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BLOW-UP OF SOLUTIONS TO A COUPLED QUASILINEAR VISCOELASTIC WAVE SYSTEM WITH NONLINEAR DAMPING AND SOURCE

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Abstract. We study the blow-up of the solution to a quasilinear viscoelastic wave system coupled by nonlinear sources. The system is of homogeneous Dirichlet boundary condition. The nonlinear damping and source are added to the equations. We assume that the relaxation functions are non-negative non-increasing functions and the initial energy is negative. The competition relations among the nonlinear principal parts are not constant functions, the viscoelasticity terms, dampings and sources are analyzed by using perturbed energy method. The blow-up result is proved under some conditions on the nonlinear principal parts, viscoelasticity terms, dampings and sources by a contradiction argument.

1. Introduction

Let Ω be a bounded domain of $R^n (n \geq 1)$ with a smooth boundary $\partial\Omega$. Consider the following nonlinear viscoelastic system

$$
\begin{gathered}
|u_t|^\rho u_{tt} - \operatorname{div}(\rho_1(|\nabla u|^2)\nabla u) + \int_0^t g(t-\tau)\Delta u(x,\tau)d\tau + u_t + |u_t|^{m-1}u_t \\
= f_1(u,v), \quad \Omega \times (0,T), \\
|v_t|^\rho v_{tt} - \operatorname{div}(\rho_2(|\nabla v|^2)\nabla v) + \int_0^t h(t-\tau)\Delta v(x,\tau)d\tau + v_t + |v_t|^{r-1}v_t \\
= f_2(u,v), \quad \Omega \times (0,T), \\
u(x,t) = v(x,t) = 0, \quad x \in \partial\Omega \times [0,T], \\
u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\
v(x,0) = v_0(x), \quad u_t(x,0) = v_1(x), \quad x \in \Omega,
\end{gathered}
\tag{1.1}
$$

where $\rho > 0$, $m, r > 1$ and $\rho_1, \rho_2, f_1, f_2, g, h$ are functions satisfying the following assumptions:

(A1) $\rho_i(s) = b_1 + b_2 s^{q_i}$ with $q_i \geq 0$ and $b_1, b_2 > 0$; $\rho_i(s) > 0$, for $s > 0$.

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(A2) The relaxation functions g and h are of class C^1 and satisfy, for $s \geq 0$,
$$g(s) \geq 0, \quad b_1 - \int_0^\infty g(s)ds = l > 0, \quad g'(s) \leq 0,$$
$$h(s) \geq 0, \quad b_1 - \int_0^\infty h(s)ds = k > 0, \quad h'(s) \leq 0.$$

(A3) Let $F(u,v) = a|u+v|^{p+1} + 2b|uv|^{\frac{p+1}{2}}$ with $a, b > 0$, $1 < p < \infty$ if $n = 1, 2$ and $1 < p < \frac{n}{n-2}$ if $n \geq 3$. Assume that
$$f_1(u,v) = \frac{\partial F}{\partial u}, \quad f_2(u,v) = \frac{\partial F}{\partial v},$$
and that there are positive constants c_0, c_1 such that
$$c_0(|u|^{p+1} + |v|^{p+1}) \leq F(u,v) \leq c_1(|u|^{p+1} + |v|^{p+1}).$$

Many studies concerning existence of global solutions or their blow-up to system (1.1) with $\rho_i \equiv 1$ are available in the literature. Georgiev and Todorova [5] considered the single equation
$$u_{tt} - \Delta u + u_t|u_t|^{m-1} = |u|^{p-1}u, \quad \text{in } \Omega \times (0,\infty), \tag{1.2}$$
and the interaction between the nonlinear damping and nonlinear source term. The authors showed that the solutions of the system with sufficient large initial data blow up in finite time if $p > m$. Messaoudi [8] extended the results of [5] to the case that the initial energy is negative. Agre and Rammaha [1] extended the results of [5] by considering an initial-boundary value problem to the coupled wave equations.

In the presence of the viscoelastic term, Messaoudi [9] considered the nonlinear viscoelastic equation
$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + au_t|u_t|^{m-1} = b|u|^{p-1}u, \quad \Omega \times (0,\infty), \tag{1.3}$$
with initial conditions and Dirichlet boundary conditions. He proved that the weak solution with negative initial energy blew up if $p > m$ when g satisfied some conditions. Messaoudi [10] considered the blow-up solution of (1.3) with $a = 1$, $b = 1$ and with small positive initial energy. Song [12] extended the results of [10] to the case that the initial energy is arbitrarily positive. For other related works on the viscoelastic wave equation, we refer the reader to [2, 4, 16].

Problem (1.1) with $\rho > 0$ has also been extensively studied. Song [13] investigated the nonexistence of global solutions to the initial-boundary value problem of the following equation with positive initial energy
$$|u_t|^\rho u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + u_t|u_t|^{m-2} = |u|^{p-2}u, \quad \Omega \times (0,\infty). \tag{1.4}$$
Liu [7] studied the general decay for the global solution and blow-up of solution to the equation
$$|u_t|^\rho u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau - \Delta u_{tt} = |u|^{p-2}u, \quad \Omega \times (0,\infty). \tag{1.5}$$
Cavalcanti et al. [3] studied the energy decay for the nonlinear viscoelastic problem
$$|u_t|^\rho u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau - \Delta u_{tt} - \gamma\Delta u_t = 0, \quad \Omega \times (0,\infty). \tag{1.6}$$

A global existence result for $\gamma \geq 0$ as well as an exponential decay for $\gamma > 0$ was established in [3]. When the source term $b|u|^{p-2}u$ appeared on the right side of system (1.6), Messaoudi et al. [11] proved that the viscoelastic term was enough to ensure existence and uniform decay of global solutions provided that the initial data were in some stable set.

For $\rho_i(s) = b_1 + b_2 s^{q_i}$ with $q_i \geq 0$ and $b_1, b_2 > 0$, Wu et al. [14] and [15] considered the blow-up of the initial boundary value problem (spatial dimension $n = 1, 2, 3$) for the system

$$\begin{aligned} u_{tt} - \operatorname{div}(\rho_1(|\nabla u|^2)\nabla u) + u_t + |u_t|^{m-1}u_t = f(u,v), \quad \Omega \times (0,T), \\ v_{tt} - \operatorname{div}(\rho_2(|\nabla v|^2)\nabla v) + v_t + |v_t|^{r-1}v_t = g(u,v), \quad \Omega \times (0,T). \end{aligned} \tag{1.7}$$

For a single wave equation with $\rho_i(s) \geq b_1 + b_2 s^{q_i}$, $q_i \geq 0$, $b_1, b_2 > 0$, Hao et al. [6] studied the global existence and blow up of the solutions.

We note that, in the literature mentioned above, only viscoelastic term was included in the equation or only nonlinear principal part (i.e. $\rho_i, i = 1, 2$, are not constant functions) was included. To the best of our knowledge, there are no papers considering the blow-up of the equation with both viscoelastic term and nonlinear principal part. The main goal of our paper is to prove that for $\rho_i(s) = b_1 + b_2 s^{q_i}$ the nonlinear coupled source terms still leads to blow-up of the solutions though there are viscoelastic terms in the equations. To be more precise, we prove that when $p > \max\{2q_1 + 1, 2q_2 + 1\}$ and the relaxation functions satisfy that $\max\{\int_0^\infty g(s)ds, \int_0^\infty h(s)ds\} < \frac{q}{q+1} b_1$, the solutions of the system will blow up. Our method is borrowed partly from [7, 14], but we must overcome some additional difficulty caused by the complex interaction among the nonlinear viscoelastic terms, the nonlinear principal parts, the coupled source terms and the nonlinear damping.

2. Preliminaries

In this section, we present some other assumptions and existence result of local solution. We use the following assumptions:

(A4) $\rho > 0$ if $n = 1, 2$ and $0 < \rho < \frac{2}{n-2}$ if $n \geq 3$.

(A5) $m < p$, $r < p$ and $\rho + 2 < p$.

Define the energy function of the system (1.1) by

$$\begin{aligned} E(t) = & \frac{1}{\rho+2}\Big(\|u_t\|_{\rho+2}^{\rho+2} + \|v_t\|_{\rho+2}^{\rho+2}\Big) + \frac{1}{2}\Big(b_1 - \int_0^t g(s)ds\Big)\|\nabla u\|^2 \\ & + \frac{1}{2}\Big(b_1 - \int_0^t h(s)ds\Big)\|\nabla v\|^2 + \frac{1}{2}(g \circ \nabla u)(t) + \frac{1}{2}(h \circ \nabla v)(t) \\ & + \frac{b_2}{2(q_1+1)}\|\nabla u\|_{2(q_1+1)}^{2(q_1+1)} + \frac{b_2}{2(q_2+1)}\|\nabla u\|_{2(q_2+1)}^{2(q_2+1)} - \int_\Omega F(u,v)\,dx. \end{aligned} \tag{2.1}$$

Combining the arguments in [5] and [3], and making some slight modification, we have the following existence of local weak solutions.

Theorem 2.1. *Let* (A1)–(A4) *hold. Then for any initial data* $u_0 \in W_0^{1,2q_1+2}(\Omega) \cap L^{p+1}(\Omega)$, $v_0 \in W_0^{1,2q_2+2}(\Omega) \cap L^{p+1}(\Omega)$, *there exists a unique local weak solution* (u,v) *to the system* (1.1) *defined on* $[0,T)$ *for some* $T > 0$, *and*

$$\begin{aligned} u \in L^\infty([0,T); W_0^{1,2q_1+2}(\Omega) \cap L^{p+1}(\Omega)), \\ v \in L^\infty([0,T]; W_0^{1,2q_2+2}(\Omega) \cap L^{p+1}(\Omega)), \end{aligned}$$

$$
\begin{gathered}
u_t \in L^\infty([0,T); W_0^{1,2q_1+2}(\Omega) \cap L^{p+1}(\Omega)),\\
v_t \in L^\infty([0,T]; W_0^{1,2q_2+2}(\Omega) \cap L^{p+1}(\Omega))\\
u_{tt} \in L^\infty([0,T); L^2(\Omega)), \quad v_{tt} \in L^\infty([0,T); L^2(\Omega))
\end{gathered}
$$

Combining the arguments of [5, 10], the following lemma can be proved easily.

Lemma 2.2. *Let* (A1)–(A4) *hold. And let* (u,v) *be a solution of* (1.1)*. Then* $E(t)$ *satisfies the inequality*

$$
\begin{aligned}
E'(t) &= -\|u_t\|^2 - \|u_t\|_{m+1}^{m+1} - \|v_t\|^2 - \|v_t\|_{r+1}^{r+1} + \frac{1}{2}(g' \circ \nabla u)(t)\\
&\quad + \frac{1}{2}(h' \circ \nabla v)(t) - \frac{1}{2}g(t)\|\nabla u\|^2 - \frac{1}{2}h(t)\|\nabla v\|^2 \le 0.
\end{aligned} \tag{2.2}
$$

Lemma 2.3 ([8])**.** *Suppose* p *satisfies* (A3)*. Then there exists a positive constant* $C(|\Omega|,p)$ *such that*

$$
\|u\|_{p+1}^s \le C(|\Omega|,p)\Big(\|\nabla u\|^2 + \|u\|_{p+1}^{p+1}\Big), \quad \forall u \in H_0^1(\Omega),
$$

where $2 \le s \le p+1$.

In this article, we use $\|\cdot\|$ and $\|\cdot\|_p$ denote the usual $L^2(\Omega)$ norm and $L^p(\Omega)$ norm, respectively. B_1 is the optimal constant of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$.

3. Blow-up results

In this section, we state and prove our main result.

Theorem 3.1. *Let* (A1)–(A5) *hold.* $q = \max\{q_1, q_2\}$*. Assume the initial energy* $E(0) < 0$ *and*

$$
\max\Big\{\int_0^\infty g(s)ds, \int_0^\infty h(s)ds\Big\} < \frac{q}{q+1}b_1, \quad p > \max\{2q_1+1, 2q_2+1\}.
$$

Then the solution of (1.1) *blows up at finite time.*

Proof. We use the contradiction method. Suppose that the solution (u,v) of (1.1) is global. Then

$$
\|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u\|^2 + \|u\|_{p+1}^{p+1} + \|v_t\|_{\rho+2}^{\rho+2} + \|\nabla v\|^2 + \|v\|_{p+1}^{p+1} \le C, \quad \forall t \ge 0. \tag{3.1}
$$

Set $M_1 = \max_{t\in[0,T]} \|u\|_{p+1}^{p+1}$, $M_2 = \max_{t\in[0,T]} \|v\|_{p+1}^{p+1}$, $M = M_1 + M_2$. Let $H(t) = -E(t)$. Then by Lemma 2.2, the function $H(t)$ is increasing. Moreover, from $E(0) < 0$ and (A3), we obtain

$$
\begin{aligned}
0 < H(0) \le H(t) &\le \int_\Omega F(u,v)\,dx\\
&\le c_1 \int_\Omega (|u|^{p+1} + |v|^{p+1})\,dx\\
&\le c_1 \max_{t\in[0,T]} \int_\Omega |u|^{p+1} + |v|^{p+1}\,dx = c_1 M.
\end{aligned} \tag{3.2}
$$

Let us introduce the auxiliary function

$$
L(t) = H^{1-\sigma}(t) + \frac{\varepsilon}{\rho+1}\Big(\int_\Omega |u_t|^\rho u_t u\,dx + \int_\Omega |v_t|^\rho v_t v\,dx\Big), \tag{3.3}
$$

where $0<\varepsilon \ll 1$ and

$$0<\sigma<\min\Big\{\frac{1}{\rho+2}-\frac{1}{p},\frac{p-m}{m(p+1)},\frac{p-r}{r(p+1)}\Big\}. \tag{3.4}$$

By differentiating $L(t)$, we obtain

$$\begin{aligned}
&L'(t)\\
&=(1-\sigma)H^{\sigma}(t)H'(t)+\frac{\varepsilon}{\rho+1}\Big(\int_\Omega |u_t|^{\rho+2}\,dx+\int_\Omega |v_t|^{\rho+2}\,dx\Big)\\
&\quad+\varepsilon\Big(\int_\Omega |u_t|^{\rho}u_{tt}u\,dx+\int_\Omega |v_t|^{\rho}v_{tt}v\,dx\Big)\\
&=(1-\sigma)H^{\sigma}(t)H'(t)+\frac{\varepsilon}{\rho+1}\Big(\|u_t\|_{\rho+2}^{\rho+2}+\|u_t\|_{\rho+2}^{\rho+2}\Big)\\
&\quad-\varepsilon\int_\Omega(\rho_1(|\nabla u|^2)|\nabla u|^2+\rho_2(|\nabla v|^2)|\nabla v|^2)\,dx\\
&\quad+\varepsilon\int_\Omega\int_0^t g(t-s)\nabla u(s)\cdot\nabla u(t)ds\,dx+\varepsilon\int_\Omega\int_0^t h(t-s)\nabla v(s)\cdot\nabla v(t)ds\,dx\\
&\quad-\varepsilon\int_\Omega(uu_t+vv_t+|u_t|^{m-1}u_tu+|v_t|^{r-1}v_tv)\,dx+\varepsilon(p+1)\int_\Omega F(u,v)\,dx\\
&=(1-\sigma)H^{\sigma}(t)H'(t)+\frac{\varepsilon}{\rho+1}\Big(\|u_t\|_{\rho+2}^{\rho+2}+\|v_t\|_{\rho+2}^{\rho+2}\Big)-\varepsilon b_1(\|\nabla u\|^2+\|\nabla v\|^2)\\
&\quad-\varepsilon b_2\Big(\|\nabla u\|_{2(q_1+1)}^{2(q_1+1)}+\|\nabla v\|_{2(q_2+1)}^{2(q_2+1)}\Big)\\
&\quad+\varepsilon\int_\Omega\int_0^t g(t-s)\nabla u(s)\cdot\nabla u(t)ds\,dx+\varepsilon\int_\Omega\int_0^t h(t-s)\nabla v(s)\cdot\nabla v(t)ds\,dx\\
&\quad-\varepsilon\int_\Omega(uu_t+vv_t+|u_t|^{m-1}u_tu+|v_t|^{r-1}v_tv)\,dx+\varepsilon(p+1)\int_\Omega F(u,v)\,dx.
\end{aligned} \tag{3.5}$$

Now, we estimate the fourth term on the right hand of (3.5). Let $\mu=\min\{l,k\}$. From the the definition of $H(t)$, it follows that

$$\begin{aligned}
&-b_2\|\nabla u\|_{2(q_1+1)}^{2(q_1+1)}-b_2\|\nabla v\|_{2(q_2+1)}^{2(q_2+1)}\\
&\geq -b_2\frac{(q+1)}{q_1+1}\|\nabla u\|_{2(q_1+1)}^{2(q_1+1)}-b_2\frac{(q+1)}{q_2+1}\|\nabla v\|_{2(q_2+1)}^{2(q_2+1)}\\
&=(q+1)\Big(2H(t)-2\int_\Omega F(u,v)\,dx+\frac{2}{\rho+2}\Big(\int_\Omega |u_t|^{\rho+2}\,dx+\int_\Omega |v_t|^{\rho+2}\,dx\Big)\\
&\quad+\Big(b_1-\int_0^t g(s)ds\Big)\|\nabla u\|^2+\Big(b_1-\int_0^t h(s)ds\Big)\|\nabla v\|^2\\
&\quad+(g\circ\nabla u)(t)+(h\circ\nabla v)(t)\Big)\\
&\geq 2(q+1)H(t)+\frac{2(q+1)}{\rho+2}\Big(\|u_t\|_{\rho+2}^{\rho+2}+\|v_t\|_{\rho+2}^{\rho+2}\Big)\\
&\quad-2(q+1)\int_\Omega F(u,v)\,dx+(q+1)\mu(\|\nabla u\|^2+\|\nabla v\|^2)\\
&\quad+(q+1)\Big((g\circ\nabla u)(t)+(h\circ\nabla v)(t)\Big).
\end{aligned} \tag{3.6}$$

By Hölder's and Young's inequalities, we estimate the fifth term on the right hand of (3.5). It yields

$$
\begin{aligned}
&\int_\Omega \int_0^t g(t-s)\nabla u(s)\cdot\nabla u(t)ds\,dx \\
&= \int_\Omega \int_0^t g(t-s)\nabla u(t)\cdot(\nabla u(s)-\nabla u(t))ds\,dx + \int_0^t g(t-s)\|\nabla u(t)\|^2 \\
&\ge -(g\circ\nabla u)(t) + \frac{3}{4}\int_0^t g(t-s)\|\nabla u(t)\|^2.
\end{aligned} \tag{3.7}
$$

Similarly, we obtain

$$
\int_\Omega \int_0^t h(t-s)\nabla v(s)\cdot\nabla v(t)ds\,dx \ge -(h\circ\nabla v)(t) + \frac{3}{4}\int_0^t h(t-s)\|\nabla v(t)\|^2. \tag{3.8}
$$

Therefore, based on (3.6), (3.7) and (3.8), we conclude that

$$
\begin{aligned}
&L'(t)\\
&\ge (1-\sigma)H^\sigma(t)H'(t) + \frac{\varepsilon}{\rho+1}\Big(\|u_t\|_{\rho+2}^{\rho+2} + \|v_t\|_{\rho+2}^{\rho+2}\Big)\\
&\quad - \varepsilon b_1(\|\nabla u\|^2 + \|\nabla v\|^2) + 2\varepsilon(q+1)H(t) + \frac{2\varepsilon(q+1)}{\rho+1}\Big(\|u_t\|_{\rho+2}^{\rho+2} + \|v_t\|_{\rho+2}^{\rho+2}\Big)\\
&\quad + \mu\varepsilon(q+1)(\|\nabla u\|^2 + \|\nabla v\|^2) + \varepsilon(q+1)\Big((g\circ\nabla u)(t) + (h\circ\nabla v)(t)\Big)\\
&\quad + \varepsilon(p-2q-1)\int_\Omega F(u,v)\,dx - \varepsilon(g\circ\nabla u)(t)\\
&\quad + \frac{3}{4}\varepsilon\int_0^t g(s)ds\|\nabla u(t)\|^2 - \varepsilon(h\circ\nabla v)(t) + \frac{3}{4}\varepsilon\int_0^t h(s)ds\|\nabla v(t)\|^2\\
&\quad - \varepsilon\int_\Omega (uu_t + vv_t + |u_t|^{m-1}u_t u + |v_t|^{r-1}v_t v)\,dx.
\end{aligned} \tag{3.9}
$$

Now we use Young's inequality and (2.2) to obtain the inequality

$$
\int_\Omega |u||u_t|\,dx \le \frac{\varepsilon_1^2}{2}\|u\|^2 + \frac{1}{2\varepsilon_1^2}\|u_t\|^2 \le \frac{\varepsilon_1^2 B_1}{2}\|\nabla u\|^2 + \frac{1}{2\varepsilon_1^2}H'(t), \tag{3.10}
$$

$$
\int_\Omega |v||v_t|\,dx \le \frac{\varepsilon_1^2}{2}\|v\|^2 + \frac{1}{2\varepsilon_1^2}\|v_t\|^2 \le \frac{\varepsilon_1^2 B_1}{2}\|\nabla v\|^2 + \frac{1}{2\varepsilon_1^2}H'(t), \tag{3.11}
$$

$$
\begin{aligned}
\int_\Omega |u_t|^{m-1}u_t u\,dx &\le \frac{\delta_1^{m+1}}{m+1}\|u\|_{m+1}^{m+1} + \frac{m\delta_1^{-\frac{m+1}{m}}}{m+1}\|u_t\|_{m+1}^{m+1}\\
&\le \frac{\delta_1^{m+1}}{m+1}\|u\|_{m+1}^{m+1} + \frac{m\delta_1^{-\frac{m+1}{m}}}{m+1}H'(t),
\end{aligned} \tag{3.12}
$$

$$
\begin{aligned}
\int_\Omega |v_t|^{r-1}v_t v\,dx &\le \frac{\delta_2^{r+1}}{r+1}\|v\|_{r+1}^{r+1} + \frac{r\delta_2^{-\frac{r+1}{r}}}{r+1}\|v_t\|_{r+1}^{r+1}\\
&\le \frac{\delta_2^{r+1}}{r+1}\|v\|_{r+1}^{r+1} + \frac{r\delta_2^{-\frac{r+1}{r}}}{r+1}H'(t),
\end{aligned} \tag{3.13}
$$

where $\varepsilon_1, \delta_1, \delta_2$ are constants depending on the time t and are specified later.

Since g and h are positive, we have, for any $t > t_0 > 0$,

$$\int_0^t g(s)ds \geq \int_0^{t_0} g(s)ds =: g_0 > 0, \quad \int_0^t h(s)ds \geq \int_0^{t_0} h(s)ds =: h_0 > 0.$$

Let $\chi = \min\left\{\frac{3}{4}g_0, \frac{3}{4}h_0\right\}$. Then $\chi > 0$. By (3.10)–(3.13), we obtain

$$\begin{aligned}
L'(t) \geq{}& \Big((1-\sigma)H^\sigma(t) - \frac{\varepsilon m \delta_1^{-\frac{m+1}{m}}}{m+1} - \frac{\varepsilon r \delta_2^{-\frac{r+1}{r}}}{r+1} - \frac{\varepsilon}{\varepsilon_1^2}\Big)H'(t)\\
&+ 2\varepsilon(q+1)H(t) + \varepsilon\Big(\frac{1}{\rho+1} + \frac{2(q+1)}{\rho+2}\Big)\Big(\|u_t\|_{\rho+2}^{\rho+2} + \|v_t\|_{\rho+2}^{\rho+2}\Big)\\
&+ \varepsilon\Big(\mu(q+1) - b_1 - \frac{B_1\varepsilon_1^2}{2} + \chi\Big)(\|\nabla u\|^2 + \|\nabla v\|^2)\\
&+ \varepsilon(p-2q-1)\int_\Omega F(u,v)\,dx - \varepsilon\Big(\frac{\delta_1^{m+1}}{m+1}\|u\|_{m+1}^{m+1} + \frac{\delta_2^{r+1}}{r+1}\|v\|_{r+1}^{r+1}\Big)\\
&+ \varepsilon q\Big((g\circ\nabla u)(t) + (h\circ\nabla v)(t)\Big).
\end{aligned} \tag{3.14}$$

Let $\varepsilon_1^{-2} = K_1H^{-\sigma}$, $\delta_1^{-\frac{m+1}{m}} = K_2H^{-\sigma}$, $\delta_2^{-\frac{r+1}{r}} = K_3H^{-\sigma}$, where $K_1, K_2, K_3 > 0$ will be chosen later. Then, by (3.2), we obtain

$$\delta_1^{m+1} = K_2^{-m}H^{\sigma m}(t) \leq K_2^{-m}c_1^{\sigma m}(\|u\|_{p+1}^{p+1} + \|v\|_{p+1}^{p+1})^{\sigma m}, \tag{3.15}$$
$$\delta_2^{r+1} = K_3^{-r}H^{\sigma r}(t) \leq K_3^{-r}c_1^{\sigma r}(\|u\|_{p+1}^{p+1} + \|v\|_{p+1}^{p+1})^{\sigma r}. \tag{3.16}$$

Hence,

$$\begin{aligned}
L'(t) \geq{}& \Big((1-\sigma)H^\sigma(t) - \frac{\varepsilon m K_2 H^{-\sigma}}{m+1} - \frac{\varepsilon r K_3 H^{-\sigma}}{r+1} - \varepsilon K_1 H^{-\sigma}\Big)H'(t)\\
&+ 2\varepsilon(q+1)H(t) + \varepsilon\Big(\frac{1}{\rho+1} + \frac{2(q+1)}{\rho+2}\Big)\Big(\|u_t\|_{\rho+2}^{\rho+2} + \|v_t\|_{\rho+2}^{\rho+2}\Big)\\
&+ \varepsilon\Big(\mu(q+1) - b_1 - \frac{B_1\varepsilon_1^2}{2} + \chi\Big)(\|\nabla u\|^2 + \|\nabla v\|^2)\\
&+ \varepsilon(p-2q-1)\int_\Omega F(u,v)\,dx - \varepsilon\Big(\frac{K_2^{-m}c_1^{\sigma m}}{m+1}\big(\|u\|_{p+1}^{p+1}\\
&+ \|v\|_{p+1}^{p+1}\big)^{\sigma m}\|u\|_{m+1}^{m+1} + \frac{K_3^{-r}c_1^{\sigma r}}{r+1}(\|u\|_{p+1}^{p+1} + \|v\|_{p+1}^{p+1})^{\sigma r}\|v\|_{r+1}^{r+1}\Big)\\
&+ \varepsilon q\Big((g\circ\nabla u)(t) + (h\circ\nabla v)(t)\Big).
\end{aligned} \tag{3.17}$$

By (A5) and the Sobolev embedding theorem, we have

$$\|u\|_{m+1}^{m+1} \leq B_2\|u\|_{p+1}^{m+1} \leq B_2(\|u\|_{p+1} + \|v\|_{p+1})^{m+1}, \tag{3.18}$$
$$\|v\|_{r+1}^{r+1} \leq B_3\|v\|_{p+1}^{r+1} \leq B_3(\|u\|_{p+1} + \|v\|_{p+1})^{r+1}. \tag{3.19}$$

Using the inequality $(a+b)^\lambda \le B_4(a^\lambda + b^\lambda)$, we have

$$\begin{aligned}
L'(t) \ge{}& \Big((1-\sigma)H^\sigma(t) - \frac{\varepsilon m K_2 H^{-\sigma}}{m+1} - \frac{\varepsilon r K_3 H^{-\sigma}}{r+1} - \varepsilon K_1 H^{-\sigma}\Big)H'(t)\\
&+ 2\varepsilon(q+1)H(t) + \varepsilon\Big(\frac{1}{\rho+1} + \frac{2(q+1)}{\rho+2}\Big)\Big(\|u_t\|_{\rho+2}^{\rho+2} + \|v_t\|_{\rho+2}^{\rho+2}\Big)\\
&+ \varepsilon\Big(\mu(q+1) - b_1 - \frac{B_1\varepsilon_1^2}{2} + \chi\Big)(\|\nabla u\|^2 + \|\nabla v\|^2)\\
&+ \varepsilon(p-2q-1)\int_\Omega F(u,v)\,dx\\
&- \varepsilon\Big(\frac{K_2^{-m}B_5 c_1^{\sigma m}}{m+1}(\|u\|_{p+1} + \|v\|_{p+1})^{\sigma m(p+1)+m+1}\\
&+ \frac{K_3^{-r}B_6 c_1^{\sigma r}}{r+1}(\|u\|_{p+1} + \|v\|_{p+1})^{\sigma r(p+1)+r+1}\Big)\\
&+ \varepsilon q\Big((g\circ\nabla u)(t) + (h\circ\nabla v)(t)\Big)
\end{aligned} \tag{3.20}$$

where $B_5 = B_2B_4, B_6 = B_3B_4$.

If we set $s = \sigma m(p+1) + m + 1$ and $\sigma r(p+1) + r + 1$, then by Lemma 2.3, there exist two positive constants B_7, B_8 depending on $|\Omega|, m, r$ such that

$$\|u\|_{p+1}^{\sigma m(p+1)+m+1} \le B_7(\|\nabla u\|^2 + \|u\|_{p+1}^{p+1}), \tag{3.21}$$

$$\|v\|_{p+1}^{\sigma r(p+1)+r+1} \le B_8(\|\nabla v\|^2 + \|v\|_{p+1}^{p+1}). \tag{3.22}$$

Thus

$$\begin{aligned}
&L'(t)\\
&\ge \Big((1-\sigma)H^\sigma(t) - \frac{\varepsilon m K_2 H^{-\sigma}}{m+1} - \frac{\varepsilon r K_3 H^{-\sigma}}{r+1} - \varepsilon K_1 H^{-\sigma}\Big)H'(t)\\
&+ 2\varepsilon(q+1)H(t) + \varepsilon\Big(\frac{1}{\rho+1} + \frac{2(q+1)}{\rho+2}\Big)\Big(\|u_t\|_{\rho+2}^{\rho+2} + \|v_t\|_{\rho+2}^{\rho+2}\Big)\\
&+ \varepsilon\Big(\mu(q+1) - b_1 - \frac{B_1\varepsilon_1^2}{2} + \chi - \frac{K_2^{-m}B_5B_7c_1^{\sigma m}}{m+1} - \frac{K_3^{-r}B_6B_8c_1^{\sigma r}}{r+1}\Big)\\
&\times\big(\|\nabla u\|^2 + \|\nabla v\|^2\big) + \varepsilon\Big((p-2q-1)c_0 - \frac{K_2^{-m}B_5B_7c_1^{\sigma m}}{m+1}\\
&- \frac{K_3^{-r}B_6B_8c_1^{\sigma r}}{r+1}\Big)(\|u\|_{p+1}^{p+1} + \|v\|_{p+1}^{p+1}) + \varepsilon q\Big((g\circ\nabla u)(t) + (h\circ\nabla v)(t)\Big).
\end{aligned} \tag{3.23}$$

Using the condition of Theorem 3.1, we obtain $\mu(q+1) - b_1 > 0$. Now, we can choose K_1, K_2, K_3 large enough so that the following inequalities hold:

$$\begin{aligned}
&\mu(q+1) - b_1 + \chi - \frac{B_1\varepsilon_1^2}{2} - \frac{K_2^{-m}B_5B_7c_1^{\sigma m}}{m+1} - \frac{K_3^{-r}B_6B_8c_1^{\sigma r}}{r+1}\\
&\ge \mu(q+1) - b_1 + \chi - \frac{B_1M^\sigma}{2K_1} - \frac{K_2^{-m}B_5B_7c_1^{\sigma m}}{m+1} - \frac{K_3^{-r}B_6B_8c_1^{\sigma r}}{r+1}\\
&\ge \frac{\mu(q+1) - b_1}{2}
\end{aligned} \tag{3.24}$$

and

$$(p-2q-1)c_0 - \frac{K_2^{-m}B_5B_7c_1^{\sigma m}}{m+1} - \frac{K_3^{-r}B_6B_8c_1^{\sigma r}}{r+1} \ge \frac{(p-2q-1)c_0}{2}\,. \tag{3.25}$$

Furthermore, for fixed K_1, K_2, K_3, $T_0 \geq t_0$, we choose ε small enough such that

$$(1-\sigma) - \frac{\varepsilon m K_2}{m+1} - \frac{\varepsilon r K_3}{r+1} - \varepsilon K_1 \geq 0, \tag{3.26}$$

$$\begin{aligned} L(T_0) = H^{1-\sigma}(T_0) + \frac{\varepsilon}{\rho+1}\Big(\int_\Omega |u_t(T_0)|^\rho u_t(T_0) u(T_0)\,dx \\ + \int_\Omega |v_t(T_0)|^\rho v_t(T_0) v(T_0)\,dx\Big) > 0. \end{aligned} \tag{3.27}$$

From the condition of Theorem 3.1, for $t > T_0$, we have

$$\begin{aligned} L'(t) \geq \varepsilon\gamma\Big[H(t) + \|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u\|^2 + \|u\|_{p+1}^{p+1} \\ + \|v_t\|_{\rho+2}^{\rho+2} + \|\nabla v\|^2 + \|v\|_{p+1}^{p+1}\Big], \end{aligned} \tag{3.28}$$

$$L(t) \geq L(T_0) > 0, \tag{3.29}$$

where

$$\gamma = \min\Big\{2(q+1), \Big(\frac{1}{\rho+1} + \frac{2(q+1)}{\rho+2}\Big), \frac{\mu(q+1) - b_1}{2}, \frac{(p-2q-1)c_0}{2}\Big\}. \tag{3.30}$$

We now estimate $L(t)^{\frac{1}{1-\sigma}}$. By Hölder's inequality and the condition (A5), we obtain

$$\big|\textstyle\int_\Omega |u_t|^\rho u_t u\,dx\big| \leq \|u_t\|_{\rho+2}^{\rho+1}\|u\|_{\rho+2} \leq B_9\|u_t\|_{\rho+2}^{\rho+1}\|u\|_{p+1}. \tag{3.31}$$

Therefore,

$$\Big|\int_\Omega |u_t|^\rho u_t u\,dx\Big|^{\frac{1}{1-\sigma}} \leq B_9\|u_t\|_{\rho+2}^{\frac{\rho+1}{1-\sigma}}\|u\|_{p+1}^{\frac{1}{1-\sigma}} \leq B_{10}(\|u_t\|_{\rho+2}^{\frac{\rho+1}{1-\sigma}\mu} + \|u\|_{p+1}^{\frac{\theta}{1-\sigma}}), \tag{3.32}$$

where $\frac{1}{\mu} + \frac{1}{\theta} = 1$. Choosing $\mu = \frac{(1-\sigma)(\rho+2)}{\rho+1} > 1$, we have

$$2 < \frac{\theta}{1-\sigma} = \frac{\rho+2}{(1-\sigma)(\rho+2) - (\rho+1)} < p+1. \tag{3.33}$$

By Lemma 2.3, taking $s = \frac{\theta}{1-\sigma}$, it follows that

$$\|u\|_{p+1}^{\frac{\theta}{1-\sigma}} \leq B_{11}(\|\nabla u\|^2 + \|u\|_{p+1}^{p+1}). \tag{3.34}$$

Hence

$$\big|\int_\Omega |u_t|^\rho u_t u\,dx\big|^{\frac{1}{1-\sigma}} \leq B_{12}\Big[\|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u\|^2 + \|u\|_{p+1}^{p+1}\Big]. \tag{3.35}$$

Similarly,

$$\big|\int_\Omega |v_t|^\rho v_t v\,dx\big|^{\frac{1}{1-\sigma}} \leq B_{13}\Big[\|v_t\|_{\rho+2}^{\rho+2} + \|\nabla v\|^2 + \|v\|_{p+1}^{p+1}\Big]. \tag{3.36}$$

Hence, combining (3.2), (3.35) and (3.36), we easily get

$$
\begin{aligned}
L(t)^{\frac{1}{1-\sigma}} &= \Big(H^{1-\sigma}(t) + \frac{\varepsilon}{\rho+1}\Big(\int_\Omega |u_t|^\rho u_t u\,dx + \int_\Omega |v_t|^\rho v_t v\,dx\Big)\Big)^{\frac{1}{1-\sigma}} \\
&\le 2^{\frac{1}{1-\sigma}}\Big(H(t) + \frac{\varepsilon}{\rho+1}\Big(\Big|\int_\Omega |u_t|^\rho u_t u\,dx\Big| + \Big|\int_\Omega |v_t|^\rho v_t v\,dx\Big|\Big)^{\frac{1}{1-\sigma}}\Big) \\
&\le 2^{\frac{1}{1-\sigma}} B_{14}\Big[H(t) + \|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u\|^2 + \|u\|_{p+1}^{p+1} \\
&\quad + \|v_t\|_{\rho+2}^{\rho+2} + \|\nabla v\|^2 + \|v\|_{p+1}^{p+1}\Big] \\
&\le \widetilde{C}\Big[\|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u\|^2 + \|u\|_{p+1}^{p+1} + \|v_t\|_{\rho+2}^{\rho+2} + \|\nabla v\|^2 + \|v\|_{p+1}^{p+1}\Big],
\end{aligned} \tag{3.37}
$$

where $\widetilde{C}$ depends on c_1, B_9—B_{14}.

Combining (3.28) and (3.37), we have

$$
L'(t) > \frac{\varepsilon\gamma}{\widetilde{C}} L^{\frac{1}{1-\sigma}}(t), \quad \text{for } t \ge T_0. \tag{3.38}
$$

The inequality above implies that $L(t)$ blows up at a finite time T^* and

$$
T^* \le \frac{\widetilde{C}(1-\sigma)}{\varepsilon\gamma L^{\sigma/(1-\sigma)}(T_0)}. \tag{3.39}
$$

Furthermore, from (3.37) we obtain

$$
\lim_{t\to T^{*-}} \Big[\|u_t\|_{\rho+2}^{\rho+2} + \|\nabla u\|^2 + \|u\|_{p+1}^{p+1} + \|v_t\|_{\rho+2}^{\rho+2} + \|\nabla v\|^2 + \|v\|_{p+1}^{p+1}\Big] = +\infty. \tag{3.40}
$$

If we choose the $T > \frac{\widetilde{C}(1-\sigma)}{\varepsilon\gamma L^{\sigma/(1-\sigma)}(T_0)}$, obviously, (3.40) contradicts (3.1). Thus, the solution of problem (1.1) blows up in finite time. □

Concluding remarks. In this paper, we considered the blow-up of solutions to a coupled quasilinear system with the nonlinear viscoelastic terms, the nonlinear principal parts, the coupled source terms and the nonlinear dampings. A sufficient condition under which the solutions of the system will blow up at finite time is given. We show that the coupled sources are enough to lead to the blow-up when the relaxation functions and the nonlinear principle parts satisfy some conditions.

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