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PARTIAL COMPACTNESS FOR THE 2-D LANDAU-LIFSHITZ FLOW

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ABSTRACT. Uniform local C^∞-bounds for Ginzburg-Landau type approximations for the Landau-Lifshitz flow on planar domains are proven. They hold outside an energy-concentration set of locally finite parabolic Hausdorff-dimension 2, which has finite times-slices. The approximations subconverge to a global weak solution of the Landau-Lifshitz flow, which is smooth away from the energy concentration set. The same results hold for sequences of global smooth solutions of the 2-d Landau-Lifshitz flow.

1. INTRODUCTION

The Ginzburg-Landau approximations $u_\epsilon : \overline{\Omega} \times \mathbb{R}_+ \to \mathbb{R}^3$ to the Landau-Lifshitz flow are solutions of

$$\gamma_1 \partial_t u_\epsilon - \gamma_2 u_\epsilon \times \partial_t u_\epsilon - \Delta u_\epsilon = -\frac{1}{\epsilon^2} f(u_\epsilon) \quad \text{in } \Omega \times \mathbb{R}_+ \tag{1.1}$$

$$u_\epsilon = u_0 \quad \text{on } \big(\Omega \times \{0\}\big) \cup \big(\partial\Omega \times \mathbb{R}_+\big)\,. \tag{1.2}$$

where $\gamma_1 > 0$ and $\gamma_2 \in \mathbb{R}$. "$\times$" denotes the usual vector product in $\mathbb{R}^3$. The domain $\Omega \subset \mathbb{R}^2$ is open, bounded and smooth. The initial and boundary data u_0 is always assumed to map a.e. into the standard sphere $S^2 \subset \mathbb{R}^3$ or an embedded manifold $N \subset \mathbb{R}^n$ (see below). For the definition of f_ϵ we distinguish two cases:
Case (I): If $\gamma_2 \neq 0$, the target is $S^2 \hookrightarrow \mathbb{R}^3$ and the right hand side is given by

$$f(u_\epsilon) := -(1 - |u_\epsilon|^2) u_\epsilon = \frac{1}{4} \frac{d}{du} \big(1 - |u_\epsilon|^2\big)^2 .$$

For small $\epsilon > 0$ the maps u_ϵ then approximate the Landau-Lifshitz flow

$$\gamma_1 \partial_t u - \gamma_2 u \times \partial_t u - \Delta u = |\nabla u|^2 u \quad \text{in } \Omega \times \mathbb{R}_+\,. \tag{1.3}$$

For sufficiently regular solutions $u : \overline{\Omega} \times \mathbb{R} \to S^2$ equation (1.3) is equivalent to

$$\partial_t u = -\alpha u \times (u \times \Delta u) + \beta u \times \Delta u \text{ in } \Omega \times \mathbb{R}_+\,, \tag{1.4}$$

where $\alpha := \frac{\gamma_1}{\gamma_1^2 + \gamma_2^2} > 0$ and $\beta := \frac{\gamma_2}{\gamma_1^2 + \gamma_2^2} \in \mathbb{R}$. This is the usual form of the Landau-Lifshitz equations known in physics. (Compare [22] and [15], [16], [17].)

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Case (II): If $\gamma_2 = 0$, the target is a smooth, closed, isometrically embedded manifold $N \hookrightarrow \mathbb{R}^n$. For small $\epsilon > 0$, the map $u_\epsilon : \overline{\Omega} \times \mathbb{R}_+ \to \mathbb{R}^n$ will then be an approximation of a harmonic map flow (compare [36]) and is defined to be a solution of

$$\partial_t u_\epsilon - \Delta u_\epsilon = -\frac{1}{2\epsilon^2}\frac{d}{du}\chi\big(\mathrm{dist}^2(u_\epsilon, N)\big) \quad \text{in } \Omega \times \mathbb{R}_+ \tag{1.5}$$

$$u_\epsilon = u_0 \quad \text{on } \big(\Omega \times \{0\}\big) \cup \big(\partial\Omega \times \mathbb{R}_+\big)\,, \tag{1.6}$$

That is, for the function $f(u_\epsilon)$ in (1.1), we choose

$$f(u_\epsilon) := \frac{1}{2}\frac{d}{du}\chi\big(\mathrm{dist}^2(u_\epsilon, N)\big)\,,$$

The cut-off function $\chi : \mathbb{R}_+ \to \mathbb{R}_+$ is smooth, non decreasing and satisfies $\chi(t) = t$ for $0 \le t \le \delta_N^2$ and $\chi(t) \equiv 2\delta_N^2$ for $t \ge 4\delta_N^2$. The parameter $\delta_N > 0$ is chosen in such a way that the nearest neighbour projection $U \ni x \mapsto \pi_N(x) \in N$ is defined and smooth on a tubular neighborhood $U \subset \mathbb{R}^n$ of N with uniform radius radius $2\delta_N > 0$. (Such a $\delta_N > 0$ always exists if N is closed. Compare [36] and [29, Section 2.12.3 p.42].)

For fixed $\epsilon > 0$, smooth solutions of (1.1)-(1.2) or (1.5)-(1.6) on $\Omega \times \mathbb{R}_+$ exist and if $u_0 \in H^{1,2}(\Omega; N) \cap H^{3/2,2}(\partial\Omega; N)$, they are unique in

$$H^{1,2}_{\mathrm{loc}} \cap L^\infty(H^{1,2}) := H^{1,2}_{\mathrm{loc}}(\overline{\Omega} \times \mathbb{R}_+; \mathbb{R}^n) \cap L^\infty(\mathbb{R}_+; H^{1,2}(\Omega; \mathbb{R}^n))\,.$$

(Compare [3],[36].) Existence is obtained by Galerkin's method, regularity (C^∞) follows from a standard bootstrap argument and uniqueness may be proven as for the two dimensional harmonic map flow (see [30] or [31] (5°) p.234 in the proof of Theorem 6.6).

The total energy of the flow at time $t \ge 0$ is defined by

$$G_\epsilon\big(u_\epsilon(t)\big) := \int_\Omega g_\epsilon(u_\epsilon)(x,t)dx \tag{1.7}$$

where

$$g_\epsilon(u_\epsilon) := \frac{1}{2}|\nabla u_\epsilon|^2 + \frac{1}{4\epsilon^2}(1 - |u_\epsilon|^2)^2 \quad \text{if } \gamma_2 \ne 0\,,$$

$$g_\epsilon(u_\epsilon) := \frac{1}{2}|\nabla u_\epsilon|^2 + \frac{1}{2\epsilon^2}\chi\big(\mathrm{dist}^2(u_\epsilon, N)\big) \quad \text{if } \gamma_2 = 0\,.$$

While the total energy of the "ϵ-approximations" always decreases (see Lemma 2.1 below), the local energy given by

$$G_\epsilon\big(u_\epsilon(t)\,, B_R^\Omega(x_0)\big) := \int_{B_R(x_0)\cap\Omega} g_\epsilon(u_\epsilon)(x,t)\,dx\,. \tag{1.8}$$

may concentrate at space-time points (x_0, t_0) as $\epsilon \searrow 0$ either for fixed $t = t_0$ or for variable $t \nearrow t_0$ or $t \searrow t_0$. It characterizes the local "asymptotic regularity behaviour" of the flow. Here asymptotic refers to the limit $\epsilon \searrow 0$.

We will show that all the derivatives of the family of maps $\{u_\epsilon\}_{\epsilon>0}$ are locally uniformly bounded on a regular set $\mathrm{Reg}\big(\{u_\epsilon\}_{\epsilon>0}\big)$ consisting of all points

$$z_0 = (x_0, t_0) \in \overline{\Omega} \times]0, \infty[$$

for which there is $R_0 = R_0(z_0) > 0$, such that

$$\limsup_{\epsilon\searrow 0} \sup_{t_0 - R_0^2 < t < t_0} G_\epsilon\big(u_\epsilon(t)\,, B_{R_0}^\Omega(x_0)\big) < \epsilon_0\,, \tag{1.9}$$

for a constant $\epsilon_0 > 0$ that will be determined later. The complement

$$\mathbb{S}\big(\{u_\epsilon\}_{\epsilon>0}\big) := \big(\overline{\Omega} \times \mathbb{R}_+\big) \setminus \mathrm{Reg}\big(\{u_\epsilon\}_{\epsilon>0}\big)$$

is referred to as the energy-concentration set. It is closed, has locally finite parabolic Hausdorff dimension two and finite slices at fixed time. The limits of converging subsequences $\{u_{\epsilon_j}\}_j$ are distributional solutions of the Landau-Lifshitz flow (or harmonic map flow if $\gamma_2 = 0$) on all $\Omega\times]0,\infty[$ in $H^{1,2}_{\mathrm{loc}} \cap L^\infty(H^{1,2})$.

Bubbling phenomena of the ϵ-approximations as $\epsilon \searrow 0$ either for fixed $t = t_0$ or for variable $t \nearrow t_0$ or $t \searrow t_0$ as described in [16] will be presented in [18]. Strong subconvergence of the harmonic map flow penalty-approximations in

$$W^{1,0}_{2,\mathrm{loc}}\Big(\mathrm{Reg}(\{u_\epsilon\}_{\epsilon>0}); \mathbb{R}^n\Big)$$

to a global distributional $H^{1,2}_{\mathrm{loc}} \cap L^\infty(H^{1,2})$-solution of the harmonic map flow was already proved by M. Struwe and Y. Chen in [36] for the case of a closed domain manifold $\Omega = M$ with $\dim M = m \geq 2$ or $M = \mathbb{R}^m$. ($W^{1,0}_{2,\mathrm{loc}}$ refers to functions f, whose restriction to any closed ball (in space-time) lies in L^2 as well as the restriction of the space-gradient ∇f.)

Struwe and Chen provided uniform local L^∞-bounds for $g_\epsilon(u_\epsilon)$ on $\mathrm{Reg}(\{u_\epsilon\}_{\epsilon>0})$. Their result was extended to compact domains with boundary by Chen and Lin in [7]. The energy-concentration set $\mathbb{S}(\{u_\epsilon\}_{\epsilon>0})$ is known to have locally finite $m(=\dim M)$-dimensional Hausdorff measure in the case of the harmonic map flow (see [36]).

X. Cheng investigates in [8] weak(*) $H^{1,2}_{\mathrm{loc}} \cap L^\infty(H^{1,2})$-limits u_* of sequences of smooth solutions of the harmonic map flow on the domain $M = \mathbb{R}^m$ and shows that the time slice $\mathbb{S}(\{u_k\}_k) \cap \big(\mathbb{R}^m \times \{t\}\big)$ has finite $(m-2)$-dimensional Hausdorff measure.

Weak(*)-subconvergence in $H^{1,2}_{\mathrm{loc}} \cap L^\infty(H^{1,2})$ of the Landau-Lifshitz ϵ-approximations from closed surfaces to a distributional $H^{1,2}_{\mathrm{loc}} \cap L^\infty(H^{1,2})$-solution of the Landau-Lifshitz flow was proven by B. Guo and M.C. Hong in [15].

Guo and Ding also studied partial convergence of the two dimensional Landau-Lifshitz penalty-approximations in [9],[10] and [11]. Their arguments however contain several gaps and inconsistencies.

2. Energy-estimates

In the case $\gamma_2 = 0$, equation (1.5) is the L^2-gradient flow of the functional $u \mapsto G_\epsilon(u)$. (1.1) is not known to be a gradient flow, but the total energy still decreases along the (smooth) flow (1.1)-(1.2).

Lemma 2.1. *Let u_ϵ be a solution of (1.1)-(1.2). Then*

$$G_\epsilon\big(u_\epsilon(T)\big) + \gamma_1 \int_0^T \int_\Omega |\partial_t u_\epsilon|^2 dx dt = G_\epsilon\big(u_\epsilon(0)\big) = E(u_0) =: E_0\,, \tag{2.1}$$

$$G_\epsilon\big(u_\epsilon(T_2), B^\Omega_R(x_0)\big) \leq G_\epsilon\big(u_\epsilon(T_1), B^\Omega_{2R}(x_0)\big) + \frac{C}{\gamma_1 R^2} \int_{T_1}^{T_2} G_\epsilon\big(u_\epsilon(t), B^\Omega_{2R}(x_0)\big) dt, \tag{2.2}$$

for $0 \le T_1 < T_2$. Also for all $\eta > 0$, there exist $T_0 > 0$ and $R_0 > 0$, such that for all $x_0 \in \Omega$ and all $\epsilon > 0$ we have

$$\sup_{0 \le t \le T_0} G_\epsilon\big(u_\epsilon(t), B^\Omega_{R_0}(x_0)\big) \le \eta\,. \tag{2.3}$$

Proof. Inequality (2.1) is obtained by multiplying (1.1) with $\partial_t u_\epsilon$. Inequality (2.2) follows by multiplying (1.1) with $\partial_t u_\epsilon \phi^2$ for an adequate cut-off function ϕ and then integrating by parts and absorbing. Note that $\partial_t u_\epsilon \equiv 0$ on $\partial\Omega \times \mathbb{R}_+$. Inequality (2.3) follows from (2.2), if we set $T_1 = 0$ and $T_2 = T_0 = \frac{\gamma_1 R_0^2}{2CE_0}$ for sufficiently small $R_0 > 0$, such that $G_\epsilon\big(u_\epsilon(0), B^\Omega_{R_0}(x_0)\big) = E\big(u_0, B^\Omega_{R_0}(x_0)\big) < \eta/2$. □

The energy estimates imply the penalty-approximations subconverge weak(*) in $H^{1,2}_{\rm loc}(\overline{\Omega} \times \mathbb{R}_+; \mathbb{R}^n) \cap L^\infty\big(\mathbb{R}_+; H^{1,2}(\Omega; \mathbb{R}^n)\big)$. This was already pointed out by B.Guo and M.C.Hong in section 4 of [15].

3. Partial compactness

In this section we show that, under the uniform smallness condition (1.9) on the local energy, all higher derivatives of u_ϵ are locally and uniformly bounded. Here "uniform" of course always means uniform in $\epsilon > 0$. In Section 3.1, estimates for linear parabolic systems that can be applied to (1.1) as soon as ∇u_ϵ is locally bounded are recalled. In Section 3.2 we show that ∇u_ϵ is necessarily locally uniformly bounded, whenever (1.9) holds. In Section 3.3 we derive estimates that will provide bounds for the right hand side of (1.1) and allow to combine the previous estimates into a bootstrap argument.

3.1. Some "standard" parabolic estimates.

Equation (1.1) may be written as

$$L_\epsilon(u_\epsilon) := \partial_t u_\epsilon - M(u_\epsilon)\Delta u_\epsilon = -\frac{1}{\epsilon^2} M(u_\epsilon) f(u_\epsilon) = f_\epsilon(u_\epsilon)\,. \tag{3.1}$$

The coefficient-matrix $M(u)$ is smooth with respect to u and also strictly elliptic:

$$\frac{\gamma_1}{\gamma_1^2 + \gamma_1^2}|\xi|^2 < \xi^T M(u)\xi = \frac{1}{\gamma_1(\gamma_1^2 + \gamma_2^2|u|^2)}\Big(\gamma_1^2|\xi|^2 + \gamma_2^2(u\cdot\xi)^2\Big) < \frac{1}{\gamma_1}|\xi|^2,$$

for all $\xi \in \mathbb{R}^3$ (See [10, p.12], [15, p.316], [9, p.37]). Note that for $\gamma_2 = 0$, we obtain $M(u) = \frac{1}{\gamma_1} Id$. The results of this section are indeed merely interesting in the case $\gamma_2 \ne 0$, where the left hand side of (1.1) is non-linear. We will therefore restrict ourselves to the case $\gamma_2 \ne 0$.

For fixed $\epsilon > 0$, the solution u_ϵ of (1.1)-(1.2) is smooth and in particular continuous. u_ϵ has the same regularity up to the boundary as the boundary data u_0. The family of solutions $\{u_\epsilon\}_{\epsilon>0}$ is also uniformly bounded in $\epsilon > 0$, since $|u_\epsilon(x,t)| \le 1$ $\forall x, t$. This follows from the Maximum Principle applied to the equation obtained by multiplying (1.1) with $(1 - |u_\epsilon|)$. L_ϵ defines a strongly parabolic system in the sense of Petrovskii (Definition 2, p.599 in [21]) but satisfies as well all the other (not necessarily equivalent) definitions of strong parabolicity for general linear parabolic systems (Definitions 3-6) in [21]. The boundary-data operators also fullfill the required conditions.

First we have estimates in the $W^{2,1}_p$-Sobolev spaces with $p > 1$ (see [21] Chapter IV, Theorem 9.10 p.342 and (10.12) p.355 but also Chapter VII, Theorem 10.4

p.621, for the generalization to parabolic systems). Let $f_\epsilon \in L^p(\Omega\times[0,T];\mathbb{R}^n)$ and $u_0 \in H^{2,p}(\Omega;\mathbb{R}^n)$. Then for any $\delta \in]0,1[$, $p > 3/2$ and for $t_0 - R^2 > 0$ a solution of

$$L_\epsilon(v) = f_\epsilon \text{ in } \Omega\times]0,T[\quad\text{and}\quad v = u_0 \text{ on } \big(\Omega\times\{0\}\big)\cup\big(\partial\Omega\times]0,T[\big)$$

satisfies

$$\|v\|_{W_p^{2,1}(\Omega\times[0,T])} \le C_p(\Omega,T,\omega_{u_\epsilon})\big(\|f_\epsilon\|_{L^p(\Omega\times[0,T])} + \|u_0\|_{H^{2,p}(\Omega)}\big), \tag{3.2}$$

$$\begin{aligned}\|v\|_{W_p^{2,1}(P^\Omega_{\delta R}(z_0))} \le \tilde{C}_p(R,\delta,\Omega,\omega_{u_\epsilon})\Big(&\|f_\epsilon\|_{L^p(P^\Omega_R(z_0))} + \|v\|_{L^q(P^\Omega_R(z_0))}\\ &+ \delta_{B_R\cap\partial\Omega}\|u_0\|_{H^{2-(1/p),p}(B^\Omega_R\cap\partial\Omega(z_0))}\Big),\end{aligned} \tag{3.3}$$

with $1 \le q \le p$. Here

$$P^\Omega_R(z_0) := (B_R(x_0)\times]t_0-R^2,t_0[)\cap(\Omega\times]0,\infty[)$$

and $\delta_{B_R\cap\partial\Omega} = 1$ if $B_R\cap\partial\Omega \ne \emptyset$ and 0 otherwise. The trace theorems of course imply

$$\|u_0\|_{H^{2-(1/p),p}(\partial\Omega)} \le \|u_0\|_{H^{2,p}(\Omega)}\,.$$

The constants C_p and $\tilde{C}_p$ depend on the indicated quantities and additionally on the uniform lower and upper bounds for the eigenvalues of $M(u_\epsilon)$, which may be chosen independent of $\epsilon > 0$. Note that the constants $C_p, \tilde{C}_p$ also depend on the moduli of continuity of the coefficients of the leading term, i.e. the modulus of continuity ω_{u_ϵ} of u_ϵ. The equation can also be written in divergence form,

$$L_\epsilon(v) := \partial_t v - div\big(M(u_\epsilon)\nabla v\big) + \big(DM(u_\epsilon)\partial_k u_\epsilon\big)\partial_k v = f_\epsilon\,.$$

If we assume in addition

$$\limsup_{\epsilon\searrow 0}\sup_{P^\Omega_R}|\nabla u_\epsilon| < \infty\,, \tag{3.4}$$

then estimates for equations in divergence form imply $v \in C^{\gamma,(\gamma/2)}(P^\Omega_{\delta R};\mathbb{R}^n)$ for some $\gamma \in]0,1[$ and any $\delta \in]0,1[$. (See [21] Chapter VII Theorem 3.1 p.582 or Chapter V, Theorem 1.1, p.419.) Indeed if the right hand side $f_\epsilon \in L^p(P^\Omega_R;\mathbb{R}^n)$ with $p > 2$, the following estimate for the mixed Hölder-norm of v on $P^\Omega_{\delta R}$ holds

$$\|v\|_{C^{\gamma,\gamma/2}(P^\Omega_{\delta R})} \le C(f_\epsilon)\,. \tag{3.5}$$

(See [21] p.7 for the definition of the mixed Hölder-spaces denoted there by $H^{\gamma,\gamma/2}$) The bound $C(f_\epsilon)$ depends on the parabolicity constants, on $0<\delta<1$, $\sup_{P^\Omega_R}|u_\epsilon|$, $\|f_\epsilon\|_{L^p(P^\Omega_R)}$, bounds for the coefficients of the equations depending on $\sup_{P^\Omega_R}|\nabla u_\epsilon|$ and also on $\|u_0\|_{C^\gamma(B_R\cap\partial\Omega)}$ if $B_R\cap\partial\Omega\ne\emptyset$.

Therefore, if (3.4) holds and $\|f_\epsilon\|_{L^p(P^\Omega_R)}$ or $\sup_{P^\Omega_R}|f_\epsilon|$ are uniformly bounded with respect to $\epsilon > 0$, then estimate (3.5) holds for u_ϵ and is uniform in $\epsilon > 0$. Now the modulus of continuity of u_ϵ on $P^\Omega_{\delta R}$ is bounded from above (by an increasing function h with $\lim_{t\searrow 0}h(t) = 0$) independently of $\epsilon > 0$. We gain uniform bounds for the modulus of continuity of u_ϵ with respect to $t \ge 0$ and estimate (3.3) is now uniform in $\epsilon > 0$.

Further by Lemma 3.3 p.80 in Chapter II of [21] for $p > m+2(=4)$ (m being the dimension of the spatial domain, in our case $m = 2$), we have

$$\|\nabla v\|_{C^\lambda(P^\Omega_R)} \le C(m,p,\lambda,\Omega)\|v\|_{W_p^{2,1}(P^\Omega_R)} \quad \text{for } \lambda = 1-(m+2)/p\,.$$

Also if (3.4) holds and $\|f_\epsilon\|_{L^p(P_R^\Omega)}$ is uniformly bounded, then (3.3) yields ϵ-uniform estimates for $\|\nabla u_\epsilon\|_{C^\lambda(P_{\delta R}^\Omega)}$.

3.2. **The main sup-estimates for the energy-density.**

3.2.1. *An interior sup-estimate for the Landau-Lifshitz-flow approximations.* In this section we derive an interior sup-estimate for the energy density in the case $\gamma_2 \neq 0$, but the proof also works if $\gamma_2 = 0$ and the target is N. The proof of the interior estimate is much simpler than in the boundary case and we therefore consider each case separately. The estimate will result from a scaling argument combined to the following higher estimates, that will be proven in the next section. Let

$$P_R(z_0) := B_R(x_0) \times]t_0 - R^2, t_0[\quad \text{for } z_0 = (x_0, t_0)\,.$$

Lemma 3.1. *Let u_ϵ be a solution of (1.1) for each $\epsilon > 0$. Assume*

$$\limsup_{\epsilon \searrow 0} \sup_{P_R(z_0)} g_\epsilon(u_\epsilon) \leq C_0$$

and $B_R(x_0) \subset \Omega$, $0 < R^2 < t_0$. Then for any $0 < \delta < 1$,

$$\limsup_{\epsilon \searrow 0} \|u_\epsilon\|_{C^k(P_{\delta R}(z_0))} \leq C_k \quad \text{and} \quad \limsup_{\epsilon \searrow 0} \|\frac{1}{\epsilon^2}(1 - |u_\epsilon|^2)\|_{C^k(P_{\delta R}(z_0))} \leq \tilde{C}_k$$

for all $k \geq 0$. The constants $C_k, \tilde{C}_k$ depend on $C_0, k, R, \delta > 0$. If $\gamma_2 = 0$ and the target is N, they also depend on the geometry of N (i.e. the metric on N and its derivatives).

We will now prove the following "ϵ_1-regularity" result.

Theorem 3.2. *There are constants $C_1 = C_1(N), \epsilon_1 = \epsilon_1(N) > 0$, such that if, for some $0 < R_0 < \min\{1, \sqrt{t_0}\}$ and $x_0 \in \Omega$ with $B_{R_0}(x_0) \subset \Omega$, a solution u_ϵ of (1.1) satisfies*

$$\sup_{t_0 - R_0^2 < t < t_0} \int_{B_{R_0}(x_0)} g_\epsilon(u_\epsilon)(x,t)dx < \epsilon_1\,,$$

then

$$\sup_{P_{\delta R_0}(z_0)} g_\epsilon(u_\epsilon) \leq \frac{C_1}{(1-\delta)^2 R_0^2}$$

for any $\delta \in]0,1[$.

In the proof we would like to consider points $z_\epsilon = (x_\epsilon, t_\epsilon) \in \overline{P_R}(z_0)$ such that $g_\epsilon(z_\epsilon) = \sup_{P_R(z_0)} g_\epsilon$. Difficulties however arise if $z_\epsilon \in \partial P_R(z_0)$, since we then do not have uniform estimates on a neighborhood of z_ϵ. This is elegantly avoided by considering

$$\max_{0 \leq \sigma \leq R_0} \left((R_0 - \sigma)^2 \sup_{P_\sigma} g_\epsilon\right).$$

This trick is initially due to R. Schoen. (See [28], proof of Theorem 2.2. Schoen's method was extended to the parabolic context in [32], [36].)

Proof of Theorem 3.2. Without loss of generality, let $(x_0, t_0) = 0$. We set $P_R := P_R(0)$. Since u_ϵ is regular, there is some $\sigma_\epsilon \in [0, R_0[$ such that

$$(R_0 - \sigma_\epsilon)^2 \sup_{P_{\sigma_\epsilon}} g_\epsilon = \max_{0 \leq \sigma \leq R_0} \Big((R_0 - \sigma)^2 \sup_{P_\sigma} g_\epsilon\Big)\,.$$

Moreover, there is some $z_\epsilon = (x_\epsilon, t_\epsilon) \in \overline{P_{\sigma_\epsilon}}$, such that $e_\epsilon = g_\epsilon(u_\epsilon(z_\epsilon)) = \sup_{P_{\sigma_\epsilon}} g_\epsilon$. Set $\rho_\epsilon := \frac{1}{2}(R_0 - \sigma_\epsilon)$. Since $P_{\rho_\epsilon}(z_\epsilon) \subset P_{\sigma_\epsilon+\rho_\epsilon} \subset P_{R_0}$, we have

$$\begin{aligned}\sup_{P_{\rho_\epsilon}(z_\epsilon)} g_\epsilon &\le \frac{1}{(R_0-(\sigma_\epsilon+\rho_\epsilon))^2}(R_0-(\sigma_\epsilon+\rho_\epsilon))^2 \sup_{P_{\rho_\epsilon+\sigma_\epsilon}} g_\epsilon \\ &\le \frac{4}{(R_0-\sigma_\epsilon)^2}(R_0-\sigma_\epsilon)^2 e_\epsilon \le 4e_\epsilon\,.\end{aligned}$$

Set $r_\epsilon := \sqrt{e_\epsilon}\rho_\epsilon$ and consider the rescaled map

$$v_\epsilon(y,s) := u(x_\epsilon + e_\epsilon^{-1/2}y, t_\epsilon + e_\epsilon{}^{-1}s) \quad \text{for } (y,s) \in P_{r_\epsilon}\,.$$

By definition v_ϵ satisfies (1.1) on P_{r_ϵ} with $\tilde\epsilon := \sqrt{e_\epsilon}\epsilon$ instead of ϵ and

$$g_{\sqrt{e_\epsilon}\epsilon}(v_\epsilon)(0,0) = 1\,, \quad \sup_{P_{r_\epsilon}} g_{\sqrt{e_\epsilon}\epsilon}(v_\epsilon) \le 4\,.$$

Now we claim $r_\epsilon \le 2$. This will prove the theorem, since by definition of r_ϵ, we then have $(R_0 - \sigma_\epsilon)^2 e_\epsilon \le 16$.

Assume $r_\epsilon > 2$. Since $B_{R_0}(x_0) \subset \Omega$, all the higher derivatives of v_ϵ are then bounded on P_1 independently of $\epsilon > 0$. Indeed if $\liminf_{\epsilon\searrow 0}\sqrt{e_\epsilon}\epsilon > 0$, the uniform estimates are immediate and if $\liminf_{\epsilon\searrow 0}\sqrt{e_\epsilon}\epsilon = 0$, they follow from Lemma 3.1. In particular

$$\sqrt{|\partial_t g_{\tilde\epsilon}(v_\epsilon)|}, |\nabla g_{\tilde\epsilon}(v_\epsilon)| \le C < \infty \text{ on } P_1 \text{ (uniformly in } \epsilon > 0)$$

and therefore,

$$\inf_{P_{r_0}} g_{\tilde\epsilon}(v_\epsilon) \ge \frac{1}{2} \quad \text{for } r_0 := \min\{\frac{1}{4C}, 1\}\,.$$

Note that C is an absolute constant in the sense that it merely depends on the radius 2, the factor $\delta = \frac{1}{2}$, the L^∞-bound 4 and the parabolicity constants and the geometry of N. This lower bound implies

$$\begin{aligned}1 = g_{\sqrt{e_\epsilon}\epsilon}(v_\epsilon)(0,0) &\le \frac{2}{\pi r_0^2}\sup_{-r_0^2<s<0}\int_{B_{r_0}} g_{\sqrt{e_\epsilon}\epsilon}(v_\epsilon)(y,s)\,dy \\ &\le C_*\sup_{t_\epsilon - r_0^2 e_\epsilon^{-1}<t<t_\epsilon}\int_{B_{e_\epsilon^{-1/2}r_0}(x_\epsilon)} g_\epsilon(u_\epsilon)(x,t)\,dx \\ &\le C_*\sup_{-(\frac{r_0^2}{e_\epsilon}+\sigma_\epsilon^2)<t<0}\int_{B_{\frac{r_0}{\sqrt{e_\epsilon}}+\sigma_\epsilon}(x_0)} g_\epsilon(u_\epsilon)(x,t)\,dx\,.\end{aligned}$$

Set $\epsilon_1 := \min\{\frac{1}{2}, \frac{1}{2C_*}\}$. Since $r_\epsilon = \sqrt{e_\epsilon}\rho_\epsilon > 2 > r_0$, we have $\frac{r_0}{\sqrt{e_\epsilon}} + \sigma_\epsilon \le \rho_\epsilon + \sigma_\epsilon \le R_0$ and $(\frac{r_0}{\sqrt{e_\epsilon}})^2 + \sigma_\epsilon^2 \le (\rho_\epsilon+\sigma_\epsilon)^2 \le R_0^2$. Then the last estimate yields a contradiction, since the right hand side is smaller than $\epsilon_1 \le \frac{1}{2}$. Therefore $r_\epsilon = \sqrt{e_\epsilon}\rho_\epsilon \le 2$ and

$$(1-\delta)^2 R_0^2 \sup_{P_{\delta R_0}} g_\epsilon \le 16\,.$$

$\square$

3.2.2. *A local boundary sup-estimate for the energy density.* Local L^p-estimates for $\nabla^3 u_\epsilon$ up to the boundary which are uniform in $\epsilon > 0$ cannot be expected, even if $u_0 \in C^\infty(\Omega; S^2)$. Indeed for fixed $\epsilon > 0$, u_ϵ is smooth up to the boundary and we may thus evaluate (1.1) at $x \in \partial\Omega$ for any $t \geq 0$. This gives $\Delta u_\epsilon = 0$ on $\partial\Omega \times \mathbb{R}_+$. As we will see later, uniform estimates imply the existence of a subsequence u_{ϵ_i} converging to a map u_*, which is a smooth solution of the Landau-Lifshitz or harmonic map flow in $\mathrm{Reg}(\{u_{\epsilon_i}\})$ and satisfies

$$-\Delta u_* = |\nabla u_*|^2 u_* \quad \text{ on } (\partial\Omega \times \mathbb{R}_+) \cap \mathrm{Reg}(\{u_{\epsilon_i}\}),$$

since $\partial_t u_* = 0$ on $\partial\Omega \times \mathbb{R}_+$. However L^p_{loc}-estimates for $\nabla^3 u_\epsilon$ would imply

$$0 = \Delta u_{\epsilon_i} \to \Delta u_* \text{ in } L^p_{\mathrm{loc}}((\partial\Omega \times \mathbb{R}_+) \cap \mathrm{Reg}(\{u_{\epsilon_i}\}); \mathbb{R}^3)$$

by compactness of the "projection" $H^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$. This is not possible unless $u_* \equiv const.$ on $(\partial\Omega \times \mathbb{R}_+) \cap \mathrm{Reg}(\{u_{\epsilon_i}\})$. (Compare [1], Remark 1 p.125 for a similar argument in the time independent case.)

The following lemma will be proven in Section 3.3.

Lemma 3.3. *Let u_ϵ be a solution of (1.1)-(1.2), with $u_0 \in H^{1,2}(\Omega; S^2) \cap H^{2,p}(\partial\Omega; S^2)$ and $p \geq 2$ for each $\epsilon > 0$. Assume*

$$\sup_{P^\Omega_R(z_0)} g_\epsilon \leq C_0$$

and $B_R(x_0) \cap \partial\Omega \neq \emptyset$, $0 < R^2 < t_0$. Then for any $\delta \in]0,1[$, we have

$$\begin{aligned} &\|u_\epsilon\|_{W^{2,1}_p(P^\Omega_{\delta R}(z_0))} \\ &\leq C_1\Big(\|\frac{1}{\epsilon^2}(1-|u_\epsilon|^2)\|_{L^p(P^\Omega_R(z_0))} + \|u_\epsilon\|_{L^2(P^\Omega_R(z_0))} + \|u_0\|_{H^{2-(1/p),p}(B^\Omega_R(z_0)\cap\partial\Omega)}\Big), \end{aligned}$$

where the constant C_1 depends on C_0, p, R, δ and Ω. Further we have for any $\delta \in]0,1[$,

$$\|\frac{1}{\epsilon^2}(1-|u_\epsilon|^2)\|_{L^p(P^\Omega_{\delta R}(z_0))} \leq C(p)\|g_\epsilon\|_{L^p(P^\Omega_R)} + \epsilon^{2/p} C\big(\|g_\epsilon\|_{L^p(P^\Omega_R)}, p, \delta, R\big),$$

$$\|\frac{1}{\epsilon^2}(1-|u_\epsilon|^2)\|_{L^\infty(P^\Omega_{\delta R}(z_0))} \leq 8C_0 + o_\delta(\epsilon),$$

where $\epsilon \mapsto o_\delta(\epsilon)$ is a function that depends on $\delta \in]0,1[$ and $\lim_{\epsilon\searrow 0} \epsilon^{-k} o_\delta(\epsilon) = 0$ for all $k \in \mathbb{N}$. All the constants also depend on the parabolicity constants. If $\gamma_2 = 0$ and the target is N, they also depend on the geometry of N.

We now prove the following result.

Theorem 3.4. *Consider $u_0 \in H^{1,2}(\Omega; S^2) \cap C^2(\partial\Omega; S^2)$. Let u_ϵ be a solution of (1.1)-(1.2) for each $\epsilon > 0$. There are constants $C_0 = C_0(\|u_0\|_{C^2(\partial\Omega)}, E_0, \Omega)$ and $\epsilon_0 = \epsilon_0(\|u_0\|_{C^2(\partial\Omega)}, E_0, \Omega) > 0$, such that if for some $z_0 = (x_0, t_0)$ and $R_0 \in]0, \min\{1, \sqrt{t_0}\}[$*

$$\limsup_{\epsilon\searrow 0} \sup_{t_0 - R_0^2 < t < t_0} \int_{B_{R_0}(x_0)\cap\Omega} g_\epsilon(u_\epsilon)dx < \epsilon_0\,,$$

then

$$\limsup_{\epsilon\searrow 0} \sup_{P^\Omega_{\delta R_0}(z_0)} g_\epsilon(u_\epsilon) \leq \frac{C_0}{(1-\delta)^2 R_0^2},$$

for any $\delta \in]0,1[$.

If the target is N, the above constants C_0 and ϵ_0 also depend on the geometry of N.

Proof. Without loss of generality let $(x_0,t_0)=0$. Since $u_0\in C^2(\partial\Omega)$ admits an extension $w_0\in C^2(\overline{\Omega})$ and since $t_0-R^2>0$, we may assume $u_0\in C^2(\overline{\Omega})$. We have $u_\epsilon\in C^{1,0}_\alpha(\overline{\Omega}\times\mathbb{R}_+;S^2)$ for any $0<\alpha<1$ and so there are $\sigma_\epsilon\in[0,R_0[$ and $z_\epsilon=(x_\epsilon,t_\epsilon)\in\overline{P^\Omega_{\sigma_\epsilon}}$, such that

$$(R_0-\sigma_\epsilon)^2\sup_{P^\Omega_{\sigma_\epsilon}}g_\epsilon=\max_{0\le\sigma\le R_0}\Big((R_0-\sigma)^2\sup_{P^\Omega_\sigma}g_\epsilon\Big),$$
$$e_\epsilon=g_\epsilon(u_\epsilon(z_\epsilon))=\sup_{P^\Omega_{\sigma_\epsilon}}g_\epsilon\,.$$

Again for $\rho_\epsilon:=\frac12(R_0-\sigma_\epsilon)$, we have $\sup_{P^\Omega_{\rho_\epsilon}(z_\epsilon)}g_\epsilon\le4e_\epsilon$. Consider the rescaled map

$$v_\epsilon(y,s):=u(x_\epsilon+e_\epsilon^{-1/2}y,t_\epsilon+e_\epsilon^{-1}s)\,.$$

By construction v_ϵ satisfies

$$\gamma_1\partial_tv_\epsilon-\gamma_2v_\epsilon\times\partial_tv_\epsilon-\Delta v_\epsilon=\frac{1}{\tilde\epsilon^2}(1-|v_\epsilon|^2)v_\epsilon \text{ on } P^{\Omega_\epsilon}_{r_\epsilon}\,, \tag{3.6}$$

with $\tilde\epsilon:=\sqrt{e_\epsilon}\epsilon$, $r_\epsilon:=\sqrt{e_\epsilon}\rho_\epsilon$, $\Omega_\epsilon:=\sqrt{e_\epsilon}\big(\Omega-x_\epsilon\big)$ and

$$P^{\Omega_\epsilon}_{r_\epsilon}:=\big(B_{r_\epsilon}\cap\Omega_\epsilon\big)\times]-r_\epsilon^2,0[\,.$$

Further by construction,

$$g_{\tilde\epsilon}(v_\epsilon)(0,0)=1 \text{ and } \sup_{P^{\Omega_\epsilon}_{r_\epsilon}}g_{\tilde\epsilon}(v_\epsilon)\le4\,. \tag{3.7}$$

The boundary data are also rescaled. Set $v_{\epsilon,0}(y):=u_0(x_\epsilon+e_\epsilon^{-1/2}y)$. Then

$$v_\epsilon(y,s)=v_{\epsilon,0}(y) \text{ on } (\partial\Omega_\epsilon\cap B_{r_\epsilon})\times]-r_\epsilon^2,0[$$

and

$$\sup_{P^{\Omega_\epsilon}_{r_\epsilon}}|\nabla v_{\epsilon,0}|\le e_\epsilon^{-1/2}\sup_{P^\Omega_{R_0}}|\nabla u_0|\,,\quad \sup_{P^{\Omega_\epsilon}_{r_\epsilon}}|\nabla^2v_{\epsilon,0}|\le e_\epsilon^{-1}\sup_{P^\Omega_{R_0}}|\nabla^2u_0|\,.$$

Now we claim that for sufficiently small $\epsilon>0$, we have

$$r_\epsilon\le C_0:=\max\{2,\tilde C(\Omega,\|u_0\|_{C^2(\overline{\Omega})})\},$$

where $\tilde C(\cdot)>0$ will be specified later. Again by definition of r_ϵ, this will prove the theorem.

Assume by contradiction $r_\epsilon>C_0\ge2$ for small $\epsilon>0$. Then

$$e_\epsilon^{-1/2}=\rho_\epsilon/r_\epsilon<R_0/(2C_0)\le1/(2C_0),$$

since $0<R_0<1$. First we claim that

$$\liminf_{\epsilon\searrow0}\sqrt{e_\epsilon}\epsilon=\liminf_{\epsilon\searrow0}\tilde\epsilon(\epsilon)=0\,.$$

Indeed if $\liminf_{\epsilon\searrow0}\sqrt{e_\epsilon}\epsilon>0$, the right hand side of (3.6) is uniformly bounded in $\tilde\epsilon=\sqrt{e_\epsilon}\epsilon>0$ and together with (3.7) we obtain uniform bounds in $C^\infty(P^{\Omega_\epsilon}_2)$. This however leads to a contradiction as in the proof of Theorem 3.2, if ϵ_0 is smaller than ϵ_1. Further if

$$\limsup_{\epsilon\searrow0}\big(\sqrt{e_\epsilon}\operatorname{dist}(x_\epsilon,\partial\Omega)\big)=\limsup_{\epsilon\searrow0}\operatorname{dist}(0,\partial\Omega_\epsilon)\ge\frac12\,,$$

we can also use uniform interior estimates in $C^\infty(P_{1/4}^{\Omega_\epsilon})$ and proceed as in the proof of Theorem 3.2 to get a contradiction, if we choose ϵ_0 sufficiently small. So far the required upper bound on ϵ_0 is universal in the sense that it only depends on the geometry of N and the parabolicity constants. We therefore have

$$\limsup_{\epsilon \searrow 0} \operatorname{dist}(0, \partial\Omega_\epsilon) < 1/2\,,$$

and in the sequel we consider sufficiently small $\epsilon > 0$, such that $\operatorname{dist}(0, \partial\Omega_\epsilon) < 1/2$.

Lemma 3.3 combined to the embedding $W_p^{2,1}(P_1) \hookrightarrow C^1(P_1)$ for $p > 4$, implies

$$\begin{aligned}
&\sup_{P_1^{\Omega_\epsilon}} |\nabla v_\epsilon|^2 \\
&\leq C\|v_\epsilon\|^2_{W_p^{2,1}(P_1^{\Omega_\epsilon})} \\
&\leq C(p, \Omega_\epsilon \cap B_2)\Big(\|\frac{1}{\tilde\epsilon^2}(1-|v_\epsilon|^2)\|^2_{L^p(P_2^{\Omega_\epsilon})} + \|v_\epsilon\|^2_{L^2(P_2^{\Omega_\epsilon})} + \|v_{\epsilon,0}\|^2_{H^{2,p}(P_2^{\Omega_\epsilon})}\Big)
\end{aligned}$$

Note that $\Omega_\epsilon \cap B_2$ has uniformly bounded curvature and so

$$0 < C(p, \Omega_\epsilon \cap B_2) < C(p, \Omega)\,.$$

Since $(1-|v_\epsilon|^2) \leq 1$ and $\sup_{P_2^{\Omega_\epsilon}} g_{\tilde\epsilon}(v_\epsilon) \leq 4$, Lemma 3.3 implies

$$\|\frac{1}{\tilde\epsilon^2}(1-|v_\epsilon|^2)\|^2_{L^p(P_2^{\Omega_\epsilon})} \leq C_p\big(o(\epsilon_0) + o(\tilde\epsilon)\big)\,,$$

where $C_p = C(p, E_0)$ and $o(\tau)$ denotes a generic function that satisfies

$$\lim_{\tau \searrow 0} o(\tau) = 0\,.$$

A Poincaré inequality on $P_2^{\Omega_\epsilon}$ leads to

$$\begin{aligned}
\|v_\epsilon\|^2_{L^2(P_2^{\Omega_\epsilon})} &\leq 2\big(\|v_{\epsilon,0}\|^2_{L^2(P_2^{\Omega_\epsilon})} + \|v_\epsilon - v_{\epsilon,0}\|^2_{L^2(P_2^{\Omega_\epsilon})}\big) \\
&\leq 2\|v_{\epsilon,0}\|^2_{L^2(P_2^{\Omega_\epsilon})} + C(\Omega)\big(\|\nabla v_{\epsilon,0}\|^2_{L^2(P_2^{\Omega_\epsilon})} + \|\nabla v_\epsilon\|^2_{L^2(P_2^{\Omega_\epsilon})}\big)\,.
\end{aligned}$$

Again $\|\nabla v_\epsilon\|^2_{L^2(P_2^{\Omega_\epsilon})} \leq o(\epsilon_0)$. Of course

$$\begin{aligned}
\|v_{\epsilon,0}\|_{H^{1,2}(B_2^{\Omega_\epsilon})} &\leq C(p)\|v_{\epsilon,0}\|_{H^{1,p}(B_2^{\Omega_\epsilon})} \\
&\leq C(p)\|v_{\epsilon,0}\|^2_{H^{2,p}(B_2^{\Omega_\epsilon})} \\
&\leq C(p,\Omega)\|v_{\epsilon,0}\|^2_{C^2(B_2^{\Omega_\epsilon})}
\end{aligned}$$

and we still need to estimate $\|v_{\epsilon,0}\|^2_{C^2(B_2^{\Omega_\epsilon})}$.

For each $\epsilon > 0$ we may chose coordinates for the target such that $v_{\epsilon,0}(0) = 0$. Then

$$\sup_{B_2^{\Omega_\epsilon}} |v_{\epsilon,0}| \leq 4 \sup_{B_2^{\Omega_\epsilon}} |\nabla v_{\epsilon,0}|\,,$$

$$\|v_{\epsilon,0}\|_{C^2(B_2^{\Omega_\epsilon})} \leq C e_\epsilon^{-1/2} \sup_{B_{R_0}^{\Omega}} |\nabla u_0| + e_\epsilon^{-1} \sup_{B_{R_0}^{\Omega}} |\nabla^2 u_0|\,.$$

The above estimates combined to the one for $\|\frac{1}{\tilde\epsilon^2}(1-|v_\epsilon|^2)\|_{L^\infty(P_1^{\Omega_\epsilon})}$ in Lemma 3.3 yield

$$\begin{aligned}1 &\le \sup_{P_1^{\Omega_\epsilon}} g_{\tilde\epsilon}(v_\epsilon)\\ &\le \sup_{P_1^{\Omega_\epsilon}} \frac{1}{2}|\nabla v_\epsilon|^2 + \tilde\epsilon^2 \sup_{P_1^{\Omega_\epsilon}}\big(\frac{1}{\tilde\epsilon^2}(1-|v_\epsilon|^2)\big)^2\\ &\le C_1\left(o(\epsilon_0)+o(\tilde\epsilon)+e_\epsilon^{-1}\|\nabla u_0\|^2_{C^1(B^\Omega_{R_0})}\right),\end{aligned}$$

where $C_1 = C_1(\Omega, E_0)$. Now if both $o(\epsilon_0) < (1/4)C_1^{-1}$ and $o(\tilde\epsilon) < (1/4)C_1^{-1}$, this leads to

$$e_\epsilon < 2C_1\|\nabla u_0\|^2_{C^1(B^\Omega_{R_0})},$$

which is in contradiction with $r_\epsilon > C_0 := \max\{2, 2C_1\|u_0\|_{C^2(\overline\Omega)}\}$ and $\sqrt{e_\epsilon} > 2C_0$. Thus $r_\epsilon \le C_0$ and by definition of r_ϵ also

$$\frac{1}{4}(R_0-\delta R_0)^2 \sup_{P^\Omega_{\delta R_0}} g_\epsilon \le C_0^2 = C(\Omega, E_0, \|u_0\|_{C^2(\overline\Omega)}).$$

Since $t_0 - R^2 > 0$, we could replace u_0 in the above by any $w_0 \in C^2(\overline\Omega)$ with $w_0 = u_0$ on $\partial\Omega \cap B_{R_0}$. Therefore the above constants merely depend on $\|u_0\|_{C^2(\partial\Omega)}$. □

3.3. **Higher estimates.** In this section, we prove Lemmata 3.1 and 3.3, for which the following uniform estimates will be needed.

3.3.1. *Uniform estimates in $\epsilon > 0$.* The "distance-to-the-target-function" $\rho_\epsilon := 1 - |u_\epsilon|^2$ satisfies

$$\gamma_1\partial_t\rho_\epsilon - \Delta\rho_\epsilon + \frac{2}{\epsilon^2}\rho_\epsilon = 2|\nabla u_\epsilon|^2 + \frac{2}{\epsilon^2}\rho_\epsilon^2\,. \tag{3.8}$$

Since $\gamma_1 > 0$, we may assume $\gamma_1 = 1$ without loss of generality. We will now derive uniform a priori estimates for this equation. Lemma 3.5 extends a comparision argument from [1] (Lemma 2, p.130) to the time dependent case and to non-positive solutions.

The parabolic boundary of $P_R := B_R(0)\times]-R^2, 0[$ is denoted as

$$\tilde\partial P_R := \big(B_R(0)\times\{-R^2\}\big)\cup\big(\partial B_R(0)\times[-R^2,0]\big)\,.$$

Lemma 3.5. *Let $a > 0$, $R \in]0, \frac14[$, $\epsilon \in]0,1[$ and $g \in C^0(\overline{P_R})$ with $\epsilon^2 \sup_{P_R}|g| \le a$. Let $f \in C^0(\overline{P_R}) \cap C^2(P_R)$ be a solution of*

$$\begin{gathered}\big(\partial_t f - \Delta f\big) + \frac{1}{\epsilon^2} f = g \quad \text{in } P_R\,,\\ |f| \le a \quad \text{on } \tilde\partial P_R\,.\end{gathered}$$

Then for any $\delta \in]0,1[$, we have

$$\frac{1}{\epsilon^2}|f| \le \sup_{P_R}|g| + \frac{2a}{\epsilon^2}e^{-\frac{1}{\epsilon}(1-\delta^2)^2R^4} \quad \text{on } P_{\delta R}\,.$$

Proof. Consider $\omega(x,t) = 2ae^{-\frac{1}{\epsilon}(R^2-|x|^2)(R^2+t)}$. Then

$$\begin{gathered}\epsilon^2\big(\partial_t\omega - \Delta\omega\big) + \omega > 0 \quad \text{in } P_R\,,\\ \omega = 2a \quad \text{on } \tilde\partial P_R\,.\end{gathered}$$

For $f_1 := f - \epsilon^2 \sup_{P_R}|g|$ and $f_2 := f + \epsilon^2 \sup_{P_R}|g|$, we have

$$|f_1| \le 2a \quad \text{and} \quad |f_2| \le 2a \text{ on } \tilde{\partial} P_R\,,$$

and hence

$$f_1 - \omega \le 0, \quad f_2 + \omega \ge 0 \quad \text{on } \tilde{\partial} P_R\,.$$

Moreover

$$\epsilon^2\big(\partial_t f_1 - \Delta f_1\big) + f_1 \le 0\,, \quad \epsilon^2\big(\partial_t f_2 - \Delta f_2\big) + f_2 \ge 0 \quad \text{in } P_R\,.$$

The Maximum Principle now implies $f_1 - \omega \le 0$ and $f_2 + \omega \ge 0$ on P_R, that is

$$-\omega - \epsilon^2 \sup_{P_R}|g| \le f \le \omega + \epsilon^2 \sup_{P_R}|g|\,.$$

□

The above lemma will yield interior estimates. If $B_R \cap \Omega \neq \emptyset$ and $f \equiv 0$ on $B_R \cap \partial\Omega$, we still obtain a local estimate up to the boundary, i.e. on $P^{\Omega}_{\delta R} = (B_{\delta R} \cap \Omega) \times] - (\delta R)^2, 0[$.

Corollary 3.6. *Consider a smooth domain $\Omega \subset \mathbb{R}^2$, $a > 0$, $R \in]0, \frac{1}{4}[$, $\epsilon \in]0, 1[$ and $g \in C^0(\overline{P^{\Omega}_R})$ with $\epsilon^2 \sup_{P_R}|g| \le a$. Let $f \in C^0(\overline{P^{\Omega}_R}) \cap C^2(P^{\Omega}_R)$ be a solution of*

$$\begin{gathered}\big(\partial_t f - \Delta f\big) + \frac{1}{\epsilon^2} f = g \quad \text{in } P^{\Omega}_R\,,\\ |f| \le a \quad \text{on } \tilde{\partial} P_R \cap \Omega\,,\\ f = 0 \quad \text{on } \partial\Omega \cap P_R\,.\end{gathered}$$

Then for any $\delta \in]0, 1[$, we have

$$\frac{1}{\epsilon^2}|f| \le \sup_{P^{\Omega}_R}|g| + \frac{2a}{\epsilon^2} e^{-\frac{1}{\epsilon}(1-\delta^2)^2 R^4} \quad \text{on } P^{\Omega}_{\delta R}\,.$$

The proof of Lemma 3.5 also applies in this case. The next interior-estimate-version of Lemma 3.5 deals with the case $B_R(x_0) \cap \Omega \neq \emptyset$ and $f \neq 0$ on $\partial B_R(x_0) \cap \Omega$. The estimate then also depends on $\operatorname{dist}(x, \partial\Omega)$. We formulate the following lemma in such a way that it readily extends to the case $\Omega = M$ is a manifold.

Corollary 3.7. *Let $U \subset \mathbb{R}^2$ be an open smooth neighborhood of 0 with $\operatorname{diam} U \le 1$ and set $P_{R,U} := U \times]-R^2, 0[$. Consider $a > 0$, $R \in]0, \frac{1}{4}[$, $\epsilon \in]0, \frac{1}{4}[$ and $g \in C^0(\overline{P_{R,U}})$ with $\epsilon^2 \sup_{P_{R,U}}|g| \le a$. Let $f \in C^0(\overline{P_{R,U}}) \cap C^2(P_{R,U})$ be a solution of*

$$\begin{gathered}\big(\partial_t f - \Delta f\big) + \frac{1}{\epsilon^2} f = g \quad \text{in } P_{R,U}\,,\\ |f| \le a \quad \text{on } \tilde{\partial} P_{R,U}\,.\end{gathered} \tag{3.9}$$

Then there is a constant $C = C(U) > 0$, such that for any $\delta \in]0, 1[$ we have

$$\frac{1}{\epsilon^2}|f(x,t)| \le \sup_{P_{R,U}}|g| + \frac{2a}{\epsilon^2} e^{-\frac{R^2}{C\epsilon}(1-\delta^2)\operatorname{dist}^2(x,\partial U)} \quad \text{on } P_{\delta R,U}\,.$$

Of course

$$\tilde{\partial} P_{R,U} := \big(U \times \{-R^2\}\big) \cup \big(\partial U \times [-R^2, 0]\big)\,.$$

Proof. Set $d(x) := \operatorname{dist}(x, \partial U)$, $C = C(U) := \max\{1, \|\Delta d^2\|_{L^\infty(U)}, \|\nabla d^2\|_{L^\infty(U)}\}$. Note that $d(x) \le 1$ on U since $\operatorname{diam} U \le 1$. We claim that

$$\omega(x,t) := 2ae^{-\frac{1}{C\epsilon} d^2(x)(R^2+t)}$$

is a supersolution of equation (3.9), if $0 < R < \frac{1}{4}$ and $0 < \epsilon < \frac{1}{4}$. Indeed

$$\begin{aligned}
\epsilon^2(\partial_t - \Delta)\omega + \omega &= \omega\Big[1 - \frac{\epsilon}{C} d^2 + \frac{\epsilon}{C}(R^2+t)\Delta d^2 - \frac{\epsilon}{C}\frac{1}{\epsilon C}(R^2+t)^2|\nabla d^2|^2\Big] \\
&\ge \omega\big[1 - \epsilon - \epsilon R^2 - R^4\big] \\
&\ge \frac{1}{4}\omega > 0 \quad \text{on } P_{R,U}\,.
\end{aligned}$$

The claim now follows just as in the proof of Lemma 3.5. □

We will also need a priori L^p-estimates for the above equation.

Lemma 3.8. *Consider a smooth domain $\Omega \subset \mathbb{R}^2$, $g \in L^1(\Omega\times]0,T[)$ and $\epsilon > 0$. Let $f \in C^1(\overline{\Omega} \times [0,T]) \cap C^2(\Omega\times]0,T[)$ be a solution of*

$$\begin{gathered}
\big(\partial_t f - \Delta f\big) + \frac{1}{\epsilon^2} f = g \quad \text{in } \Omega\times]0,T[\,, \\
f = 0 \quad \text{on } \Omega \times \{0\} \cup \partial\Omega\times]0,T[\,.
\end{gathered}$$

For $f \ge 0$, we only need to assume

$$\begin{gathered}
\big(\partial_t f - \Delta f\big) + \frac{1}{\epsilon^2} f \le g \quad \text{in } \Omega\times]0,T[\,, \\
f = 0 \quad \text{on } \Omega \times \{0\} \cup \partial\Omega\times]0,T[\,.
\end{gathered}$$

Then

$$\|\frac{1}{\epsilon^2} f\|_{L^1(\Omega\times]0,T[)} \le \|g\|_{L^1(\Omega\times]0,T[)}\,. \tag{3.10}$$

and for any $R, \rho > 0$ and $z_0 = (x_0, t_0) \in \Omega\times]0,T]$ with $R^2 + \rho^2 < t_0$,

$$\int_{P_R^\Omega(z_0)} \frac{1}{\epsilon^2}|f|dz \le \int_{P_{R+\rho}^\Omega(z_0)} \Big(|g| + \frac{C}{\rho^2}|f|\Big) dz\,. \tag{3.11}$$

Proof. (i) Multiplication of the equation for f by $\frac{f}{\sqrt{f^2+\delta^2}}$ leads to

$$\frac{\partial_t |f||f|}{\sqrt{f^2+\delta^2}} + \frac{|\nabla f|^2}{\sqrt{f^2+\delta^2}}\big(1 - \frac{f^2}{f^2+\delta^2}\big) + \frac{1}{\epsilon^2}\frac{f^2}{\sqrt{f^2+\delta^2}} = \frac{gf}{\sqrt{f^2+\delta^2}} + \Delta\sqrt{f^2+\delta^2}\,.$$

Now integrate over $\Omega\times]0,t[$ for any $t \in]0,T]$ and let $\delta \to 0$ to obtain

$$\sup_{0\le t\le T} \int_\Omega |f(x,t)|\,dx + \int_0^T \int_\Omega \frac{1}{\epsilon^2}|f| \le \int_0^T \int_\Omega |g|\,dx\,dt\,.$$

(ii) We multiply the equation by f with

$$\Big(\frac{f}{\sqrt{f^2+\delta^2}}\Big)(x,t)\phi(x)\eta(t)\,.$$

The cut-off function ϕ satisfies $0 \le \phi \in C_c^\infty(\mathbb{R}^2)$ with $\operatorname{spt}\phi \subset B_{R+\rho}(x_0)$ and $\phi \equiv 1$ on $B_R(x_0)$, whereas $\eta \in C^\infty(\mathbb{R}_+)$ with $0 \le \eta(t) \le 1$, $\eta(t_0 - R^2 - \rho^2) = 0$ and $\eta(t) \equiv 1$ if $t \ge t_0 - R^2$. We may assume

$$|\nabla\phi| \le \frac{C}{\rho}\,, \quad |\nabla^2\phi| \le \frac{C}{\rho^2} \quad \text{and} \quad |d_t\eta| \le \frac{C}{\rho^2}\,.$$

This leads to

$$\begin{aligned}
&\frac{\partial_t\big(|f|\phi^2\eta)|f|}{\sqrt{f^2+\delta^2}}+\frac{|\nabla f|^2\phi^2\eta}{\sqrt{f^2+\delta^2}}\big(1-\frac{f^2}{f^2+\delta^2}\big)+\frac{1}{\epsilon^2}\frac{f^2\phi^2\eta}{\sqrt{f^2+\delta^2}}\\
&=\frac{gf\phi^2\eta}{\sqrt{f^2+\delta^2}}+\operatorname{div}\big(\nabla f\frac{f\phi^2\eta}{\sqrt{f^2+\delta^2}}\big)+\frac{f^2\phi^2\partial_t\eta}{2\sqrt{f^2+\delta^2}}-2\eta\phi\nabla\phi\nabla\sqrt{f^2+\delta^2}\,.
\end{aligned}$$

Of course

$$\int_\Omega \phi\nabla\phi\eta\nabla\sqrt{f^2+\delta^2}\,dx=-\int_\Omega\eta\sqrt{f^2+\delta^2}\big(\phi\nabla^2\phi+|\nabla\phi|^2\big)\,dx\,.$$

After integrating (4.2) and letting $\delta\to 0$, we obtain

$$\sup_{t_0-(R^2+\rho^2)<t<t_0}\int_{B_R^\Omega}\frac{1}{2}|f|\,dx+\int_{P_R^\Omega}\frac{1}{\epsilon^2}|f|\,dz\le\int_{P_{R+\rho}^\Omega}\Big(|g|+\frac{C}{\rho^2}|f|\Big)\,dz\,.$$

□

Lemma 3.9. *Consider a smooth domain $\Omega\subset\mathbb{R}^2$, $g\in L^1\cap L^p(\Omega\times]0,T[)$ for $p\ge 2$ and $\epsilon>0$. Let $f\in C^1(\overline{\Omega}\times[0,T])\cap C^2(\Omega\times]0,T[)$ be a solution of*

$$\begin{gathered}
\big(\partial_t f-\Delta f\big)+\frac{1}{\epsilon^2}f=g\quad\text{in }\Omega\times]0,T[\,,\\
f=0\quad\text{on }\partial\Omega\times]0,T[\,.
\end{gathered}$$

For $f\ge 0$, we only need to assume

$$\begin{gathered}
\big(\partial_t f-\Delta f\big)+\frac{1}{\epsilon^2}f\le g\quad\text{in }\Omega\times]0,T[\,,\\
f=0\quad\text{on }\partial\Omega\times]0,T[\,.
\end{gathered}$$

(i) For any $\delta\in]0,1[$ and $z_0=(x_0,t_0)\in\Omega\times]0,T]$ with $0<R^2<t_0$, we have

$$\|\frac{1}{\epsilon^2}f\|_{L^p(P^\Omega_{\delta R}(z_0))}\le C_1\|g\|_{L^p(P^\Omega_R(z_0))}+\epsilon^{2/p}C_2\,,$$

where $C_1=C_1(p)$ and $C_2=C_2\big(\|g\|_{L^p(P^\Omega_R(z_0))},\|f\|_{L^{2p-1}(P^\Omega_R(z_0))},p,\delta,R\big)$.
(ii) The same bound as in (i) holds for

$$\|\big(\frac{1}{\epsilon^2}\big)^{(1-\frac{1}{p})}f\|_{L^{2p}(P^\Omega_{\delta R}(z_0))}\,,\|\big(\frac{1}{\epsilon^2}\big)^{(1-1/p)}f\|_{L^\infty\big([t_0-R^2,t_0];L^p(B^\Omega_{\delta R}(x_0))\big)}$$

and

$$\|\frac{1}{\epsilon}\nabla f\|_{L^2(P^\Omega_{\delta R}(z_0))}\,.$$

Proof. We multiply the equation for f by $f|f|^{2s-2}(x,t)\phi^2(x)\eta(t)$, where $s\ge 1$. The cut-off functions ϕ and η are the same as in the proof of Lemma 3.8. Then

$$\begin{aligned}
&\frac{1}{2s}\partial_t\big(|f|^{2s}(x,t)\phi^2(x)\eta(t)\big)+\frac{2s-1}{s^2}|\nabla|f|^s|^2\phi^2\eta+\frac{1}{\epsilon^2}|f|^{2s}\phi^2\eta\\
&=-\operatorname{div}\big(\nabla ff|f|^{2s-2}\phi^2\eta\big)+gf|f|^{2s-2}\phi^2\eta+\frac{1}{2s}|f|^{2s}\phi^2d_t\eta-\nabla|f||f|^{2s-1}2\nabla\phi\phi\eta\,,
\end{aligned}$$

for any $s\ge 1$. By Young's inequality $ab\le\delta^{-p}\frac{a^p}{p}+\delta^q\frac{b^q}{q}$ for $p,q>1$ with $\frac{1}{q}+\frac{1}{p}=1$ and $a,b,\delta>0$, we have

$$\big||g||f|^{2s-1}\big|\le\frac{1}{2\epsilon^2}\frac{2s-1}{2s}|f|^{2s}+(2\epsilon^2)^{(2s-1)}\frac{1}{2s}|g|^{2s}$$

and

$$\begin{aligned}\big|\nabla|f||f|^{2s-1}2\nabla\phi\phi\eta\big| &= 2|(\frac{1}{s}\nabla|f|^s\phi)(|f|^s\nabla\phi)\eta| \\ &\le \frac{2s-1}{2s^2}|\nabla|f|^s|^2\phi^2\eta+\frac{2}{2s-1}|f|^{2s}|\nabla\phi|^2\eta\,.\end{aligned}$$

This leads to

$$\begin{aligned}&\frac{1}{2s}\partial_t\big(|f|^{2s}(x,t)\phi^2(x)\eta(t)\big)+\frac{2s-1}{2s^2}|\nabla|f|^s|^2\phi^2\eta+\frac{1}{2\epsilon^2}|f|^{2s}\phi^2\eta \\ &\le -\operatorname{div}\big(\nabla f f|f|^{2s-2}\phi^2\eta\big)+(2\epsilon^2)^{(2s-1)}\frac{1}{2s}|g|^{2s}\phi^2\eta \\ &\quad+\frac{2}{2s-1}|f|^{2s}\big(|\nabla\phi|^2\eta+\phi^2|d_t\eta|\big)\,.\end{aligned}\tag{3.12}$$

For notational ease we relabel the domain as $\Omega\times]-T,0[$ and assume $z_0=(0,0)\in\Omega\times]-T,0[$ and $0<R^2+\rho^2<T$. As always $P_R:=P_R(0)$.

(i) Set $p=2s$. After multiplying (3.12) with $\big(\frac{1}{\epsilon^2}\big)^{p-1}$ and integrating, we get, for $p\ge 2$,

$$\begin{aligned}&\sup_{t\ge -R^2-\rho^2}\int_{B^\Omega_{R+\rho}}\big(\frac{1}{\epsilon^2}\big)^{p-1}|f|^p(x,t)\phi^2(x)\eta(t)\,dx \\ &\quad+\int_{P^\Omega_{R+\rho}}\big(\frac{1}{\epsilon^2}\big)^{p-1}|\nabla f^{p/2}|^2\phi^2\eta\,dz\Big) \\ &\quad+\int_{P^\Omega_{R+\rho}}\big(\frac{1}{\epsilon^2}\big)^{p}|f|^p\phi^2\eta\,dz \\ &\le C(p)\Big(\int_{P^\Omega_{R+\rho}}|g|^p\phi^2\eta dz+\big(\frac{1}{\epsilon^2}\big)^{p-1}\int_{P^\Omega_{R+\rho}}|f|^p\big(|\nabla\phi|^2\eta+\phi^2|d_t\eta|\big)dz\Big)\,.\end{aligned}\tag{3.13}$$

In particular we have

$$\int_{P^\Omega_R}\big(\frac{1}{\epsilon^2}\big)^p|f|^pdz\le C(p)\Big(\int_{P^\Omega_{R+\rho}}|g|^p\,dz+\epsilon^2\frac{C}{\rho^2}\int_{P^\Omega_{R+\rho}}\big(\frac{1}{\epsilon^2}\big)^p|f|^pdz\Big)\,.\tag{3.14}$$

Let $p=\frac{k}{2}+1$ for $k\in\mathbb{N}$. Hölder's inequality for $q_1=(2p-1)/(2p-2)$ and $q_2=2p-1$ implies

$$\int_{P^\Omega_R}\big(\frac{1}{\epsilon^2}|f|\big)^{p-1}|f|dz\le\|\frac{1}{\epsilon^2}f\|^{p-1}_{L^{p-(1/2)}}\|f\|_{L^{2p-1}}\,.$$

Now (3.13) leads to

$$\|\frac{1}{\epsilon^2}f\|^p_{L^p(P^\Omega_R)}\le C(p)\Big(\|g\|^p_{L^p(P^\Omega_{R+\rho})}+\frac{C}{\rho^2}\|\frac{1}{\epsilon^2}f\|^{p-1}_{L^{p-(1/2)}(P^\Omega_{R+\rho})}\|f\|_{L^{2p-1}(P^\Omega_{R+\rho})}\Big)\,.$$

An iteration combined either to (3.11) from Lemma 3.8 after the k-th step yields an estimate of the form

$$\|\frac{1}{\epsilon^2}f\|_{L^p(P^\Omega_R)}\le C\big(\|g\|_{L^p(P^\Omega_{R+(k+1)\rho})},\|f\|_{L^{2p-1}(P^\Omega_{R+(k+1)\rho})},p,\rho,R\big)\,.\tag{3.15}$$

If we set $\rho:=\frac{\delta R}{k+1}$ and insert (3.15) into (3.14), we obtain for any $\delta>0$

$$\|\frac{1}{\epsilon^2}f\|^p_{L^p(P^\Omega_R)}\le C(p)\|g\|^p_{L^p(P^\Omega_{(1+\delta)R})}+\epsilon^2C_2\,,\tag{3.16}$$

where
$$C_2 = C_2\big(\|g\|_{L^p(P^{\Omega}_{(1+\delta)R})}, \|f\|_{L^{2p-1}(P^{\Omega}_{(1+\delta)R})}, p, \delta, R\big)\,.$$
Claim (i) follows by setting $R_{new} = (1+\delta)R$ and $\delta_{new} = \frac{1}{1+\delta}$, i.e. $\delta_{new}R_{new} = R$.
(ii) By applying the estimate
$$\Big(\int_{[a,b]}\int_{B_R} |u|^4 dxdt\Big)^{\frac12} \le C\Big(\max_{t\in[a,b]}\int_{B_R}|u|^2dxdt + \int_{[a,b]}\int_{B_R}|\nabla u|^2 dxdt\Big)$$
(see Theorem 6.9 p.110 in [23]) to $u := f^{p/2}\phi\sqrt{\eta}$, we find that the expression
$$(\frac{1}{\epsilon^2})^{p-1}\Big(\int_{P^{\Omega}_R}|f|^{2p}\phi^4\eta^2dz\Big)^{\frac12} + \int_{P^{\Omega}_R}(\frac{1}{\epsilon^2})^p|f|^p\phi^2\eta dz \tag{3.17}$$
admits the same bound as (3.13) with a different constant $C(p) > 0$. By combining (3.17) with (3.13), we see that the same bounds as in (3.16) also holds for
$$(\frac{1}{\epsilon^2})^{p-1}\Big(\sup_{-R^2<t<0}\int_{B^{\Omega}_R}|f|^p(x,t)\,dx + \int_{P^{\Omega}_R}|\nabla f^{p/2}|^2\,dz\Big)$$
and
$$(\frac{1}{\epsilon^2})^{p-1}\Big(\int_{P^{\Omega}_R}|f|^{2p}\,dz\Big)^{\frac12}.$$
□

3.3.2. *Higher estimates.* By considering the flow equation (1.1) and equation (3.8) for $\rho_\epsilon := 1 - |u_\epsilon|^2$ as a coupled system, the uniform estimates from the previous paragraph and parabolic estimates for (1.1) can be combined in a standard bootstrap argument to prove Lemma 3.1 and 3.3.

In the case of the harmonic map flow $d_\epsilon := \mathrm{dist}(u_\epsilon, N)$ replaces ρ_ϵ. The corresponding equation is then
$$\partial_t d_\epsilon - \Delta d_\epsilon + |\nabla\nu_\epsilon|^2 d_\epsilon + \frac{1}{\epsilon^2}\chi'(d_\epsilon^2)d_\epsilon = \Delta v_\epsilon\cdot\nu_\epsilon \le C|\nabla u_\epsilon|^2\,, \tag{3.18}$$
whenever $u_\epsilon \in U$. Here we decomposed $u_\epsilon = v_\epsilon + d_\epsilon\nu_\epsilon$, where $v_\epsilon := \pi_N(u_\epsilon)$ and $\nu_\epsilon := \nu(u_\epsilon)$ is the unit normal in $\big(T_{\pi_N(u_\epsilon)}N\big)^\perp$, whereas $d_\epsilon := dist(u_\epsilon, N)$. Remember that $\pi_N : U \to N$ denotes the nearest neighbour projection from a tubular neighbourhood $U \subset \mathbb{R}^n$ of N onto N.

4. Towards characterizing the limits

We start with alternative characterisations of the "regular set" $\mathrm{Reg}\big(\{u_\epsilon\}_\epsilon\big)$. Remember that by Lemma 2.1, we have for any $0 \le s < t$
$$G_\epsilon\big(u_\epsilon(t), B^{\Omega}_R(x_0)\big) \le G_\epsilon\big(u_\epsilon(s), B^{\Omega}_{2R}(x_0)\big) + \frac{C(t-s)E_0}{\gamma_1 R^2}\,. \tag{4.1}$$
Set $\delta_0 := \frac{\gamma_1\epsilon_0}{2CE_0}$, where ϵ_0 is the constant from Theorem 3.4. After increasing C if necessary, we may assume $0 < \delta_0 < 1$.

Lemma 4.1. *Let u_ϵ be a solution of* (1.1)-(1.2) *with $u_0 \in H^{1,2}(\Omega; S^2)\cap C^2(\partial\Omega; S^2)$ for each $\epsilon > 0$. Then the following assertions are equivalent:*
(i) $z_0 = (x_0, t_0) \in \mathrm{Reg}\big(\{u_\epsilon\}_{\epsilon>0}\big)$.
(ii) $\exists\delta, R > 0 :\quad \limsup_{\epsilon\searrow 0}\sup_{t_0-\delta<t<t_0} G_\epsilon\big(u_\epsilon(t), B^{\Omega}_R(x_0)\big) < \epsilon_0$.
(iii) $\exists\delta > 0 : \lim_{R\searrow 0}\limsup_{\epsilon\searrow 0}\sup_{t_0-\delta<t<t_0} G_\epsilon\big(u_\epsilon(t), B^{\Omega}_R(x_0)\big) = 0$.

(iv) $\exists R>0 : \limsup_{\epsilon\searrow 0} \frac{1}{R^2}\int_{t_0-R^2}^{t_0}\int_{B_R^\Omega(x_0)} g_\epsilon(u_\epsilon)\,dx\,dt < \frac14\delta_0\epsilon_0$.

(v) $\exists \delta, R>0 : \limsup_{\epsilon\searrow 0}\sup_{t_0-\delta<t<t_0+\delta} G_\epsilon\big(u_\epsilon(t), B_R^\Omega(x_0)\big) < \epsilon_0$.

Proof. "$(i)\Leftrightarrow(ii)$" is obvious.
"$(ii)\Rightarrow(iii)$" follows from Theorem 3.4 in Section 3.2.2.
"$(iii)\Rightarrow(iv)$" is obvious.
"$(iv)\Rightarrow(ii)$": Assume (iv) holds. By (4.1) and the above choice of δ_0, we have for sufficiently small $\epsilon>0$,

$$
\begin{aligned}
&\sup_{t_0-(1/2)\delta_0R^2<t<t_0} G_\epsilon\big(u_\epsilon(t), B^\Omega_{(1/2)R}(x_0)\big)\\
&\le \inf_{t_0-\delta_0R^2<s<t_0-(1/2)\delta_0R^2} G_\epsilon\big(u_\epsilon(s), B_R^\Omega(x_0)\big) + \frac{C\delta_0R^2E_0}{\gamma_1R^2}\\
&\le \frac{2}{\delta_0R^2}\int_{t_0-\delta_0R^2}^{t_0-(1/2)\delta_0R^2} G_\epsilon\big(u_\epsilon(t), B_R^\Omega(x_0)\big)\,dt + \frac12\epsilon_0\\
&< \frac{2}{\delta_0}\frac14\delta_0\epsilon_0 + \frac12\epsilon_0 < \epsilon_0\,.
\end{aligned}
$$

"$(v)\Rightarrow(ii)$" is obvious.
"$(iii)\Rightarrow(v)$": Assume (iii) holds. Then there are $R,\delta>0$, such that

$$
\limsup_{\epsilon\searrow 0}\sup_{t_0-\delta\le t\le t_0}\int_{B_R(x_0)\cap\Omega} g_\epsilon(u_\epsilon(x,t))dx < \epsilon_0/2\,.
$$

On the other hand by (4.1), we have for $\delta_{new} := \frac{\epsilon_0\gamma_1R^2}{2CE_0} = \delta_0R^2$,

$$
\sup_{t_0\le t\le t_0+\delta_{new}}\int_{B_{\frac12 R}(x_0)\cap\Omega} g_\epsilon(u_\epsilon(x,t))dx \le \int_{B_R(x_0)\cap\Omega} g_\epsilon(u_\epsilon(x,t_0))dx + \frac{\delta_{new}CE_0}{\gamma_1R^2}\,.
$$

Now (v) holds for $\frac12 R$ and $\min\{\delta,\delta_{new}\}$. □

Corollary 4.2. *Let u_ϵ be a solution of (1.1)-(1.2) with u_0 in $H^{1,2}(\Omega;S^2)\cap C^2(\partial\Omega;S^2)$ for each $\epsilon>0$. Let $\{\epsilon_i\}_i$ be a sequence with $\epsilon_i\searrow 0$ as $i\to\infty$. Then the following holds:*
(i) $\mathrm{Reg}(\{u_\epsilon\}_\epsilon)$ *and* $\mathrm{Reg}(\{u_{\epsilon_i}\}_i)$ *are open in* $\overline{\Omega}\times\mathbb{R}_+$.
(ii) There is some $T_0>0$, such that $\overline{\Omega}\times[0,T_0[\subset \mathrm{Reg}(\{u_\epsilon\}_\epsilon)$.

Proof. (i) follows from Lemma 4.1 (v).
(ii) The existence of T_0 immediately follows from Lemma 2.1 (2.3). □

Set

$$
Q_R(z) := B_R(x)\times]t-R^2, t+R^2[\quad \text{for } z=(x,t)\,.
$$

and let $\mathfrak{H}^2$ denote the 2-dimensional parabolic Hausdorff measure.

Proposition 4.3. *Let u_ϵ be a solution of (1.1)-(1.2) with $u_0\in H^{1,2}(\Omega;S^2)\cap C^2(\partial\Omega;S^2)$ for each $\epsilon>0$. Then the following holds:*
(i) $\mathbb{S}(\{u_\epsilon\}_\epsilon)$ has locally finite two dimensional parabolic Hausdorff-measure. More precisely there is a constant $K_1 = K_1(E_0,\epsilon_0)>0$, such that for any compact intervall $I\subset\mathbb{R}_+$

$$
\mathfrak{H}^2\big(\mathbb{S}(\{u_\epsilon\}_\epsilon)\cap(\overline{\Omega}\times I)\big) \le K_1|I|\,.
$$

(ii) There is a constant $K_2 = K_2(E_0,\epsilon_0)>0$, such that for any $t>0$ the set $\mathbb{S}^t(\{u_\epsilon\}_\epsilon) := \mathbb{S}(\{u_\epsilon\}_\epsilon)\cap(\overline{\Omega}\times\{t\})$ consists of at most K_2 points.

Proof. (i) By (iv) of Lemma 4.1, we have for any $z_0 = (x_0, t_0) \in \mathbb{S}(\{u_\epsilon\}_\epsilon)$, any $R > 0$ and sufficiently small $0 < \epsilon \le \epsilon(z_0)$

$$\frac{1}{R^2}\int_{t_0-R^2}^{t_0}\int_{B_R^\Omega(x_0)} g_\epsilon(u_\epsilon)dxdt > \frac{1}{4}\delta_0\epsilon_0\,. \tag{4.2}$$

Fix a compact intervall $I \subset \mathbb{R}_+$ and $\delta > 0$. By compactness and Vitali's Covering Theorem any covering of $\mathbb{S}(\{u_\epsilon\}_\epsilon) \cap \big(\overline{\Omega} \times I\big)$ by parabolic cylinders $Q_R^\Omega(z)$ with $0 < R^2 < \delta$ and $z \in \mathbb{S}(\{u_\epsilon\}_\epsilon) \cap \big(\overline{\Omega} \times I\big)$ contains a finite covering $\bigcup_j Q_{5R_j}^\Omega(z_j) \supset \mathbb{S}(\{u_\epsilon\}_\epsilon) \cap \big(\overline{\Omega} \times I\big)$, such that the cylinders $Q_{R_j}^\Omega(z_j)$ are pairewise disjoint. By (4.2) and the energy estimate, we obtain for $0 < \epsilon \le \min_j\{\epsilon(z_j)\}$

$$\sum_j \omega_2(5R_j)^2 \le 25\frac{4\omega_2}{\delta_0\epsilon_0}\sum_j \int_{t_j-R_j^2}^{t_j}\int_{B_{R_j}^\Omega(x_j)} g_\epsilon(u_\epsilon)dxdt < \frac{100\omega_2}{\delta_0\epsilon_0}(|I| + \delta)E_0\,.$$

By letting $\delta \searrow 0$, we find

$$\mathfrak{H}^2\big(\mathbb{S}(\{u_\epsilon\}_\epsilon) \cap I\big) \le \frac{100\omega_2 E_0}{\delta_0\epsilon_0}|I|\,.$$

(ii) Pick any $(x_1, T), \dots, (x_k, T) \in \mathbb{S}(\{u_\epsilon\})$. By assumption we have for

$$\forall R, \delta, \gamma > 0, \exists \epsilon \in]0, \gamma[: \quad \sup_{T-\delta<t<T}\int_{B_R^\Omega(x_l)} g_\epsilon(u_\epsilon(x,t))dx \ge \frac{\epsilon_0}{2} \text{ for } 1 \le l \le k\,.$$

We may choose $R > 0$, such that the $B_R^\Omega(x_l)(1 \le l \le k)$ are pairewise disjoint. Choose $\delta \in]0, \frac{\gamma_1 R^2\epsilon_0}{4CE_0}[$, where C is the constant from Lemma 2.1 (2.1) and $\epsilon \in]0, \gamma[$ as above. Since $t \mapsto \int_{B_R^\Omega(x_l)} g_\epsilon(u_\epsilon(x,t))dx$ is continuous, we may find $t_\delta^l \in]T - \delta, T[$ such that

$$\int_{B_R^\Omega(x_l)} g_\epsilon(u_\epsilon(x, t_\delta^l))dx \ge \frac{\epsilon_0}{2} \quad \text{ for } 1 \le l \le k\,.$$

The energy estimate and the local energy inequality, Lemma 2.1,(2.1) and (2.2) now imply

$$\begin{aligned} E_0 &\ge \sum_{l=1}^k \int_{B_R^\Omega(x_l)} g_\epsilon(u_\epsilon(x, T-\delta))\,dx \\ &\ge \sum_{l=1}^k\Big(\int_{B_R^\Omega(x_l)} g_\epsilon\Big(u_\epsilon\big(x, t_\delta^l\big)\Big)dx - \frac{C}{\gamma_1 R^2}\int_{T-\delta}^T\int_{B_R^\Omega(x_l)} |\nabla u_\epsilon(x,t)|^2 dxdt\Big). \end{aligned}$$

Thus $E_0 \ge k\big(\frac{\epsilon_0}{2} - \frac{CE_0}{R^2}\delta\big)$. Now since $\delta < \frac{R^2\epsilon_0}{4CE_0}$, this implies $k \le \frac{8E_0}{\epsilon_0} =: K_2$. (Compare [32] and [31] (1°) of the proof of Theorem 6.6 p.229 for a similar argument in the case of the harmonic map flow.) $\square$

Theorem 4.4. *Let u_ϵ be a solution of* (1.1)-(1.2) *with u_0 in $H^{1,2}(\Omega; S^2) \cap H^{3/2,2}(\partial\Omega; S^2)$ for each $\epsilon > 0$. Then the following holds:*
There is at least one sequence $\{\epsilon_i\}_i$, with $\epsilon_i \to 0$ as $i \to \infty$ and

$$u_* \in H^{1,2}_{\mathrm{loc}}(\overline{\Omega} \times \mathbb{R}_+; S^2) \cap L^\infty(\mathbb{R}_+; H^{1,2}(\Omega; S^2)),$$

such that $u_{\epsilon_i} \rightharpoonup u_*$ weakly in $H^{1,2}_{\mathrm{loc}}(\overline{\Omega} \times \mathbb{R}_+; \mathbb{R}^3)$ and weak in $L^\infty(\mathbb{R}_+; H^{1,2}(\Omega; \mathbb{R}^3))$.*

In addition: (i) For any such sequence $\{u_{\epsilon_i}\}_i$, we have

$$\lim_{i\to\infty} u_{\epsilon_i} = u_* \quad \text{in } C^\infty(\mathrm{Reg}(\{u_{\epsilon_i}\}_i) \cap (\Omega \times \mathbb{R}_+); \mathbb{R}^3)$$

and $\frac{1}{\epsilon^2}(1-|u_\epsilon|^2) \to |\nabla u_*|^2$ *in* $C^\infty(\mathrm{Reg}(\{u_{\epsilon_i}\}_i) \cap (\Omega \times \mathbb{R}_+))$.
(ii) u_* *is a smooth solution of* (1.3) *in* $\mathrm{Reg}(\{u_{\epsilon_i}\}_i) \cap (\Omega \times \mathbb{R}_+)$ *and a distributional solution in*

$$H^{1,2}_{\mathrm{loc}}(\overline{\Omega} \times \mathbb{R}_+) \cap L^\infty\big(\mathbb{R}_+; H^{1,2}(\Omega; \mathbb{R}^n)\big)$$

on all $\Omega \times \mathbb{R}_+$. *Further* $\lim_{t\searrow 0} u_*(.,t) = u_0$ *in* $H^{1,2}(\Omega; \mathbb{R}^3)$ *and*

$$u_*(.,t)_{|\partial\Omega} = u_{0|\partial\Omega} \text{ as a } H^{2,2}(\Omega;\mathbb{R}^3)\text{-trace for a.e. } t > 0\,.$$

(iii) If u_* *is regular at* $z_0 = (x_0, t_0) \in \overline{\Omega} \times \mathbb{R}_+$ *in the sense that*

$$\lim_{R\searrow 0} \sup_{t_0 - R^2 \le t \le t_0} \int_{B_R(x_0)} |\nabla u_*|^2 dx = 0$$

and if z_0 *is parabolically isolated for* $\{u_{\epsilon_i}\}_i$, *i.e.*

$$B_{R_0}(x_0) \times]t_0 - R_0^2, t_0[\subset \mathrm{Reg}(\{u_{\epsilon_i}\}) \text{ for some } R_0 > 0\,,$$

then $z_0 \in \mathrm{Reg}(\{u_{\epsilon_i}\})$. *(In particular* u_* *cannot (backwards) concentrate energy and (backwards) bubble at* z_0 *as* $t \searrow t_0$. *Compare* [17], [18]*)*

Proof. (i) The convergence statements follow from the energy estimate (Lemma 2.1), Theorem 3.2 and Lemma 3.1.
(ii) For the case $\gamma_2 = 0$ and $f(u_\epsilon) = \frac{1}{2}\frac{d}{du}\chi\big(\mathrm{dist}^2(u_\epsilon, N)\big)$, this is proven in [36] III p.95. We will prove it in the case $\gamma_2 \neq 0$. If we apply "$u_{\epsilon_i} \times .$" from the left to (1.1) and pass to the limit $\epsilon_i \to 0$ on $\mathrm{Reg}(\{u_{\epsilon_i}\}) \cap (\Omega \times \mathbb{R}_+)$, we obtain

$$\gamma_1 u_* \times \partial_t u_* - \gamma_2 u_* \times (u_* \times \partial_t u_*) - u_* \times \Delta u_* = 0\,. \tag{4.3}$$

Since $(1 - |u_{\epsilon_i}|^2) \to 0$ smoothly, we also have

$$|u_*(x,t)| = 1 \quad \text{in } \mathrm{Reg}(\{u_{\epsilon_i}\}) \cap (\Omega \times \mathbb{R}_+).$$

Now we use $a \times (b \times c) = (ac)b - (ab)c$ and $|u_*| \equiv 1$ while applying "$u_* \times .$" from the left to (4.3), to obtain

$$\gamma_1 \partial_t u_* - \gamma_2 u_* \times \partial_t u_* - \Delta u_* = |\nabla u_*|^2 u_* \quad \text{in } \Omega \times \mathbb{R}_+\,. \tag{4.4}$$

In particular, since the left side of (1.1) converges to the left side of (4.4), we have

$$\frac{1}{\epsilon_i^2}(1 - |u_{\epsilon_i}|^2) \to |\nabla u_*|^2 \text{ in } C^\infty(\mathrm{Reg}(\{u_{\epsilon_i}\}) \cap (\Omega \times \mathbb{R}_+))\,.$$

We now prove that u_* is a distributional $H^{1,2}_{\mathrm{loc}} \cap L^\infty(H^{1,2})$-solution of (1.3) on all $\Omega \times \mathbb{R}_+$. Note that the sequence $\{u_{\epsilon_i}\}_i$ converges weakly in $H^{1,2}(\Omega \times \mathbb{R}_+; S^2)$ and smoothly on $\mathrm{Reg}(\{u_{\epsilon_i}\}) \cap (\Omega \times \mathbb{R}_+)$. Further since $\mathbb{S}^t(\{u_{\epsilon_i}\}) := \mathbb{S}(\{u_{\epsilon_i}\}) \cap (\overline{\Omega} \times \{t\})$ is finite for all $t \ge 0$, we have both $u_{\epsilon_i} \to u_*$ pointwise a.e. in $\Omega \times \mathbb{R}_+$ and $u_{\epsilon_i}(\cdot,t) \to u_*(\cdot,t)$ pointwise a.e. in Ω for all $t \in \mathbb{R}_+$. Since $\int_0^\infty \int_\Omega |\partial_t u_{\epsilon_i}|^2 dx dt \le E_0$, by Fatou's Lemma the complement of

$$A := \{t \ge 0 | \liminf_{\epsilon_i \searrow 0} \int_\Omega |\partial_t u_{\epsilon_i}|^2(x,t) dx < \infty\}$$

has measure 0. Pick $t_0 \in A$. Then there is a subsequence still denoted by u_{ϵ_i}, such that $\partial_t u_{\epsilon_i}(\cdot, t_0) \rightharpoonup \partial_t u_*(\cdot, t_0)$ weakly in $L^2(\Omega; \mathbb{R}^3)$. By the local energy estimate, we may assume that, for the same subsequence, we also have $u_{\epsilon_i}(\cdot, t_0) \rightharpoonup u_*(\cdot, t_0)$ weakly in $H^{1,2}(\Omega; S^2)$. By pointwise a.e. uniqueness of the limit, the whole sequence converges. Also

$$u_*(\cdot, t_0) \in H^{1,2}(\Omega; S^2) \quad \text{and} \quad \partial_t u_*(\cdot, t_0) \in L^2(\Omega; \mathbb{R}^3) \quad \text{for all } t_0 \in A\,.$$

Now

$$-\Delta u_*(\cdot,t_0) = (|\nabla u_*|^2 u_*)(\cdot,t_0) + f\,,$$

where

$$f = -\gamma_1 \partial_t u_*(\cdot,t_0) + \gamma_2 u_* \times \partial_t u_*(\cdot,t_0) \in L^2(\Omega;\mathbb{R}^3)$$

and by a regularity result of T.Rivière (see [27] Lemma p.3), we have

$$u_*(\cdot,t_0) \in H^{2,2}(\Omega;S^2) \quad \text{if} \quad u_0 \in H^{3/2,2}(\partial\Omega;S^2) \cap H^{1,2}(\Omega;S^2)\,.$$

This in particular implies $u_*(.,t)_{|\partial\Omega} = u_{0|\partial\Omega}$ as a $H^{2,2}(\Omega)$-trace for any $t \in A$. Further since $\mathbb{S}^{t_0}(\{u_{\epsilon_i}\}_i)$ consists of finitely many points, it has vanishing 2-capacity in $\mathbb{R}^2$, i.e.

$$Cap_2\big(\mathbb{S}^{t_0}(\{u_{\epsilon_i}\})\big) = 0$$

(see [12]). Therefore, there is a sequence $\{\eta_k\}_k = \{\eta_{k,q}\}_k \subset C_c^\infty(\mathbb{R}^2)$ with

$$\eta_k(x) = 1 \forall x \in \mathbb{S}^{t_0}(\{u_{\epsilon_i}\}_i) \quad \text{and} \quad \|\eta_k\|_{H^{1,2}(\mathbb{R}^2)} \overset{(k\to\infty)}{\to} 0$$

(see [12] 4.7.1). For $\phi \in C_c^\infty(\Omega)$, we may test equation (4.3) with the cut-off function $(1-\eta_k)\phi$, which has support in $\mathrm{Reg}(\{u_{\epsilon_i}\}_i)$. After passing to the limit $k \to \infty$, we find that for any $t \in A$,

$$\begin{aligned}&\int_\Omega \gamma_1 \partial_t u_*(x,t)\phi(x) - \gamma_2 (u_* \times \partial_t u_*)(x,t)\phi(x) + \nabla u_*(x,t)\nabla\phi(x)\,dx\\ &= \int_\Omega (|\nabla u_*|^2 u_*)(x,t)\phi(x)\,dx\,.\end{aligned}$$

This equation holds for a.e. $t \geq 0$. On the other hand, we have $u_* \in H^{1,2}(\Omega \times [0,T];S^2)$ for any $T > 0$ and so both sides of the above equation are locally integrable on $\mathbb{R}_+$. Therefore we may multiply the equation with $\psi \in C_c^\infty([0,\infty[)$ and integrate over $\mathbb{R}_+$. Moreover linear combinations $\sum_k a_k\phi_k(x)\psi_k(t)$ with $\phi_k \in C_c^\infty(\Omega)$ and $\psi_k \in C_c^\infty([0,\infty[)$ are dense in $C_c^\infty(\Omega \times [0,\infty[)$ and so

$$\begin{aligned}&\int_0^\infty \int_\Omega \gamma_1 \partial_t u_*(x,t)\phi(x,t) - \gamma_2(u_* \times \partial_t u_*)(x,t)\phi(x,t) + \nabla u_*(x,t)\nabla\phi(x,t)dx\,dt\\ &= \int_0^\infty \int_\Omega (|\nabla u_*|^2 u_*)(x,t)\phi(x,t)dx\,dt\,,\end{aligned}$$

for any $\phi \in C_c^\infty(\Omega \times [0,\infty[)$. Finally $\lim_{t\searrow 0} u_*(.,t) = u_0$ in $H^{1,2}(\Omega;S^2)$ immediately follows from $E(u_*(t_0)) \leq E(u_0)$, since we have weak convergence as $t \searrow 0$.
(iii): By assumption there is $R > 0$, such that

$$\sup_{t_0 - R^2 \leq t \leq t_0} \int_{B_R(x_0)\cap\Omega} \frac{1}{2}|\nabla u_*(x,t)|^2 dx < \frac{\epsilon_0}{4}\,.$$

Set $\delta := \min\{R^2, \frac{\gamma_1 R^2 \epsilon_0}{2CE_0}\}$ and $s_0 := t_0 - \frac{1}{2}\delta$. We may assume we have for the same $R > 0$ $P_{2R}^\Omega(z_0) \setminus \{z_0\} \subset \mathrm{Reg}(\{u_{\epsilon_i}\})$. Then Theorem 3.2 and 3.4 and Lemma 3.1 and 3.3 imply

$$\lim_{i\to\infty} \int_{B_R(x_0)\cap\Omega} g_{\epsilon_i}(u_{\epsilon_i}(x,s_0))dx = \int_{B_R(x_0)\cap\Omega} \frac{1}{2}|\nabla u_*(x,s_0)|^2 dx < \frac{\epsilon_0}{4}\,.$$

Now by Lemma 2.1 (2.2), we have

$$\sup_{s_0\le t\le s_0+\delta}\int_{B_{\frac12 R}(x_0)\cap\Omega} g_{\epsilon_i}(u_{\epsilon_i}(x,t))\,dx \le \int_{B_R(x_0)\cap\Omega} g_{\epsilon_i}(u_{\epsilon_i}(x,s_0))\,dx + \frac{\delta C E_0}{\gamma_1 R^2} < \frac{\epsilon_0}{2}+\frac{\epsilon_0}{2}\,,$$

for i sufficiently large. Since by construction $t_0 \in]s_0, s_0+\delta[$, the claim follows. $\square$

By Theorem 4.4, Corollary 4.2 (ii) and uniqueness of smooth solutions, we obtain the following.

Remark 4.5. There is $T_0 > 0$, such that

$$\lim_{\epsilon\searrow 0} u_\epsilon = u_* \quad \text{in } C^\infty(\Omega\times]0,T_0[;S^2),$$

where u_* is the unique smooth solution of (1.3) with initial and boundary data u_0. (Compare [17].)

If the energy of a (sub-)limit u_* was everywhere decreasing, A.Freire's uniqueness result [13] would imply that u_* is (globally) the Struwe-solution. However all we can say about the energy of sublimits u_* is the following Lemma 4.6. In particular extension u_* after the maximal smooth existence time T_0 with backward bubbling cannot be excluded. (See [18].)

Lemma 4.6. *Let u_ϵ be a solution of* (1.1)-(1.2) *for fixed $\epsilon > 0$ and assume $u_* =$ weak-$H^{1,2}$-$\lim_{i\to\infty} u_{\epsilon_i}$ for a sequence $0 < \epsilon_i \searrow 0$. If $s < t$ and $\mathbb{S}^s(\{u_{\epsilon_i}\}) := (\overline{\Omega}\times\{s\})\cap\mathbb{S}(\{u_{\epsilon_i}\}_i) = \emptyset$ and $\mathbb{S}^t(\{u_{\epsilon_i}\}) \neq \emptyset$, then*

$$\int_\Omega \frac12|\nabla u_*|^2(x,s)\,dx \ge \int_\Omega \frac12|\nabla u_*|^2(x,\tau)\,dx \quad \forall \tau > s\,,$$

and

$$\int_\Omega \frac12|\nabla u_*|^2(x,s)\,dx \ge \int_\Omega \frac12|\nabla u_*|^2(x,t)\,dx + \epsilon_0\,,$$

where $\epsilon_0 > 0$ is the constant from Theorem 3.4.

Proof. Set $\overline{x} := (x_1,\dots,x_K)$ if $\mathbb{S}^t(\{u_{\epsilon_i}\}) = \{x_1,\dots,x_K\}$ and $B_R(\overline{x}) := \bigcup_{j=1}^K B_R(x_j)$. Then

$$\begin{aligned}
E(u_*(s),\Omega) &:= \int_\Omega \frac12|\nabla u_*|^2(x,s)\,dx\\
&= \lim_{i\to\infty}\int_\Omega g_{\epsilon_i}(u_{\epsilon_i})(x,s)\,dx\\
&\ge \limsup_{i\to\infty}\int_\Omega g_{\epsilon_i}(u_{\epsilon_i})(x,\tau)\,dx \quad \forall\tau > s \qquad \text{(by Lemma 2.1)}\\
&\ge \int_\Omega \frac12|\nabla u_*|^2(x,\tau)\,dx \quad \forall \tau > s \quad \text{(by weak lower semi-continuity)}
\end{aligned}$$

Also

$$E(u_*(s),\Omega) \ge \limsup_{i\to\infty}\Big(\int_{\Omega\setminus B_R(\overline{x})} g_{\epsilon_i}(u_{\epsilon_i})(x,\tau)\,dx + \int_{B_R(\overline{x})} g_{\epsilon_i}(u_{\epsilon_i})(x,\tau)\,dx\Big),$$

for all $\tau \in]s,t]$. Now for any $\delta \in]0,1[$ and $R > 0$, there are sequences $s < t_i \nearrow t$ and $0 < \delta_i \searrow 0$, such that

$$\int_{B_R(\overline{x})} g_{\epsilon_i}(u_{\epsilon_i})(x,t_i)dx = \sup_{\delta_i<\tau<t} \int_{B_R(\overline{x})} g_{\epsilon_i}(u_{\epsilon_i})(x,\tau)dx \geq \delta\epsilon_0$$

and so

$$\begin{aligned} E(u_*(s)) &\geq \limsup_{i\to\infty} \int_{\Omega\smallsetminus B_R(\overline{x})} g_{\epsilon_i}(u_{\epsilon_i})(x,t_i)dx + \delta\epsilon_0 \\ &\geq \int_{\Omega\smallsetminus B_R(\overline{x})} \frac{1}{2}|\nabla u_*|^2(x,t)dx + \delta\epsilon_0 \quad \forall R > 0\,,\ \delta \in]0,1[\,. \end{aligned}$$

Since the last inequality holds for any $R > 0$ and $\delta \in]0,1[$, the claim follows. $\square$

Theorem 4.4 provides an alternative version of the construction of the "Struwe-solution" (see [17]).

Corollary 4.7. *Let $u_0 \in H^{1,2}(\Omega; S^2)$. Then there is a global distributional solution $u \in H^{1,2}_{\rm loc}(\overline{\Omega}\times]0,\infty[; S^2) \cap L^\infty(]0,\infty[; H^{1,2}(\Omega; S^2))$ with $\partial_t u \in L^2(\Omega\times]0,\infty[; \mathbb{R}^3)$ of* (1.3) *with initial and boundary data u_0, which is smooth on $\Omega\times]0,\infty[$ except at finitely many points and has decreasing and right continuous energy. If in addition $u_0 \in H^{3/2,2}(\partial\Omega; S^2)$, then u is unique among the solutions of* (1.3) *with initial and boundary data u_0 which are smooth except for isolated singular points and with $\lim_{t\searrow s} E(u(t)) < E(u(s)) + \epsilon_0$ for all $s \geq 0$. (It is also unique among the $H^{1,2}_{\rm loc}$-solutions with decreasing energy by Freire's result.)*

Proof. By Theorem 4.4 the ϵ-approximation scheme provides a smooth short time solution

$$u \in C^\infty\big(\Omega\times]0,T_0[; S^2\big)$$

to (1.3) with boundary data u_0 and $\lim_{t\searrow 0} u(\cdot,t) = u_0$ in $H^{1,2}(\Omega;\mathbb{R}^3)$. Also there are $\{x_1,\dots,x_K\} \subset \Omega$, such that

$$\lim_{t\nearrow T_0} u(\cdot,t) = u(\cdot,T_0) \quad \text{in } C^\infty\big(\Omega\smallsetminus\{x_1,\dots,x_K\},\mathbb{R}^3\big)$$

and

$$\|\nabla u(\cdot,T_0)\|^2_{L^2(\Omega)} \leq \liminf_{t\nearrow T_0} \|\nabla u(\cdot,t)\|^2_{L^2(\Omega)} \leq 2E_0\,.$$

In particular $u(\cdot,T_0) \in H^{1,2}(\Omega)$. If we now set $\tilde{u}_0 := u(\cdot,T_0)$ and repeat the same procedure with $\tilde{u}_0$ instead of u_0, we obtain step by step a global solution with point singularities. To see that $\partial_t u \in L^2(\Omega\times\mathbb{R}_+;\mathbb{R}^3)$, we sum up the energy inequalities of each time intervall $]t_k,t_{k+1}[$ on which u is regular and use that the energy is right continuous, whereas

$$\limsup_{t\nearrow t_{k+1}} E(u(t)) \geq E(u(t_{k+1})) + \epsilon_0$$

by Lemma 4.6. This yields

$$\int_0^\infty \int_\Omega |\partial_t u|^2 dxdt \leq E_0 - \sum_k \epsilon_0$$

and also shows that there can only be finitely many "singular times" t_k.

Now assume we have two solutions u_1 and u_2 of (1.3) with initial and boundary data u_0 and both with finitely many point singularities and $\lim_{t\searrow s} E(u(t)) < E(u(s)) + \epsilon_0$ for all $s \geq 0$. By Remark 4.5, we have $u_1 = u_2$ on $\Omega \times [0,T_1[$, where

T_1 is the maximal commun smooth existence time, i.e. either u_1 or u_2 has point singularities at T_1. However by Corollary 4 on the existence of smooth extensions in [17], if u_1 admits a smooth extension up to T_1, then so does u_2 and conversely. Moreover, since the criterion for the existence of a smooth extension is local, both solutions have the same singularities $x_1, \dots, x_K$ at time T_1 and $u_1(\cdot, T_1) = u_2(\cdot, T_1)$ on $\Omega \setminus \{x_1, \dots, x_K\}$. By Theorem 6 in [17], and the assumption on the energy, the extension of u_1 and u_2 after T_1 is again unique "for a short time" and an iteration of the previous argument leads to the claimed uniqueness. $\square$

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