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Existence of periodic solutions for a semilinear ordinary differential equation *

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Abstract

Dancer [3] found a necessary and sufficient condition for the existence of periodic solutions to the equation

$$\ddot{x} + g_1(\dot{x}) + g_0(x) = f(t)\,.$$

His condition is based on a functional that depends on the solution to the above equation with $g_0 = 0$. However, that solution is not always explicitly known which makes the condition unverifiable in practical situations. As an alternative, we find computable bounds for the functional that provide a sufficient condition and a necessary condition for the existence of solutions.

1 Introduction

In this paper, we study the existence and the non-existence of solutions of the semilinear boundary-value problem

$$\ddot{x}(t) + g_1(\dot{x}(t)) + g_0(x(t)) = f(t)\,, \tag{1}$$

$$x(0) = x(T),\ \dot{x}(0) = \dot{x}(T)\,. \tag{2}$$

Although a necessary and sufficient condition is already known [2], it can not be verified in practical situations because the condition is given by a related nonlinear boundary-value problem. In this article we give, on the one hand, a sufficient condition, and on the other hand a necessary condition, which can be verified for any continuous function f. In the first part of this article, we present a survey of known results and their physical interpretation. And in the second part, we present our main result, which is stated as Theorem 2.

Overall, we will suppose that g_0, g_1, f are continuous real-valued functions, and f is T–periodic. For a given $k \geq 0$, let

$$\begin{aligned} C_T^k \quad = \quad & \big\{u : u \text{ is } k\text{-times continuously differentiable on } [0,T], \text{ with} \\ & u(0) = u(T), u'(0) = u'(T), \dots, u^{(k)}(0) = u^{(k)}(T)\big\}\,. \end{aligned}$$

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In these spaces the maximum norm will be denoted by $\|\cdot\|_{C^k_T}$, and C^0_T will be denoted by C_T. The subspace consisting of functions with mean value zero will be denoted by

$$\tilde{C}^k_T = \big\{u \in C^k_T : \int_0^T u(t)\,dt = 0\big\}\,.$$

For functions with domain $[0,T]$, with distributional derivatives, we define:

$$\begin{gathered} L^p = \big\{u : \textstyle\int_0^T |u(t)|^p dt < \infty\big\},\ 1 \le p < +\infty\,; \\ L^\infty = \big\{u : \text{ess sup}_{t\in[0,T]}\,|u(t)| < \infty\big\}, \\ W^{1,2}_T = \big\{u \in L^2 : u' \in L^2,\ u(0) = u(T)\big\}\,, \\ W^{2,\infty}_T = \big\{u \in L^\infty : u'' \in L^\infty,\ u(0) = u(T),\ u'(0) = u'(T)\big\}\,. \end{gathered}$$

For a subset X on the space of integrable functions on $[0,T]$, we define $\tilde{X} = \big\{u \in X : \int_0^T u(t)dt = 0\big\}$. For integrable functions, we use the decomposition

$$f = \tilde{f} + \bar{f}\,, \quad \text{with } \bar{f} = \frac{1}{T}\int_0^T f(t)\,dt\,.$$

We will assume that f, the right-hand side of (1), belongs to C_T, and the solution x belongs to C^2_T. Although the results in [2] assume that f is in a certain Lebesgue space and that x is in a certain Sobolev space, it is not hard to get analogous results for f in C_T and x in C^2_T. So when we cite results from [2], we do a conversion to our function spaces (except in section 2).

When $g_0 = 0$, for each $\tilde{f} \in \tilde{C}_T$ there exists a value $s(\tilde{f})$ such that

$$\ddot{x}(t) + g_1(\dot{x}(t)) = \tilde{f}(t) + s(\tilde{f}) \tag{3}$$

has a periodic solution, [2, Theorem 1]. Equivalently, the range of the operator $H_1 : C^2_T \to C_T$, $H_1(x) = \ddot{x} + g_1 \circ \dot{x}$, can be written as

$$\mathcal{R}_1 = \{\tilde{f} + s(\tilde{f}) : \tilde{f} \in \tilde{C}_T\}\,. \tag{4}$$

Under the assumption that g_0 is bounded, continuous, and satisfies

$$g_0(-\infty) := \lim_{\xi\to-\infty} g_0(\xi) < \lim_{\xi\to+\infty} g_0(\xi) =: g_0(+\infty)\,,$$

Dancer [2, Theorem 2] showed that a function $f \in C_T$ belongs to $\mathcal{R}$, the range of $H : C^2_T \to C_T$, $H(x) = \ddot{x} + g_1 \circ \dot{x} + g_0 \circ x$, if

$$g_0(-\infty) < \bar{f} - s(\tilde{f}) < g_0(+\infty)\,. \tag{5}$$

Thus, (5) is a sufficient condition for (1) to posses a periodic solution. However, if we also have

$$g_0(-\infty) < g_0(\xi) < g_0(+\infty) \quad \forall \xi \in \mathbb{R}\,,$$

then (5) is also a necessary condition, [2, Theorem 4]. Since we do not know the functional $s(\tilde{f})$ explicitly, we can not verify condition (5) in practical situations.

The aim of our work is to find estimates for the functional $s(\tilde{f})$. In particular we find functionals $a : \tilde{C}_T \to \mathbb{R}$, and $A : \tilde{C}_T \to \mathbb{R}$, such that $a(\tilde{f}) \le s(\tilde{f}) \le A(\tilde{f})$ for all $\tilde{f} \in C_T$ (see Theorem 2). Using these bounds we define the sets:

$$\mathcal{A}_1 = \left\{ f : f \in C_T \text{ and } a(\tilde{f}) \le \bar{f} \le A(\tilde{f}) \right\} ,$$
$$\mathcal{A} = \left\{ f : f \in C_T \text{ and } g_0(-\infty) + a(\tilde{f}) < \bar{f} < g_0(+\infty) + A(\tilde{f}) \right\} ,$$
$$\mathcal{B} = \left\{ f : f \in C_T \text{ and } g_0(-\infty) + A(\tilde{f}) < \bar{f} < g_0(+\infty) + a(\tilde{f}) \right\} .$$

Main result With the above definitions, $\mathcal{R} = H(C_T^2)$, and $\mathcal{R}_1 = H_1(C_T^2)$, our main result is stated as

$$\mathcal{R}_1 \subset \mathcal{A}_1, \quad \text{and} \quad \mathcal{B} \subset \mathcal{R} \subset \mathcal{A} .$$

This means that f being in $\mathcal{A}$ is a necessary condition, and that f being in $\mathcal{B}$ is a sufficient condition for the existence of solutions to (1).

2 Related results

In this section, we present some known results, and give a physical interpretation for particular cases of equation (1). We want to emphasize the fact that although the conditions come from abstract methods of functional analysis, they have physical interpretations (For various physical examples see e.g. [7] or [8]).

If the function g_0 is bounded and $g_1(\xi) = \lambda\xi$ for some $\lambda \in \mathbb{R}$, then (1) becomes the "classical" Landesman-Lazer equation.

$$\ddot{x}(t) + \lambda\dot{x}(t) + g_0(x(t)) = f(t) . \tag{6}$$

A short review of applicable results for this equation with boundary conditions (2) is as follows:

- A sufficient condition for (6) to have a T–periodic solution is the so called *Landesman–Lazer condition* [4],
$$g_0(-\infty) < \bar{f} < g_0(+\infty) . \tag{7}$$
- Condition (7) is also necessary when $g_0(-\infty) < g_0(\xi) < g_0(+\infty)$ for all $\xi \in \mathbb{R}$.
- The range of the operator $\ddot{x} + \lambda\dot{x} + g_0 \circ x : C_T^2 \to C_T$ is a set of functions in C_T, enclosed by two parallel hyper-planes.

From a physical point of view, when $g_0(-\infty) < 0 < g_0(+\infty)$, this boundary-value problem is a model for vibrations with linear damping and nonlinear restoring force. When λ is equal to zero, we have a conservative oscillator. Condition (7) can be interpreted as representing the restoring force being able to overcome the mean value of the external forcing term f.

For $g_0 = 0$, a brief summary of results is as follows:

- For g_1 continuous, Dancer [2] proved that for all $\tilde{f} \in \tilde{L}^\infty$ there exists exactly one $s(\tilde{f}) \in \mathbb{R}$ such that (3) has solution x in $W_T^{2,\infty}$, in the sense of distributions. Furthermore, the functional $s : \tilde{L}^\infty \to \mathbb{R}$ is continuous.

- For g_1 continuous, Mawhin [5] showed that for all $\tilde{f} \in \tilde{L}^1$ there exists $s(\tilde{f}) \in \mathbb{R}$ such that (3) has a strong solution.

- For g_1 continuously differentiable, Cañada, Drábek [1] proved that for all $\tilde{f} \in \tilde{C}_T$ there exists exactly one $s(\tilde{f}) \in \mathbb{R}$ such that (3) has a classical solution. Furthermore, the functional $s : \tilde{C}_T \to \mathbb{R}$ is continuously differentiable, and the range of H_1 can be written as in (4).

- The functional s gives the necessary and sufficient condition for the solvability of the boundary-value problem, namely

  $$\bar{f} = s(\tilde{f})\,.$$

  But $s(\tilde{f})$ is given in terms of the solution, which we do not know a priori. Thus, we can not formulate the condition explicitly as is done in the Landesman–Lazer result.

From a physical point of view, (3) describes the periodically forced vibration of a mass on a damper. The damping term makes the system unbalanced and $s(\tilde{f})$ represents a constant force which tends to compensate for the damping term. In this example, we consider the dissipative case: $\dot{x}g_1(\dot{x}) > 0$, or $\dot{x}g_1(\dot{x}) < 0$, which represents a self–excitation (positive damping).

For general functions g_1 and g_0, with g_0 bounded as in the Landesman–Lazer case, Dancer [2] proved that the range $H(W_T^{2,\infty})$ is enclosed by two manifolds parallel to the range $H_1(W_T^{2,\infty})$. A sufficient condition for the solvability of Problem (1)–(2) is given by (5). Note that if $g_0(-\infty) < 0 < g_0(+\infty)$, then from (5) it follows that the range $\mathcal{R}_1$ of the operator H_1 is a subset of the range $\mathcal{R}$ of the operator H. In this case (1) is a model for vibrations with nonlinear damping and nonlinear restoring force.

3 Bounds for $s(\tilde{f})$

Estimates for $s(\tilde{f})$ are derived from the study of equation (3). Putting $w = \dot{x}$, problem (3) subject to (2) becomes

$$\dot{w}(t) + g_1(w(t)) = \tilde{f}(t) + s(\tilde{f})\,, \tag{8}$$

$$w(0) = w(T)\,, \quad \int_0^T w(\tau)\,d\tau = 0\,. \tag{9}$$

Theorem 1 *Let g_1 be a continuously differentiable function satisfying $|g_1(\xi)| \leq K$ for all $\xi \in \mathbb{R}$. Then for each $\tilde{f} \in \tilde{C}_T$ there exists precisely one $s(\tilde{f})$ such that*

(3) has a periodic solution. In this case problem (3) has a family of solutions $x_c(t) = x(t) + c$, where $c \in \mathbb{R}$ is arbitrary and

$$x(t) = \int_0^t w_{s(\tilde{f})}(\tau)\, d\tau\,,$$

with $w_{s(\tilde{f})}$ the unique solution of (8) subject to (9). Moreover, the map $s : \tilde{f} \mapsto s(\tilde{f})$ from $\tilde{C}_T$ to $\mathbb{R}$ is continuously differentiable and $-K \le s(\tilde{f}) \le K$.

The proof of the above theorem can be found in [1]. Existence results considering a continuous function g_1 are studied in [2]. The analogous a priori bound for $\|w\|_{C_T}$ as in the following Lemma is also given in [2].

Lemma 1 *Let g_1 be a continuous function, and w be the solution of (8) subject to (9). Then*

$$\|w\|_{C_T} \le \|\tilde{f}\|_2 \sqrt{\frac{T}{12}}\,.$$

Proof Multiplying both sides of equation (8) by $\dot{w}$ and integrating from 0 to T, using the boundary condition $w(0) = w(T)$, we see that $\|\dot{w}\|_2^2 = \langle \tilde{f}, \dot{w}\rangle_2$. The Cauchy-Schwartz inequality yields that $\|\dot{w}\|_2^2 \le \|\tilde{f}\|_2 \|\dot{w}\|_2$, so that $\|\dot{w}\|_2 \le \|\tilde{f}\|_2$. Since $w \in C_T^1 \subset W_T^{1,2}$ and $\int_0^T w(t)dt = 0$, we can use a Sobolev inequality, [6, Prop. 1.3], to obtain

$$\|w\|_\infty \le \|\dot{w}\|_2 \sqrt{\frac{T}{12}} \le \|\tilde{f}\|_2 \sqrt{\frac{T}{12}}\,.$$

Since w is continuous, $\|w\|_{C_T} = \|w\|_\infty$ which is the desired inequality. ♢

As a consequence of the above lemma, Theorem 1 can be applied for a function g_1 that is not necessarily bounded. This is so because the argument of g_1 lies on a bounded interval.

An estimate for $s(\tilde{f})$ is obtained as follows: Integrate each term in (8) from 0 to T, use the boundary conditions (9) to eliminate terms with $\dot{w}, cw, \tilde{f}$, and divide by T, to obtain

$$\frac{1}{T} \int_0^T g_1(w(t))\, dt = s(\tilde{f})\,.$$

As in [2, Theorem 1], the minimum and the maximum values of g_1 provide bounds for $s(\tilde{f})$. To obtain the basic estimate we use the fact that $\int_0^T w = 0$. First subtract cw in the integrand above, and then compute the infimum and the supremum over $c \in \mathbb{R}$:

$$\sup_{c\in\mathbb{R}} \min_{|\xi|\le b} (g_1(\xi) - c\xi) \le s(\tilde{f}) \le \inf_{c\in\mathbb{R}} \max_{|\xi|\le b} (g_1(\xi) - c\xi)\,, \tag{10}$$

where $\|w\|_{C_T} \le b$ (for instance we can set $b = \|\tilde{f}\|_2 \sqrt{T/12}$ due to Lemma 1).

Remarks If the function g_1 is a polynomial of degree 1, then $a(\tilde{f}) = A(\tilde{f}) = g_1(0)$, and this is the exact value of $s(\tilde{f})$. On the other hand if $g_1(w) = w^2$, then $a(\tilde{f}) < A(\tilde{f})$ and the direct estimate in Dancer [2, Theorem 1] is the same as the basic estimate.

Finding the infimum and the supremum over all real numbers is not amenable for computations; hence, we need to find a finite set of suitable values for c. For example, the upper bound can be interpreted as an error in a minimax polynomial approximation. In which case, we are looking for a polynomial $q(\xi) = c\xi + d$ such that $\|g_1 - q\|_\infty$ is as small as possible. With interpolation nodes $\{-b, 0, b\}$, we obtain $c = (g_1(b) - g_1(-b))/(2b)$, and we avoid calculating the supremum over $\mathbb{R}$.

Notice that the upper bound minus the lower bound in (10) is an increasing function of b, the bound for $\|\dot{x}\|_\infty$. Therefore, our strategy is to decrement b, which is done by using the following two lemmas.

Lemma 2 *Let k and K be positive constants, and $w \in \tilde{C}_T$ be absolutely continuous with $-k \le \dot{w}(\xi) \le K$ a.e. on $[0,T]$. Then*

$$\|w\|_{C_T} \le \frac{TkK}{2(k+K)}\,.$$

Proof On the contrary, suppose that $\|w\|_{C_T} > \frac{TkK}{2(k+K)}$. Without lost of generality, we may assume that the maximum norm is attained at a point $t_0 = \frac{kT}{2(k+K)}$, $0 \le t_0 \le T/2$. If necessary multiply w by -1, interchange the roles of k and K, and shift w suitably in time. Then

$$w(t_0) = \|w\|_{C_T} > \frac{TkK}{2(k+K)} = Kt_0\,.$$

Our strategy is to prove the following two inequalities for $t \in [0, \frac{T}{2}]$:

$$w(t) > \min\left\{Kt, k(\tfrac{T}{2} - t)\right\}, \tag{11}$$

$$w(t + \tfrac{T}{2}) > -\min\left\{kt, K(\tfrac{T}{2} - t)\right\}. \tag{12}$$

Which lead us to the contradiction that $\int_0^T w = 0$ and

$$\int_0^T w(t)\,dt = \int_0^{T/2} w(t)\,dt + \int_0^{T/2} w(t + \frac{T}{2})\,dt > 0\,.$$

For (11), we consider the two cases: If $0 < t \le t_0$, then

$$w(t) = w(t_0) + \int_t^{t_0} \big(-\dot{w}(\tau)\big)\,d\tau > Kt_0 + (t_0 - t)(-K) = Kt\,.$$

and if $t_0 < t \le \frac{T}{2}$, then

$$w(t) = w(t_0) + \int_{t_0}^t \dot{w}(\tau)\,d\tau > Kt_0 + (t - t_0)(-k) = k(\frac{T}{2} - t)\,.$$

For (12), we put $u(t) = w(t + \frac{T}{2})$ and notice that $u(0) = w(\frac{T}{2}) > 0$ and $u(\frac{T}{2}) = w(T) = w(0) > 0$. For t in $[0, \frac{T}{2}]$ we have

$$w\left(t + \tfrac{T}{2}\right) = u(t) = u(0) + \int_0^t \dot{u}(\tau)\, d\tau > -kt$$

and

$$w\left(t + \tfrac{T}{2}\right) = u(t) = u\left(\tfrac{T}{2}\right) + \int_t^{T/2} \left(-\dot{u}(\tau)\right) d\tau > -K(\frac{T}{2} - t)\,.$$

Hence

$$w(t + \tfrac{T}{2}) > \max\left\{ -kt, -K(\frac{T}{2} - t)\right\} = -\min\left\{kt, K(\frac{T}{2} - t)\right\}.$$

Which concludes the present proof. ♢

Lemma 3 *Let w be a solution to Problem (8)-(9), with $\|w\|_{C_T} \le b$ and $\tilde{f} \not\equiv 0$. Then*

$$-k \le \dot{w}(t) \le K\,,$$

where k and K are the positive constants: $-k = \min_{t\in[0,T]} \tilde{f}(t) + m$ and $K = \max_{t\in[0,T]} \tilde{f}(t) + M$, where

$$\begin{aligned} m &= \sup\nolimits_{c\in\mathbb{R}} \min\nolimits_{|\xi|\le b}(g_1(\xi) - c\xi) - \max\nolimits_{|\xi|\le b} g_1(\xi)\,, \\ M &= \inf\nolimits_{c\in\mathbb{R}} \max\nolimits_{|\xi|\le b}(g_1(\xi) - c\xi) - \min\nolimits_{|\xi|\le b} g_1(\xi)\,. \end{aligned}$$

Proof From (8) we obtain

$$\min_{t\in[0,T]} \left(\tilde{f}(t) + s(\tilde{f}) - g_1(w(t))\right) \le \dot{w}(t) \le \min_{t\in[0,T]} \left(\tilde{f}(t) + s(\tilde{f}) - g_1(w(t))\right)\,.$$

Using the estimates for $s(\tilde{f})$ in (10), we obtain the desired inequality. Notice that because g_1 is continuous and the extrema is computed on a bounded interval, then

$$\begin{aligned} -\infty &< \min\nolimits_{|\xi|\le b}(g_1(\xi) - 0\cdot\xi) - \max\nolimits_{|\xi|\le b} g_1(\xi) \le m\,, \\ M &\le \max\nolimits_{|\xi|\le b}(g_1(\xi) - 0\cdot\xi) - \min\nolimits_{|\xi|\le b} g_1(\xi) < \infty\,. \end{aligned}$$

It is left only to check that k and K are positive. This follows from the fact that $-k < \dot{w}(t) < K$ on [0,T], $\int_0^T \dot{w}(\tau) d\tau = w(T) - w(0) = 0$ and w is not a constant function if $\tilde{f} \not\equiv 0$. ♢

Iterated estimates As an initial value put $b_0 > 0$, such that $\|w\|_{C_T} \le b_0$ (for instance: $b_0 = \|\tilde{f}\|_2 \sqrt{T/12}$ due to Lemma 1). Then for $n = 0, 1, 2, \ldots$, let k_n, K_n be the constants obtained in Lemma 3 with $b = b_n$, and let

$$b_{n+1} = \frac{T k_n K_n}{2(k_n + K_n)}\,.$$

Lemma 4 *Let b_n, k_n, K_n be defined as above. If $b_1 \le b_0$, then $b_{n+1} \le b_n$ for all $n \ge 1$.*

Proof We proceed by induction. First notice that $b_1 \le b_0$ is one of the hypotheses. Now assume that $b_n \le b_{n-1}$. Then in the statement of Lemma 3 we see that

$$0 \ge m_n \ge m_{n-1} \quad \text{and} \quad 0 \le M_n \le M_{n-1}\,.$$

Thus, $k_n \le k_{n-1}$ and $K_n \le K_{n-1}$. Since $\frac{TkK}{2(k+K)}$ is a decreasing function of k, and of K, we have $b_{n+1} \le b_n$. ♢

From the above lemma, iterations can be repeated indefinitely. However, in practice the process should stop when the decrement in b_n is less than a predetermined value. Now, we define the lower and upper bounds for $s(\tilde{f})$.

Theorem 2 *Let b_n be as defined above. Put $b = \inf\{b_0, b_1, \dots\}$, and*

$$\begin{aligned} a(\tilde{f}) &= \sup\nolimits_{c\in\mathbb{R}} \min\nolimits_{|\xi|\le b} \left(g_1(\xi) - c\xi\right)\,, \\ A(\tilde{f}) &= \inf\nolimits_{c\in\mathbb{R}} \max\nolimits_{|\xi|\le b} \left(g_1(\xi) - c\xi\right)\,. \end{aligned}$$

Then the functional $s(\tilde{f})$ satisfies $a(\tilde{f}) \le s(\tilde{f}) \le A(\tilde{f})$.

Proof Notice that by Lemma 2, $\|w\|_\infty \le b_n$ for all n. Therefore, from the basic estimate (10), the statement of this theorem follows. Notice that even if $A(\tilde{f})$ is not the absolute infimum over c, the equality in this Theorem is still valid. The same statement applies for $a(\tilde{f})$. ♢

Computational experiments show that the iteration method refines estimates if the ratio $-\max(\tilde{f})/\min(\tilde{f})$ is much larger than one, or very close to zero. To illustrate this case, we study the following boundary-value problem

Example 1 Consider $\dot{w}(t) + g_1(w(t)) = \tilde{f}(t) + s(\tilde{f})$, where

$$\tilde{f}(t) = \begin{cases} -\sin(t)/20 & \text{if } 0 \le t \le \pi \\ \sin(20t) & \text{if } \pi < t \le 21\pi/20\,. \end{cases}$$

Notice that the ratio $-\max\tilde{f}/\min\tilde{f}$ is large. The period is $T = 21\pi/20$, $\|\tilde{f}\|_2^2 = \pi/800 + \pi/40$, and the estimate for $\|w\|_\infty$ is $b_0 = \|\tilde{f}\|_2\sqrt{T/12}$.

To avoid computing the maximum and the minimum over $c \in \mathbb{R}$, we use $c = (g_1(b) - g_1(-b))/(2b)$; see the remark after (10). The following table shows the estimates obtained for several functions g_1.

	$g_1(\xi) = \xi^2$	$g_1(\xi) = \xi^3$	$0.1\arctan(\xi)$
min-max g	$0 \le s \le$ 2.2669e-2	$\lvert s\rvert \le$ 3.4131e-3	$\lvert s\rvert \le$ 1.4944e-2
basic est.	$0 \le s \le$ 2.2669e-2	$\lvert s\rvert \le$ 1.3137e-3	$\lvert s\rvert \le$ 4.3011e-5
iterated	$0 \le s \le$ 8.2592e-3	$\lvert s\rvert \le$ 1.9403e-4	$\lvert s\rvert \le$ 1.0002e-5

Example 2 For $\alpha > 0$, consider the equation

$$\dot{w}(t) + \arctan(w(t)) = \alpha\sin(t) + s(\tilde{f})\,.$$

Notice that $\max \tilde{f} = -\min \tilde{f} = 1$, the period is $T = 2\pi$, $\|\tilde{f}\|_2 = \alpha\sqrt{\pi}$, and the estimate for $\|w\|_\infty$ is $b_0 = \alpha\pi/\sqrt{6}$. The following table shows the estimates obtained for several values of α.

	$\alpha = 0.01$	$\alpha = 0.1$	$\alpha = 1$
min-max g	$\lvert s\rvert \le$ 1.2824e-2	$\lvert s\rvert \le 0.12756$	$\lvert s\rvert \le 0.90856$
basic est.	$\lvert s\rvert \le$ 2.7064e-7	$\lvert s\rvert \le$ 2.6716e-4	$\lvert s\rvert \le 0.11593$
iterated	$\lvert s\rvert \le$ 2.7064e-7	$\lvert s\rvert \le$ 2.6716e-4	$\lvert s\rvert \le 0.11593$

Remark For all functions g_1 and all $\alpha \neq 0$ in $\dot{w}(t) + g_1(w(t)) = \alpha \sin(t) + s(\tilde{f})$ the iterated method fails to improve the basic estimate.

To prove this statement, notice that $\max \tilde{f} = -\min \tilde{f} = |\alpha|$, the period is $T = 2\pi$, $\|\tilde{f}\|_2 = |\alpha|\sqrt{\pi}$, and the estimate for $\|w\|_\infty$ is $b_0 = |\alpha|\pi/\sqrt{6}$. As in Lemma 3, m_0 and M_0 are non-negative quantities; thus, $k_0 \ge |\alpha|$ and $K_0 \ge |\alpha|$. Since b_1 is an increasing function of k_0 and of K_0, it follows that

$$b_1 \ge \frac{\pi}{2}|\alpha| > \frac{\pi}{\sqrt{6}}|\alpha| = b_0\,.$$

Which indicates that the iteration method is unsuccessful in this case.

Example 3 Consider $\dot{w}(t) + g_1(w(t)) = \tilde{f}(t) + s(\tilde{f})$ with

$$g_1(\xi) = 2\big(\arctan\big(10000(\xi + 0.12)\big) + \arctan\big(10000(\xi - 0.12)\big)\big)$$

and $\tilde{f}$ defined as in Example 1. Note that g_1 varies significantly only in the neighbourhood of several points (namely -0.12 and 0.12). In such a case it is better no to apply the iteration method directly, but apply the iteration method with $b_0 = \|\tilde{f}\|_2\sqrt{T/12}$ to

$$\dot{v}(t) + d(v(t)) = \tilde{f}(t) + s_d(\tilde{f})\,,$$

where $d(\xi) = g_1(\xi)$ for $|\xi| < \delta$ and $d(\xi) = g_1(\delta\operatorname{sgn}(\xi))$ otherwise with some $0 < \delta \le b_0$. If $b = \inf\{b_0, b_1, \dots\} \le \delta$ then considering $\|v\|_\infty \le b$ and $d(\xi) = g_1(\xi)$ for $|\xi| \le \delta$ we get $w = v$. Thus $\|w\|_\infty \le b$ and $s(\tilde{f}) = s_d(\tilde{f})$. The following table shows the estimates obtained for direct application of iteration method and different values of δ.

	direct application	$\delta = 0.11$	$\delta = 0.1$
b	1.5056e-1	1.1946e-1>0.11	9.0409e-2<0.1
min-max g	$\lvert s\rvert \le 6.2759$	$\lvert s\rvert \le 6.2759$	$\lvert s\rvert \le$ 9.0908e-3
basic est.	$\lvert s\rvert \le 4.8202$	$\lvert s\rvert \le 4.8202$	$\lvert s\rvert \le$ 1.4010e-3
iterated	$\lvert s\rvert \le 4.8202$	$\lvert s\rvert \le 4.8202$	$\lvert s\rvert \le$ 1.6342e-3

Remark Note that for $\delta = 0.1$ the basic estimate yields better result than iterated although $b =$ 9.0409e-2 $< b_0 =$ 1.5056e-1. The reason is that we avoid calculating supremum or infimum over c and use $c = \big(g_1(b) - g_1(b)\big)/2b$ in formulas for $a(\tilde{f})$ and $A(\tilde{f})$ in Theorem 2.

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