

REALITY CONDITIONS OF LOOP SOLITONS GENUS g : HYPERELLIPTIC AM FUNCTIONS

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ABSTRACT. This article is devoted to an investigation of a reality condition of a hyperelliptic loop soliton of higher genus. In the investigation, we have a natural extension of Jacobi am-function for an elliptic curves to that for a hyperelliptic curve. We also compute winding numbers of loop solitons.

1. INTRODUCTION

In this article, we investigate a reality condition of loop solitons with genus g . Here the loop soliton is defined as follows.

Definition 1.1. For a real parameter $t_2 \in \mathbb{R}$, let us consider a smooth immersion of a curve in \mathbb{C} parameterized by $t_1 \in \mathbb{R}$ and its smooth deformation by t_2 , $Z_{t_2} : \mathbb{R} \hookrightarrow \mathbb{C}$ ($t_1 \mapsto Z(t_1, t_2) := Z_{t_2}(t_1) = X^1 + \sqrt{-1}X^2$) with $\partial_{t_1} Z = e^{\sqrt{-1}\phi(t_1, t_2)}$. We call the deformation of the curve *loop soliton* if its real tangential angle $\phi(t_1, t_2)$ is characterized by a solution of MKdV equation

$$\partial_{t_2}\phi + \frac{1}{4}(\partial_{t_1}\phi)^3 + \partial_{t_1}^3\phi = 0. \quad (1.1)$$

The loop soliton or geometry of MKdV equation has been studied by several researchers from viewpoints of a connection between integrable system and classical differential geometry, and a relation between algebraic geometry and differential geometry ([9] and references therein). From a historical point of view, simple loop solitons appeared in Euler's book [2] as solutions of an elastica problem which was proposed by James Bernoulli as a problem in mathematical science [11]. In [4], we have proposed a problem of statistical mechanics of elasticas as a generalization of the elastica problem, which we sometimes call quantized elasticas using similarity between quantum mechanics and statistical mechanics. The new problem is related to large polymers in a heat bath. In [4] we show that the equi-energy state of quantized elasticas is given by the loop soliton. It means that the loop soliton is directly related to (low energy) physics. Thus we have studied the loop soliton and quantized elasticas in a series of works [4, 5, 6].

In [5], we gave explicit solutions of loop solitons in terms of hyperelliptic functions based upon theories of Baker's [1, 5] and Weierstrass's [13] as follows. For a

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hyperelliptic curve C_g given by an affine equation,

$$\begin{aligned} C_g : \quad y^2 &= x^{2g+1} + \lambda_{2g}x^{2g} + \lambda_{2g-1}x^{2g-1} + \cdots + \lambda_2x^2 + \lambda_1x + \lambda_0 \\ &= (x - e_1)(x - e_2)(x - e_3) \cdots (x - e_{2g})(x - e_{2g+1}), \end{aligned} \quad (1.2)$$

where each e_a is a complex number \mathbb{C} , we have a coordinate system in a complex vector space $J_g^\infty := \mathbb{C}^g$ as maps from Abelian universal covering of symmetric product of C_g , $\mathcal{USym}^g(C_g)$ to J_g^∞ :

$$u_{g-1} = \sum_{i=1}^g u_{g-1}^{(i)}, \quad u_g = \sum_{i=1}^g u_g^{(i)}, \quad (1.3)$$

$$u_{g-1}^{(i)} = \int_{\infty}^{(x^{(i)}, y^{(i)})} \frac{x^{g-2} dx}{2y}, \quad u_g^{(i)} = \int_{\infty}^{(x^{(i)}, y^{(i)})} \frac{x^{g-1} dx}{2y}. \quad (1.4)$$

Proposition 1.2. *A hyperelliptic solution of the loop soliton of genus g is give by*

$$\partial_{t_1} Z^{(a)} = \prod_{i=1}^g (x^{(i)} - e_a), \quad (1.5)$$

where $t_1 = Ku_g$ and $t_2 = K(u_{g-1} - (\lambda_{2g} + e_a)^{-1}u_g)$ for a constant positive number K , if the curve (1.2) and integrals contours which satisfy the reality condition,

- (1) $|\partial_{u_g} Z^{(a)}| = R$ for a constant positive number R ,
- (2) $u_g \in \mathbb{R}$.

The proof of this proposition can be found in [5, Proposition 3.4].

However, we did not deal with explicit expression of its reality conditions in [5]. Thus we will concentrate on the reality condition of loop soliton in this article. The reality condition of soliton equations were investigated well [[8, 3] and references therein] but these investigations can not be directly applied to our problem. On the other hand in [7], Mumford gave natural results on the reality condition of the elastica and a loop soliton of genus one. In other words, he showed the moduli of loop solitons of genus one as elasticas in terms of θ functions, or the geometry of the Abelian varieties of genus one. However when one considers its straightforward extension to general genus case, he encounters a difficulty. In the higher genus case, there appears a problem that the moduli of the Abelian varieties differs from the moduli of Jacobian varieties, *i.e.*, a problem that there are excess parameters in the Abelian varieties. On the other hand, on the investigation of loop soliton even with higher genus, we have chosen the strategy that we use only the data of curves themselves to avoid the problem of excess parameters, and give some explicit results in [5, 6]. Thus we will go on to follow the strategy to investigate the reality condition.

To use the strategy, we will, first, interpret the results of Mumford in terms of the language of the curve in the case of genus one. After then, we will apply the scheme to the reality condition of higher genus case. Section two is devoted to the reinterpretation of Mumford results. Section three gives the moduli of the loop solitons of genus two, which can be easily generalized to higher genus cases as in §4. As we will show in Theorem 4.4, the reality condition is reduced to the following conditions.

Theorem 1.3. *Let a set of the zero points e_b of y in (1.2) be denoted by \mathcal{B} . $Z^{(a)}$ satisfies the reality condition if and only if the following conditions satisfy,*

- (1) each $e_c \in \mathcal{B}$ is real,
- (2) there exists g pairs $(e_{c_j}, e_{d_j})_{j=1, \dots, g}$ satisfies $(e_{c_j} - e_a)(e_{d_j} - e_a) = e_a^2$ for negative e_a ,
- (3) the contour in the integral u_g in (1.3) satisfies a certain condition.

Using our result of this article, we give in principle explicit solutions of the loop solitons, even though the numerical problems might remain to illustrate its shape graphically. Though [12] illustrated shapes of large polymers in terms of elliptic functions as approximations, our results of this article promises to steps to exact solutions of such shapes.

In the investigation, we have a natural extension of Jacobi am-function for an elliptic curves to that for a hyperelliptic curve. We also compute winding numbers of loop soliton.

As there are so many open problems related to this as in [6, 9], this result could be applied to them.

2. GENUS ONE

First we consider the genus one case using data from the curve given by

$$\begin{aligned} y^2 &= x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0 \\ &= (x - e_1)(x - e_2)(x - e_3). \end{aligned} \quad (2.1)$$

The coordinate u of the complex plane $J_1^\infty := \mathbb{C}$ is given by,

$$\int^{(x,y)} du, \quad du = \frac{dx}{2y}. \quad (2.2)$$

It is known that a shape of the (classical) elastica, *i.e.*, a loop soliton with genus one, $Z : \mathbb{R} \hookrightarrow \mathbb{C}$ ($u \mapsto Z(u) = X^1(u) + \sqrt{-1}X^2(u)$) with $\partial_u Z = e^{\sqrt{-1}\phi}$ satisfies the differential equation,

$$a\partial_u(\phi) + \frac{1}{3}(\partial_u\phi)^3 + \partial_u^3\phi = 0, \quad (2.3)$$

where $\partial_u := d/du$.

Proposition 2.1 (Euler [2]). *A solution of (2.3) is given by*

$$\partial_u Z^{(a)} = (x - e_a),$$

for an elliptic curve given by the form (2.1). If it is a loop soliton if and only if it satisfies the reality condition:

- (1) $|\partial_u Z^{(a)}| = 1$.
- (2) $u \in \mathbb{R}$.

For a proof of the above propositions, see [5, Proposition 3.4].

Proposition 2.2 (Mumford [7]). *The moduli Λ of elastica or loop soliton of genus one is given by the following subspace in the upper half plane $\mathbb{H} := \{z \in \mathbb{C} \mid \Im z > 0\}$ modulo $\text{PSL}(2, \mathbb{Z})$,*

$$\Lambda := \sqrt{-1}\mathbb{R}_{>0} \cup \left(\frac{1}{2} + \sqrt{-1}\mathbb{R}_{>0} \right) \cup \infty \quad \text{modulo} \quad \text{PSL}(2, \mathbb{Z}).$$

Here $\mathbb{R}_{>0}$ is $\{x \in \mathbb{R} \mid x > 0\}$.

Though Mumford led this result using the geometry of Abelian variety of genus one [7], we will give another proof only using the language of curve itself as mentioned in Introduction. The purpose of this section is to give its proof using only the data of the curve itself.

Lemma 2.3. *For different numbers a, b and c in $\{1, 2, 3\}$, let $e^{2\sqrt{-1}\varphi_a} := (x - e_a)/c_{cba}$, $e_{ab} := e_a - e_b$ and $c_{cba} := \sqrt{e_{ca}e_{ba}}$. The elliptic differential of the first kind (2.2) up to sign is*

$$du = \frac{d\varphi_a}{\sqrt{(\sqrt{e_{ba}} - \sqrt{e_{ca}})^2 + 4\sqrt{e_{ba}e_{ca}} \sin^2 \varphi_a}}.$$

Proof. Direct computations give

$$\begin{aligned} dx &= 2c_{cba}\sqrt{-1}e^{2\sqrt{-1}\varphi_a}d\varphi_a, \\ y &= c_{cba}\sqrt{-1}e^{2\sqrt{-1}\varphi_a}\sqrt{e_{ba}(e^{-2\sqrt{-1}\varphi_a} - c_{cba}e_{ba}^{-1})(e^{2\sqrt{-1}\varphi_a} - c_{cba}^{-1}e_{ca})} \\ &= c_{cba}\sqrt{-1}e^{2\sqrt{-1}\varphi_a}\sqrt{e_{ba} + e_{ca} - 2\sqrt{e_{ba}e_{ca}} \cos 2\varphi_a}, \end{aligned}$$

up to sign. The addition formula $\cos(2\varphi) = 1 - 2\sin^2 \varphi$ leads the result. \square

Let us use the standard representations,

$$k := \frac{2\sqrt{-1}\sqrt[4]{e_{ba}e_{ca}}}{\sqrt{e_{ba}} - \sqrt{e_{ca}}}$$

and then

$$du = \frac{d\varphi_a}{(\sqrt{e_{ba}} - \sqrt{e_{ca}})\sqrt{1 - k^2 \sin^2 \varphi_a}}. \quad (2.4)$$

By letting $w := \sin(\varphi_a)$, (2.4) becomes

$$du = \frac{dw}{(\sqrt{e_{ba}} - \sqrt{e_{ca}})\sqrt{(1 - w^2)(1 - k^2 w^2)}} \quad (2.5)$$

Remark 2.4. (1) Due to the (2.4), we have the following elliptic integral $u(\varphi_a)$

$$u(\varphi_a) = \int_0^{\varphi_a} \frac{d\varphi}{H_a^{[1]}(\varphi)},$$

and its inverse function $\varphi_a(u)$ gives

$$\exp(\sqrt{-1}\varphi_a(u)) = \sqrt{x - e_a}.$$

As $\sqrt{(e_3 - e_1)/(x - e_3)}$ is sn-function, $\varphi_a(u)$ is essentially the same as Jacobi-am function $\text{am}(u)$ [10], though we need Landen-transformation.

(2) Behind (2.5), there is a kinematic system with an energy

$$E = \dot{w}^2 + (1 - w^2)(1 - k^2 w^2).$$

Due to the reality condition, Proposition 2.1 (1), φ_a belongs to a subregion of a real number. For any φ_a in a certain region $[\varphi_l, \varphi_u]$, the reality condition, Proposition 2.1 (2), requires that the denominator in (2.5) should be real and thus that k^2 , or $\sqrt{e_{ba}e_{ca}}$ and $(\sqrt{e_{ba}} - \sqrt{e_{ca}})^2$, should be real;

$$\Im\sqrt{e_{ba}} = \Im\sqrt{e_{ca}}, \quad \arg(e_{ba}) = -\arg(e_{ca}),$$

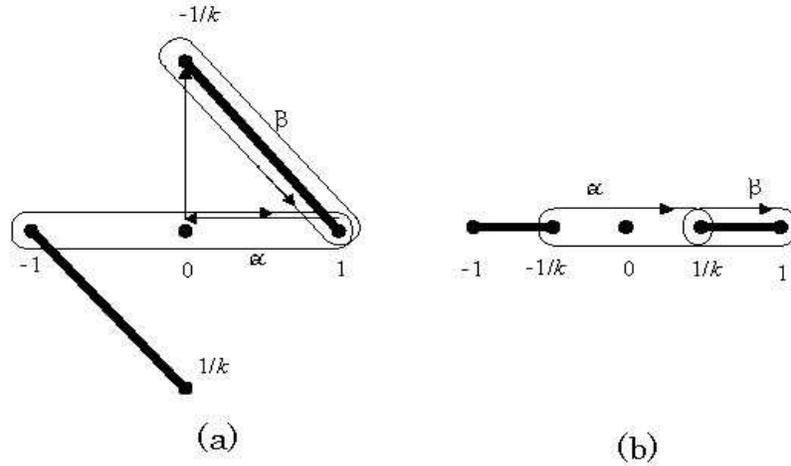


FIGURE 1. Geometry of Contours: α and β are Homology basis of the elliptic curves.

where $\arg(a) := \Im \log(a)$ for $a \in \mathbb{C}$. Accordingly introducing an expression $e_{ba} =: \beta_{ba} e^{\sqrt{-1}\alpha_{ba}}$, using $\alpha_{ba} \in [0, \pi)$ and $\beta_{ba} \in \mathbb{R}$,¹ the reality condition of the loop soliton $Z^{(a)}$ require alternative cases:

- (1) α_{ba} and α_{ca} vanish, i.e., e_{ba} and e_{ca} belong to \mathbb{R} , or
- (2) $\alpha_{ba} = -\alpha_{ca}$ and $\beta_{ba} = \beta_{ca}$.

However, the second case means that $(\sqrt{e_{ba}} - \sqrt{e_{ca}})^2$ vanishes and corresponds to $k = \infty$.² Thus we find the following lemma.

Lemma 2.5. *The reality condition of the loop soliton $Z^{(a)}$ is reduced to two alternative cases:*

- I-1 $e_{ba} > 0$ and $e_{ca} > 0$, i.e., $k \in \sqrt{-1}\mathbb{R}_{\geq 0}$, $w \equiv \sin \varphi_a \in [-1, 1]$.
- I-2 $e_{ba} \leq 0$ and $e_{ca} \leq 0$, i.e., $k > 1$ and $w \equiv \sin \varphi_a \in [1/k, 1]$ or $w \equiv \sin \varphi_a \in [-1, -1/k]$.

Proof. For general $\varphi_a \in \mathbb{R}$, u must be real. Hence the candidates of e_{ba} 's are followings: (I-0) $e_{ba} < 0$ and $e_{ca} > 0$, or $e_{ba} > 0$ and $e_{ca} < 0$, (I-1) $e_{ba} > 0$ and $e_{ca} > 0$, and (I-2) $e_{ba} \leq 0$ and $e_{ca} \leq 0$.

In (I-0) case $(\sqrt{e_{ba}} - \sqrt{e_{ca}})$ has a non-trivial angle in the complex plane, which cannot be cancelled by the other factors. We remove (I-0) case. (I-1) is obvious. The region of $\sin \phi_a$ must be a subset of $[-1, 1]$. On the case (I-2), noting that prefactor $1/(\sqrt{e_{ba}} - \sqrt{e_{ca}})$ generates the factor $\sqrt{-1}$, we conclude that $k > 1$ and $\sin \phi_a \in [1/k, 1]$ or $\sin \phi_a \in [-1, -1/k]$. \square

¹Here we defined $\beta_{ba} \in \mathbb{R}$ rather than $\beta_{ba} \in \mathbb{R}_{\geq 0}$ due to the domain of α_{ba} .

²Though it is not important, it is interesting that the second case can be reduced to the first case, i.e., $\alpha_{ca} = 0$, by transforming φ_a to $\varphi_a - \alpha_{ca}$ due to the formula in the proof in Lemma 2.3.

Proof of Proposition 2.2. Let us consider the geometry of the integration. Fig.1 gives an illustration of our situations, where Fig.1 (a) corresponds to case I-1 and (b) to case I-2 in Lemma 2.5

I-1: The periodicity $(4\omega, 2\omega')$ of $\sqrt{(x - e_a)}$ is given by

$$\omega = \int_0^1 \frac{dw}{\sqrt{(1-w^2)((\sqrt{e_{ba}} - \sqrt{e_{ca}})^2 + 4\sqrt{e_{ba}e_{ca}}w^2)}},$$

$$\omega' = \left(\int_1^0 + \int_0^{\sqrt{-1}/|k|} \right) \frac{dw}{\sqrt{(1-w^2)((\sqrt{e_{ba}} - \sqrt{e_{ca}})^2 + 4\sqrt{e_{ba}e_{ca}}w^2)}}.$$

Thus $\omega' = \omega + \sqrt{-1}L[k]$ for general k with a certain real valued function L . On the other hand, for $k \rightarrow 0$, $L \rightarrow \infty$ and for $k \rightarrow \infty$, L vanishes. Further $L[k]$ is a continuous function of k and its range is $\mathbb{R}_{>0}$. Hence $\tau = 2\omega'/4\omega \in (1/2 + \sqrt{-1}\mathbb{R}_{>0})$.

I-2: The periodicity $(4\omega, 2\omega')$ of $\sqrt{(x - e_a)}$ is given by

$$\omega = 2 \int_0^{1/k} \frac{dw}{\sqrt{(1-w^2)((\sqrt{e_{ba}} - \sqrt{e_{ca}})^2 + 4\sqrt{e_{ba}e_{ca}}w^2)}},$$

$$\omega' = \int_{1/k}^1 \frac{dw}{\sqrt{(1-w^2)((\sqrt{e_{ba}} - \sqrt{e_{ca}})^2 + 4\sqrt{e_{ba}e_{ca}}w^2)}}.$$

On the other hand, for $k \rightarrow 0$, $\omega \rightarrow \infty$ and for $k \rightarrow \infty$, ω vanishes while ω' is a finite number. Further $\omega[k]$ and $\omega'[k]$ are continuous in k . Hence $\tau = 2\omega'/4\omega \in \sqrt{-1}\mathbb{R}_{>0}$.

Since theory of the Jacobi elliptic functions gives the fact that $k' := \sqrt{1-k^2}$ gives the inversion of moduli $\tau \rightarrow -1/\tau$, the constraint $k > 1$ in Lemma 2.5 is less important.

We note that the periodicity of $\sqrt{(x - e_a)}$ differs from $\partial_u Z^{(a)}$ by twice but the difference is not so significant. Hence we have a complete proof of Proposition 2.2 based upon geometry of elliptic curve itself instead of geometry of Abelian variety as a domain of elliptic theta function. \square

Remark 2.6. (1) We list its special cases for $a = 1$:

- (a) $k = 0$ in I-1: its shape is a circle and its related curve is $y^2 = (x - e_1)^2(x - e_2)$
- (b) $k = \infty$ in I-2: its shape is a loop soliton solution, and its related curve is $y^2 = (x - e_1)(x - e_2)^2$

(2) Since $\partial_s Z \equiv e^{\sqrt{-1}\phi}$ can be regarded as a harmonic map: $\partial_s Z : S^1 \rightarrow S^1$ with energy

$$E = \oint ds |\partial_s \phi|^2.$$

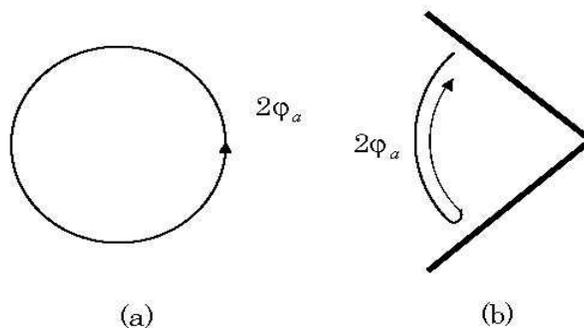
(3) Above Lemma 2.5, we argued the angle of e_{ba} 's. However the geometry of the integrals depends only on $\sqrt{e_{ba}e_{ca}}$ and $\sqrt{e_{ba}} - \sqrt{e_{ca}}$ rather than e_{ba} 's themselves.

For the map $\partial_u Z : S^1 \rightarrow S^1$, we can find index as a winding number as shown in Fig.2. We call it $\text{index}(\partial_u Z)$.

Corollary 2.7. *The index $(\partial_u Z)$ is given as follows.*

- I-1 $\text{index}(\partial_u Z) = \pm 1$.
- I-2 $\text{index}(\partial_u Z) = 0$.

Proof. In the case I-1, since the contours $w \equiv \sin \varphi_a$ is $[-1, 1]$ which is identified with the range of sine function, φ_a becomes a monotonic increasing function of

FIGURE 2. The behavior of φ

u . In fact passing by $w = \pm 1$ changes the sign of $\sqrt{1-w^2}$ or $\cos \varphi_a$. By paying attentions on the orientation of the contour, we have the sign of the index. On the other hand, in the case I-2, φ does not wind around S^1 like Fig. 2(b). The branch point $(1/k, 0)$ does not have an effect of the sign of $\sqrt{1-w^2}$. \square

3. GENUS TWO

In this section, we will investigate the reality condition associated with a hyperelliptic curve C_2 of genus two expressed by

$$\begin{aligned} y^2 &= x^5 + \lambda_4 x^4 + \lambda_3 x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0 \\ &= (x - e_1)(x - e_2)(x - e_3)(x - e_4)(x - e_5), \end{aligned} \quad (3.1)$$

where each e_a is a complex number \mathbb{C} . We have the coordinate system of the complex vector space $J_2^\infty := \mathbb{C}^2$;

$$u_1 = u_1^{(1)} + u_1^{(2)}, \quad u_2 = u_2^{(1)} + u_2^{(2)}, \quad (3.2)$$

$$u_1^{(i)} = \int_\infty^{(x^{(i)}, y^{(i)})} \frac{dx}{2y}, \quad u_2^{(i)} = \int_\infty^{(x^{(i)}, y^{(i)})} \frac{x dx}{2y}. \quad (3.3)$$

Let the Abelian map $\text{Sym}^2(C_2) \rightarrow J_2 := J_2^\infty / \Lambda$ be denoted by ω'_A where Λ is a lattice in J_2^∞ associated with C_2 . Considering winding numbers, we will denote the Abelian universal covering of $\text{Sym}^2(C_2)$ by $\mathcal{U}\text{Sym}^2(C_2)$ and its map from $\mathcal{U}\text{Sym}^2(C_2)$ to J_2^∞ by ω_A .

The loop soliton solution of (3.1) is given by $\partial_{t_1} Z^{(a)} = (x^{(1)} - e_a)(x^{(2)} - e_a)$ if it satisfies the reality condition.

Lemma 3.1. *For different numbers a, b and c of $\{1, 2, 3, 4, 5\}$, let $e^{2\sqrt{-1}\varphi_a^{(i)}} := (x^{(i)} - e_a)/c_{cba}$, $e_{ab} := e_a - e_b$ and $c_{cba} := \sqrt{e_{ba}e_{ca}}$. In general, the following*

relation up to sign holds:

$$du_2^{(i)} = \frac{\sqrt{-1}(c_{cba}e^{\sqrt{-1}\varphi_a^{(i)}} + e_a e^{-\sqrt{-1}\varphi_a^{(i)}})d\varphi_a^{(i)}}{\sqrt{((\sqrt{e_{ba}} - \sqrt{e_{ca}})^2 + 4\sqrt{e_{ba}e_{ca}}\sin^2\varphi_a^{(i)})c_{cba}e_{da}(e^{-2\sqrt{-1}\varphi_a^{(i)}} - c_{cba}e_{da}^{-1})}} \\ \times \frac{1}{\sqrt{(e^{2\sqrt{-1}\varphi_a^{(i)}} - c_{cba}^{-1}e_{ea})}}.$$

Proof. Direct computations lead the formula. \square

We will find a subspace $(\Gamma, \omega_A(\Gamma)) \subset \mathcal{USym}^2(C_2) \times J_2^\infty$ which satisfies the reality condition. We note that since the reality condition is local, we need not pay attentions upon the difference between $\text{Sym}^2(C_2)$ and $\mathcal{USym}^2(C_2)$.

Lemma 3.2. *The reality condition of the loop soliton $Z^{(a)}$ satisfies if and only if $(x^{(1)}, x^{(2)}) \in \mathcal{USym}^2(C_2)$ and λ 's satisfy the following relations:*

- (1) $|(x^{(i)} - e_a)| = K_i$ of a real constant K_i , ($i = 1, 2$),
- (2) $u_2^{(i)} \in \mathbb{R}$ for $i = 1, 2$.

Proof. Proposition 1.2 leads to $(x^{(1)}, x^{(2)}) \in \Gamma \subset \mathcal{USym}^2(C_2)$ satisfying the reality conditions is given by

$$|x^{(2)} - e_a| = \frac{K}{|x^{(1)} - e_a|}, \quad (3.4)$$

for a real constant K and

$$\Im u_2^{(2)}(x^{(2)}) = -\Im u_2^{(1)}(x^{(1)}). \quad (3.5)$$

When (3.4) is trivial, *i.e.*, $|(x^{(i)} - e_a)| = K_i$ of a real constant K_i , ($i = 1, 2$), and both sides in (3.5) vanish, we obtain above conditions as sufficient conditions.

Thus we will consider its necessity condition. Assume that both conditions (3.4) and (3.5) are not trivial, *i.e.*, $x^{(2)}$ and $x^{(1)}$ are not independent. Since these conditions (3.4) and (3.5) are real analytic ones, we must also deal with their complex conjugate $\bar{x}^{(1)}$, $\bar{x}^{(2)}$, and so on. Due to the conditions, for example, $x^{(2)}$ is a function of $x^{(1)}$, $\bar{x}^{(1)}$, and $\bar{x}^{(2)}$. Of course, there is no guarantee whether there exists such a function $x^{(2)}(x^{(1)}, \bar{x}^{(1)}, \bar{x}^{(2)})$ and even continuity but we can assume that they exist, at least, locally. The reality condition locally determines an open subspace $\omega_A(\Gamma)$ in J_2^∞ . Due to the dependence between $x^{(1)}$ and $x^{(2)}$ or $u_2^{(1)}$ and $u_2^{(2)}$, u_1 , u_2 , \bar{u}_1 and \bar{u}_2 are neither independent over $\omega_A(\Gamma)$. Hence $\partial/\partial x^{(1)}|_{x^{(2)}}$ nor $\partial/\partial u_1|_{u_2}$ do not behave well as differential operators among sections over Γ and $\omega_A(\Gamma)$, and should be replaced with covariant derivatives. For example, $\partial/\partial u_1|_{u_2}$ is replaced with $\partial/\partial u_1 - A_{u_1}(u_1, u_2, \bar{u}_1, \bar{u}_2)$ using an appropriate connection A_{u_1} .

On the other hand, the loop soliton $\partial_{t_1} Z^{(a)}$ is a meromorphic function over Γ and $\omega_A(\Gamma)$. However it is a restricted section of the J_2^∞ at $\omega_A(\Gamma)$ in (3.2) and satisfies the MKdV equation (1.1) with respect *only* to the differentials of u_1 and u_2 over there as mentioned in Proposition 1.2. However the connection A_{u_1} prevents that the angle part of $\partial_{t_1} Z^{(a)}$ does satisfy the MKdV equation (1.1). Hence A_{u_1} and A_{u_2} must vanish.

However, the condition that A_{u_1} vanishes means that $\omega_A(\Gamma)$ is a flat real plane in $J_2^\infty = \mathbb{C}^2$ and $x^{(2)}$ is independent of $x^{(1)}$. Hence we prove this Lemma. \square

Remark 3.3. By letting an appropriate immersion $\iota : S^1 \hookrightarrow C_2$, $\partial_{u_2} Z \circ \omega_A \circ \iota$ is an analytic map from S^1 to S^1 .

Lemma 3.4. For the situation of Lemma 3.1, the reality condition of the loop soliton $Z^{(a)}$ needs $e_a = -c_{cba}$ and $c_{eda} = c_{cba}$, and then we have the relation up to sign,

$$du_2^{(i)} = \frac{2\sqrt{c_{cba}} \sin \varphi_a^{(i)} d\varphi_a^{(i)}}{\sqrt{((\sqrt{e_{ba}} - \sqrt{e_{ca}})^2 + 4\sqrt{e_{ba}e_{ca}} \sin^2 \varphi_a^{(i)})((\sqrt{e_{da}} - \sqrt{e_{ea}})^2 + 4\sqrt{e_{da}e_{ea}} \sin^2 \varphi_a^{(i)})}}.$$

Proof. Due to the Lemma 3.2, $\varphi_a^{(i)}$ is real and each factor must be real. Hence the imaginary parts should be canceled locally. It means the conditions. \square

Let us introduce a representation as an extension of the standard representation (2.4),

$$k_1 := \frac{2\sqrt{-1}\sqrt[4]{e_{ba}e_{ca}}}{\sqrt{e_{ba}} - \sqrt{e_{ca}}}, \quad k_2 := \frac{2\sqrt{-1}\sqrt[4]{e_{da}e_{ea}}}{\sqrt{e_{da}} - \sqrt{e_{ea}}},$$

and then

$$du_2^{(i)} = \frac{2\sqrt[4]{e_{ba}e_{ca}} \sin \varphi_a^{(i)} d\varphi_a^{(i)}}{(\sqrt{e_{ba}} - \sqrt{e_{ca}})(\sqrt{e_{da}} - \sqrt{e_{ea}})\sqrt{1 - k_1^2 \sin^2 \varphi_a^{(i)}}\sqrt{1 - k_2^2 \sin^2 \varphi_a^{(i)}}}. \tag{3.6}$$

By letting $w := \sin(\varphi_a^{(i)})$, we have

$$du_2^{(i)} = \frac{\sqrt[4]{e_{ba}e_{ca}} w dw}{\sqrt{(1 - w^2)((\sqrt{e_{ba}} - \sqrt{e_{ca}})^2 + 4\sqrt{e_{ba}e_{ca}} w^2)((\sqrt{e_{da}} - \sqrt{e_{ea}})^2 + 4\sqrt{e_{da}e_{ea}} w^2)}} = \frac{2\sqrt[4]{e_{ba}e_{ca}} w dw}{(\sqrt{e_{ba}} - \sqrt{e_{ca}})(\sqrt{e_{da}} - \sqrt{e_{ea}})\sqrt{(1 - w^2)(1 - k_1^2 w^2)(1 - k_2^2 w^2)}}. \tag{3.7}$$

Remark 3.5. (1) (3.7) is an elliptic integral by $u = w^2$ due to a specialty of genus two. It cannot be generalized to higher genus case.

(2) Due to the remark 2.4, we should regard that (3.6) gives the integral as a function $u_2^{(i)}$ of $\varphi_a^{(i)}$,

$$u_2^{(i)} = \int_0^{\varphi_a^{(i)}} \frac{d\varphi_a^{(i)'}}{H_a^{[2]}(\varphi_a^{(i)'})}$$

for an appropriate function $H_2^{[2]}$. Hence the inverse function $\varphi_a^{(i)}(u_2^{(i)})$ gives the relation,

$$\exp(\sqrt{-1}\varphi_a^{(i)}(u_2^{(i)})) = \sqrt{(x^{(i)} - e_a)/c_{cba}}.$$

Further $\varphi_a := \varphi_a^{(1)}(u_2^{(1)}) + \varphi_a^{(2)}(u_2^{(2)})$ gives the al-function of $u_2 := u_2^{(1)} + u_2^{(2)}$ [1, 13],

$$\exp(\sqrt{-1}\varphi_a(u_2)) = \text{al}_a(u_2).$$

Accordingly, we should regard this φ_a as a hyperelliptic am-function of genus two.

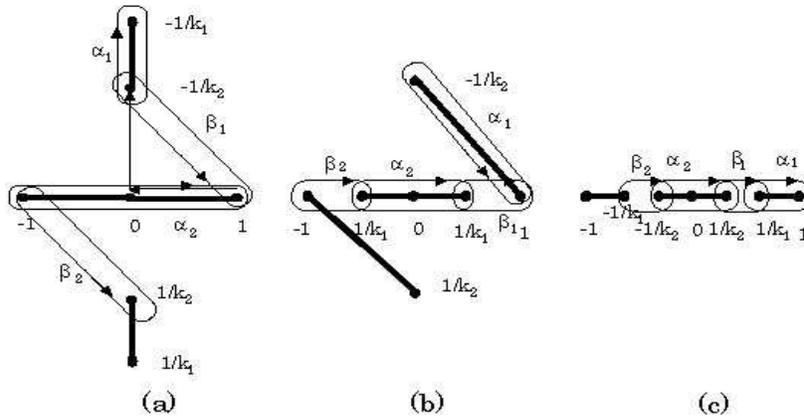


FIGURE 3. Geometry of Contours: $\alpha_1, \beta_1, \alpha_2$ and β_2 are Homology basis of the hyperelliptic curves.

(3) Behind the hyperelliptic am-functions, there is also kinematic system with a hamiltonian:

$$E = \dot{w}^2 + (1 - w^2)((\sqrt{e_{ba}} - \sqrt{e_{ca}})^2 + 4\sqrt{e_{ba}e_{ca}}w^2)((\sqrt{e_{da}} - \sqrt{e_{ea}})^2 + 4\sqrt{e_{da}e_{ea}}w^2).$$

For each $\varphi_a^{(i)}$ in a region $[\varphi_l, \varphi_u]$, the reality condition of the loop soliton $Z^{(a)}$, Lemma 3.2 (2), requires that the denominator should be real and thus that k_d^2 , or $\sqrt{e_{ba}e_{ca}}$ and $\sqrt{e_{ba}} - \sqrt{e_{ca}}$ should be also real.

Theorem 3.6. *The reality condition of the loop soliton $Z^{(a)}$ of genus two is reduced to the conditions: $e_a = -c_{cba}$ and $c_{eda} = c_{cba}$ with three alternative cases:*

- II-1. $e_{ba} > 0, e_{ca} > 0, e_{ea} > 0, e_{da} > 0$, i.e., $k_1, k_2 \in \sqrt{-1}\mathbb{R}$ and $\sin \varphi_a \in [-1, 1]$.
- II-2. $e_{ba} > 0, e_{ca} > 0, e_{ea} \leq 0$ and $e_{da} \leq 0$, i.e., $k_1 \in \sqrt{-1}\mathbb{R}$ and $k_2 \in \mathbb{R}$ $\sin \varphi_a \in [1/k_2, 1]$ or $\sin \varphi_a \in [-1, -1/k_2]$.
- II-3. $e_{ba} \leq 0, e_{ca} \leq 0, e_{ea} \leq 0, e_{da} \leq 0$, i.e., $k_1, k_2 \in \mathbb{R}, (k_1 < k_2)$,
 - (a) if $k_2 < 1, \sin \varphi_a \in [-1, 1]$.
 - (b) if $k_2 > 1, \sin \varphi_a \in [-1/k_2, 1/k_2]$.
 - (c) if $k_1 > 1, \sin \varphi_a \in [1/k_1, 1]$ or $\sin \varphi_a \in [-1, -1/k_1]$.

Proof. As in the case of the elliptic curves, we have the results. □

Fig.3 gives an illustration of our situation, where Fig.3 (a) corresponds to II-1 and (b) does to II-2 and (c) to II-3.

In this case, we show the $\text{index}(\partial_{t_1} Z)$.

Corollary 3.7. *The $\text{index}(\partial_{t_1} Z)$ as a winding number of the map $\iota(S^1)$ to S^1 is*

- II-1. $\text{index}(\partial_{t_1} Z) = 0$ or ± 2 ,
- II-2. $\text{index}(\partial_{t_1} Z) = 0$,
- II-3. (a) $\text{index}(\partial_{t_1} Z) = 0$ or ± 2 , and (b) (c) $\text{index}(\partial_{t_1} Z) = 0$.

Proof. These indexes consist of those of each $2\varphi_a^{(i)}$. If the index of $2\varphi_a^{(i)}$ is one, that of $2\varphi_a$ is sum over $i = 1, 2$, $\varphi_a = \pm\varphi_a^{(1)} \pm \varphi_a^{(2)}$. Here \pm depends upon the orientation of contours. The computations of φ_a are essentially the same as the genus one illustrated in Fig. 2. \square

4. GENUS g

The computations of genus two are easily extended to higher genus loop solitons. Let us introduce the sets, $A := \{1, 2, 3, \dots, 2g + 1\}$, $A_a := A - \{a\}$ for $a \in A$, $O_1 := \{3, 5, \dots, 2g - 1\}$, and a bijection $\sigma_a : \{1, 2, \dots, 2g\} \rightarrow A_a$ for $a \in A$ which determines the order. We will fix the order σ_a for an $a \in A$.

Recalling the facts in genus two case, the direct computations give the following lemmas.

Lemma 4.1. For $a \in A$, let $e^{2\sqrt{-1}\varphi_a^{(i)}} := (x^{(i)} - e_a)/c_{cba}$, $e_{ba} := e_{\sigma_a(b)} - e_a$ and $c_{cba} := \sqrt{e_{ba}e_{ca}}$,

$$D_{a,\sigma_a}^{(i)}(\varphi_a) := \left((\sqrt{e_{1a}} - \sqrt{e_{2a}})^2 + 4\sqrt{e_{1a}e_{2a}} \sin^2 \varphi_a^{(i)} \right) \times \prod_{d \in O_1, e=d+1} c_{12a}e_{da} (e^{-2\sqrt{-1}\varphi_a^{(i)}} - c_{12a}e_{da}^{-1}) (e^{2\sqrt{-1}\varphi_a^{(i)}} - c_{12a}^{-1}e_{ea})^{1/2},$$

$$N_{a,\sigma_a}^{(i)}(\varphi_a) := \left(\sqrt{-1}(c_{12a}e^{\sqrt{-1}\varphi_a^{(i)}} + e_a e^{-\sqrt{-1}\varphi_a^{(i)}}) \right)^{g-1}.$$

In general, (1.4) up to sign becomes

$$du_g^{(i)} = \frac{N_{a,\sigma_a}^{(i)} d\varphi_a}{D_{a,\sigma_a}^{(i)}}.$$

Lemma 4.2. For the situations of Lemma 4.1, the reality condition of the loop soliton $Z^{(a)}$ requires the conditions that $e_a = -c_{cba}$ for any $c \in O_1$, $b = c + 1$ and then we have

$$D_{a,\sigma_a}^{(i)}(\varphi_a) = \left((\sqrt{e_{1a}} - \sqrt{e_{2a}})^2 + 4\sqrt{e_{1a}e_{2a}} \sin^2 \varphi_a^{(i)} \right) \times \prod_{d \in O_1, e=d+1} \left((\sqrt{e_{da}} - \sqrt{e_{ea}})^2 + 4\sqrt{e_{da}e_{ea}} \sin^2 \varphi_a^{(i)} \right)^{1/2}, \tag{4.1}$$

$$N_{a,\sigma_a}^{(i)}(\varphi_a) = \left(2\sqrt{c_{12a}} \sin \varphi_a^{(i)} \right)^{g-1}.$$

These lemma can be proved along the line of the arguments for the case of genus two.

Corresponding to Remark 3.5, we have the following remarks:

Remark 4.3. (1) Let $\varphi_a := \varphi_a^{(1)} + \varphi_a^{(2)} + \dots + \varphi_a^{(g)}$ and then (1.5) is expressed by

$$\partial_{t_1} Z^{(a)} = e^{2\sqrt{-1}\varphi_a},$$

as a function of $u_g := u_g^{(1)} + u_g^{(2)} + \dots + u_g^{(g)}$. The hyperelliptic al-function is written by

$$\text{al}_a(u) = e^{\sqrt{-1}\varphi_a(u)},$$

(2) φ_a can be regarded as hyperelliptic am-function of genus g .

We will state our main theorem as follows, which is also proved along the line of the same arguments in the case of genus two.

Theorem 4.4. *The reality condition of the loop soliton $Z^{(a)}$ in (1.5) can be reduced to the conditions that there are g pairs $(e_{b,a}, e_{b+1,a})_{b \in O_1} \in \mathbb{R}^2$ satisfying $-e_a = \sqrt{e_{b,a}e_{b+1,a}} \geq 0$, and the contour of integral of each $u_g^{(i)}$ of $i = 1, \dots, g$ should be chosen so that $u_g^{(i)}$ is real.*

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