PROPERTIES RELATING INDEPENDENT CHANCE VARIABLES, DISTRIBUTION FUNCTIONS, AND MOMENT-GENERATING FUNCTIONS

THESIS

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CHAPTER I

INTRODUCTION

Most authors of probability and statistics books introduce the definition of a joint-distribution function to prove certain properties involving moment-generating functions for distribution functions of independently distributed chance variables.

The purpose of this paper is to show that some of these properties can be established without using the definition of a joint-distribution function.

First, some properties involving sets and independent chance variables will be established. These arguments will then be used to obtain the distribution function for the sum of two independent chance variables.

CHAPTER II

PRELIMINARY THEOREMS, NOTATIONS, AND DEFINITIONS

NOTATION 2.1. The symbol ε will be used to denote the phrase belongs to.

NOTATION 2.2. If A and B are sets, then the statement that A is a subset of B will be denoted by $A \subseteq B$.

DEFINITION 2.1. The statement that $A \subseteq B$ means that if $x \in A$, then $x \in B$.

DEFINITION 2.2. If A and B are sets, then the following statements are equivalent:

(1) $A = B$ and

(2) $A \subseteq B$ and $B \subseteq A$.

DEFINITION 2.3. The statement that the set C is the common part of the set A and the set B, denoted by $C = A \cap B$ or $C = (A,B)$, means that C is the set such that $x \in C$ if and only if $x \in A$ and $x \in B$.

DEFINITION 2.4. The statement that C is the union of the set A and the set B, denoted by $C = A \cup B$, means that C is the set such that $x \in C$ if and only if $x \in A$ or $x \in B$.

NOTATION 2.3. If each of A_1, A_2, A_3, ... is a set, then $\bigcap_{i=1}^{\infty} A_i$ denotes the set such that $x \in \bigcap_{i=1}^{\infty} A_i$ if and only if $x \in A_i$ for each integer $i > 0$.

NOTATION 2.4. If each of A_1, A_2, A_3, ... is a set, then $\bigcup_{i=1}^{\infty} A_i$ denotes the set such that $x \in \bigcup_{i=1}^{\infty} A_i$ if and only if there is an integer $n > 0$ such that $x \in A_n$.

NOTATION 2.5. The symbols $<$, $\leq$, $>$, and $\geq$ will be used to denote the phrases *less than*, *less than or equal*, *greater than*, and *greater than or equal* respectively.

NOTATION 2.6. If A is a set, then the symbol A^c will be used to denote the complement of A.

DEFINITION 2.5. The statement that R is a probability domain means that R is a collection of sets such that the following statements are true:

(1) there is a set S belonging to R such that if $A \in R$, then $A \subseteq S$;

(2) there is a set ϕ belonging to R such that if $A \in R$, then $\phi \subseteq A$;

(3) if $A \in R$ and $B \in R$, then $A \cap B \in R$; and

(4) if $A \in R$, then there is a set A^c in R such that $A \cap A^c = \phi$ and $A \cup A^c = S$.

THEOREM 2.1. Suppose that D is a probability domain and A is a set in D such that if $s \in S$, then $s \notin A$, then $A = \phi$.

Henceforth ϕ will be called the empty set.

<u>DEFINITION 2.6.</u> The statement that the set A and the set B are disjoint means that $A \cap B = \phi$.

<u>THEOREM 2.2.</u> If R is a probability domain, $A \varepsilon R$, $B \varepsilon R$, and $C \varepsilon R$, then each of the following statements is true:

(1) $(A \cap B) \cap C = (A \cap C) \cap B$;

(2) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$;

(3) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$;

(4) $A^c \cap B^c = (A \cup B)^c$; and

(5) $A^c \cup B^c = (A \cap B)^c$.

<u>DEFINITION 2.7.</u> If u and v are sets, then the statement that f is a function with domain u and range v means

(1) f is a collections of ordered pairs such that if $(x,y) \varepsilon f$, then $x \varepsilon u$ and $y \varepsilon v$,

(2) if $(x,y) \varepsilon f$ and $(x,z) \varepsilon f$, then $y = z$, and

(3) if $x \varepsilon u$, then there is a pair (x,y) such that $(x,y) \varepsilon f$.

If u is a collection of sets, then f is called a set function.

<u>DEFINITION 2.8.</u> If f is a set function with domain R and range the real numbers, then the statement that f is additive means that if $A \varepsilon R$ and $B \varepsilon R$, then $f(A) + f(B) = f(A \cup B) + f(A \cap B)$.

<u>DEFINITION 2.9.</u> The statement that R is a complete probability domain means that R is a probability domain such that if A_1, A_2, A_3, ... is a

sequence of sets, each of which belongs to R, $A_{i+1} \subseteq A_i$ for $i \in \{1,2,3, \ldots\}$, and A is the common part of the sets $A_1, A_2, A_3, \ldots$, then $A \in R$.

<u>DEFINITION 2.10.</u> The statement that (R,P) is a probability distribution means that

(1) R is a complete probability domain and

(2) P is a function with domain R and range the real numbers such that

(a) if $A \in R$, then $P(A) \geq 0$,

(b) P is additive,

(c) $P(S) = 1$ and $P(\phi) = 0$, and

(d) if $A_1, A_2, A_3, \ldots$ is a sequence of sets, each of which belongs to R, $A_{i+1} \subseteq A_i$ for $i \in \{1,2,3,\ldots\}$, A is the common part of $A_1, A_2, A_3, \ldots$, and $\xi > 0$, then there is a positive number N such that if n is an integer and $n \geq N$, then $|P(A_n) - P(A)| < \xi$. If $A \in R$, then P(A) denotes the probability of the set A.

<u>THEOREM 2.3.</u> If $A \in R$, $B \in R$, and $A \subseteq B$, then $P(A) \leq P(B)$.

<u>DEFINITION 2.11.</u> The statement that T is a chance variable means that there is a probability domain R such that

(1) T is a function with domain S and range the real numbers and

(2) if t is a real number and $(T \leq t)$ denotes the set such that $x \in (T \leq t)$ if and only if $x \in S$ and $T(x) \leq t$, then $(T \leq t) \in R$.

THEOREM 2.4. If T is a chance variable for a probability domain R and t is a real number, then the set $(T = t) \varepsilon R$. Moreover, if $t' < t$, then $(t' < T \leq t) \varepsilon R$. The set $(t' < T \leq t)$ denotes the set $[(T \leq t) \cap (T \leq t')^c]$.

DEFINITION 2.12. If T is a chance variable and (R,P) is a probability distribution, then the statement that F is the distribution function for T means that if t is a real number, then $F(t) = P(T \leq t)$.

DEFINITION 2.13. The statement that the function f is continuous from the right at c means if $\xi > 0$, then there is a real number $\partial > 0$ such that if x is a real number and $c < x < c + \partial$, then $|f(x) - f(c)| < \xi$.

DEFINITION 2.14. The statement that F is a nondecreasing function means that if $a < b$ and a and b are in the domain of F, then $F(a) \leq F(b)$.

THEOREM 2.5. If T is a chance variable and F is the distribution function for T, then each of the following statements is true:

(1) F is nondecreasing;

(2) if t is a real number, then $0 \leq F(t) \leq 1$;

(3) if $\xi > 0$, then there is a real number X such that if $x \leq X$, then $F(x) < \xi$;

(4) if $\xi > 0$, then there is a real number Y such that if $y \geq Y$, then $F(y) > 1 - \xi$; and

(5) F is continuous from the right.

DEFINITION 2.15. If $n > 1$ is an integer and $T_1, T_2, T_3, \ldots, T_n$ is an n-term sequence of chance variables with distribution function F, then the statement that $T_1, T_2, T_3, \ldots, T_n$ are independently distributed means that if $t_1, t_2, t_3, \ldots, t_n$ is an n-term sequence of real numbers, then $P(T_1 \leq t_1, T_2 \leq t_2, \ldots, T_n \leq t_n) = P(T_1 \leq t_1) \cdot P(T_2 \leq t_2) \cdot \ldots \cdot P(T_n \leq t_n)$.

<u>NOTATION 2.7.</u> If T is a chance variable, then $(T \leq \infty)$ will be used to denote the set such that $x \in (T \leq \infty)$ if and only if $x \in S$ or $T(x)$ is a real number.

<u>DEFINITION 2.16.</u> The statement that D is a subdivision of the number interval $[a,b]$ means that

(1) D is a finite subset of $[a,b]$ and

(2) $a \in D$ and $b \in D$.

<u>DEFINITION 2.17.</u> The statement that E is a refinement of the subdivision D of $[a,b]$ means that

(1) E is a subdivision of $[a,b]$ and

(2) $D \subseteq E$.

<u>NOTATION 2.8.</u> If $[a,b]$ is a number interval and D is a subdivision of $[a,b]$, then the symbol $\sum_D$ will be used to denote a summation ranging over D.

<u>DEFINITION 2.18.</u> If x and y are functions and $[a,b]$ is a real number interval, then the statement that $\int_a^b y(t)\, d\, x(t) = A$ means if $\xi > 0$, then there is a subdivision D of $[a,b]$ such that if E is a refinement of D, then $\left| \sum_E \frac{1}{2}[y(t_i) + y(t_{i+1})][x(t_{i+1}) - x(t_i)] - A \right| < \xi$.

NOTATION 2.9. If f is a function, then $\Delta f(t_i)$ will be used to denote $f(t_{i+1}) - f(t_i)$.

DEFINITION 2.19. The statement that the function f is quasi-continuous on the real number interval [a,b] means that if $\xi > 0$, then there is a subdivision $D = \{a_0, a_1, a_2, \ldots, a_n\}$ of [a,b] such that if a_i, $a_{i+1} \varepsilon D$ and s and t are numbers such that $a_i < s < t < a_{i+1}$, then $|f(s) - f(t)| < \xi$.

DEFINITION 2.20. The statement that the function f is of bounded variation on the real number interval [a,b] means that there is a number $M \geq 0$ such that if D is a subdivision of [a,b], then $\sum_D |f(x_{i+1}) - f(x_i)| \leq M$.

THEOREM 2.6. If (R,P) is a probability distribution, T is a chance variable, F is the distribution function for T, and [a,b] is a real number interval, then F is quasi-continuous on [a,b] and F has bounded variation on [a,b].

THEOREM 2.7. If [a,b] is a real number interval and f and g are functions such that f is quasi-continuous on [a,b] and g has bounded variation on [a,b], then ${}_a\int^b fdg$ exists.

THEOREM 2.8. If [a,b] and [c,d] are real number intervals, f is a continuous, nondecreasing function such that f(c) = a and f(d) = b, and

${}_a\int^b y(t)\,dx(t)$ exists, then ${}_a\int^b y(t)dx(t) = {}_c\int^d y[f(t)]\,d\,x[f(t)]$.

<u>DEFINITION 2.21.</u> The statement that the number set A is bounded above means there is a number M such that if $x \in A$, then $x \leq M$.

<u>DEFINITION 2.22.</u> The statement that the number set A has a least upper bound means there is a number M such that

(1) if $x \in A$, then $x \leq M$ and

(2) if $p < M$, then there is a number q in A such that $q > p$.

<u>AXIOM 2.1.</u> If a nonempty number set A is bounded above, then A has a least upper bound.

<u>DEFINITION 2.23.</u> The statement that $\int_{-\infty}^{\infty} y(t)\, d\, X(t) = I$ means if $\xi > 0$, then there is a real number interval $[a,b]$ such that if $A \leq a$ and $b \leq B$, then $\left|\int_A^B y(t)\, d\, X(t) - I\right| < \xi$.

<u>THEOREM 2.9.</u> If A is a real number, then $\lim_{p\to\infty} \dfrac{A}{2^{p-1}} = 0$.

<u>THEOREM 2.10.</u> If A and B are real numbers, then the following statements are equivalent:

(1) $A = B$ and

(2) if $\xi > 0$, then $|A - B| < \xi$.

<u>NOTATION 2.10.</u> If F is a distribution function for the chance variable T, then $F(-\infty)$ denotes $P(T \leq -\infty) = P(\phi)$.

CHAPTER III

THE DISTRIBUTION FUNCTION FOR THE SUM OF TWO INDEPENDENT CHANCE VARIABLES

LEMMA 3.1. If t is a real number, A is a positive integer, $D_n = \{a_{ni} = -A + \frac{iA}{2^{n-1}}\}$ where $i \in \{0,1,2,3,\ldots,2^n\}$ and $n \in \{1,2,3,\ldots\}$, T_1 and T_2 are independently distributed chance variables, and A_n denotes the set such that $A_n = \bigcup_{i=0}^{2^n-1} (t-A < T_2 \leq t-a_{ni},\ a_{ni} < T_1 \leq a_{ni+1})$, then $\bigcap_{n=1}^{\infty} A_n = (t-2A < T_1 + T_2 \leq t,\ T_1 > -A,\ T_2 > t-A)$.

Proof:

Let $x \in \bigcap_{n=1}^{\infty} A_n$. $x \in \bigcap_{n=1}^{\infty} A_n$ means if $p \geq 1$ is an integer, then $x \in A_p$. Let p be an integer such that $p \geq 1$. Thus, $x \in A_p$. Since $x \in A_p$, then there is an integer q where $0 \leq q \leq 2^p - 1$ such that $x \in (t-A < T_2 \leq t-a_{pq},\ a_{pq} < T_1 \leq a_{pq+1})$. Since $x \in (t-A < T_2 \leq t-a_{pq},\ a_{pq} < T_1 \leq a_{pq+1})$, then $t - A < T_2(x) \leq t - a_{pq}$ and $a_{pq} < T_1(x) \leq a_{pq+1}$. Since $T_1(x) > a_{pq}$ and $a_{pq} \geq -A$, then $T_1(x) > -A$. Since $T_1(x) \leq a_{pq+1}$ and $T_2(x) \leq t - a_{pq}$, then $T_1(x) + T_2(x) \leq t + a_{pq+1} - a_{pq}$. Since

$$a_{pq+1} - a_{pq} = -A + \frac{(q+1)A}{2^{p-1}} - (-A + \frac{qA}{2^{p-1}})$$

$$= -A + \frac{qA}{2^{p-1}} + \frac{A}{2^{p-1}} + A - \frac{qA}{2^{p-1}}$$

$$= \frac{A}{2^{p-1}}$$, then $T_1(x) + T_2(x) \leq t + \frac{A}{2^{p-1}}$ for each integer $p \geq 1$.

Since $T_1(x) + T_2(x) \leq t - \frac{A}{2^{p-1}}$ for each integer $p \geq 1$ and $\lim_{p\to\infty} \frac{A}{2^{p-1}} = 0$, then $T_1(x) + T_2(x) \leq t$. Since $T_1(x) > -A$ and $T_2(x) > t - A$, then $T_1(x) + T_2(x) > t - 2A$. Since $t - 2A < T_1(x) + T_2(x) \leq t$, $T_1(x) > -A$, and $T_2(x) > t - A$, then $x \in (t-2A < T_1+T_2 \leq t, T_1 > -A, T_2 > t-A)$.

Hence, $\bigcap_{n=1}^{\infty} A_n \subseteq (t-2A < T_1+T_2 \leq t, T_1 > -A, T_2 > t-A)$.

Suppose that $x \in (t-2A < T_1+T_2 \leq t, T_1 > -A, T_2 > t-A)$, then $t - 2A < T_1(x) + T_2(x) \leq t$, $T_1(x) > -A$, and $T_2(x) > t - A$. Since $T_1(x) + T_2(x) \leq t$, then $T_1(x) \leq t - T_2(x)$. Since $T_2(x) > t-A$, then it follows that $T_1(x) < A$. Let j be an integer such that $j \geq 1$. Since $-A < T_1(x) < A$ and D_j is a subdivision of $[-A,A]$, then there is an integer k where $0 \leq k \leq 2^j - 1$ such that a_{jk}, $a_{jk+1} \in D_j$ and $a_{jk} < T_1(x) \leq a_{jk+1}$. Since $T_1(x) > a_{jk}$ and $T_2(x) \leq t - T_1(x)$, then $T_2(x) < t - a_{jk}$. Since $t - A < T_2(x) < t - a_{jk}$ and $a_{jk} < T_1(x) \leq a_{jk+1}$, then $x \in (t-A < T_2 \leq t-a_{jk}, a_{jk} < T_1 \leq a_{jk+1})$. Since there is an integer k where $0 \leq k \leq 2^j-1$ such that $x \in (t-A < T_1 \leq t-a_{jk}, a_{jk} < T_1 \leq a_{jk+1})$, then $x \in A_j$. Since j is an integer where $j \geq 1$ and $x \in A_j$, then $x \in \bigcap_{n=1}^{\infty} A_n$. Hence, $(t-2A < T_1+T_2 \leq t, T_1 > -A, T_2 > t-A) \subseteq \bigcap_{n=1}^{\infty} A_n$.

Since $\bigcap_{n=1}^{\infty} A_n \subseteq (t-2A < T_1+T_2 \leq t, T_1 > -A, T_2 > t-A)$ and $(t-2A < T_1+T_2 \leq t, T_1 > -A, T_2 > t-A) \subseteq \bigcap_{n=1}^{\infty} A_n$, then $\bigcap_{n=1}^{\infty} A_n = (t-2A < T_1+T_2 \leq t, T_1 > -A, T_2 > t-A)$.

<u>LEMMA 3.2.</u> If a and b are real numbers such that $a < b$, then $P(a < T_1 \leq b) = P(T_1 \leq b) - P(T_1 \leq a)$.

Proof:

$$
\begin{aligned}
P(a < T_1 \leq b) &= P[(T_1 \leq b) \cap (T_1 \leq a)^c] \\
&= P(T_1 \leq b) + P(T_1 \leq a)^c - P[(T_1 \leq b) \cup (T_1 \leq a)^c] \\
&\quad \text{because } P \text{ is an additive set function,} \\
&= P(T_1 \leq b) + P(T_1 \leq a)^c - P(T_1 \leq \infty) \text{ by Notation 2.7,} \\
&= P(T_1 \leq b) + 1 - P(T_1 \leq a) - P(T_1 \leq \infty) \\
&\quad \text{by definition 2.5,} \\
&= P(T_1 \leq b) + 1 - P(T_1 \leq a) - 1 \\
&= P(T_1 \leq b) - P(T_1 \leq a).
\end{aligned}
$$

<u>LEMMA 3.3.</u> If a, b, c, and d are real numbers such that $a < b$ and $c < d$ and T_1 and T_2 are independently distributed chance variables, then $P(a < T_1 \leq b,\ c < T_2 \leq d) = P(a < T_1 \leq b) \cdot P(c < T_2 \leq b)$.

Proof:

$$
\begin{aligned}
&P(a < T_1 \leq b,\ c < T_2 \leq d) \\
&\quad = P[(T_1 \leq b) \cap (T_1 \leq a)^c \cap (c < T_2 \leq d)] \\
&\quad = P[(T_1 \leq b) \cap (c < T_2 \leq d) \cap (T_1 \leq a)^c] \text{ by Theorem 2.2,} \\
&\quad = P\{[(T_1 \leq b) \cap (c < T_2 \leq d) \cap (T_1 \leq a)^c] \cup [\phi]\} \\
&\quad = P\{[(T_1 \leq b) \cap (c < T_2 \leq d) \cap (T_1 \leq a)^c] \cup [(T_1 \leq b) \cap (c < T_2 \leq d) \cap \\
&\qquad (c < T_2 \leq d)^c]\} \\
&\quad = P\{[(T_1 \leq b) \cap (c < T_2 \leq d)] \cap [(T_1 \leq a)^c \cup (c < T_2 \leq d)^c]\} \\
&\qquad \text{by Theorem 2.2,} \\
&\quad = P[(T_1 \leq b) \cap (c < T_2 \leq d)] + P[(T_1 \leq a)^c \cup (c < T_2 \leq d)^c] \\
&\qquad - P\{[(T_1 \leq b) \cap (c < T_2 \leq d)] \cup [(T_1 \leq a)^c \cup (c < T_2 \leq d)^c]\}.
\end{aligned}
$$

Since $P\{[(T_1 \leq b) \cap (c < T_2 \leq d)] U[(T_1 \leq a)^c \; U(c < T_2 \leq d)^c]\}$

$= P\{[(T_1 \leq b) U(T_1 \leq a)^c \; U(c < T_2 \leq d)^c] \cap [(c < T_2 \leq d) U(T_1 \leq a)^c$

$U(c < T_2 \leq d)^c]\}$ by Theorem 2.2,

$= P\{[(T_1 \leq \infty) U(c < T_2 \leq d)^c] \cap [(T_1 \leq a)^c \; U(T_2 \leq \infty)]\}$

$= P[(T_1 \leq \infty) \cap (T_2 \leq \infty)]$

$= P(T_1 \leq \infty) \cdot P(T_2 \leq \infty)$ because T_1 and T_2 are independent,

$= 1,$

therefore, $P(a < T_1 \leq b, \; c < T_2 \leq d)$

$= P[(T_1 \leq b) \cap (c < T_2 \leq d)] + P[(T_1 \leq a)^c \; U(c < T_2 \leq d)^c] - 1$

$= P[(T_1 \leq b) \cap (c < T_2 \leq d)] - 1 + P[(T_1 \leq a) \cap (c < T_2 \leq d)]^c$

by Theorem 2.2,

$= P(T_1 \leq b, \; c < T_2 \leq d) + P(T_1 \leq a, \; c < T_2 \leq d)$ by Definition 2.5,

$= P(T_1 \leq b, \; c < T_2 \leq d) + P(T_1 \leq b, \; T_2 \leq c) - P(T_1 \leq b, \; T_2 \leq c)$

$-P(T_1 \leq a, \; c < T_2 \leq d) + P(T_1 \leq a, \; T_2 \leq c) - P(T_1 \leq a, \; T_2 \leq c)$

$= P[(T_1 \leq b, \; T_2 \leq c) U(T_1 \leq b, \; c < T_2 \leq d)] - P(T_1 \leq b, \; T_2 \leq c)$

$-P[(T_1 \leq a, \; T_2 \leq c) U(T_1 \leq a, \; c < T_2 \leq d)] + P(T_1 \leq a, \; T_2 \leq c)$

because $(T_1 \leq b, \; T_2 \leq c)$ and $(T_1 \leq b, \; c < T_2 \leq d)$, and $(T_1 \leq a,$

$T_2 \leq c)$ and $(T_1 \leq a, \; c < T_2 \leq d)$ are disjoint,

$= P\{[T_1 \leq b] \cap [(T_2 \leq c) U(c < T_2 \leq d)]\} - P(T_1 \leq b, \; T_2 \leq c)$

$-P\{[T_1 \leq a] \cap [(T_2 \leq c) U(c < T_2 \leq d)]\} + P(T_1 \leq a, \; T_2 \leq c)$

by Theorem 2.2,

$= P(T_1 \leq b, \; T_2 \leq d) - P(T_1 \leq b, \; T_2 \leq c) - P(T_1 \leq a, \; T_2 \leq d)$

$+P(T_1 \leq a, \; T_2 \leq c)$

$= P(T_1 \leq b) \cdot P(T_2 \leq d) - P(T_1 \leq b) \cdot P(T_2 \leq c) - P(T_1 \leq a) \cdot P(T_2 \leq d)$

$+P(T_1 \leq a) \cdot P(T_2 \leq c)$ because T_1 and T_2 are independent,

$= [P(T_1 \le b) - P(T_1 \le a)][P(T_2 \le d) - P(T_2 \le c)]$

$= P(a < T_1 \le b)\cdot P(c < T_2 \le d)$ by Lemma 3.2.

<u>LEMMA 3.4.</u> If $n \ge 1$ is an integer, $A > 0$ is a real number, and A_n is the set defined in Lemma 3.1, then

$$P(A_n) = \sum_{D_n} [F_2(t-a_{ni}) - F_2(t-A)][F_1(a_{ni+1}) - F_1(a_{ni})].$$

Proof:

$$P(A_n) = P[\bigcup_{i=0}^{2^n-1} (t-A < T_2 \le t-a_{ni},\ a_{ni} < T_1 \le a_{ni+1})]$$

$= \sum_{D_n} P(t-A < T_2 \le t-a_{ni},\ a_{ni} < T_1 \le a_{ni+1})]$ because the function P is additive and the sets $(t-A < T_2 \le t-a_{ni})$ and $(a_{ni} < T_1 \le a_{ni+1})$ are disjoint for $i \in \{0,1,2,3,\ldots,2^n-1\}$,

$= \sum_{D_n} P(t-A < T_2 \le t-a_{ni})\cdot P(a_{ni} < T_1 \le a_{ni+1})$ by Lemma 3.3,

$= \sum_{D_n} [P(T_2 \le t-a_{ni}) - P(T_2 \le t-A)][P(T_1 \le a_{ni+1}) - P(T_1 \le a_{ni})]$ by Lemma 3.2,

$= \sum_{D_n} [F_2(t-a_{ni}) - F_2(t-A)][F_1(a_{ni+1}) - F_1(a_{ni})]$ by Definition 2.12.

<u>LEMMA 3.5.</u> If n is an integer such that $n \ge 1$ and E is a refinement of D_n where D_n is the same as in Lemma 3.1, then

$$\sum_{E} [F_2(t-a_i) - F_2(t-A)][F_1(a_{i+1}) - F_1(a_i)] \le \sum_{D_n} [F_2(t-a_{ni}) - F_2(t-A)]$$

$$[F_1(a_{ni+1}) - F_1(a_{ni})].$$

Proof:

In Lemma 3.4 we have shown that $\sum_{D_n} [F_2(t-a_{ni}) - F_2(t-A)]$ $[F_1(a_{ni+1}) - F_1(a_{ni})] = P[\bigcup_{i=0}^{2^n-1} (t-A < T_2 \leq t-a_{ni},\ a_{ni} < T_1 \leq a_{ni+1})]$. Suppose that $n \geq 1$ is an integer. Let E be a refinement of D_n such that $E = \{a_0, a_1, a_2, \ldots, a_m\}$. Let $x \in \bigcup_{i=0}^{m-1} (t-A < T_2 \leq t-a_i,\ a_i < T_1 \leq a_{i+1})$.

Since $x \in \bigcup_{i=0}^{m-1} (t-A < T_2 \leq t-a_i,\ a_i < T_1 \leq a_{i+1})$, then there exists an integer p such that $0 \leq p \leq m-1$ and $x \in (t-A < T_2 \leq t-a_p,\ a_p < T_1 \leq a_{p+1})$. Therefore, $t - A < T_2(x) \leq t - a_p$ and $a_p < T_1(x) \leq a_{p+1}$. Since $-A \leq a_p < T_1(x) \leq a_{p+1} \leq A$ and D_n is a subdivision of $[-A,A]$, then there is an integer k such that $0 \leq k \leq 2^n-1$ and $a_{nk} < T_1(x) \leq a_{nk+1}$. Since E is a refinement of D_n, then $a_{nk} \leq a_p$. Since $T_2(x) \leq t - a_p$ and $a_{nk} \leq a_p$, then $T_2(x) \leq t - a_{nk}$. Since $t - A < T_2(x) \leq t - a_{nk}$ and $a_{nk} < T_1(x) \leq a_{nk+1}$, then $x \in (t-A < T_2 \leq t-a_{nk},\ a_{nk} < T_1 \leq a_{nk+1})$. Since there is an integer k such that $0 \leq k \leq 2^n-1$ and $x \in (t-A < T_2 \leq t-a_{nk},\ a_{nk} < T_1 \leq a_{nk+1})$, then $x \in \bigcup_{i=0}^{2^n-1} (t-A < T_2 \leq t-a_{ni},\ a_{ni} < T_1 \leq a_{ni+1})$. Consequently, $\bigcup_{i=0}^{m-1} (t-A < T_2 \leq t-a_i,\ a_i < T_1 \leq a_{i+1}) \subseteq \bigcup_{i=0}^{2^n-1} (t-A < T_2 \leq t-a_{ni},\ a_{ni} < T_1 \leq a_{ni+1})$. Thus, from Theorem 2.3 $P[\bigcup_{i=0}^{m-1} (t-A < T_2 \leq t-a_i,\ a_i < T_1 \leq a_{i+1})] \leq P[\bigcup_{i=0}^{2^n-1} (t-A < T_2 \leq t-a_{ni},\ a_{ni} < T_1 \leq a_{ni+1})]$ and from

Lemma 3.4 $\sum_E [F_2(t-a_i) - F_2(t-A)][F_1(a_{i+1}) - F_1(a_i)] \leq \sum_{D_n} [F_2(t-a_{ni}) - F_2(t-A)][F_1(a_{ni+1}) - F_1(a_{ni})]$.

<u>LEMMA 3.6.</u> If D is a subdivision of $[-A,A]$, then $\sum_D [F_2(t-x_i) - F_2(t-A)][\Delta F_1(x_i)] \geq P(t-2A < T_1+T_2 \leq t,\ T_1 > -A,\ T_2 > t-A)$.

Proof:

Let D be a subdivision of $[-A,A]$ such that $D = \{x_0, x_1, x_2, \ldots, x_n\}$. It was shown in Lemma 3.4 that $\sum_D [F_2(t-x_i) - F_2(t-A)][\Delta F_1(x_i)] = P[\bigcup_{i=0}^{n-1} (t-A < T_2 \leq t-x_i,\ x_i < T_1 \leq x_{i+1})]$. Let $x \in (t-2A < T_1+T_2 \leq t,\ T_1 > -A,\ T_2 > t-A)$. Then, $t - 2A < T_1(x) + T_2(x) \leq t$, $T_1(x) > -A$, and $T_2(x) > t - A$. Since $T_1(x) \leq t - T_2(x)$ and $T_2(x) > t - A$, then $T_1(x) < A$. Since $-A < T_1(x) < A$ and D is a subdivision of $[-A,A]$, then there is an integer k such that $0 \leq k \leq n - 1$ and $x_k < T_1(x) \leq x_{k+1}$. Since $T_2(x) \leq t - T_1(x)$ and $T_1(x) > x_k$, then $T_2(x) < t - x_k$. Since there is an integer k such that $0 \leq k \leq n - 1$, $t - A < T_2(x) < t - x_k$, and $x_k < T_1(x) \leq x_{k+1}$, then $x \in \bigcup_{i=0}^{n-1} (t-A < T_2 \leq t-x_i,\ x_i < T_1 \leq x_{i+1})$. Thus, $(t-2A < T_1+T_2 \leq t,\ T_1 > -A,\ T_2 > t-A) \subseteq \bigcup_{i=0}^{n-1} (t-A < T_2 \leq t-x_i,\ x_i < T_1 \leq x_{i+1})$, and by Theorem 2.3 $P(t-2A < T_1+T_2 \leq t,\ T_1 > -A,\ T_2 > t-A)$

$\leq P[\bigcup_{i=0}^{n-1}(t-A < T_2 \leq t-x_i,\ x_i < T_1 \leq x_{i+1})]$. Since $\sum_D [F_2(t-x_i) - F_2(t-A)]$

$[\Delta F_1(x_i)] = P[\bigcup_{i=0}^{n-1}(t-A < T_2 \leq t-x_i,\ x_i < T_1 \leq x_{i+1})]$, then $P(t-2A <$

$T_1+T_2 \leq t,\ T_1 > -A,\ T_2 > t-A) \leq \sum_D [F_2(t-x_i) - F_2(t-A)][\Delta F_1(x_i)]$.

<u>LEMMA 3.7.</u> If $[a,b]$ is a real number interval, T_1 is a chance variable, and $\xi > 0$, then there is a subdivision D of $[a,b]$ such that if $a_i,\ a_{i+1} \in D$, then $|P(T_1 < a_{i+1}) - P(T_1 \leq a_i)| < \xi$.

Proof:

Let $[a,b]$ be a real number interval and $\xi > 0$. Let J denote the set such that $x \in J$ if and only if $a < x \leq b$ and there exists a subdivision D of $[a,x]$ such that if $a_i,\ a_{i+1} \in D$, then $P(a_i < T_1 < a_{i+1}) < \xi$.

Let F_1 denote the distribution function for the chance variable T_1. Since F_1 is continuous from the right at a and $\xi > 0$, then there is a positive number ∂ such that if $a < x < a + \partial$, then $|F_1(x) - F_1(a)| < \xi$. Let t be a real number such that $a < t < a + \partial$. Let D_1 be a subdivision of $[a,a+\partial]$. Therefore, if $a_i,\ a_{i+1} \in D_1$, then

$P(a_i < T_1 < a_{i+1}) = P(T_1 < a_{i+1}) - P(T_1 \leq a_i)$ by Lemma 3.2,

$\leq P(T_1 \leq a_{i+1}) - P(T_1 \leq a_i)$ because $(T_1 < a_{i+1}) \subseteq (T_1 \leq a_{i+1})$,

$\leq P(T_1 \leq a+\partial) - P(T_1 \leq a_i)$ because $(T_1 \leq a_{i+1}) \subseteq (T_1 \leq a+\partial)$,

$\leq P(T_1 \leq a+\partial) - P(T_1 \leq a)$ because $(T_1 \leq a) \subseteq (T_1 \leq a_i)$,

$= |P(T_1 \leq a+\partial) - P(T_1 \leq a)|$ because $P(T_1 \leq a+\partial) \geq P(T_1 \leq a)$,

$< \xi$.

Thus, $a + \partial \in J$ and, therefore, J is nonempty. Since J is bounded above by b, then J has a least upper bound. Let L denote the least upper bound of J.

Suppose $z \in \bigcap_{i=1}^{\infty} (T_1 \geq L - \frac{1}{2^i})$. Suppose that $T_1(z) < L$. Then, there exists a positive integer n such that $L - T_1(z) > \frac{1}{2^n}$. Thus, $T_1(z) < L - \frac{1}{2^n}$, which contradicts the fact that $z \in \bigcap_{i=1}^{\infty} (T_1 \geq L - \frac{1}{2^i})$. Therefore, the assumption that $T_1(z) < L$ is false and $T_1(z) \geq L$. Since $T_1(z) \geq L$, then $z \in (T_1 \geq L)$. Consequently, $\bigcap_{i=1}^{\infty} (T_1 \geq L - \frac{1}{2^i}) \subseteq (T_1 \geq L)$.

Suppose that $z \in (T_1 \geq L)$, then $T_1(z) \geq L$. Since $L \geq L - \frac{1}{2^i}$ for each integer $i \geq 1$, then $T_1(z) \geq L - \frac{1}{2^i}$ for each integer $i \geq 1$. Thus, $z \in \bigcap_{i=1}^{\infty} (T_1 \geq L - \frac{1}{2^i})$ and, as a result, $(T_1 \geq L) \subseteq \bigcap_{i=1}^{\infty} (T_1 \geq L - \frac{1}{2^i})$. Since $\bigcap_{i=1}^{\infty} (T_1 \geq L - \frac{1}{2^i}) \subseteq (T_1 \geq L)$ and $(T_1 \geq L) \subseteq \bigcap_{i=1}^{\infty} (T_1 \geq L - \frac{1}{2^i})$, then $\bigcap_{i=1}^{\infty} (T_1 \geq L - \frac{1}{2^i}) = (T_1 \geq L)$.

Since $(T_1 \geq L - \frac{1}{2})$, $(T_1 \geq L - \frac{1}{4})$, $(T_1 \geq L - \frac{1}{8})$, ... is a sequence of sets, each of which belongs to R, $(T_1 \geq L - \frac{1}{2^{i+1}}) \subseteq (T_1 \geq L - \frac{1}{2^i})$ for each integer $i \geq 1$, and $\xi > 0$, then there is a positive number N such that if n is an integer and $n \geq N$, then

$$\begin{aligned}
\xi &> \left|P(T_1 \geq L - \frac{1}{2^n}) - P(T_1 \geq L)\right| \\
&= \left|1 - P(T_1 < L - \frac{1}{2^n}) - 1 + P(T_1 < L)\right| \\
&= \left|P(T_1 < L) - P(T_1 < L - \frac{1}{2^n})\right| \\
&\geq \left|P(T_1 < L) - P(T_1 \leq L - \frac{1}{2^n})\right| \text{ because } (T_1 < L - \frac{1}{2^n}) \subseteq \\
&\quad (T_1 \leq L - \frac{1}{2^n}), \\
&= \left|P(L - \frac{1}{2^n} < T_1 < L)\right| \text{ by Theorem 3.2,} \\
&= P(L - \frac{1}{2^n} < T_1 < L) \text{ because } P(L - \frac{1}{2^n} < T_1 < L) \geq 0.
\end{aligned}$$

Since $L - \frac{1}{2^n} < L$ and L is the least upper bound of J, then there is an $x \in J$ such that $x > L - \frac{1}{2^n}$. Since $x \in J$ and $\xi > 0$, then there is a subdivision D_2 of $[a,x]$ such that if a_i, $a_{i+1} \in D_2$, then $P(a_i < T_1 < a_{i+1}) < \xi$. Let $D = D_2 U\{L\}$. Since $P(L - \frac{1}{2^n} < T_1 < L) < \xi$ and $x > L - \frac{1}{2^n}$, then $P(x < T_1 < L) < \xi$. Therefore, if a_i, $a_{i+1} \in D$, then $P(a_i < T_1 < a_{i+1}) < \xi$. Thus, it follows that $L \in J$.

Since J is bounded above by b and L is the least upper bound of J, then $L \leq b$. Suppose that $L < b$. Since F_1 is continuous from the right at L and $\xi > 0$, then there is a real number $\partial_L > 0$ such that if $L < t < \partial_L + L$, then $|F_1(t) - F_1(L)| < \xi$. Let r be the minimum of $\{\partial_L + L, b\}$ and let $D_r = DU\{r\}$. Thus, if a_i, $a_{i+1} \in D_r$, then

$|F_1(a_{i+1}) - F_1(a_i)| < \xi$. Thus, $r \in J$. If $r = \partial_L + L$ or if $r = b$, then $r > L$, which is a contradiction to the fact that L is the least upper bound of J. Consequently, $L = b$. Therefore, if $[a,b]$ is an interval and $\xi > 0$, then there is a subdivision D of $[a,b]$ such that if $a_i, a_{i+1} \in D$, then $|P(T_1 < a_{i+1}) - P(T_1 \leq a_i)| < \xi$.

LEMMA 3.8. If C_n denotes the set such that $C_n = \bigcup_{i=0}^{2^n-1} (t-x_{i+1} < T_2 \leq t - x_i) \cap (T_1+T_2 = t)^c \cap (X_i < T_1 \leq x_{i+1})$ where $x_i = -A + \frac{iA}{2^{n-1}}$, $i \in \{0,1,2,3,\ldots,2^n-1\}$, and $n \in \{1,2,3,\ldots\}$, then

(1) $\bigcap_{n=1}^{\infty} C_n = \phi$ and

(2) if $\xi > 0$, then there is a positive number N such that if n is an integer and $n \geq N$, then $|P(C_n)| < \xi$.

Proof:

Part (1). Suppose that $s \in \bigcap_{n=1}^{\infty} C_n$. Since $s \in \bigcap_{n=1}^{\infty} C_n$ and if $n \geq 1$ is an integer, then $s \in C_n$. Let $n \geq 1$ be an integer. Since $s \in C_n$, then there is an x_i, $x_{i+1} \in D_n$ where D_n is the same as in Lemma 3.1 such that $t - x_{i+1} < T_2(s) \leq t - x_i$, $T_1(s) + T_2(s) > t$ or $T_1(s) + T_2(s) < t$, and $x_i < T_1(s) \leq x_{i+1}$.

Suppose that $T_1(s) + T_2(s) > t$. Then, there exists a positive number N such that $T_1(s) + T_2(s) = t + N$ and $T_1(s) - N + T_2(s) = t$. Since D_{n+1} is a refinement of D_n where $n \in \{1,2,3,\ldots\}$ such that if

x_i, $x_{i+1} \varepsilon D_n$, then there is a number $\frac{x_i + x_{i+1}}{2} \varepsilon D_{n+1}$ and since $T_1(s) - N < T_1(s)$, then there exist integers $k \geq 1$ and p where $0 \leq p \leq 2^k - 1$ such that x_p, $x_{p+1} \varepsilon D_k$ and $T_1(s) - N < x_p < T_1(s) \leq x_{p+1} \leq x_{i+1}$. Since $t = T_2(s) + T_1(s) - N$ and $T_1(s) - N < x_p$, then $t < T_2(s) + x_p$. Thus, $T_2(s) > t - x_p$. Since $x_p < T_1(s) \leq x_{p+1}$ and $T_2(s) > t - x_p$, then $s \notin C_k$. Since $k \geq 1$ is an integer and $s \notin C_k$, then $s \notin \bigcap_{n=1}^{\infty} C_n$. Hence, the assumption that $\bigcap_{n=1}^{\infty} C_n$ is nonempty is false and $\bigcap_{n=1}^{\infty} C_n = \phi$. If we consider the case where $T_1(s) + T_2(s) < t$, then a similar proof will show that $\bigcap_{n=1}^{\infty} C_n = \phi$.

Part (2). Let $\varepsilon > 0$. Since C_1, C_2, C_3, ... is a sequence of sets, each of which belongs to R, $C_{n+1} \subseteq C_n$ for each integer $n \geq 1$, and $\xi > 0$, then there is a positive number N such that if n is an integer and $n \geq N$, then $\xi > |P(C_n) - P(\phi)| = |P(C_n)|$.

THEOREM 3.1. If T_1 and T_2 are independently distributed chance variables, F_1 and F_2 are the distribution functions for T_1 and T_2, and t is a real number, then $P(T_1+T_2 \leq t) = \int_{-\infty}^{\infty} F_2(t-x)dF_1(x)$.

Proof:

Let t be a real number, $\xi > 0$, and let A be a positive integer. For each positive integer n, define $D_n = \{a_{ni} = -A + \frac{iA}{2^{n-1}} |$ i ε

$\{0,1,2,3,\ldots,2^n\}\}$. Also, let A_n denote the set such that $A_n = \bigcup_{i=1}^{2^n-1} (t-A < T_2 \leq t-a_{ni}, a_{ni} < T_1 \leq a_{ni+1})$. Since $A_1, A_2, A_3, \ldots$ is a sequence of sets, each of which belongs to R, $A_{n+1} \subseteq A_n$ for $n \in \{1,2,3,\ldots\}$, and $\frac{\xi}{3} > 0$, then there is a positive number N such that if m is an integer and $m \geq N$, then $|P(A_m) - P(\bigcap_{n=1}^{\infty} A_n)| < \frac{\xi}{3}$.

Since $[-A,A]$ is a real number interval and F_1 and F_2 are distribution functions, then by Theorem 2.6 F_1 is quasi-continuous on $[-A,A]$ and F_2 is of bounded variation on $[-A,A]$.

Since F_1 is quasi-continuous on $[-A,A]$ and F_2 is of bounded variation on $[-A,A]$, then by Theorem 2.7 $\int_{-A}^{A} [F_2(t-x) - F_2(t-A)]dF_1(x)$ exists.

Since $\int_{-A}^{A}[F_2(t-x) - F_2(t-A)]dF_1(x)$ exists and $\frac{\xi}{3} > 0$, then there is a subdivision D_1 of $[-A,A]$ such that if E_1 is a refinement of D_1, then $\left| \int_{-A}^{A}[F_2(t-x) - F_2(t-A)]dF_1(x) - \sum_{E_1} \frac{1}{2}[F_2(t-x_{i+1}) - F_2(t-A) + F_2(t-x_i) - F_2(t-A)][\Delta F_1(x_i)]\right| < \frac{\xi}{3}$.

Since $[-A,A]$ is a real number interval and $\frac{\xi}{3} > 0$, then by Lemma 3.7 there is a subdivision D_2 of $[-A,A]$ such that if $a_i, a_{i+1} \in D_2$, then $|P(T_1 < a_{i+1}) - P(T_1 \leq a_i)| < \frac{\xi}{3}$.

Let $C_n = \bigcup_{i=0}^{2^n-1} (t-x_{i+1} < T_2 \leq t-x_i) \cap (T_1+T_2 = t)^c \cap (X_i < T_1 \leq x_{i+1})$.

Since $\frac{\xi}{3} > 0$, then by Lemma 3.8 there is a positive number N such that if n is an integer and $n \geq N$, then $|P(C_n)| < \xi$. Let p be an integer such that $p \geq N$, and let D_p denote the subdivision defined by p.

Let $E = D_m \cup D_1 \cup D_2 \cup D_p$.

We must next consider the following absolute value:

$$\Big| \sum_E [F_2(t-x_i) - F_2(t-A)]\Delta F_1(x_i) - \sum_E \tfrac{1}{2}[F_2(t-x_{i+1}) - F_2(t-A) + F_2(t-x_i) - F_2(t-A)]\Delta F_1(x_i)\Big|$$

$$= \Big|\sum_E F_2(t-x_i)\Delta F_1(x_i) - \sum_E \tfrac{1}{2}[F_2(t-x_{i+1}) + F_2(t-x_i)]\Delta F_1(x_i)\Big|$$

$$= \Big|\sum_E F_2(t-x_i)\Delta F_1(x_i) - \tfrac{1}{2}\sum_E F_2(t-x_{i+1})\Delta F_1(x_i) - \tfrac{1}{2}\sum_E F_2(t-x_i)\Delta F_1(x_i)\Big|$$

$$= \Big|\tfrac{1}{2}\sum_E F_2(t-x_i)\Delta F_1(x_i) - \tfrac{1}{2}\sum_E F_2(t-x_{i+1})\Delta F_1(x_i)\Big|$$

$$= \tfrac{1}{2}\Big|\sum_E F_2(t-x_i)\Delta F_1(x_i) - \sum_E F_2(t-x_{i+1})\Delta F_1(x_i)\Big|$$

$$= \tfrac{1}{2}\Big|\sum_E [F_2(t-x_i) - F_2(t-x_{i+1})][F_1(x_{i+1}) - F_1(x_i)]\Big|$$

$$= \tfrac{1}{2}\Big|\sum_E P(t-x_{i+1} < T_2 \leq t-x_i)P(x_i < T_1 \leq x_{i+1})\Big| \text{ by Lemma 3.2,}$$

$$= \tfrac{1}{2}\Big|\sum_E P(t-x_{i+1} < T_2 \leq t-x_i,\ x_i < T_1 \leq x_{i+1})\Big| \text{ by Lemma 3.3,}$$

$$= \tfrac{1}{2}\Big|\sum_E P\{[(t-x_{i+1} < T_2 \leq t-x_i) \cap (T_1+T_2 = t)^c \cap (x_i < T_1 \leq x_{i+1})] \cup [(t-x_{i+1} < T_2 \leq t-x_i) \cap (T_1+T_2 = t) \cap (x_i < T_1 \leq x_{i+1})]\}\Big|$$

$$= \tfrac{1}{2}\Big|\sum_E \{P[(t-x_{i+1} < T_2 \leq t-x_i) \cap (T_1+T_2 = t)^c \cap (x_i < T_1 \leq x_{i+1})] + P[(t-x_{i+1} < T_2 \leq t - x_i) \cap (T_1+T_2 = t) \cap (x_i < T_1 \leq x_{i+1})]\}\Big| \text{ because}$$

the two sets in brackets are disjoint,

$$= \frac{1}{2}\Big| \sum_E P[(t-x_{i+1} < T_2 \leq t-x_i) \cap (T_1+T_2 = t)^c \cap (x_i < T_1 \leq x_{i+1})] +$$

$$\sum_E P[(t-x_{i+1} < T_2 \leq t-x_i) \cap (T_1+T_2 = t) \cap (x_i < T_1 \leq x_{i+1})]\Big|$$

$$\leq \frac{1}{2}\Big| \sum_E P[(t-x_{i+1} < T_2 \leq t-x_i) \cap (T_1+T_2 = t)^c \cap (x_i < T_1 \leq x_{i+1})]\Big|$$

$$+ \frac{1}{2}\Big| \sum_E P[(t-x_{i+1} < T_2 \leq t-x_i) \cap (T_1+T_2 = t) \cap (x_i < T_1 \leq x_{i+1})]\Big|$$

$$= \frac{1}{2}\Big| \sum_E P[(t-x_{i+1} < T_2 \leq t-x_i) \cap (T_1+T_2 = t)^c \cap (x_i < T_1 \leq x_{i+1})\Big|$$

$$+ \frac{1}{2}\Big| \sum_E P(t-x_{i+1} < T_2 \leq t-x_i)P(T_1+T_2 = t)P(x_i < T_1 \leq x_{i+1})\Big|$$

$$< \frac{1}{2}\left(\frac{\xi}{3}\right) + \frac{1}{2} \sum_E |P(t-x_{i+1} < T_2 \leq t-x_i)||P(T_1+T_2 = t)||P(x_i < T_1 \leq$$

$x_{i+1})|$ from Lemma 3.8,

$$< \frac{\xi}{6} + \frac{1}{2} \sum_E |P(t-x_{i+1} < T_2 \leq t-x_i)||P(T_1+T_2 = t)| \frac{\xi}{3}$$ by Lemma 3.7,

$$\leq \frac{\xi}{6} + \frac{1}{2}\left(\frac{\xi}{3}\right)$$ because P is a probability function,

$$= \frac{\xi}{3}.$$

Since $|P(A_m) - P(\bigcap_{n=1}^{\infty} A_n)| < \frac{\xi}{3}$, then by Lemma 3.4 $|\sum_{D_m} [F_2(t-a_{mi}) - F_2(t-A)][F_1(a_{mi+1}) - F_1(a_{mi})] - P(\bigcap_{n=1}^{\infty} A_n)| < \frac{\xi}{3}$. Since E is a refinement of D_m, then by Lemma 3.5 $\sum_E [F_2(t-x_i) - F_2(t-A)][F_1(x_{i+1}) - F_1(x_i)] \leq \sum_{D_m} [F_2(t-a_{mi}) - F_2(t-A)][F_1(a_{mi+1}) - F_1(a_{mi})]$. Since E is a subdivision of $[-A,A]$, then by Lemma 3.6 $\sum_E [F_2(t-x_i) - F_2(t-A)][F_1(x_{i+1}) - F_1(x_i)]$ $\geq P(\bigcap_{n=1}^{\infty} A_n)$. Since $|\sum_{D_m} [F_2(t-a_{mi}) - F_2(t-A)]\Delta F_1(a_{mi}) - P(\bigcap_{n=1}^{\infty} A_n)| < \frac{\xi}{3}$ and $\sum_{D_m} [F_2(t-a_{mi}) - F_2(t-A)]\Delta F_1(a_{mi}) \geq \sum_E [F_2(t-x_i) - F_2(t-A)]\Delta F_1(x_i)$ $\geq P(\bigcap_{n=1}^{\infty} A_n)$, then $|\sum_E [F_2(t-x_i) - F_2(t-A)]\Delta F_1(x_i) - P(\bigcap_{n=1}^{\infty} A_n)| < \frac{\xi}{3}$. Since E is a refinement of D_1, then $|\int_{-A}^{A}[F_2(t-x) - F_2(t-A)]dF_1(x) -$ $\sum_E \frac{1}{2}[F_2(t-x_{i+1}) - F_2(t-A) + F_2(t-x_i) - F_2(t-A)]\Delta F_1(x_i)| < \frac{\xi}{3}$. Since $|\sum_E [F_2(t-x_i) - F_2(t-A)]\Delta F_1(x_i) - P(\bigcap_{n=1}^{\infty} A_n)| < \frac{\xi}{3}$, $|\sum_E \frac{1}{2} [F_2(t-x_{i+1}) -$ $F_2(t-A) + F_2(t-x_i) - F_2(t-A)]\Delta F_1(x_i) - \sum_E [F_2(t-x_i) - F_2(t-A)]\Delta F_1(x_i)| < \frac{\xi}{3}$, and $|\int_{-A}^{A}[F_2(t-x) - F_2(t-A)]dF_1(x) - \sum_E \frac{1}{2} [F_2(t-x_{i+1}) + F_2(t-x_i) - 2\cdot F_2(t-A)]$ $\Delta F_1(x_i)| < \frac{\xi}{3}$, then $|\int_{-A}^{A}[F_2(t-x) - F_2(t-A)]dF_1(x) - P(\bigcap_{n=1}^{\infty} A_n)| < \xi$. Since $\xi > 0$ and $|\int_{-A}^{A}[F_2(t-x) - F_2(t-A)]dF_1(x) - P(\bigcap_{n=1}^{\infty} A_n)| < \xi$, then by Theorem 2.10 $\int_{-A}^{A} [F_2(t-x) - F_2(t-A)]dF_1(x) = P(\bigcap_{n=1}^{\infty} A_n)$. Since $P(\bigcap_{n=1}^{\infty} A_n) = P(t-2A <$ $T_1+T_2 \leq t,\ T_1 > -A,\ T_2 > t-A)$, then $\int_{-A}^{A}[F_2(t-x) - F_2(t-A)]dF_1(x) =$ $P(t-2A < T_1+T_2 \leq t,\ T_1 > -A,\ T_2 > t-A)$.

We must next prove that the $\lim_{A\to\infty} P(t-2A < T_1+T_2 \leq t,\ T_1 > -A,$ $T_2 > t-A) = P(T_1+T_2 \leq t)$. Let $\xi > 0$ and B_A denote the set such that

$B_A = (t-2A < T_1+T_2 \leq t,\ T_1 > -A,\ T_2 > t-A)$ where $A \in \{1,2,3,\ldots\}$.

Let $x \in \bigcup_{A=1}^{\infty} B_A$. Thus, there is an integer $n \geq 1$ such that $x \in B_n$. Since $x \in B_n$, then $T_1(x) + T_2(x) \leq t$. Therefore, $x \in (T_1+T_2 \leq t)$ and $\bigcup_{i=1}^{\infty} B_i \subseteq (T_1+T_2 \leq t)$.

If $x \in (T_1+T_2 \leq t)$, then $T_1(x) + T_2(x) \leq t$. Since $T_1(x)$ and $T_2(x)$ are real numbers, then there are positive integers p and q such that $T_1(x) > -p$ and $T_2(x) > t - q$. Let r be the larger of p and q. Hence, $T_1(x) > -r$, $T_2(x) > t - r$, and $T_1(x) + T_2(x) > t - 2r$. Therefore, $x \in (t-2r < T_1+T_2 \leq t,\ T_1 > -r,\ T_2 > t-r)$. Since there is an integer $r \geq 1$ such that $x \in (t-2r < T_1+T_2 \leq t,\ T_1 > -r,\ T_2 > t-r)$, then $x \in \bigcup_{A=1}^{\infty} B_A$. Thus, $(T_1+T_2 \leq t) \subseteq \bigcup_{A=1}^{\infty} B_A$. Since $(T_1+T_2 \leq t) \subseteq \bigcup_{A=1}^{\infty} B_A$ and $\bigcup_{A=1}^{\infty} B_A \subseteq (T_1+T_2 \leq t)$, then $\bigcup_{A=1}^{\infty} B_A = (T_1+T_2 \leq t)$. From Theorem 2.2 it follows that $(\bigcup_{A=1}^{\infty} B_A)^c = \bigcap_{A=1}^{\infty} B_A{}^c$. Since $B_1{}^c, B_2{}^c, B_3{}^c, \ldots$ is a sequence of sets, each of which belongs to R, $B_{A+1}{}^c \subseteq B_A{}^c$ for $A \in \{1,2,3,\ldots\}$, and $\xi > 0$, then there is a positive number N such that if n is an integer and $n \geq N$, then

$$\xi > \left|P(B_n{}^c) - P(\bigcap_{A=1}^{\infty} B_A{}^c)\right|$$

$$= \left|P(B_n{}^c) - 1 + P(\bigcap_{n=1}^{\infty} B_A{}^c)^c\right|$$

$$= \left|1 - P(B_n{}^c)^c - 1 + P(\bigcap_{A=1}^{\infty} B_A{}^c)^c\right|$$

$$= \left|P(\bigcup_{A=1}^{\infty} B_A) - P(B_n)\right|$$

$= \left|P(T_1+T_2 \leq t) - P(B_n)\right|$. Since $P(B_A)$ where $A \in \{1,2,3,\ldots\}$ is a sequence of numbers, $\xi > 0$, and there is a number N such that if $n \geq N$, then $\left|P(B_n) - P(T_1+T_2 \leq t)\right| < \xi$, then $\lim_{A\to\infty} P(B_A) = P(T_1+T_2 \leq t)$. Since $\int_{-A}^{A} [F_2(t-x) - F_2(t-A)]dF_1(x) = P(B_A)$ for every positive integer A and $\lim_{A\to\infty} P(B_A)$ exists, then $\lim_{A\to\infty} \int_{-A}^{A}[F_2(t-x) - F_2(t-A)]dF_1(x) = \lim_{A\to\infty} P(B_A)$. Since $\lim_{A\to\infty} \int_{-A}^{A}[F_2(t-x) - F_2(t-A)]dF_1(x) = \int_{-\infty}^{\infty} [F_2(t-x) - F_2(-\infty)]dF_1(x)$ and $F_2(-\infty) = 0$, then $P(T_1+T_2 \leq t) = \int_{-\infty}^{\infty} F_2(t-x)dF_1(x)$ for every real number t.

CHAPTER IV

THE MOMENT-GENERATING FUNCTION FOR THE DISTRIBUTION FUNCTION OF AN N-TERM SEQUENCE OF INDEPENDENTLY DISTRIBUTED CHANCE VARIABLES

<u>LEMMA 4.1.</u> Suppose that (R,P) is a probability distribution, and F_1 and F_2 are the distribution functions for the independent chance variables T_1 and T_2. If z is a real number such that $\int_{-\infty}^{\infty} e^{-zt}\, d \int_{-\infty}^{\infty} F_2(t-x)dF_1(x)$ exists and $\xi > 0$, then there is an interval [a,b] such that if $A \leq a$ and $b \leq B$, then $\left| \int_{-\infty}^{\infty} e^{-zt}\, d \int_{-\infty}^{\infty} F_2(t-x)dF_1(x) - \int_{2A}^{2B} e^{-zt}\, d \int_{A}^{B} F_2(t-x)dF_1(x) \right| < \xi$.

Proof:

Let z be a real number such that $\int_{-\infty}^{\infty} e^{-zt}\, d \int_{-\infty}^{\infty} F_2(t-x)dF_1(x)$ exists, and let $\xi > 0$. Since $\int_{-\infty}^{\infty} e^{-zt}\, d \int_{-\infty}^{\infty} F_2(t-x)dF_1(x)$ exists and $\frac{\xi}{2} > 0$, then there is an interval [j,k] such that if $J \leq j$ and $k \leq K$, then $\left| \int_{-\infty}^{\infty} e^{-zt}\, d \int_{-\infty}^{\infty} F_2(t-x)dF_1(x) - \int_{J}^{K} e^{-zt}\, d \int_{-\infty}^{\infty} F_2(t-x)dF_1(x) \right| < \frac{\xi}{2}$. Let [J,K] be an interval such that $J \leq j$ and $k \leq K$.

Let Q denote the set such that $q \in Q$ if and only if $q = e^{-zt}$ and $J \leq t \leq K$. Since Q is bounded above, then Q has a least upper bound. Let M be the least upper bound of Q.

Since $\xi > 0$ and $M > 0$, then $\frac{\xi}{4M} > 0$. Since $\frac{\xi}{4M} > 0$ and F_1 is a distribution function, then by Theorem 2.5 there exists real numbers c and d such that if $C \leq c$ and $D \geq d$, then $F_1(C) < \frac{\xi}{4M}$ and $\frac{\xi}{4M} > 1 - F_1(D)$.

Thus, $\left| \int_{-\infty}^{\infty} e^{-zt}\, d \int_{-\infty}^{\infty} F_2(t-x)dF_1(x) - \int_J^K e^{-zt}\, d \int_C^D F_2(t-x)dF_1(x) \right|$

$$= \left| \int_{-\infty}^{\infty} e^{-zt}\, d \int_{-\infty}^{\infty} F_2(t-x)dF_1(x) - \int_J^K e^{-zt}\, d \int_{-\infty}^{\infty} F_2(t-x)dF_1(x) + \int_J^K e^{-zt}\, d \int_{-\infty}^{C} F_2(t-x)dF_1(x) + \int_J^K e^{-zt}\, d \int_D^{\infty} F_2(t-x)dF_1(x) \right|$$

$$\leq \left| \int_{-\infty}^{\infty} e^{-zt}\, d \int_{-\infty}^{\infty} F_2(t-x)dF_1(x) - \int_J^K e^{-zt}\, d \int_{-\infty}^{\infty} F_2(t-x)dF_1(x) \right| + \left| \int_J^K e^{-zt}\, d \int_{-\infty}^{C} F_2(t-x)dF_1(x) \right| + \left| \int_J^K e^{-zt}\, d \int_D^{\infty} F_2(t-x)dF_1(x) \right|$$

$$< \frac{\xi}{2} + \left| \int_J^K e^{-zt}\, d \int_{-\infty}^{C} F_2(t-x)dF_1(x) \right| + \left| \int_J^K e^{-zt}\, d \int_D^{\infty} F_2(t-x)dF_1(x) \right|$$

because $J \leq j$ and $k \leq K$,

$$= \frac{\xi}{2} + \int_J^K e^{-zt}\, d \int_{-\infty}^{C} F_2(t-x)dF_1(x) + \int_J^K e^{-zt}\, d \int_D^{\infty} F_2(t-x)dF_1(x)$$

because the integrals above are nonnegative,

$$\leq \frac{\xi}{2} + \int_J^K M\, d \int_{-\infty}^{C} F_2(t-x)dF_1(x) + \int_J^K M\, d \int_D^{\infty} F_2(t-x)dF_1(x)$$

$$= \frac{\xi}{2} + M \int_J^K d \int_{-\infty}^{C} F_2(t-x)dF_1(x) + M \int_J^K d \int_D^{\infty} F_2(t-x)dF_1(x)$$

$$= \frac{\xi}{2} + M\left[\int_{-\infty}^{C} F_2(K-x)dF_1(x)dF_1(x) - \int_{-\infty}^{C} F_2(J-x)dF_1(x)\right] + M\left[\int_D^{\infty} F_2(K-x)dF_1(x) - \int_D^{\infty} F_2(J-x)dF_1(x)\right]$$

$$= \frac{\xi}{2} + M \int_{-\infty}^{C} [F_2(K-x) - F_2(J-x)]dF_1(x) + M \int_D^{\infty} [F_2(K-x) - F_2(J-x)]dF_1(x)$$

$\leq \frac{\xi}{2} + M \int_{-\infty}^{C} dF_1(x) + M \int_{D}^{\infty} dF_1(x)$ because F_2 is a distribution function,

$= \frac{\xi}{2} + M[F_1(C) - F_1(-\infty)] + M[F_1(\infty) - F_1(D)]$

$= \frac{\xi}{2} + MF_1(C) + M[1-F_1(D)]$

$< \frac{\xi}{2} + M \cdot \frac{\xi}{4M} + M \cdot \frac{\xi}{4M}$

$= \xi.$

Therefore, $\left| \int_{-\infty}^{\infty} e^{-zt}\, d \int_{-\infty}^{\infty} F_2(t-x)dF_1(x) - \int_{J}^{K} e^{-zt}\, d \int_{C}^{D} F_2(t-x)dF_1(x) \right| < \xi.$

Choose $[a,b]$ to be a real number interval such that $a < 0$, $a \leq J$, $a \leq C$, $b > 0$, $K \leq b$, and $D \leq b$. Let $[A,B]$ be a real number interval such that $A \leq a$ and $b \leq B$. Since $e^{-zt} > 0$ for every real number t, $\int_{-\infty}^{\infty} F_2(t-x)dF_1(x)$ is nondecreasing as t increases, $2B \geq K$, $B \geq D$, $2A \leq J$, and $A \leq C$, then $\int_{2A}^{2B} e^{-zt}\, d \int_{A}^{B} F_2(t-x)dF_1(x) \geq \int_{J}^{K} e^{-zt}\, d \int_{C}^{D} F_2(t-x)dF_1(x)$. Likewise, $\int_{-\infty}^{\infty} e^{-zt}\, d \int_{-\infty}^{\infty} F_2(t-x)dF_1(x) \geq \int_{2A}^{2B} e^{-zt}\, d \int_{A}^{B} F_2(t-x)dF_1(x)$. Since $\left| \int_{-\infty}^{\infty} e^{-zt}\, d \int_{-\infty}^{\infty} F_2(t-x)dF_1(x) - \int_{J}^{K} e^{-zt}\, d \int_{C}^{D} F_2(t-x)dF_1(x) \right| < \xi$ and $\int_{-\infty}^{\infty} e^{-zt}\, d \int_{-\infty}^{\infty} F_2(t-x)dF_1(x) \geq \int_{2A}^{2B} e^{-zt}\, d \int_{A}^{B} F_2(t-x)dF_1(x) \geq \int_{J}^{K} e^{-zt}\, d \int_{C}^{D} F_2(t-x)dF_1(x)$, then $\left| \int_{-\infty}^{\infty} e^{-zt}\, d \int_{-\infty}^{\infty} F_2(t-x)dF_1(x) - \int_{2A}^{2B} e^{-zt}\, d \int_{A}^{B} F_2(t-x) dF_1(x) \right| < \xi.$

<u>THEOREM 4.1.</u> Suppose (R,P) is a probability distribution, $\{T_i\}_{i=1}^{n}$ is an n-term sequence of independently distributed chance variables with the distribution functions F_1, F_2, F_3, ..., F_n, and $T_1 + T_2 + \ldots + T_n$ is the chance variable for the distribution function F. G_1, G_2, ..., G_n, and G are the moment-generating functions for the distribution functions, F_1, F_2, ..., F_n, and F respectively.

If z belongs to the domain of G_1, G_2, ..., G_n, and G such that each of the moment-generating functions exists, then $G(z) = G_1(z) \cdot G_2(z) \cdot \ldots \cdot G_n(z)$.

Proof:

Let $\xi > 0$, and let z belong to the domain of G_1, G_2, G_3, ..., G_n, and G such that each moment-generating function exists.

Consider the case where $n = 2$. Since $F(t) = P(T_1+T_2 \leq t)$ for every real number t and from Theorem 3.1 $P(T_1+T_2 \leq t) = \int_{-\infty}^{\infty} F_2(t-x)dF_1(x)$, then $F(t) = \int_{-\infty}^{\infty} F_2(t-x)dF_1(x)$. Since $G(z)$ exists and $G(z) = \int_{-\infty}^{\infty} e^{-zt}dF(t)$, then $\int_{-\infty}^{\infty} e^{-zt}\, d\int_{-\infty}^{\infty} F_2(t-x)dF_1(x)$ exists. Since $\int_{-\infty}^{\infty} e^{-zt}\, d\int_{-\infty}^{\infty} F_2(t-x)dF_1(x)$ exists and $\frac{\xi}{6} > 0$, then from Lemma 4.1 there is a real number interval $[a,b]$ such that if $A \leq a$ and $b \leq B$, then $\left| \int_{-\infty}^{\infty} e^{-zt}\, d\int_{-\infty}^{\infty} F_2(t-x)dF_1(x) - \int_{2A}^{2B} e^{-zt}\, d\int_{A}^{B} F_2(t-x)dF_1(x) \right| < \frac{\xi}{6}$. Since $G_1(z) \geq 0$ and $\frac{\xi}{6} > 0$, then $\frac{\xi}{6[G_1(z)+1]} > 0$.

Since $\int_{-\infty}^{\infty} e^{-zy}dF_2(y)$ exists and $\frac{\xi}{6[G_1(z)+1]} > 0$, then there is an interval $[c,d]$ such that if $C \leq c$ and $d \leq D$, then $\left| \int_{-\infty}^{\infty} e^{-zy}dF_2(y) - \int_{C}^{D} e^{-zy}dF_2(y) \right| < \frac{\xi}{6[G_1(z)+1]}$. Since $G_1(z) \cdot G_2(z)$ exists and $\frac{\xi}{6} > 0$, then there is an interval $[w,v]$ such that if $W \leq w$ and $v \leq V$, then $\left|G_1(z) \cdot G_2(z) - G_2(z) \int_{W}^{V} e^{-zx}dF_1(x)\right| < \frac{\xi}{6}$. Let $[A,B]$ be an interval such that A is less than or equal to a, c, and w; and B is greater than or equal to b, d, and v. Consequently, the following statements are true:

(1) $\left| \int_{-\infty}^{\infty} e^{-zt} \, d \int_{-\infty}^{\infty} F_2(t-x)dF_1(x) - \int_{2A}^{2B} e^{zt} \, d \int_{A}^{B} F_2(t-x)dF_1(x) \right| < \frac{\xi}{6}$;

(2) $\left| \int_{-\infty}^{\infty} e^{-zy}dF_2(y) - \int_{A}^{B} e^{-zy}dF_2(y) \right| < \frac{\xi}{6[G_1(z)+1]}$; and

(3) $\left|G_1(z) \cdot G_2(z) - G_2(z) \int_{A}^{B} e^{-zx}dF_1(x)\right| < \frac{\xi}{6}$.

Since $\int_{2A}^{2B} e^{-zt} \, d \int_{A}^{B} F_2(t-x)dF_1(x)$ exists and $\frac{\xi}{6} > 0$, then there is a subdivision D_1 of $[2A,2B]$ such that if E_1 is a refinement of D_1, then

$$\left| \int_{2A}^{2B} e^{-zt} \, d \int_{A}^{B} F_2(t-x)dF_1(x) - \sum_{E_1} \frac{1}{2} \left(e^{-zt_{i+1}} + e^{-zt_i}\right) \int_{A}^{B} \Delta F_2(t_i-x)dF_1(x) \right| < \frac{\xi}{6}$$

and $\left| \int_{2A}^{2B} e^{-zt} \, d \int_{A}^{B} F_2(t-x)dF_1(x) - \int_{A}^{B} \sum_{E_1} \frac{1}{2} \left(e^{-zt_{i+1}} + e^{-zt_i}\right)\Delta F_2(t_i-x)dF_1(x) \right| < \frac{\xi}{6}$. Let $x \in [A,B]$. Since $\int_{-\infty}^{\infty} e^{-zy}dF_2(y)$ exists, then $\int_{2A}^{2B} e^{-z(t-x)}dF_2(t-x)$ exists. Since $\int_{2A}^{2B} e^{-z(t-x)}dF_2(t-x)$ exists and $\frac{\xi}{6[G_1(z)+1]} > 0$, then there is a subdivision D_x of [2A,2B] such that if E_x is a refinement of D_x, then $\left|\int_{2A}^{2B} e^{-z(t-x)} dF_2(t-x) - \sum_{E_x} \frac{1}{2}\left[e^{-z(t_{i+1}-x)} + e^{-z(t_i-x)}\right]\Delta F_2(t_i-x)\right|$

$< \frac{\xi}{6[G_1(z)+1]}$. Let $D = D_1 \cup D_x$ where $A \leq x \leq B$. Then,

$$\left|\int_{2A}^{2B} e^{-z(t-x)} dF_2(t-x) - \sum_{D} \frac{1}{2}\left[e^{-z(t_{i+1}-x)} + e^{-z(t_i-x)}\right]\Delta F_2(t_i-x)\right|$$

$< \frac{\xi}{6[G_1(z)+1]}$ and

$$(4)\quad \left|\int_{2A}^{2B} e^{-zt}\, d\int_A^B F_2(t-x)dF_1(x) - \int_A^B \sum_D \frac{1}{2}\left(e^{-zt_{i+1}} + e^{-zt_i}\right)\right.$$

$$\left.\Delta F_2(t_i-x)dF_1(x)\right| < \frac{\xi}{6}.$$

Since $y = t - x$, then by Theorem 2.8 $\int_{2A}^{2B} e^{-z(t-x)} dF_2(t-x) =$

$\int_{2A-x}^{2B-x} e^{-zy} dF_2(y)$. Since $x \in [A,B]$, then $2B - x \geq B$ and $2A - x \leq A$.

Therefore, since e^{-zy} is positive for every real number y and F_2 is nondecreasing, then $\int_{-\infty}^{\infty} e^{-zy} dF_2(y) \geq \int_{2A-x}^{2B-x} e^{-zy} dF_2(y) \geq \int_A^B e^{-zy} dF_2(y)$.

Hence, since $\left|\int_{-\infty}^{\infty} e^{-zy} dF_2(y) - \int_A^B e^{-zy} dF_2(y)\right| < \frac{\xi}{6[G_1(z)+1]}$, then

$\left|\int_{2A-x}^{2B-x} e^{-zy} dF_2(y) - \int_A^B e^{-zy} dF_2(y)\right| < \frac{\xi}{6[G_1(z)+1]}$ and $\left|\int_A^B e^{-zy} dF_2(y) -\right.$

$\int_{2A}^{2B} e^{-z(t-x)} dF_2(t-x) \bigg| < \frac{\xi}{6[G_1(z)+1]}$. Since $\bigg| \int_{A}^{B} e^{-zy} dF_2(y) -$

$\int_{2A}^{2B} e^{-z(t-x)} dF_2(t-x) \bigg| < \frac{\xi}{6[G_1(z)+1]}$ and $\bigg| \int_{2A}^{2B} e^{-z(t-x)} dF_2(t-x) -$

$\sum_{D} \frac{1}{2} [e^{-z(t_{i+1}-x)} + e^{-z(t_i-x)}] \Delta F_2(t_i-x) \bigg| < \frac{\xi}{6[G_1(z)+1]}$, then

$\bigg| \int_{A}^{B} e^{-zy} dF_2(y) - \sum_{D} \frac{1}{2} [e^{-z(t_{i+1}-x)} + e^{-z(t_i-x)}] \Delta F_2(t_i-x) \bigg| < \frac{2\xi}{6[G_1(z)+1]}$.

Since $\bigg| \int_{-\infty}^{\infty} e^{-zy} dF_2(y) - \int_{A}^{B} e^{-zy} dF_2(y) \bigg| < \frac{\xi}{6[G_1(z)+1]}$, then

$\bigg| \int_{-\infty}^{\infty} e^{-zy} dF_2(y) - \sum_{D} \frac{1}{2} [e^{-z(t_{i+1}-x)} + e^{-z(t_i-x)}] \Delta F_2(t_i-x) \bigg| < \frac{3\xi}{6[G_1(z)+1]}$.

Therefore, since $\int_{-\infty}^{\infty} e^{-zy} dF_2(y) = G_2(z)$, then $-\frac{\xi}{6[G_1(z)+1]} < G_2(z) -$

$\sum_{D} \frac{1}{2} [e^{-z(t_{i+1}-x)} + e^{-z(t_i-x)}] \Delta F_2(t_i-x) < \frac{3\xi}{6[G_1(z)+1]}$. Since $e^{-zx} > 0$

for each x such that $A \leq x \leq B$, then $-\frac{3\xi}{6[G_1(z)+1]} e^{-zx} < G_2(z) e^{-zx} -$

$\sum_{D} \frac{1}{2} (e^{-zt_{i+1}} + e^{-zt_i}) \Delta F_2(t_i-x) < \frac{3\xi}{6[G_1(z)+1]} e^{-zx}$. Therefore, since

F_1 is nondecreasing, then $-\frac{3\xi}{6[G_1(z)+1]} \int_{A}^{B} e^{-zx} dF_1(x) \leq G_2(z) \int_{A}^{B} e^{-zx} dF_1(x) -$

$\int_{A}^{B} \sum_{D} \frac{1}{2} (e^{-zt_{i+1}} + e^{-zt_i}) \Delta F_2(t_i-x) dF_1(x) \leq \frac{3\xi}{6[G_1(z)+1]} \int_{A}^{B} e^{-zx} dF_1(x)$.

Since $-\frac{3\xi}{6[G_1(z)+1]}\int_A^B e^{-zx}dF_1(x) \geq -\frac{3\xi}{6[G_1(z)+1]}\int_{-\infty}^{\infty} e^{-zx}dF_1(x)$

$$= -\frac{3\xi}{6[G_1(z)+1]}G_1(z)$$

$$> -\frac{3\xi}{6[G_1(z)+1]}[G_1(z)+1]$$

$= -\frac{3\xi}{6}$ and

$$\frac{3\xi}{6[G_1(z)+1]}\int_A^B e^{-zx}dF_1(x) \leq \frac{3\xi}{6[G_1(z)+1]}\int_{-\infty}^{\infty} e^{-zx}d\,F_1(x)$$

$$= \frac{3\xi}{6[G_1(z)+1]}G_1(z)$$

$$< \frac{3\xi}{6[G_1(z)+1]}[G_1(z)+1]$$

$= \frac{3\xi}{6}$, then

(5) $$\left|G_2(z)\int_A^B e^{-zx}dF_1(x) - \int_A^B \sum_D \frac{1}{2}\left(e^{-zt_{i+1}} + e^{-zt_i}\right)\Delta F_2(t_i-x)dF_1(x)\right|$$

$< \frac{3\xi}{6}$.

Hence, from inequalities 1, 4, 5, and 3 it follows that $|G_1(z)\cdot G_2(z) - G(z)| < \xi$. Since $\xi > 0$ and $|G_1(z)\cdot G_2(z)-G(z)| < \xi$, then from Theorem 2.10 $G_1(z)\cdot G_2(z) = G(z)$.

If n is an integer such that $n > 2$, then by mathematical induction we can show that $G_1(z)\cdot G_2(z)\cdot\ldots\cdot G_n(z) = G(z)$.

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