POWER DOMINATION ON PERMUTATION GRAPHS

THESIS

Presented to the Graduate Council of
Texas State University-San Marcos
in Partial Fulfillment
of the Requirements

for the Degree

Master of SCIENCE

by

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San Marcos, Texas
May 2013

ACKNOWLEDGMENTS

Thanks to the members of my committee – Dr. Ferrero, Dr. Jia, and Dr. Tamir – as well as the whole of the Mathematics Department. Thanks also to my friends and family for their support on my way to this Master's degree; I couldn't have gotten anywhere near this far without you.

This manuscript was submitted on December 13th, 2012.

TABLE OF CONTENTS

Page

LIST OF FIGURES

ABSTRACT

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For the purposes of monitoring power networks, power companies use devices known as phase measurement units (PMUs); these monitor the waveform of the various nodes in a network. The cost of these units makes it worthwhile to minimize the number required. When a power network is modeled by a graph, the question of precisely how many are necessary to observe a given network, and of where they should be placed, is known as the power domination problem. We will consider this problem as it relates to permutation graphs on cycles, and suggest upper bounds for the power domination numbers on such graphs.

I. INTRODUCTION

In order to maintain effective and efficient service, it is necessary for power companies to monitor their power networks so as to react to changing conditions in usage and availability. Without real-time data, networks must be given settings that err on the side of caution, and which are therefore inefficient; otherwise, momentary spikes and shifts could cause major problems. As such, many power companies make use of devices known as phase measurement units – PMUs – to provide real-time analysis of their power networks. Whereas older solutions could only poll a network infrequently, and took several seconds to do so, PMUs can collect data continuously and simultaneously, allowing networks to react to short-lived phenomena as they appear.

However, these PMUs were and are prohibitively expensive, and it is both inefficient and unnecessary to install one on every bus in a network. Instead, by making use of a PMU's ability to monitor not only the waveform of its own bus, but of all lines connected to that bus, and applying Kirchhoff's law (which states that the sum of the currents entering a node is equivalent to the sum of the currents leaving) to calculate others, an entire network may be monitored by only a handful of units. This has given rise to the graph theoretic problem of power domination: how many PMUs are needed to observe every node in a given network, and where should they be placed? In this thesis we will consider the power domination problem as it pertains to permutation graphs on cycles.

II. DEFINITIONS

The graphs we consider in this thesis are connected, simple graphs – i.e. those which have only a single component, are undirected, and contain no loops. We call the set of neighbors of **S** the *open neighborhood* of **S**, $N(\mathbf{S})$, and $N(\mathbf{S}) \bigcup \mathbf{S}$ the *closed neighborhood* of **S**, $N[\mathbf{S}]$.

Given a simple graph $\mathbf{G} = (V, E)$ and $\mathbf{S} \subseteq V(\mathbf{G})$, we say that **S** *dominates* each vertex in $N_{\mathbf{G}}[\mathbf{S}]$; i.e. a vertex $v \in V(\mathbf{G})$ is dominated if it lies in **S**, or is adjacent to some vertex in **S**. We say that **S** is a *dominating set* for **G** if **S** dominates every vertex in **G**, and thus $V(\mathbf{G}) = N_{\mathbf{G}}[\mathbf{S}]$. The *domination number* $\gamma(\mathbf{G})$ is the minimal cardinality amongst subsets with this property.

As an extension of domination, we can consider the vertices *observed* by **S**:

- Any vertex in **S** is observed.
- Any vertex adjacent to **S** is observed.
- If a vertex v with $deg(v) > 1$ is observed, and all but one of the vertices adjacent to v are also observed, then the remaining vertex adjacent to v becomes observed. This rule is applied until no such vertex v exists.

In Figure 1, the two vertices marked with diamonds represet the set **S**; we can immediately say that they observe themselves as well as their neighbors. Then the vertex

a is observed, and has three neighbors, two of which are observed. Thus, a observes b. However, c remains unobserved, as its only neighbor, d, fails to meet the criteria of the propagation rule (it has two unobserved neighbors).

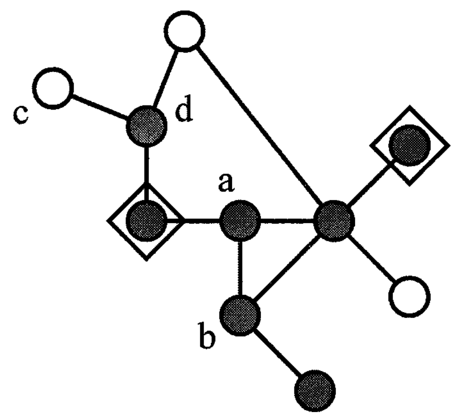

Figure 1: Power domination

Following the same pattern used for domination, we say that **S** is a *power dominating set* (PDS) for **G**, or simply that **S** observes **G**, if it observes every vertex in $V(\mathbf{G})$, and the *power domination number* $\gamma_P(G)$ is the minimal cardinality among power dominating sets for **G**. The question, called the power dominating set (PDS) problem, of whether a given graph has a power dominating set of size k has been shown to be NP-complete, even on certain classes of graphs, such as split and bipartite graphs; however, linear-time algorithms exist for finding minimal PDSs of several specific types of graphs.

This thesis focuses on a class of graphs known as permutation graphs; some examples of this class are shown in Figure 2. Given a graph **G** and a permutation $f: V(\mathbf{G}) \to V(\mathbf{G})$, the *permutation graph* of f on **G**, written $P(\mathbf{G},f)$, is constructed by taking two copies, $\mathbf{G}_1$ and $\mathbf{G}_2$, of **G**, and then adding an edge from each vertex u in $\mathbf{G}_1$ to its image $f(u)$ in $\mathbf{G}_2$.

In other words, $P(\mathbf{G},f) = \mathbf{G}_1 \bigcup \mathbf{G}_2 \bigcup \{(u,v) \mid u \in \mathbf{G}_1, v \in \mathbf{G}_2, \text{and } f(u) = v\}$. The graph $\mathbf{G}$ is called the *underlying graph* of $P(\mathbf{G},f)$.

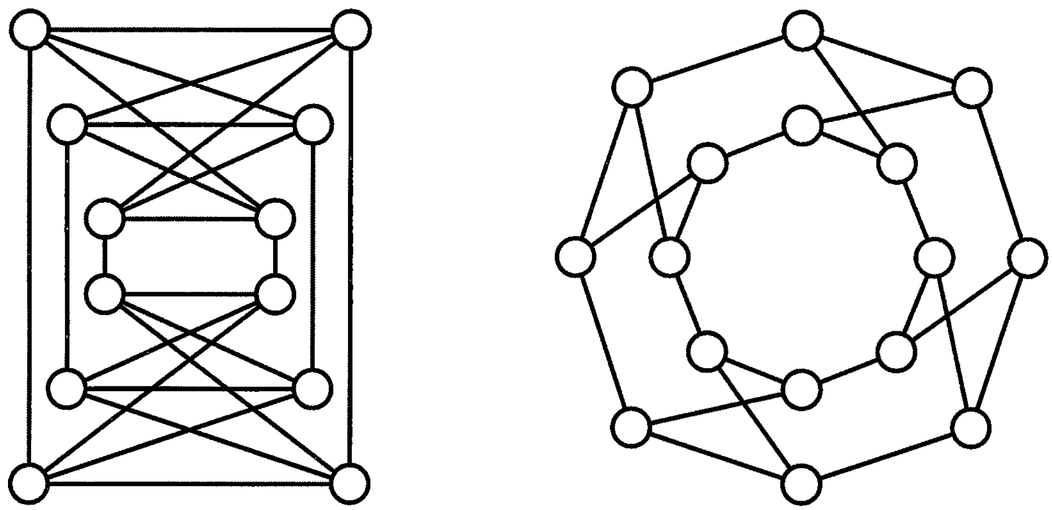

Figure 2: Permutation graphs: $P(K_{3,3}, id)$ and $P(\mathbf{C}_8, (1\ 2)(3\ 4)(5\ 6)(7\ 8))$

In particular, we will consider permutation graphs whose underlying graphs are cycles; these have the form $P(\mathbf{C}_n,f), f \in S_n$, where $n \geq 3$. This type of graph is notable in that each such graph is cubic: each vertex u is adjacent only to the two vertices adjacent to it in its cycle and to the single vertex $f(u)$. In the context of the power domination problem, any vertex of degree 1 or 2 is effectively "unnecessary" in the sense that no PMU need ever be placed there so long as a vertex of degree three or higher exists in the same connected component; thus, cubic graphs are of particular interest. The generalized Petersen graphs are a subset of permutation graphs on cycles; for example, the original Petersen graph $G(5,2)$ is isomorphic to the permutation graph $P(\mathbf{C}_5, (1\ 3\ 5\ 2\ 4))$.

III. LITERARY REVIEW

The first prototypes of PMUs were constructed in the 1970s at Virginia Tech, and production models began to appear in the mid-1980s. They were constructed to solve a serious flaw in the power systems of the day: individual nodes in the network can alter their voltages to fit a given voltage profile, but without accurate real-time data on the state of the network, it is difficult to set a profile which is both safe and efficient. Existing systems only checked the status of the network every few minutes, and it took several seconds for data from all the nodes to be recorded. This level of precision proved insufficient to set the profile in real-time, so instead the optimal settings were generally calculated ahead of time and then fixed. Engineers were thus forced to trade efficiency for stability, as the system was unable to react to short-term changes and voltage spikes (Mili et al., 1990).

Power domination as a graph theoretic problem seems to have been first considered by Mili, Baldwin, and Adapa in a 1990 paper titled *Phasor Measurement Placement for Voltage Stability Analysis of Power Systems*. At this point in time, monitoring schemes involving PMUs were already in place in several locations, notably France and Italy. Mili et al. (1990) note that the problem of where to place them had already been considered by several previous authors, and describe solutions created by Begovic and Phadke, Schleuter, and Ilic, all based around the idea of dividing the network into "coherent regions." These would be areas whose nodes had similar reactions to changes in the state of the system, and which could therefore be effectively monitored simply by assigning to each region a single PMU. The various approaches differ primarily on how to define the coherent regions of a network, and which point within each region should have a PMU (Mili et al., 1990).

This design suffers from a few flaws, however. Most importantly, it assumes that coherent regions can be found, and that they are large enough that the scheme is efficient; in reality, regions can be very small and are not necessarily stable over time. Also, the given definitions of coherency require only that nodes in a certain region have similar phase angles – they can differ substantially in magnitude, which poses problems in constructing the voltage profile (Mili et al., 1990).

The alternative to this approach is to install PMUs on various nodes in the network such that every edge's current is either directly observed by some PMU, or can be accurately calculated. Mili et al. (1990) give two rules used to determine which parts of the network are monitored by a set of PMUs:

- Assign a current phasor measurement to all the branches incident to a bus provided with a PMU,
- Assign a calculated current phasor to any branch connecting two buses with known voltages.

Power networks are usually modeled by graphs where the generators, substations, and consumers are represented as vertices, and the power lines as edges (Jenelius, E., 2004). Note that, in this context, finding a minimal set of locations where PMUs should be placed in order to monitor the entire network according to the two above rules is, in fact, the *standard* domination problem; without the use of Kirchhoff's law, observation does not spread beyond the closed neighborhoods of the PMUs. Also interesting is the fact that this early definition focuses on observation of the edges of the graph, rather than the vertices; Brueni and Heath (2005) showed that the more modern vertex-based definition is equivalent and edges can thus be ignored.

The solution given to this problem by Mili et al. (1990) is an algorithm which finds a minimal dominating tree:

1. Place a PMU at an arbitrarily chosen root node;
2. Expand the tree by placing a PMU on a node which (a) is adjacent to a node which already has a PMU, and (b) has the greatest number of unobserved neighbors;
3. Repeat until the whole graph is observed (note that the nodes with PMUs form a tree);
4. Try to improve this solution by a binary search approach: take the first $\frac{m}{2}$ of the m nodes in your solution, reallocate them using a probabilistic process called simulated annealing, then repeat with $\frac{m}{4}$ or $\frac{3m}{4}$ nodes as appropriate and so on.

Power domination diverged from standard graph domination, and thus reached its present form, with the application of Kirchhoff's law to the process. Kirchhoff's first law states that, at any given node, the sum of the incoming currents is equal to the sum of the outgoing currents; when we know the currents of all but one of the edges incident to a given node, and the total current at the node itself, this law gives us immediately the current on the one remaining edge. Thus we have the third rule given in the standard power domination problem, sometimes referred to as the "propagation rule":

- Whenever there exists an observed node v such that $deg(v) > 1$, and all but one of the nodes adjacent to v are observed, the remaining node adjacent to v becomes observed.

This allows PMUs to observe nodes at arbitrarily long distances under the right conditions, and gives rise to the power dominating set problem (Baldwin et al., 1993).

Kirchhoff's law is first referenced with regards to PMU placement in 1993, by Baldwin et al. Where Mili et al. (1990) place the PMUs in such a way as to form a spanning tree, Baldwin et al. (1993) note that we do not need this restriction, and can use any arbitrary subset; they present a modification of the earlier algorithm which simply chooses at each step the node with the largest number of unobserved neighbors, regardless of its position in the graph. Several statistics about the problem are given as well, such as the fact that the power domination number of a graph is generally between %20 and %30 of the graph's size, as shown by several test cases on real-world networks. More applicable to graph theory is the conjecture that, for any graph **G**, $\gamma_P(\mathbf{G}) \leq \left\lceil \frac{V(\mathrm{G})+E(\mathrm{G})/2}{3} \right\rceil$ (Baldwin et al., 1993). This bound is supported by a worst-case example, shown in Figure 3; as the size of the graph grows, the power domination number approaches $\frac{V(G)}{2}$.

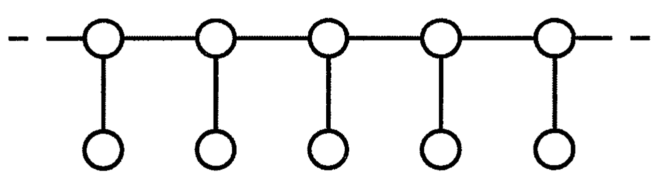

Figure 3: A graph illustrating the worst case for power domination

Over the next decade, power domination came to be studied more and more from a graph theoretical perspective. One of the landmark papers in the development of the problem is *Domination in Graphs Applied to Electric Power Networks* (Haynes et al., 2002), a paper which provides the groundwork for much of the work that has been done since. Among other results, it provides a number of basic, and highly useful, facts about power domination numbers (PDNs):

- Any dominating set is also a power dominating set, and thus, for any graph **G**, $1 \leq \gamma_P(\mathbf{G}) \leq \gamma(\mathbf{G})$.
- If **G** is complete, a path, a cycle, or bipartite with one of the partitions containing only two vertices, then $\gamma_P(\mathbf{G}) = 1$.
- There exist graphs for which $\gamma(\mathbf{G}) = \gamma_P(\mathbf{G})$ (e. g. the corona $(P_n \circ \overline{K_2})$), as well as graphs where $\gamma(\mathbf{G}) - \gamma_P(\mathbf{G})$ is arbitrarily large: as above, any path P_n has a PDN of exactly 1, but domination number $\lceil \frac{n}{3} \rceil$. Further, there is no forbidden subgraph for graphs which have $\gamma(\mathbf{G}) = \gamma_P(\mathbf{G})$.
- For any graph with at least one node of degree 3 or higher, there must exist a minimal PDS **S** such that $deg(s) \geq 3$ for all $s \in \mathbf{S}$; essentially, having any node of degree two or less in a PDS is pointless, as any such node can always be dominated by a node of degree 3 or higher. Conversely, nodes of low degree are extremely important to the problem as a whole; subdividing a single edge is often enough to change the PDN of a graph significantly.

Further, it is shown that the power domination problem – specifically, the question of whether a given graph has a PDS of a given size – is NP-complete. This is done by converting it into the famous NP-complete problem 3SAT, which involves finding a truth assignment which satisfies a number of clauses. This result was later clarified and expanded by several other papers; one noted that the original proof was flawed, and constructed a new proof based on split graphs (Liao and Lee, 2005), while another provided an improved algorithm for finding power dominating sets on trees (Guo et al., 2005). Kneis et al. described the problem in terms of "parametrized complexity," showing that it is at least $W[2]$ hard, but is definitely within $W[P]$. They note, however, that this is still not very exact: the same result is true of standard domination (Kneis et al., 2006).

Haynes et al. (2002) also began the process of finding bounds on the PDN of various classes of graph, specifically by considering trees. The PDN of a tree is 1 if and only if the tree is a spider, and in fact any tree has PDN precisely equal to its spider number (Haynes et al., 2002). A linear time algorithm for finding a minimal PDS on a tree is given; this was expanded to cover graphs of finite treewidth by Kneis et al. (2006)

Results for many different types of graphs followed: Dorfling and Henning (2006) considered the bounds of the PDN of grid graphs – the Cartesian product of two cycles – and their work inspired studies on cylinders and tori (Barrera and Ferrero, 2011), as well as the direct, strong, and lexicographic products of two cycles (Dorbec et al., 2008). Liao and Lee (2005) focused on split graphs, as well as on interval graphs and circular-arc graphs; the PDS problem is NP-complete on the former, but linear time algorithms are given for the latter two.

Barrera and Ferrero (2011) also considered generalized Petersen graphs, and as these are a subset of permutation graphs on cycles, these results are the most directly relevant to this thesis. They showed that, for a generalized Petersen graph $P(m, k)$, $\gamma_P(P(m, k)) \leq k$; taking a k-length path on one cycle and placing a PMU on each vertex of the second cycle adjacent to that path will observe the entire graph (Barrera and Ferrero, 2011). In particular, this shows that there are arbitrarily large generalized Petersen graphs having a PDN of 2: for $m \geq 5$, $\gamma_P(P(m, 2)) = 2$.

We finally note that the basic power dominating set problem has been generalized in several different ways. One such extension is known as k-power domination; here the propagation rule is to read, "If there exists an observed node v such that $deg(v) > k$, and all but k of the nodes adjacent to v are observed, the remaining adjacent nodes of v become observed" (Chang et al., 2012). This is equivalent to power domination for $k = 1$ and to domination for $k = 0$, and thus represents a way to connect the two, allowing us to generalize some results about domination (which has been more widely studied) to power domination. For example, barring graphs with maximum degree $\leq k + 1$, any vertex in a minimal k-PDS with degree $\leq k + 1$ can be replaced with one of higher degree, and there always exists a minimal k-PDS such that every vertex has degree $\geq k + 2$ and at least $k + 1$ private neighbors.

Several other authors created generalizations designed with real-world considerations in mind. Nuqui and Phadke (2005) suggested that the number of PMU's required to monitor the network could be kept low by allowing some small subset of the graph to go unobserved; unobserved vertices' waveforms can be estimated from the data of those of their neighbors which are observed. This could be useful in particular when the PMUs are being installed incrementally, as small numbers of units could be added in stages, gradually decreasing the depth of unobservability until the whole graph is observed (Nuqui and Phadke, 2005).

The same paper poses the question of what can be done when some vertices are restricted from receiving PMUs, for example because they lack communications lines. This inspired the creation of another generalization: for a subset $\mathbf{Z}$ of $\mathbf{G}$, a $\mathbf{Z}$-restricted PDS of $\mathbf{G}$ is defined as a PDS of $\mathbf{G}$ which contains no vertex of $\mathbf{Z}$ (Pai et al., 2010). It is of course perfectly possible that no such set exists, such as in the degenerate case that $\mathbf{Z} = \mathbf{G}$. Related to this is a fault-tolerant variant, which requires that a set continue to observe all vertices even if k of the PMUs fail. Both of these subsume the standard PDS problem, when $\mathbf{Z}$ is empty or $k = 0$ respectively.

IV. PRELIMINARY RESULTS

We first present some basic results about permutation graphs on cycles. Let $\mathbf{G} = P(\mathbf{C}_n, f)$ be any such graph, where f is the related permutation, n the cardinality of the underlying cycle (so that the order of $\mathbf{G}$ is $2n$), and $\mathbf{G}_1$ and $\mathbf{G}_2$ the two images of $\mathbf{C}_n$ contained in P. We say $V(\mathbf{G}_1) = \{u_0, u_1, u_2, u_{n-1}\}$ and $V(\mathbf{G}_2) = \{v_0, v_1, v_2, v_{n-1}\}$, where u_k is adjacent to u_{k-1} and u_{k+1} where the sum is modulo n; similarly, v_k is adjacent to v_{k-1} and v_{k+1} where the sum is modulo n.

Proposition 1. There exist arbitrarily large permutation graphs on cycles with power domination number equal to 2.

This follows immediately from the same result about generalized Petersen graphs (Barrera and Ferrero, 2011). One notable example of this type of graph is the case where f is the identity permutation, as shown in Figure 4.

In this case, simply placing one PMU at any node u_k and a second at $f(u_k) = v_k$ is sufficient to observe every vertex: u_{k+1} is observed, and has two observed neighbors (u_k and v_{k+1}), so it observes u_{k+2}, and similarly v_{k+1} observes v_{k+2}. Then u_{k+2} is observed, and has two observed neighbors, so it observes u_{k+3}, and similarly v_{k+2} observes v_{k+3}. This continues until the entire graph is observed.

Proposition 2. If a set $\mathbf{S}$ observes every vertex in $\mathbf{G}_1$ or every vertex in $\mathbf{G}_2$, then $\mathbf{S}$ observes $\mathbf{G}$.

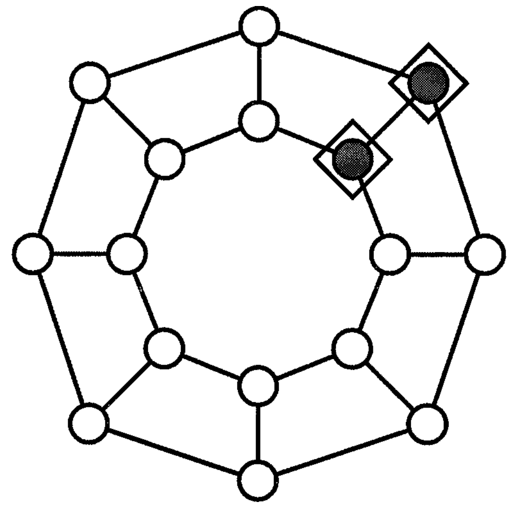

Figure 4: $P(\mathbf{C}_8, id)$, and a minimal PDS thereon

Suppose without loss of generality that **S** observes $\mathbf{G}_1$, and let p be any vertex in $\mathbf{G}_2$. Then say $f^{-1}(v) = u_k$, and note that **S** observes u_k as well as u_{k-1} and u_{k+1}. Then u_k observes p. Thus **S** observes every vertex of **G**.

Proposition 3. $\gamma_P(\mathbf{G}) \leq \lceil \frac{n}{3} \rceil$, and $\gamma_P(G) = 1$ if and only if $n = 3$.

Let $\mathbf{S} = u_1, u_4, u_7, u_{1+3k}$ where $k = \lfloor \frac{n-1}{3} \rfloor$. Then each vertex in $\mathbf{G}_1$ is either in **S**, or adjacent to an element of **S**. Thus **S** observes all of $\mathbf{G}_1$ and thus all of **G**. For the second part, note that if $n = 3$, any single vertex will dominate its entire cycle, and thus the whole graph. Conversely, suppose $n \geq 4$ and $|\mathbf{S}| = 1$; then none of the neighbors of the single PMU will be adjacent, so no vertices beyond $N(\mathbf{S})$ will be observed. Since $|N(\mathbf{S})| = 4$ and $|\mathbf{G}| \geq 8$, **S** cannot observe all of **G**.

V. MAIN RESULT

We will attempt to prove the following conjecture:

Conjecture. Given any permutation graph $\mathbf{G} = P(C_n, f)$ for some cycle C_n and a permutation $f \in S_n$:

$$\gamma_P(\mathbf{G}) = 1, \qquad n = 3$$
$$\gamma_P(\mathbf{G}) = 2, \qquad 4 \leq n \leq 8$$
$$\gamma_P(\mathbf{G}) \leq \left\lceil \tfrac{n}{4} \right\rceil, \quad n \geq 9$$

Further, we will show that the bound given for $n \geq 9$ is sharp, at least whenever $n \not\equiv 1 \pmod 4$.

The first case, when $n = 3$, was proven in Proposition 3. The cases where $4 \leq n \leq 6$ are similarly immediate, since $2 \leq \gamma_P(\mathbf{G}) \leq \left\lceil \frac{n}{3} \right\rceil = 2$.

Thus, we proceed to prove the cases where $n = 7$ or $n = 8$. For these, we will need the following lemma:

Lemma 1. If **S** observes all but one vertex of $\mathbf{G}_1$ and all but two vertices of $\mathbf{G}_2$, or vice versa, then **S** observes all of **G**.

Proof: Assume to the contrary that **S** observes all but one vertex of $\mathbf{G}_1$ and all but one of $\mathbf{G}_2$, but fails to observe the remaining three vertices (the reverse follows symmetrically). Note that this implies that $n \geq 4$, as any smaller graph will be completely observed by the first PMU placed. Call the unobserved vertex in $\mathbf{G}_1$ u_k.

First, suppose $f(u_{k-1})$ is observed. Then u_{k-1} is observed, and so are $f(u_{k-1})$ and u_{k-2}. But then u_{k-1} has two observed neighbors, and therefore observes u_k, a contradiction. A similar argument holds for $f(u_{k+1})$. Thus it must be that the two unobserved vertices in $\mathbf{G}_2$ are $f(u_{k-1})$ and $f(u_{k+1})$, and in particular $f(u_k)$ is observed.

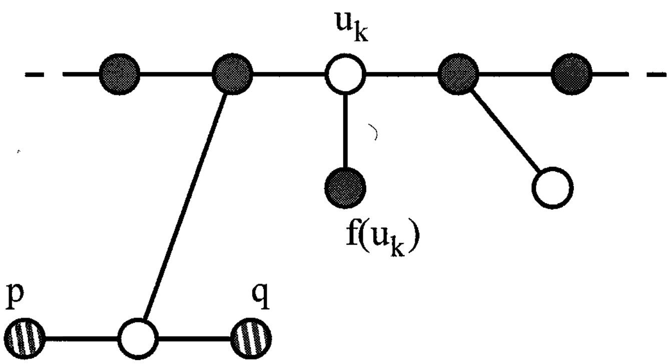

Figure 5: Lemma 1

Now, consider the two neighbors of $f(u_k)$ in $\mathbf{G}_2$. If both are observed, then $f(u_k)$ observes u_k, contradicting the hypothesis. On the other hand, if neither are observed, then $f(u_k)$ is an observed vertex which has only unobserved neighbors, which is clearly an impossibility. Thus it must be that precisely one is observed.

As such, we assume without loss of generality that $f(u_{k-1})$ is not adjacent to $f(u_k)$, and consider the neighbors p and q in $\mathbf{G}_2$ of $f(u_{k-1})$; see Figure 5. If one of them is $f(u_{k+1})$, then the other is an observed vertex with two observed neighbors (its pre-image in $\mathbf{G}_1$ cannot be u_k and is thus observed, and its other neighbor in $\mathbf{G}_2$ cannot be $f(u_{k+1})$ since

$n \geq 4$), and thus observes $f(u_{k-1})$, which is a contradiction since $f(u_{k-1})$ was assumed to be unobserved. So both p and q must be observed. Note that they cannot both be adjacent to $f(u_{k+1})$, as the only way this could occur would be if $n = 4$, and then $f(u_k)$ would have to be one of p or q and thus adjacent to $f(u_{k-1})$.

We can therefore see that the only remaining case is when at least one of p or q is observed, but not adjacent to $f(u_{k+1})$; that one has two observed neighbors and observes $f(u_{k-1})$. This is again a contradiction, so it must be the case that **S** observes all of **G**.

□

Theorem. $\gamma_P(P(C_7, f)) = 2$ for any permutation $f \in S_7$.

Proof: First note that, by Proposition 3, $2 \leq \gamma_P(\mathbf{G}) \leq 3$. So it suffices to show that $\gamma_P(\mathbf{G}) < 3$.

Suppose $\mathbf{G} = P(C_7, f)$ for some $f \in S_7$. Then let $\mathbf{S} = u_1, u_4$. Note that u_2 and u_3 are observed, and each has two observed neighbors, so then $f(u_2)$ and $f(u_3)$ are observed. As $f(u_1)$ and $f(u_4)$ are also observed, there are at least four observed vertices in $\mathbf{G}_2$. Since $\mathbf{G}_2$ consists of only seven vertices, at least two of those four must be adjacent, and assuming not every vertex in $\mathbf{G}_2$ is observed, some vertex p in this block of observed vertices must be adjacent to some as-yet unobserved vertex q, as shown in Figure 6.

Then p is the image of one of $\{u_1, u_2, u_3, u_4\}$, all of which are observed, and p is adjacent to some other observed vertex. Thus p observes q. Then all but one vertex in $\mathbf{G}_1$ and all but two vertices in $\mathbf{G}_2$ are observed, so by the lemma **S** observes **G**.

□

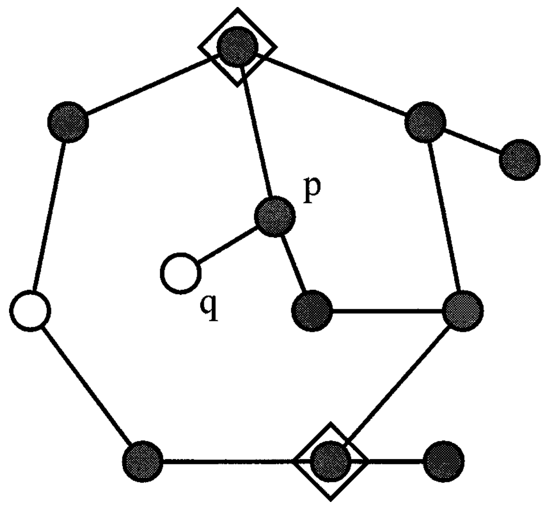

Figure 6: $P(\mathbf{C}_7, f)$

For the case where $n = 8$, we need a second lemma:

Lemma 2. Consider the following graphs, as in Figure 7:

$$D_4 = P(P_4, (1\ 2)(3\ 4))$$
$$D_4' = P(P_4, (1\ 3)(2\ 4))$$
$$D_5 = P(P_5, (1\ 2)(4\ 5))$$
$$D_5' = P(P_5, (1\ 4)(2\ 5))$$

Let **G** be a permutation graph on a cycle such that one of the above graphs appears in **G** as an induced subgraph, and let D be that subgraph. Then any set that observes **G** must include at least one vertex of D.

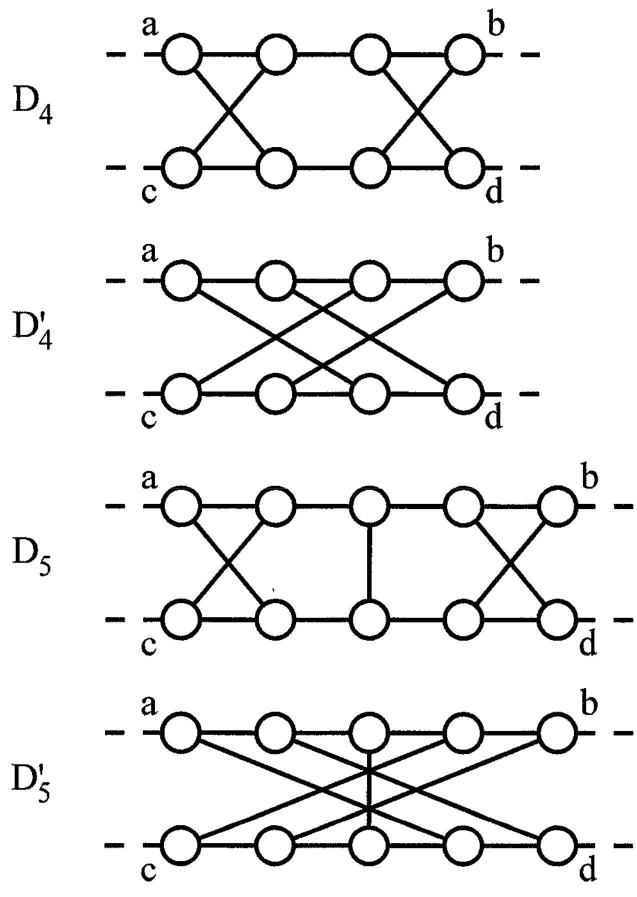

Figure 7: Subgraphs D_4, D_4', D_5, and D_5'

Proof. To show this, let $\mathbf{S} \subset \mathbf{G}$ be a subset such that $\mathbf{S} \bigcap D = \emptyset$, and consider the vertices of $\mathbf{G}$ observed by $\mathbf{S}$. In each case, $\mathbf{S}$ can dominate at most the outer vertices of D (labeled a, b, c, and d in Figure 5): even if $\mathbf{S}$ consists of every vertex of $\mathbf{G}$ not in D, a, b, c, and d will each still have two unobserved vertices, and the inner vertices of D will remain unobserved. Thus $\mathbf{S}$ cannot observe $\mathbf{G}$, and the lemma holds.

□

Theorem. $\gamma_P(P(C_8,f)) = 2$ for any permutation $f \in S_8$.

Proof: Let $\mathbf{G} = P(C_8,f)$ for some $f \in S_8$, and again let $\mathbf{S} = u_1, u_4$. As before, there are at least four observed vertices in $\mathbf{G}_2$. Unfortunately, unlike the case where $n = 7$, the distribution of these vertices is relevant, so we let $T = \{u_1, u_2, u_3, u_4\}$ and consider the possible cases. There are six possible configurations, shown in Figure 8.

In the following proofs, when we mention a "block" of vertices in T, we mean a subset of T such that each vertex in the block is adjacent to another vertex in the block, and none is adjacent to a vertex of T not in the block. Note that a block of two or more vertices in T must have two currently unobserved neighbors p and q. Since each vertex in that block has two observed neighbors – its preimage in $\mathbf{G}_1$ and another vertex in the block – the block must observe p and q.

1. No two vertices in T are adjacent; i.e. the vertices in $\mathbf{G}_2$ alternate between observed and unobserved. In this case we redefine $\mathbf{S}$ as u_2, u_5; this will still observe $f(u_2)$, $f(u_3)$, and $f(u_4)$, as well as $f(u_5) \neq f(u_1)$. Then, since in the previous attempt each unobserved vertex in $\mathbf{G}_2$ was adjacent to one of $f(u_2), f(u_3)$, and $f(u_4), f(u_5)$ must be adjacent to one of them. This leads us to one of the other cases – for ease of description we rotate the vertex labels so that the elements of $\mathbf{S}$ are u_1 and u_4 again.

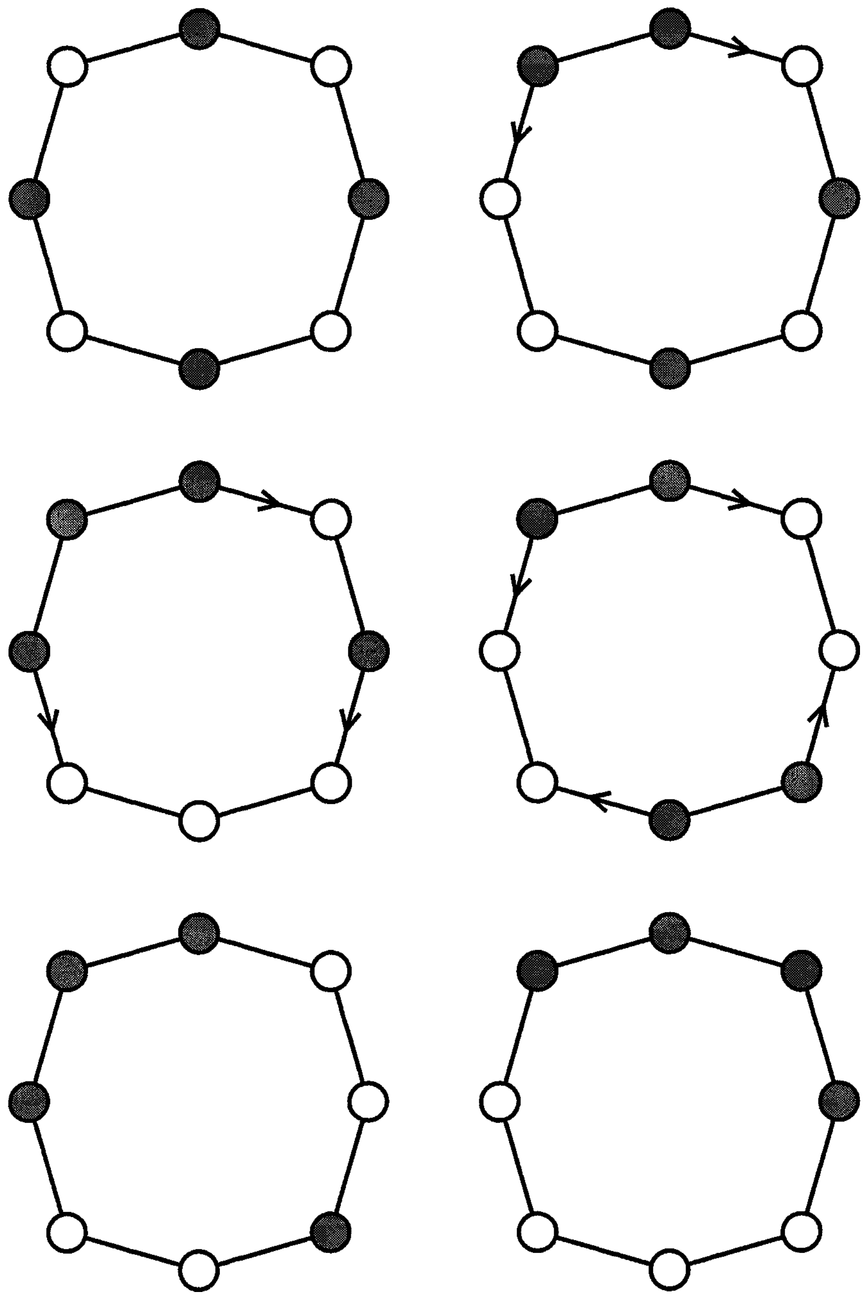

Figure 8: Possible configurations for $\mathbf{G}_2$ in $P(\mathbf{C}_8, f)$

2. T consists of three blocks of vertices; then precisely one of the blocks contains two vertices. This block observes its two neighbors in $\mathbf{G}_2$, and due to the fact that the other two vertices of T cannot be adjacent to each other, one must be adjacent to one of the newly observed vertices. Then it observes its other neighbor, giving us seven observed vertices in $\mathbf{G}_2$ and six in $\mathbf{G}_1$, so Lemma 1 applies and we are done.

3. T consists of two blocks of vertices (either two and two, or three and one) with a single vertex not in T between them. In this case the vertices of T observe all three of T's neighbors in $\mathbf{G}_2$; then we have seven observed vertices in $\mathbf{G}_2$ and six in $\mathbf{G}_1$, so we are done again.

4. T consists of two blocks of two vertices each such that there are two vertices not in T between them (in each direction). Then T observes all of $\mathbf{G}_2$ and thus all of $\mathbf{G}$.

5. T consists of two blocks, one with three vertices and one with a single vertex, such that there are two vertices not in T between them (in each direction). In this case we redefine $\mathbf{S}$ to be the set $\{u_5, u_8\}$. This adjustment means that the new $\mathbf{S}$ observes precisely those vertices of $\mathbf{G}_2$ which are not in T; these form two blocks of two with a single vertex between them, so we end up in case 3 and are done.

6. The final case is when T consists of only a single block. In this case, let p and q be the vertices of $\mathbf{G}_2$ adjacent to T, and consider the preimages $f^{-1}(p)$ and $f^{-1}(q)$. Suppose one of these is actually u_5; then u_5 observes u_6. Similarly, if one of the preimages is $u_0 = u_8$, it will observe u_7. In either case, we have six observed vertices in $\mathbf{G}_2$ and seven in $\mathbf{G}_1$, so by Lemma 1 $\mathbf{S}$ observes $\mathbf{G}$. So assume that $\{f^{-1}(p), f^{-1}(q)\} = \{u_6, u_7\}$. Then the four vertices in $\mathbf{G}_2 - T$ form a path with p and q as its endpoints, and each point is connected to its preimage on the path $\{u_5, u_6, u_7, u_8\}$; thus, this is a permutation graph on P_4. In fact, since $\{f^{-1}(p), f^{-1}(q)\} = \{u_6, u_7\}$, we can see that it is actually either D_4 or D_4'. Finally, redefine $\mathbf{S}$ as $\{u_5, u_8\}$; as in case 5 this will cause $\mathbf{S}$ to observe $\mathbf{G}_2 - T$. Since this is

of course again a single block, we are in case 6 again; repeating the above argument will show that, if **S** still fails to observe **G**, the second half of **G** must consist of a second copy of D_4 or D'_4. Then **G** is one of the graphs shown in Figure 9, and since each has a power dominating set of order 2, we are again done.

□

For the case where $n \geq 9$, we will proceed by induction, so we first need the following:

Definition. Let **G** be a permutation graph on a cycle of length n, where $n \geq 7$, and let **C** be a path of length 3 in one of the two cycles $\mathbf{G}_1$ and $\mathbf{G}_2$. We define $\mathbf{G} \div \mathbf{C}$ to be the graph obtained by the following procedure (see Figure 10):

- Delete from **G** each of the edges $(c,f(c))$, where $c \in V(\mathbf{C})$.
- Smooth out each of the eight vertices that were adjacent to those edges – each of these vertices has lost one incident edge, and thus degree 2, so we replace it with a edge connecting its neighbors. Call w the single edge in $\mathbf{G} \div \mathbf{C}$ produced by smoothing out **C**.

Then $\mathbf{G} \div \mathbf{C}$ consists of two cycles, $\mathbf{G} \div \mathbf{C}_1$ and $\mathbf{G} \div \mathbf{C}_2$, each of length $n - 4$ (each of the cycles in **G** lost four vertices), plus edges such that each vertex in $\mathbf{G} \div \mathbf{C}_1$ is connected to precisely one vertex in $\mathbf{G} \div \mathbf{C}_2$. Thus $\mathbf{G} \div \mathbf{C}$ is a permutation graph on a cycle of length $n - 4$.

Finally, we need the following conjecture; it is surmised that it holds for every permutation graph on a cycle.

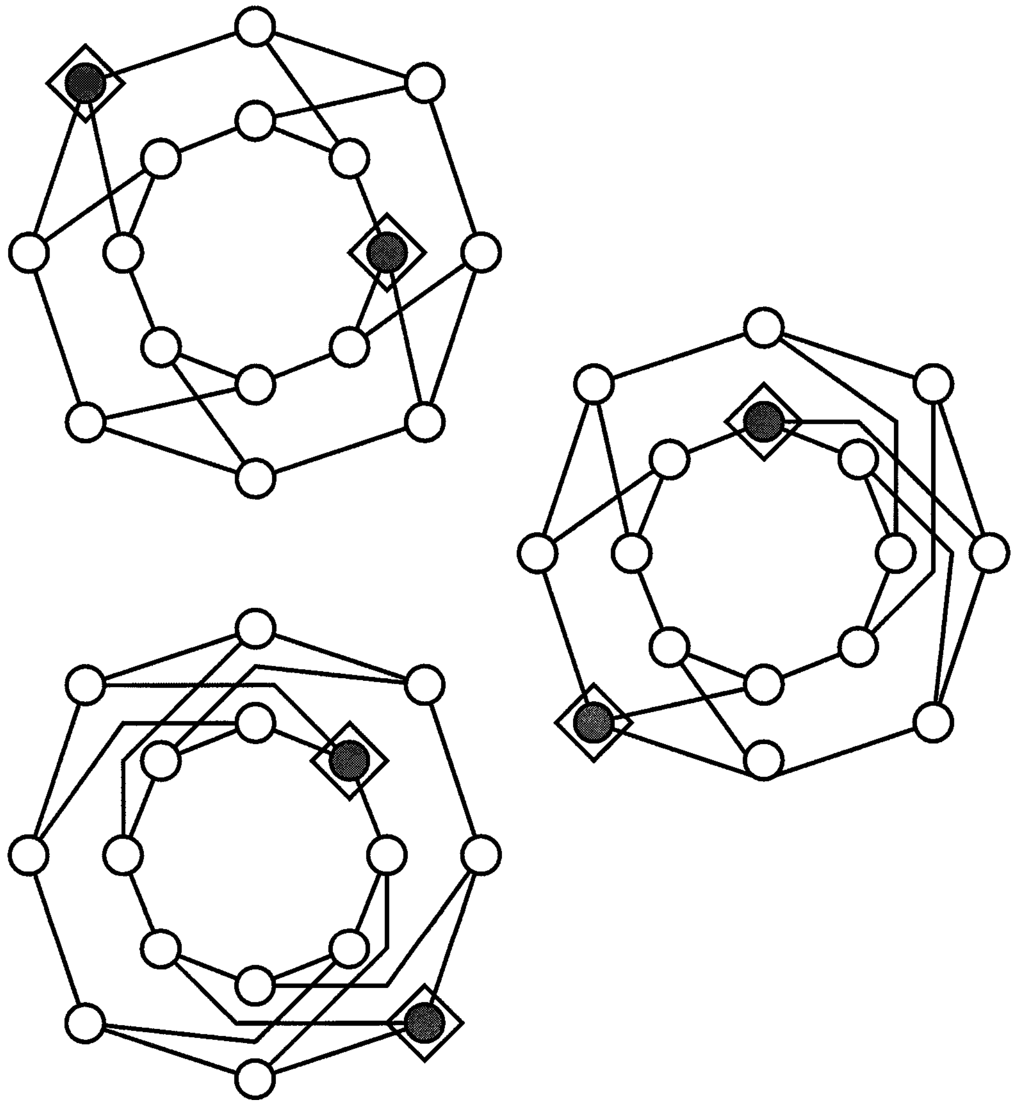

Figure 9: Case 6 for $P(\mathbf{C}_8,f)$

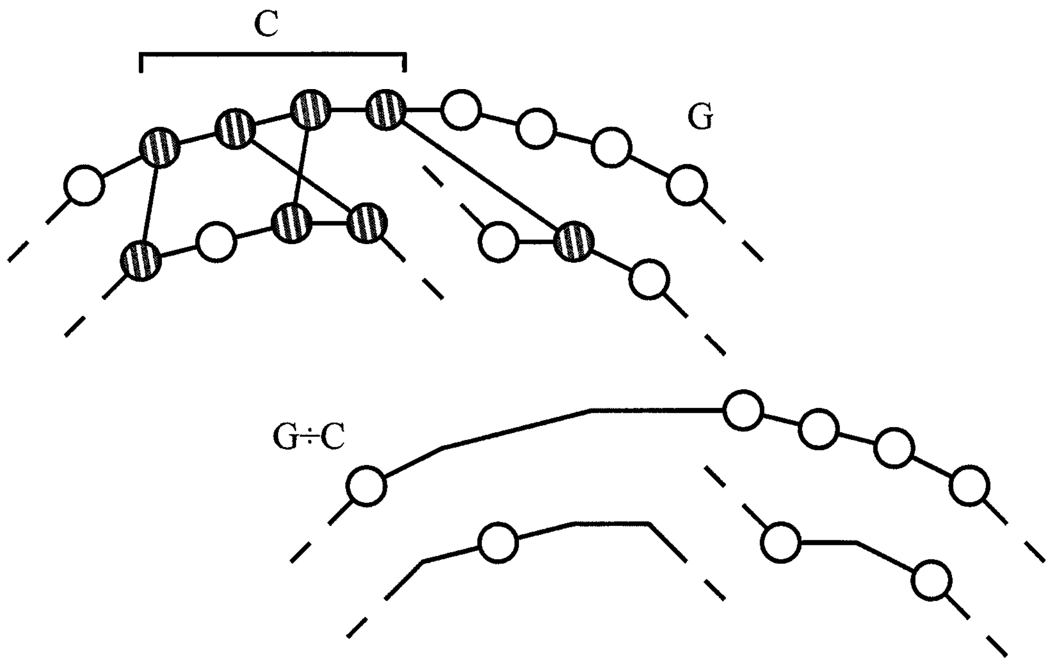

Figure 10: Deleting vertices to generate $\mathbf{G} \div \mathbf{C}$

Conjecture. Given any permutation graph on a cycle of length n, where $n \geq 9$, there exists some path $\mathbf{C}$ of length 3 in one of the two cycles $\mathbf{G}_1$ and $\mathbf{G}_2$ such that $\mathbf{G} \div \mathbf{C}$ has a minimal power dominating set $\mathbf{S}$ including a vertex incident to w.

In support of this claim, we provide an example. Consider the graphs $\mathbf{K}_n = P(C_n, (1\ 2)(3\ 4)(5\ 6))$ for $n \geq 7$ – i.e. graphs based on a permutation which maps all but six of the elements of the underlying cycle to themselves, and the remaining six in such a way as to form two overlapping instances of D_4, as defined in Lemma 2. These graphs have the structure illustrated in Figure 11.

These graphs are a particularly bad case: if $n \geq 12$, choosing $\mathbf{C}$ such that no vertex of $\mathbf{C}$ is adjacent to any of the vertices in the instances of D_4 – note that, for large n, this makes up the vast majority of possible $\mathbf{C}$s – gives us $\mathbf{K}_n \div \mathbf{C} = \mathbf{K}_{n-4}$. The set $\{u_2, v_5\}$ is a power dominating set for every $\mathbf{K}_n$, and as it has size 2 it must be minimal. Now, if there existed some minimal power dominating set $\mathbf{S}$ on $\mathbf{K}_n \div \mathbf{C}$ such that a vertex of $\mathbf{S}$ were incident to w, that vertex would lie in neither of the instances of D_4. Then, in order to have a vertex in each instance of D_4, as required by Lemma 2, the second vertex must be one of u_3, v_3, v_4, and v_4; none of these possibilities allows $\mathbf{S}$ to observe $\mathbf{K}_n \div \mathbf{C}$, so this cannot occur, and there are indeed no minimal power dominating sets of the type we hoped for.

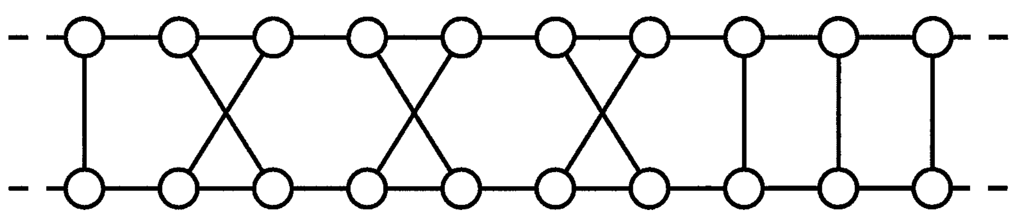

Figure 11: An example of the class of graphs $\mathbf{K}_n$

However, despite the existence of arbitrarily many possible $\mathbf{C}$s for which the above occurs, the conjecture is nonetheless true for $\mathbf{K}_n$. Choosing $\mathbf{C}$ to be the path from u_1 to u_4 gives us $\mathbf{K}_n \div \mathbf{C} = P(C_{n-4}, (1\ 2))$; $\{u_1, v_1\}$ is a minimal power dominating set for this graph, and u_1 is incident to w in $\mathbf{K}_n \div \mathbf{C}$, which is what we needed.

Theorem. $\gamma_P(P(C_n,f)) \leq \lceil \frac{n}{4} \rceil$ for any $n \geq 9$ and any $f \in S_n$ such that the above conjecture holds.

Proof: Let $n \geq 5$ and say $n = a + 4m$, where $a \in \{5, 6, 7, 8\}$ and $m \in \mathbb{Z}^+$. We proceed by induction on m.

The base case is when $m = 0$. In this case, $5 \leq n \leq 8$, so by the previous theorems $\gamma_P(\mathbf{G}) = 2 = \lceil \frac{n}{4} \rceil$. Thus, the inductive hypothesis applies.

Now let **G** be any permutation graph on a cycle of length $n \geq 9$. By the inductive hypothesis, we can assume that, for any permutation graph on a cycle of size $n - 4$, there exists a power dominating set of size at most $\lceil \frac{n-4}{4} \rceil = \lceil \frac{n}{4} \rceil - -1$.

By the conjecture, there exists some path $\mathbf{C} = (c_1, c_2, c_3, c_4)$ in either $\mathbf{G}_1$ or $\mathbf{G}_2$ such that $\mathbf{G} \div \mathbf{C}$ has a minimal power dominating set **S** containing a vertex incident to the edge w. Without loss of generality, we assume **C** lies in G_1. Call $\mathbf{S}'$ the set of vertices in **G** which were mapped to **S** by the deletion of **C** (the mapping was of course bijective). Obviously $\mathbf{S}'$ is disjoint from **C**, and $|\mathbf{S}'| = |\mathbf{S}| \leq \lceil \frac{n}{4} \rceil - -1$. We claim that there exists a vertex $a \in \mathbf{C}$ such that the set $\mathbf{S}' \bigcup \{a\}$ observes all of **G**.

Since **S** contained a vertex incident to w in $\mathbf{G} \div \mathbf{C}$, $\mathbf{S}'$ contains some vertex s adjacent to one of the endpoints of **C**; suppose without loss of generality that s is adjacent to c_1. Then let $a = c_3$, the point in **C** that lies at distance 3 from s. Then $\mathbf{S}' \bigcup \{a\}$ observes s, c_1, c_2, c_3, c_4, and the images of all of these except c_4. It may seem possible to immediately make use of Lemma 1, but we must first consider the possibility that the edge in $\mathbf{G} \div \mathbf{C}$ that resulted from the smoothing out of $f(c_4)$ was part of a path that observed some other vertices.

At this point we will need the following fact: if **G** is any graph with power dominating set **S**, and we subdivide any edge u of **G**, it may be that **S** fails to observe all of **G**. If, however, we can observe (say, by adding vertices to **S**) the vertex x created by the subdivision, as well as any neighbors of x not incident to u, **S** will indeed observe **G** – vertices that were observed via the first or second rules will remain the same, while if the third rule allowed one vertex incident to u to observe the other, it will instead allow x to observe the second vertex, due to x and all but one of its neighbors being observed.

Using this fact, pick any vertex y in $\mathbf{G} \div \mathbf{C}$ and consider any path starting with y whose last edge is w. Then compare this with the respective path in $\mathbf{G}$ that starts from y's preimage and ends at $f(c_4)$. Note that the second can be formed from the first via subdivision, but that every vertex thus added is observed – it must be either in $\mathbf{C}$ or be the image of some vertex in $\mathbf{C}$, and cannot be $f(c_4)$.

Thus, if in $\mathbf{G} \div \mathbf{C}$ any vertex in the path observed any other vertex (except for $f(c_4)$) via an edge of the path, it will still do so in $\mathbf{G}$. Since this is true of every vertex y and every path, and since in fact each vertex in $\mathbf{G} \div \mathbf{C}$ was observed by $\mathbf{S}$, we can see that every vertex other than $f(c_4)$ in $\mathbf{G} \div \mathbf{C}$ is observed by $\mathbf{S}' \bigcup \{a\}$. Then we can indeed apply Lemma 1, and $\mathbf{S}' \bigcup \{a\}$ observes $\mathbf{G}$.

□

Finally, we show that the bound given for $n \geq 9$ is sharp, at least in the cases where $n \not\equiv 1 \pmod 4$. To prove this, we construct permutation graphs on cycles which require precisely $\lceil \frac{n}{4} \rceil$ PMUs.

First, consider the following graphs, each with power domination number exactly 2: $B_5 = P(\mathbf{C}_5, (1\ 2)(3\ 4)(5))$, $B_6 = P(\mathbf{C}_6, (1\ 2)(3\ 4)(5\ 6))$, $B_7 = P(\mathbf{C}_7, (1\ 2)(3\ 4)(5\ 6))$, $B_8 = P(\mathbf{C}_8, (1\ 2)(3\ 4)(5\ 6)(7\ 8))$, ; see Figure 12.

To extend this, we use the graphs D_4 and D_5, as defined in Lemma 2: let $n \geq 10$, and write $n = a + 4m$, where $a \in \{6, 7, 8, 9\}$ and $m \in \mathbb{Z}^+$. Then take B_a and insert m copies of D_4, replacing the edges (u_a, u_1) and (v_a, v_1); call this new graph B_n.

This new graph is a permutation graph on a cycle of length n; we want to show that it requires $2 + m = \lceil \frac{n-1}{4} \rceil$ PMUs. So suppose $\mathbf{S}$ were a power dominating set for B_n with

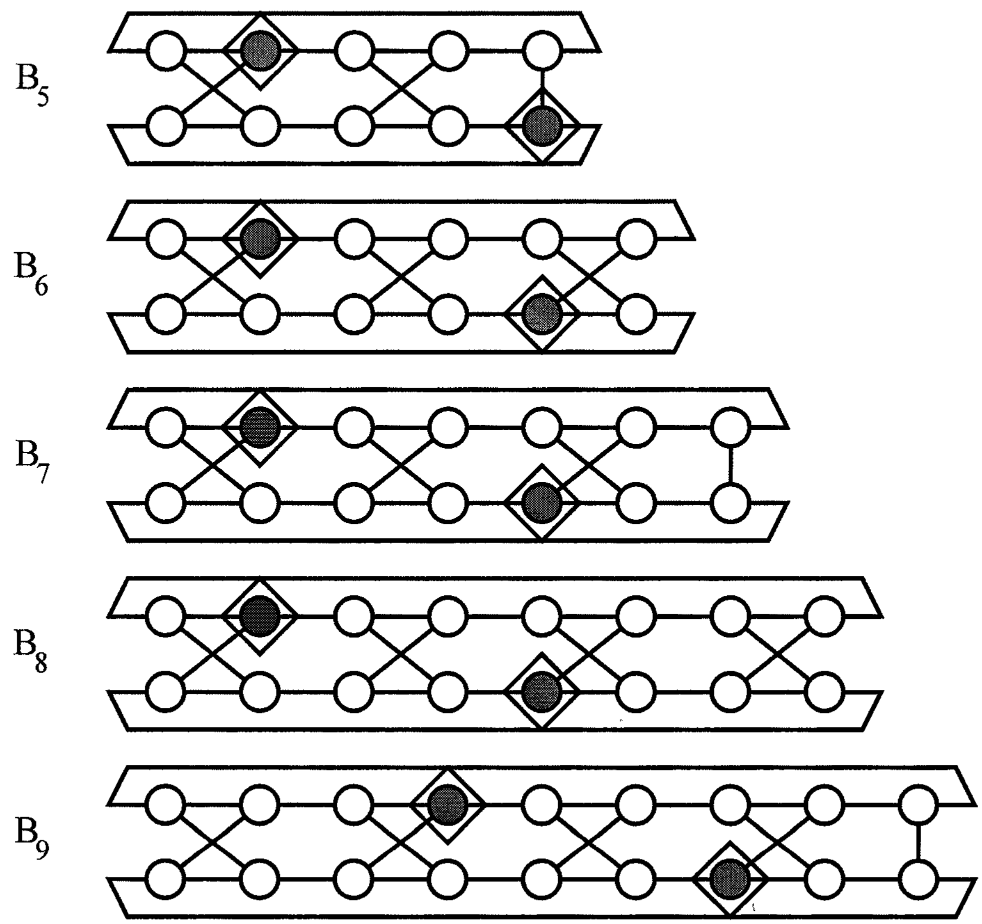

Figure 12: Worst-case graphs and minimal power dominating sets thereon

$|S| \leq 1 + m$. By the lemma, every time D_4 or D_5 appears as in induced subgraph of B_n, it must contain an element of **S**. However, there are a total of $\lfloor \frac{n}{2} \rfloor$ distinct (though not disjoint) occurrances of D_4 and D_5. Further, any vertex of B_n can lie in at most two distinct occurances. Thus, to ensure every copy of D_4 or D_5 contains at least one element of **S**, at least $\frac{\lfloor \frac{n}{2} \rfloor}{2} > 1 + m$ elements are needed. Thus it must be that any power dominating set on such a graph must have at least $\lceil \frac{n-1}{4} \rceil$ elements.

VI. EXTENSIONS

Of course, an immediate extension of the work done above would be a proof of the conjecture regarding $\mathbf{G} \div \mathbf{C}$. Such a proof might be combinatorial in nature, as the sheer number of possible configurations for **C** may prove sufficient to ensure that at least one has the desired property.

In addition, we note that the reason for the exclusion of $n \not\equiv 1 \pmod 4$ in the sharpness proof is that it appears, in fact, that $\gamma_P(C_9,f) = 2 < \lceil \frac{n}{4} \rceil$; vis the graph $B_9 = P(\mathbf{C}_9, (1\ 2)(3\ 4)(5\ 6)(7\ 8))$, which would otherwise be the obvious continuation of the pattern, but which has a PDN of 2. This case warrants further study; we surmise that perhaps our upper bound for $\gamma_P(C_n,f)$, $n \geq 9$, could be improved from $\lceil \frac{n}{4} \rceil$ to $\lceil \frac{n-1}{4} \rceil$.

Finally, there are several interesting classes of graphs (including, as already mentioned, the generalized Petersen graphs) which are sub- or supersets of, or closely related to, the class of graph considered in this paper; the results given here could perhaps be extended to these other types. For example, general permutation graphs can have significantly more complex structures, but the inductive process used here is possible for other permutation graphs so long as their underlying graphs contain vertices of degree 2, and could perhaps be adapted to fit other situations.

In addition, consider the class of graph produced by constructing a permutation graph on a cycle, deleting $j > 1$ edges in $\mathbf{G}_2$, and then reconnecting the vertices of $\mathbf{G}_2$ so as to form j smaller cycles. The resulting graph remains cubic, and most of the vertices retain their original neighbors; thus, it presumably shares many of the properties of the graph from which it was constructed.

An analysis of the power domination numbers of such graphs, and how they relate to those of the original permutation graph, could produce interesting results.

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VITA

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