \leq a} |u_0^{(n-2)}(s+\theta)| \\ &\leq \sup_{-\tau \leq t \leq b} |u^{(n-2)}(t)| + \sup_{-\tau \leq t \leq b} |u_0^{(n-2)}(t)| = \|u\|_{[-\tau,b]} + \|u_0\|_{[-\tau,b]} \\ &\leq M_1 + M_0 := M_2 \end{aligned} \tag{2.9}$$

for $u \in B$ and $s \in [0,1]$. Hence, there exists a constant $M_3 > 0$ such that

$$|f(s, u_s + (u_0)_s)| \leq M_3, \quad \text{on } [0,1] \times C^+_{[0,M_3]}, \tag{2.10}$$

since f is continuous on $[0,1] \times C^+$. For $u \in B$ we have

$$(A_m u)^{(n-2)}(t) := \begin{cases} (B_{1m}u)^{(n-2)}(t), & -\tau \leq t \leq 0, \\ \int_0^1 g(t,s)h_m(s)f(s, u_s + (u_0)_s)ds, & 0 \leq t \leq 1, \\ (B_{2m}u)^{(n-2)}(t), & 1 \leq t \leq b, \end{cases}$$

where

$$(B_{1m}u)^{(n-2)}(t) := \begin{cases} e^{\frac{\alpha}{\beta}t}\int_0^1 g(0,s)h_m(s)f(s,u_s+(u_0)_s)ds, & t\in[-\tau,0],\ \beta\neq 0,\\ 0, & t\in[-\tau,0],\ \beta=0,\end{cases}$$

and

$$(B_{2m}u)^{(n-2)}(t) := \begin{cases} e^{-\frac{\gamma}{\delta}(t-1)}\int_0^1 g(1,s)h_m(s)f(s,u_s+(u_0)_s)ds, & t\in[1,b],\ \delta\neq 0,\\ 0, & t\in[1,b],\ \delta=0.\end{cases}$$

These and (2.10) imply $(A_mu)^{(n-2)}(t)$ is continuous and uniformly bounded for $u\in B$. So $(A_mu)'(t)$ is continuous and uniformly bounded for $u\in B$ also. The Ascoli-Arzela Theorem implies that A_m is a completely continuous operator on K for any $m\geq 2$.

Moreover, A_m converges uniformly to A as $m\to\infty$ on any bounded subset of K. To see this, note that if $u\in K$ with $\|u\|_{[-\tau,b]}\leq M$, then from (H3) and $0\leq h_n(s)\leq h(s)$,

$$\begin{aligned}|(A_mu)^{(n-2)}(t)-(Au)^{(n-2)}(t)| &= \Big|\int_0^{\frac{1}{m}} g(t,s)[h(s)-h_m(s)]f(s,u_s+(u_0)_s)ds\\ &\quad+\int_{\frac{m-1}{m}}^1 g(t,s)[h(s)-h_m(s)]f(s,u_s+(u_0)_s)ds\Big|\\ &\leq \int_0^{\frac{1}{m}} g(s,s)|h(s)-h_m(s)|f(s,u_s+(u_0)_s)ds\\ &\quad+\int_{\frac{m-1}{m}}^1 g(s,s)|h(s)-h_m(s)|f(s,u_s+(u_0)_s)ds\\ &\leq 2\overline{M}\Big[\int_0^{\frac{1}{m}} g(s,s)h(s)ds+\int_{\frac{m-1}{m}}^1 g(s,s)h(s)ds\Big]\\ &\to 0 \quad \text{as } m\to\infty,\end{aligned}$$

where $\overline{M} := \max_{t\in[0,1],\varphi\in C^+_{[0,M+M_0]}} f(t,\varphi)$. Thus, we have

$$\|A_mu-Au\|_{[-\tau,b]} = \|A_mu-Au\|_{[0,1]}\to 0,\quad n\to\infty,$$

for each $u\in K$ with $\|u\|_{[-\tau,b]}\leq M$. Hence, A_m converges uniformly to A as $m\to\infty$ and therefore A is completely continuous also. This completes the proof of Lemma 2.4. $\square$

3. Main results

For convenience, we introduce the following notation. Let

$$\omega = \Big(\int_0^1 g(s,s)h(s)ds \Big)^{-1}; \quad N = \Big(\min_{\varepsilon \le t \le 1-\varepsilon} \int_\varepsilon^{1-\varepsilon} g(t,s)h(s)ds \Big)^{-1};$$

$$f_{\overline{\sigma}\rho}^{\rho} = \inf \Big\{ \min_{t\in[\varepsilon,1-\varepsilon]} \frac{f(t,\varphi)}{\rho} : \varphi \in C^+_{[\overline{\sigma}\rho, \rho+M_0]} \Big\};$$

$$f_0^{\rho} = \sup \Big\{ \max_{t\in[0,1]} \frac{f(t,\varphi)}{\rho} : \varphi \in C^+_{[0,\rho+M_0]} \Big\};$$

$$f^{\mu} = \lim_{\|\varphi\|_{[-\tau,a]}\to\mu} \sup \max_{t\in[0,1]} \frac{f(t,\varphi)}{\|\varphi\|_{[-\tau,a]}};$$

$$f_{\mu} = \lim_{\|\varphi\|_{[-\tau,a]}\to\mu} \inf \min_{t\in[\varepsilon,1-\varepsilon]} \frac{f(t,\varphi)}{\|\varphi\|_{[-\tau,a]}}, \quad (\mu := \infty \text{ or } 0^+).$$

Now, we impose conditions on f which we assure that $i_K(A, K_\rho) = 1$.

Lemma 3.1. *Assume that*

$$f_0^{\rho} \le \omega \quad \textit{and} \quad u \ne Au \quad \textit{for } u \in \partial K_\rho. \tag{3.1}$$

Then $i_K(A, K_\rho) = 1$.

Proof. For $u \in \partial K_\rho$, we have $\|u_s + (u_0)_s\|_{[-\tau,a]} \le \rho + M_0$, for all $s \in [0,1]$, i.e., $u_s + (u_0)_s \in C_{[0,\rho+M_0]}$ for any $s \in [0,1]$. It follows from (3.1) that for $t \in [0,1]$,

$$\begin{aligned} (Au)^{(n-2)}(t) &= \int_0^1 g(t,s)h(s)f(s, u_s + (u_0)_s)ds \\ &\le \int_0^1 g(s,s)h(s)f(s, u_s + (u_0)_s)ds \\ &< \rho\omega \int_0^1 g(s,s)h(s)ds \\ &= \rho = \|u\|_{[-\tau,b]}. \end{aligned}$$

This implies that $\|Au\|_{[-\tau,b]} < \|u\|_{[-\tau,b]}$ for $u \in \partial K_\rho$. By Lemma 2.2 (1), we have $i_K(A, K_\rho) = 1$. □

Let $u \in \partial\Omega_\rho$, then for any $s \in [\varepsilon, 1-\varepsilon]$, we have by Lemma 2.1(c) that

$$\begin{aligned} \|u_s + (u_0)_s\|_{[-\tau,a]} &= \sup_{\theta\in[-\tau,a]} (u^{(n-2)}(s+\theta) + (u_0)^{(n-2)}(s+\theta)) \\ &\ge \sup_{\theta\in[-\tau,a]} u^{(n-2)}(s+\theta) \text{ (since } (u_0)^{(n-2)}(t) \ge 0 \text{ for } t \in [-\tau,b]) \\ &\ge u^{(n-2)}(s) \\ &\ge \min_{t\in[\varepsilon,1-\varepsilon]} u^{(n-2)}(t) = \overline{\sigma}\rho. \end{aligned}$$

By Lemma 2.1(b), we have $\overline{\Omega}_\rho \subset \overline{K}_\rho$, that is $\|u\|_{[-\tau,b]} \le \rho$. Thus, from (2.9) we get

$$\|u_s + (u_0)_s\|_{[-\tau,a]} \le \|u\|_{[-\tau,b]} + \|u_0\|_{[-\tau,b]} \le \rho + M_0, \quad \forall s \in [0,1].$$

Hence

$$u_s + (u_0)_s \in C^+_{[\overline{\sigma}\rho,\ \rho+M_0]}, \quad \text{for } u \in \partial\Omega_\rho,\ s \in [\varepsilon, 1-\varepsilon]. \tag{3.2}$$

Next, we impose conditions on f which assure that $i_K(A, \Omega_\rho) = 0$.

Lemma 3.2. *If f satisfies the condition*

$$f^{\rho}_{\overline{\sigma}\rho} \geq N\overline{\sigma} \quad \text{and} \quad u \neq Au \quad \text{for } u \in \partial\Omega_{\rho}. \tag{3.3}$$

Then $i_K(A, \Omega_\rho) = 0$.

Proof. Let

$$e(t) = \begin{cases} -\frac{t^{n-1}}{(1+\tau)(n-1)!}, & n \text{ is odd}, \ -\tau \leq t \leq 0, \\ \frac{t^n}{(1+\tau)^2 n!}, & n \text{ is even }, \ -\tau \leq t \leq 0, \\ \frac{t^{n-1}}{b(n-1)!}, & 0 \leq t \leq b. \end{cases}$$

It is easy to verify that $e \in C^{(n-2)}([-\tau, b], \mathbb{R})$, $e(t) \geq 0$ for $t \in [-\tau, b]$, and

$$e^{(n-2)}(t) = \begin{cases} -\frac{t}{1+\tau}, & n \text{ is odd}, \ -\tau \leq t \leq 0, \\ \frac{t^2}{2(1+\tau)^2}, & n \text{ is even}, \ -\tau \leq t \leq 0, \\ \frac{t}{b}, & 0 \leq t \leq b, \end{cases}$$

which implies that $e \in K$ and $\|e\|_{[-\tau,b]} = 1$, that is $e \in \partial K_1$. We claim that

$$u \neq Au + \lambda e, \quad u \in \partial\Omega_\rho, \quad \lambda > 0.$$

In fact, if not, there exist $u \in \partial\Omega_\rho$ and $\lambda_0 > 0$ such that $u = Au + \lambda_0 e$. Then by (3.2) for $t \in [\varepsilon, 1-\varepsilon]$, we get

$$\begin{aligned} u^{(n-2)}(t) &= (Au)^{(n-2)}(t) + \lambda_0 e^{(n-2)}(t) \\ &= \int_0^1 g(t,s)h(s)f(s, u_s + (u_0)_s)ds + \lambda_0 e^{(n-2)}(t) \\ &\geq \int_\varepsilon^{1-\varepsilon} g(t,s)h(s)f(s, u_s + (u_0)_s)ds + \lambda_0 \frac{\varepsilon}{b} \\ &\geq \min_{\varepsilon \leq t \leq 1-\varepsilon} \int_\varepsilon^{1-\varepsilon} g(t,s)h(s)f(s, u_s + (u_0)^s)ds + \lambda_0 \overline{\sigma} \\ &\geq \rho N\overline{\sigma} \min_{\varepsilon \leq t \leq 1-\varepsilon} \int_\varepsilon^{1-\varepsilon} g(t,s)h(s)ds + \lambda_0\overline{\sigma} \\ &\geq \overline{\sigma}\rho + \lambda_0\overline{\sigma}. \end{aligned}$$

This implies that $\overline{\sigma}\rho \geq \overline{\sigma}\rho + \lambda_0\overline{\sigma}$, a contradiction. Moreover, it is easy to check that $u \neq Au$ for $u \in \partial\Omega_\rho$ from (3.3). Hence, by Lemma 2.2 (2), it follows that $i_K(A, \Omega_\rho) = 0$. □

Theorem 3.3. *If one of the following conditions holds:*

(H7) *There exist $\rho_1, \rho_2, \rho_3 \in (0, \infty)$ with $\rho_1 < \overline{\sigma}\rho_2$ and $\rho_2 < \rho_3$ such that*

$$f_0^{\rho_1} \leq \omega, \quad f^{\rho_2}_{\overline{\sigma}\rho_2} \geq N\overline{\sigma}, \quad u \neq Au \quad \text{for } u \in \partial\Omega_{\rho_2} \quad \text{and} \quad f_0^{\rho_3} \leq \omega.$$

(H8) *There exist $\rho_1, \ \rho_2, \ \rho_3 \in (0, \infty)$ with $\rho_1 < \rho_2 < \overline{\sigma}\rho_3$ such that*

$$f^{\rho_1}_{\overline{\sigma}\rho_1} \geq N\overline{\sigma}, \quad f_0^{\rho_2} \leq \omega, \quad u \neq Au \quad \text{for } u \in \partial K_{\rho_2} \quad \text{and} \quad f^{\rho_3}_{\overline{\sigma}\rho_3} \geq N\overline{\sigma}.$$

Then BVP (1.1)-(1.4) has two positive solutions. Moreover, if in (H7), $f_0^{\rho_1} \leq \omega$ is replaced by $f_0^{\rho_1} < \omega$, then (1.1)-(1.4) has a third positive solution $u_3 \in K_{\rho_1}$.

Proof. Suppose that (H7) holds. We show that either A has a fixed point u_1 in ∂K_{ρ_1} or in $\Omega_{\rho_2} \setminus \overline{K_{\rho_1}}$. If $u \neq Au$ for $u \in \partial K_{\rho_1} \cup \partial K_{\rho_3}$, by Lemmas 3.1-3.2, we have $i_K(A, K_{\rho_1}) = 1$, $i_K(A, \Omega_{\rho_2}) = 0$ and $i_K(A, K_{\rho_3}) = 1$. Since $\rho_1 < \overline{\sigma}\rho_2$, we have $\overline{K_{\rho_1}} \subset K_{\overline{\sigma}\rho_2} \subset \Omega_{\rho_2}$ by Lemma 2.1 (b). It follows from Lemma 2.2 that A has a fixed point $u_1 \in \Omega_{\rho_2} \setminus \overline{K_{\rho_1}}$. Similarly, A has a fixed point $u_2 \in K_{\rho_3} \setminus \overline{\Omega_{\rho_2}}$. The proof is similar when (H8) holds. □

As a special case of Theorem 3.3 we obtain the following result.

Corollary 3.4. *Let $\xi(t) \equiv 0$, $\eta(t) \equiv 0$. If there exists $\rho > 0$ such that one of the following conditions holds:*

(H9) $0 \le f^0 < \omega$, $f^{\rho}_{\overline{\sigma}\rho} \ge N\overline{\sigma}$, *$u \neq Au$ for $u \in \partial\Omega_\rho$ and* $0 \le f^\infty < \omega$.

(H10) $N < f_0 \le \infty$, $f^{\rho}_{0} \le \omega$, *$u \neq Au$ for $u \in \partial K_\rho$ and* $N < f_\infty \le \infty$.

Then BVP (1.1)-(1.4) *has two positive solutions.*

Proof. From $\xi(t) \equiv 0, \eta(t) \equiv 0$, it is clear that $u_0(t) \equiv 0$ for $t \in [-\tau, b]$, thus $M_0 = 0$. We now show that (H_9) implies (H_7). It is easy to verify that $0 \le f^0 < \omega$ implies that there exists $\rho_1 \in (0, \overline{\sigma}\rho)$ such that $f^{\rho_1}_0 < \omega$. Let $k \in (f^\infty, \omega)$. Then there exists $r > \rho$ such that $\max_{t\in[0,1]} f(t,\varphi) \le k\|\varphi\|_{[-\tau,a]}$ for $\varphi \in C^+_{[r,\infty)}$ since $0 \le f^\infty < \omega$. Let

$$l = \max\{\max_{t\in[0,1]} f(t,\varphi) : \varphi \in C^+_{[0,r]}\}, \quad \text{and} \quad \rho_3 > \max\big\{\frac{l}{\omega - k},\ \rho\big\}.$$

Then we have

$$\max_{t\in[0,1]} f(t,\varphi) \le k\|\varphi\|_{[-\tau,a]} + l \le k\rho_3 + l < \omega\rho_3 \quad \text{for } \varphi \in C^+_{[0,\rho_3]}.$$

This implies that $f^{\rho_3}_0 < \omega$ and (H7) holds. Similarly, (H10) implies (H8). □

By a similar argument to that of Theorem 3.3, we obtain the following results on existence of at least one positive solution of (1.1)-(1.4).

Theorem 3.5. *If one of the following conditions holds:*

(H11) *There exist $\rho_1, \rho_2 \in (0,\infty)$ with $\rho_1 < \overline{\sigma}\rho_2$ such that*

$$f^{\rho_1}_0 \le \omega \quad \text{and} \quad f^{\rho_2}_{\overline{\sigma}\rho_2} \ge N\overline{\sigma}.$$

(H12) *There exist $\rho_1, \rho_2 \in (0,\infty)$ with $\rho_1 < \rho_2$ such that*

$$f^{\rho_1}_{\overline{\sigma}\rho_1} \ge N\overline{\sigma} \quad \text{and} \quad f^{\rho_2}_0 \le \omega.$$

Then BVP (1.1)-(1.4) *has a positive solution.*

As a special case of Theorem 3.5, we obtain the following result.

Corollary 3.6. *Let $\xi(t) \equiv 0, \eta(t) \equiv 0$. If one of the following conditions holds:*

(H13) $0 \le f^0 < \omega$ *and* $N < f_\infty \le \infty$.

(H14) $0 \le f^\infty < \omega$ *and* $N < f_0 \le \infty$.

Then BVP (1.1)-(1.4) *has a positive solution.*

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