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OSCILLATION OF HIGH ORDER LINEAR FUNCTIONAL DIFFERENTIAL EQUATION WITH IMPULSES

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Abstract. We study the solutions to high-order linear functional differential equations with impulses. We improve previous results in the oscillation theory for ordinary differential equations and obtain new criteria on the oscillation of solutions.

1. Introduction

In the past years, the theory of the oscillatory behavior of impulsive ordinary differential equation (IODE) and impulsive functional differential equation (IFDE) has been investigated by many authors; see for example [1, 2, 4, 5, 6, 8, 10, 11, 12, 13]. However, most of these articles concern first-order or second-order IODE and IFDE [1, 2, 5, 6, 8, 11, 13]. Just a few of them have studied third and the fourth-order IODE [4, 10, 12]. Recently, in [3], the authors studied oscillatory criteria for even order IODE

$$
\begin{gathered}
x^{(2n)}(t)+p(t)x(t)=0, \quad t\geq t_0,\ t\neq t_k,\\
x^{(i)}(t_k^+)=a_k^{(i)}x^{(i)}(t_k), \quad i=0,1,\dots,2n-1;\ k=1,2,\dots
\end{gathered}
\tag{1.1}
$$

and obtained some important results. To the best of our knowledge, paper [3] is probably the first publication on the high order IODE. However, there are somethings worth further consideration. Firstly, the results of [3] is invalid for odd order IODE and IFDE; Secondly, in order to assure the oscillatory behavior of (1.1), the following condition is required:

$$
\begin{aligned}
&\int_{t_0}^{t_1}p(s)ds+\frac{a_1^{(0)}}{a_1^{(2n-1)}}\int_{t_1}^{t_2}p(s)ds+\frac{a_1^{(0)}a_2^{(0)}}{a_1^{(2n-1)}a_2^{(2n-1)}}\int_{t_2}^{t_3}p(s)ds\\
&+\dots+\frac{a_1^{(0)}a_2^{(0)}\dots a_k^{(0)}}{a_1^{(2n-1)}a_2^{(2n-1)}\dots a_k^{(2n-1)}}\int_{t_k}^{t_{k+1}}p(s)ds+\dots=+\infty\,,
\end{aligned}
\tag{1.2}
$$

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which is parallel assumption to

$$\int^{+\infty} p(s)ds = +\infty \tag{1.3}$$

in ODE. We know, however, in the corresponding oscillation theory of ODE, it is sufficient to assume that

$$\int^{+\infty} s^{n-2}p(s)ds = +\infty\,. \tag{1.4}$$

So it is natural to ask if it is possible to improved (1.2) to a better form? Moreover, what is the result about the odd order differential equations with impulses? In the present article, we deal with a more general linear IFDE and establish several useful criteria for it. We believe our approach is simple and is also helpful to be used in other systems.

Consider the impulsive delay differential equation

$$\begin{gathered} x^{(n)}(t) + p(t)x(t-\tau) = 0, \quad t \geq t_0, t \neq t_k, \\ x^{(i)}(t_k^+) = a_k^{(i)} x^{(i)}(t_k), \quad i = 0, 1, \dots, n-1;\ k = 1, 2, \dots\,, \end{gathered} \tag{1.5}$$

where n is a natural number with $n \geq 2$, $0 \leq t_0 < t_1 < t_2 < \dots < t_k < \dots$, $\lim_{k\to\infty} t_k = +\infty$, $t_{k+1} - t_k > 3\tau$, $x^{(0)}(t) = x(t)$,

$$\begin{aligned} x^{(i)}(t_k) &= \lim_{h\to -0} \frac{x^{(i-1)}(t_k + h) - x^{(i-1)}(t_k)}{h}, \\ x^{(i)}(t_k^+) &= \lim_{h\to +0} \frac{x^{(i-1)}(t_k + h) - x^{(i-1)}(t_k^+)}{h}, \end{aligned}$$

where $i = 1, 2, \dots, n$ and $x^{(0)}(t_k^+) = x(t_k^+)$.

For the rest of this paper, assume the following conditions:

- $a_k^{(i)} > 0$, $i = 0, 1, \dots, n-1$; $k = 1, 2, \dots$.
- $p(t)$ is continuous in $[t_0 - \tau, \infty)$; $p(t) \geq 0$ and for any $T \geq t_0$, $p(t)$ is not identically zero in $[T, +\infty)$.

Definition. A function $x : [t_0 - \tau, a) \to R(a > t_0)$ is said to be a solution of (1.5) on $[t_0 - \tau, a)$ satisfying the initial-value condition $x^{(i)}(t) = \phi^{(i)}(t)$ for $i = 0, 1, \dots, n-1$ and $t \in [t_0 - \tau, t_0)$, if

(i) $x^{(i)}(t)$ is continuous for $t \in [t_0, a)$ and $t \neq t_k$, $i = 0, 1, \dots n-1$; $k = 1, 2, \dots$
(ii) $x(t)$ satisfies $x^{(n)}(t) + p(t)x(t-\tau) = 0$ for $t \in [t_0, a)$, $t \neq t_k$, $k = 1, 2, \dots$
(iii) $x^{(i)}(t) = \phi^{(i)}(t)$, $t \in [t_0 - \tau, t_0]$, $i = 0, 1, \dots, n-1$
(iv) $x^{(i)}(t_k^+) = a_k^{(i)} x(t_k)$, for $t_k \in [t_0, a)$, $i = 0, 1, \dots, n-1$.

It is clear that (1.5) can be transformed into a first-order linear impulsive differential systems. Theorems on existence of solutions, on uniqueness, and on existence of global solutions of the first order linear differential equation with impulses can be found in [7, 9]. There, we can find the existence of global solutions under some simple conditions.

In the following, we assume that the solutions of (1.5) exist on $[t_0 - \tau, +\infty)$.

Definition. A solution of (1.5) is said to be non-oscillatory if this solution is eventually positive or eventually negative. Otherwise, this solution is said to be oscillatory.

2. Main Results

The following conditions are assumed in this paper:

(H1) For $i = 1, 2, \dots, n-1$,
$$(t_1 - t_0) + \frac{a_1^{(i)}}{a_1^{(i-1)}}(t_2 - t_1) + \frac{a_1^{(i)}a_2^{(i)}}{a_1^{(i-1)}a_2^{(i-1)}}(t_3 - t_2) + \dots + \frac{a_1^{(i)}a_2^{(i)}\dots a_m^{(i)}}{a_1^{(i-1)}a_2^{(i-1)}\dots a_m^{(i-1)}}(t_{m+1} - t_m) + \dots = +\infty$$

(H2) $\liminf_{k\to+\infty}(a_k^{(i)}a_{k-1}^{(i)}\dots a_2^{(i)}a_1^{(i)}) = \delta_i > 0$, $i = 0, 1, \dots, n-1$

(H3) If $n \geq 3$, then $w_m^L(k) \geq W$ holds for k large enough, where W is a constant, and
$$\begin{aligned} w_m^L(k) =&(a_k^{(L-m-1)} - 1)(t_k^m - t_{k-1}^m) + (a_k^{(L-m-1)}a_{k-1}^{(L-m-1)} - 1)(t_{k-1}^m - t_{k-2}^m) \\ &+ \dots + (a_k^{(L-m-1)}a_{k-1}^{(L-m-1)}\dots a_1^{(L-m-1)} - 1)(t_1^m - t_0^m), \end{aligned}$$
where $L = 2, 3, \dots, n-1$ and $m = 1, 2, \dots L-1$.

We remark that if $a_k^{(i)} \geq 1$, then (H2) and (H3) are satisfied.

Lemma 2.1. *Let $x(t)$ be a solution of (1.5). Suppose (H1) holds and for some $i \in \{1, 2, \dots, n-1\}$, there exists a constant $T \geq t_0$ such that $x^{(i)}(t) > 0 (< 0)$, $x^{(i+1)}(t) \geq 0 (\leq 0)$ for $t \geq T$. Then $x^{(i-1)}(t) > 0 (< 0)$ holds for sufficiently large t.*

Proof. We prove only the conclusion under the assumptions that $x^{(i)}(t) > 0$, $x^{(i+1)}(t) \geq 0$. The case that $x^{(i)}(t) < 0$, $x^{(i+1)}(t) \leq 0$ can be proved similarly. Without loss of generality, suppose $T = t_0$. By $x^{(i)}(t) > 0$, $x^{(i+1)}(t) \geq 0$, we know that $x^{(i)}(t)$ is monotonically nondecreasing in each interval (t_k, t_{k+1}), $k = 0, 1, 2, \dots$. Hence
$$x^{(i)}(t) \geq x^{(i)}(t_k^+), \quad \text{for } t \in (t_k, t_{k+1}].$$
Integrating the above inequality, we have
$$x^{(i-1)}(t_{k+1}) \geq x^{(i-1)}(t_k^+) + x^{(i)}(t_k^+)(t_{k+1} - t_k).$$
Then
$$x^{(i-1)}(t_2) \geq x^{(i-1)}(t_1^+) + x^{(i)}(t_1^+)(t_2 - t_1),$$
and thus
$$\begin{aligned} x^{(i-1)}(t_3) &\geq x^{(i-1)}(t_2^+) + x^{(i)}(t_2^+)(t_3 - t_2) \\ &= a_2^{(i-1)}x^{(i-1)}(t_2) + a_2^{(i)}x^{(i)}(t_2)(t_3 - t_2) \\ &\geq a_2^{(i-1)}[x^{(i-1)}(t_1^+) + x^{(i)}(t_1^+)(t_2 - t_1)] + a_2^{(i)}x^{(i)}(t_2)(t_3 - t_2) \\ &\geq a_2^{(i-1)}[x^{(i-1)}(t_1^+) + x^{(i)}(t_1^+)(t_2 - t_1) + \frac{a_2^{(i)}}{a_2^{(i-1)}}x^{(i)}(t_1^+)(t_3 - t_2)]. \end{aligned}$$
By induction, we find that
$$\begin{aligned} x^{(i-1)}(t_k) \geq a_{k-1}^{(i-1)}\dots a_3^{(i-1)}a_2^{(i-1)}\Big\{&x^{(i-1)}(t_1^+) + x^{(i)}(t_1^+)\Big[(t_2 - t_1) \\ &+ \frac{a_2^{(i)}}{a_2^{(i-1)}}(t_3 - t_2) + \dots + \frac{a_2^{(i)}a_3^{(i)}\dots a_{k-1}^{(i)}}{a_2^{(i-1)}a_3^{(i-1)}\dots a_{k-1}^{(i-1)}}(t_k - t_{k-1})\Big]\Big\}. \end{aligned}$$

Since $a_k^{(i)} > 0$, it follows from (H1) that for sufficiently large k, $x^{(i-1)}(t_k) > 0$. i.e., there exists some N such that $x^{(i-1)}(t_k) > 0$ for $k \geq N$. Since $x^{(i)}(t) > 0$,we have

$$x^{(i-1)}(t) > x^{(i-1)}(t_k^+) > 0, \quad \text{for } t \in (t_k, t_{k+1}], t_k \geq t_N.$$

Thus, for sufficiently large $t, x^{(i-1)}(t) > 0$, which completes the proof. □

Lemma 2.2. *Let $x(t)$ be a solution of* (1.5)*. Suppose (H1) holds, and for some $i \in \{1, 2, \dots, n\}$, there exists a constant $T(T \geq t_0)$ such that $x(t) > 0, x^{(i)}(t) \leq 0(t \geq T)$. Furthermore, $x^{(i)}(t)$ is not identically zero in any interval $[t', +\infty)$. Then $x^{(i-1)}(t) > 0$ holds for sufficiently large t.*

Proof. Without loss of generality, assume $T = t_0$. We will show that for any $t_k \geq t_0$, $x^{(i-1)}(t_k) > 0$ holds. Suppose that there exists some $t_j \geq t_0$ such that $x^{(i-1)}(t_j) \leq 0$. Since $x^{(i)}(t) \leq 0$, it is obvious that $x^{(i-1)}(t)$ is monotonically non-increasing in any interval $(t_k, t_{k+1}]$ If $k \geq j$. By the condition that $x^{(i)}(t)$ is not identically zero in any interval $[t', +\infty)$, we obtain that there exists some $t_l \geq t_j$ such that $x^{(i)}(t)$ is not identically zero in$(t_l, t_{l+1}]$. For convenient, assume $l = j$. So

$$x^{(i-1)}(t_{j+1}) < x^{(i-1)}(t_j^+) = a_j^{(i-1)} x^{(i-1)}(t_j) \leq 0$$

and

$$x^{(i-1)}(t) \leq x^{(i-1)}(t_{j+1}^+) = a_{j+1}^{(i-1)} x^{(i-1)}(t_{j+1}) < 0, \quad \text{for } t \in (t_{j+1}, t_{j+2}].$$

By induction, $x^{(i-1)}(t) < 0$ holds for $t \in (t_{j+m}, t_{j+m+1}]$, where m is a natural number. Then $x^{(i-1)}(t) < 0, x^{(i)}(t) \leq 0, t \in (t_{j+1}, \infty)$. Thus, by Lemma 2.1, we obtain $x^{(i-2)}(t) < 0$ for all sufficiently large t.

Making use of Lemma 2.1 repeatedly, we eventually obtain that $x(t) < 0$ for t large enough, which contradicts $x(t) > 0(t \geq T)$. Therefore, $x^{(i-1)}(t_k) > 0$ for any t_k. Since $a_k^{(i-1)} > 0$ and that $x^{(i-1)}(t)$ is monotonically non-increasing in $(t_k, t_{k+1}]$, we have $x^{(i-1)}(t) > 0$ for sufficiently large t. Thus the proof is complete. □

Lemma 2.3. *Let $x(t)$ be a solution of* (1.5)*. Suppose that (H1) holds and that there exists a constant $T \geq t_0$ such that $x(t) > 0$ for $t \geq T$. Then, there exists a T' and an integer L, $0 \leq L \leq n$, with $n + L$ odd, such that*

$$\begin{gathered} x^{(i)}(t) > 0, \quad i = 0, 1, \dots L, \\ (-1)^{i+L} x^{(i)}(t) > 0, \quad i = L+1, \dots, n-1, t \geq T'. \end{gathered} \tag{2.1}$$

Proof. By the assumption that $x(t) > 0(t \geq T)$, we have $x^{(n)}(t) = -p(t)x(t-\tau) \leq 0$ for $t \geq T + \tau$ and that $x^{(n)}(t)$ is not identically zero in any interval $[t', +\infty)$. According to Lemma 2.2, there exists some $T_0 \geq T + \tau$ such that $x^{(n-1)}(t) > 0$ holds for $t \geq T_0$.

Therefore, $x^{(n-2)}(t)$ is monotonically nondecreasing in $(t_k, t_{k+1}](t_k \geq T_0)$. If $x^{(n-2)}(t_k) < 0$ holds for all $t_k \geq T_0$, then it is obvious that $x^{(n-2)}(t) < 0(t \geq T_0)$. If there is some j such that $x^{(n-2)}(t_j) \geq 0$, then, by the monotonicity of $x^{(n-2)}(t)$ and $a_k^{(n-2)} > 0$, we obtain $x^{(n-2)}(t) > 0$ for sufficiently large t. So, in any case, there exists a T_1 such that one of the following statements is true:

(A1) $x^{(n-1)}(t) > 0$, $x^{(n-2)}(t) > 0$, $t \geq T_1$
(B1) $x^{(n-1)}(t) > 0$, $x^{(n-2)}(t) < 0$, $t \geq T_1$.

If (A1) is true, Lemma 2.1 shows that $x^{(n-3)}(t) > 0$ holds for sufficiently large t. By using Lemma 2.1 repeatedly, we finally arrive at that

$$x^{(n-1)}(t) > 0, \quad x^{(n-2)}(t) > 0, \quad \dots, \quad x'(t) > 0, \quad x(t) > 0.$$

If (B1) holds, Lemma 2.2 suggests that $x^{(n-3)}(t) > 0$ for sufficiently large t. So, there exists a $T_2 \geq T_1$ such that one of the following statements is true:

(A2) $x^{(n-3)}(t) > 0$, $x^{(n-4)}(t) > 0$, $t \geq T_2$

(B2) $x^{(n-3)}(t) > 0$, $x^{(n-4)}(t) < 0$, $t \geq T_2$.

Proceeding as in the above argument, we obtain that there exists a $T' \geq T$ and $L : 0 \leq L \leq n-1$, with $n + L$ odd, such that (2.1) holds. $\square$

Lemma 2.4. *Let $x(t)$ be a solution of* (1.5). *Assume that (H2) and (H3) are satisfied and there exist a natural number $L \geq 1$ and a $T' \geq t_0$, such that $x^{(i)}(t) > 0$ holds for $t \geq T'$ and $i = 0, 1, \dots L$. Then there exist constants M and T such that*

$$x(t) \geq M t^{L-1}, \quad t \geq T. \tag{2.2}$$

Proof. Without loss of generality, let $T' = t_0$. At first, we claim that there exists a constant $a > 0$ such that

$$x^{(L-1)}(t) \geq a. \tag{2.3}$$

holds for sufficiently large t. Suppose it is not true, then $\liminf_{t\to+\infty} x^{(L-1)}(t) = 0$. Since $x^{(L)}(t) > 0$, this implies $\liminf_{k\to+\infty} x^{(L-1)}(t_k^+) = 0$. Note that

$$\begin{aligned} x^{(L-1)}(t_k^+) &= a_k^{(L-1)} x^{(L-1)}(t_k) \geq a_k^{(L-1)} x^{(L-1)}(t_{k-1}^+) \\ &= a_k^{(L-1)} a_{k-1}^{(L-1)} x^{(L-1)}(t_{k-1}) \geq \dots \geq a_k^{(L-1)} a_{k-1}^{(L-1)} \dots a_1^{(L-1)} x^{(L-1)}(t_0^+), \end{aligned}$$

hence $\liminf_{t\to+\infty}(a_k^{(L-1)} a_{k-1}^{(L-1)} \dots a_1^{(L-1)}) = 0$, which contradicts condition (H2). The claim is proved.

If $L = 1$, by (2.3) we find the Lemma 2.4 has been proved. Now, suppose $L \geq 2$. We will show that

$$x^{(L-2)}(t) \geq \frac{a}{2} t \tag{2.4}$$

holds for sufficiently large t.

In order to simplify the sign, we assume that (2.3) holds for $t \geq t_0$. Consequently, for $t \in (t_0, t_1]$, we have

$$x^{(L-2)}(t) = x^{(L-2)}(t_0^+) + \int_{t_0}^{t} x^{(L-1)}(s) ds \geq a(t - t_0).$$

Particularly, $x^{(L-2)}(t_1) \geq a(t_1 - t_0)$. For $t \in (t_1, t_2]$, we have

$$\begin{aligned} x^{(L-2)}(t) &= x^{(L-2)}(t_1^+) + \int_{t_1}^{t} x^{(L-1)}(s) ds \\ &\geq x^{(L-2)}(t_1^+) + a(t - t_1) \\ &= a_1^{(L-2)} x^{(L-2)}(t_1) + a(t - t_1) \\ &\geq a_1^{(L-2)} a(t_1 - t_0) + at - at_1 \\ &= a[t + (a_1^{(L-2)} - 1)t_1 - a_1^{(L-2)} t_0]. \end{aligned}$$

In particular, $x^{(L-2)}(t_2) \geq a[t_2 + (a_1^{(L-2)} - 1)t_1 - a_1^{(L-2)}t_0]$. For $t \in (t_2, t_3]$, we obtain

$$\begin{aligned} x^{(L-2)}(t) &\geq x^{(L-2)}(t_2^+) + a(t - t_2) \\ &= a_2^{(L-2)} x^{(L-2)}(t_2) + a(t - t_2) \\ &\geq a_2^{(L-2)} a[t_2 + (a_1^{(L-2)} - 1)t_1 - a_1^{(L-2)}t_0] + at - at_2 \\ &= a[t + (a_2^{(L-2)} - 1)t_2 + a_2^{(L-2)}(a_1^{(L-2)} - 1)t_1 - a_2^{(L-2)}a_1^{(L-2)}t_0]. \end{aligned}$$

By induction, for $t \in (t_k, t_{k+1}]$, we get

$$\begin{aligned} &x^{(L-2)}(t) \\ &\geq a[t + (a_k^{(L-2)} - 1)t_k + a_k^{(L-2)}(a_{k-1}^{(L-2)} - 1)t_{k-1} + \dots \\ &\quad + a_k^{(L-2)}a_{k-1}^{(L-2)} \dots a_2^{(L-2)}(a_1^{(L-2)} - 1)t_1 - a_k^{(L-2)}a_{k-1}^{(L-2)} \dots a_2^{(L-2)}a_1^{(L-2)}t_0] \\ &= a\big[t + (a_k^{(L-2)} - 1)(t_k - t_{k-1}) + (a_k^{(L-2)}a_{k-1}^{(L-2)} - 1)(t_{k-1} - t_{k-2}) \\ &\quad + \dots + (a_k^{(L-2)}a_{k-1}^{(L-2)} \dots a_1^{(L-2)} - 1)(t_1 - t_0) - t_0\big] \\ &= a[t + w_1^L(k) - t_0]. \end{aligned}$$

From (H3), we find $x^{(L-2)}(t) \geq \frac{a}{2}t$ holds for sufficiently large t. To complete the proof, we prove the inequality

$$x^{(L-j)}(t) \geq \frac{a}{p_j}t^{j-1}, \quad j = 1, 2, \dots L, \tag{2.5}$$

where $p_j = 2^{j-1}(j-1)!$ and that t is sufficiently large. From the above argument, it is clear that (2.5) holds for $j = 1, 2$. We suppose (2.5) holds for $j (j < L)$ and $t > t_0$. Then for $t \in (t_0, t_1]$,

$$x^{(L-j-1)}(t) \geq x^{(L-j-1)}(t_0^+) + \int_{t_0}^t \frac{a}{p_j}s^{j-1}ds \geq \frac{a}{p_j}\int_{t_0}^t s^{j-1}ds = \frac{a}{jp_j}(t^j - t_0^j).$$

In particular, $x^{(L-j-1)}(t_1) \geq \frac{a}{jp_j}(t_1^j - t_0^j)$. For $t \in (t_1, t_2]$, we get

$$\begin{aligned} x^{(L-j-1)}(t) \geq& x^{(L-j-1)}(t_1^+) + \int_{t_1}^t \frac{a}{p_j}s^{j-1}ds \\ \geq& a_1^{(L-j-1)}\frac{a}{jp_j}(t_1^j - t_0^j) + \frac{a}{jp_j}(t^j - t_1^j) \\ =& \frac{a}{jp_j}[t^j + (a_1^{(L-j-1)} - 1)t_1^j - a_1^{(L-j-1)}t_0^j]. \end{aligned}$$

In particular, $x^{(L-j-1)}(t_2) \geq \frac{a}{jp_j}[t_2^j + (a_1^{(L-j-1)} - 1)t_1^j - a_1^{(L-j-1)}t_0^j]$. By induction, we find that for $t \in (t_k, t_k + 1]$,

$$\begin{aligned} &x^{(L-j-1)}(t) \\ &\geq \frac{a}{jp_j}\big[t^j + (a_k^{(L-j-1)} - 1)t_k^j + a_k^{(L-j-1)}(a_{k-1}^{(L-j-1)} - 1)t_{k-1}^j + \dots \\ &\quad + a_k^{(L-j-1)}a_{k-1}^{(L-j-1)} \dots a_2^{(L-j-1)}(a_1^{(L-j-1)} - 1)t_1^j - a_k^{(L-j-1)}a_{k-1}^{(L-j-1)} \\ &\quad \dots a_2^{(L-j-1)}a_1^{(L-2)}t_0^j\big] \\ &= \frac{a}{jp_j}\big[t^j + (a_k^{(L-j-1)} - 1)(t_k^j - t_{k-1}^j) + (a_k^{(L-j-1)}a_{k-1}^{(L-j-1)} - 1)(t_{k-1}^j - t_{k-2}^j) \end{aligned}$$

$$+\dots+(a_k^{(L-j-1)}a_{k-1}^{(L-j-1)}\dots a_1^{(L-j-1)}-1)(t_1^j-t_0^j)-t_0^j]$$
$$=\frac{a}{jp_j}[t^j+w_j^L(k)-t_o^j].$$

Using (H3) again, we obtain that

$$x^{(L-j-1)}(t)\ge\frac{a}{2jp_j}t^j=\frac{a}{p_{j+1}}t^j$$

holds for sufficiently large t. Therefore, (2.5) is as well satisfied for $j+1$. Thus the proof is complete. □

We are now able to state and show the main results, using the assumption

(H4)

$$\int_{t_0}^{t_1}p(s)s^{n-2}ds+\frac{1}{b_1}\int_{t_1}^{t_2}p(s)s^{n-2}ds+\frac{1}{b_1b_2\dots b_k}\int_{t_k}^{t_{k+1}}p(s)s^{n-2}ds+\dots=+\infty,$$

where $b_i=\max\{a_i^{(1.5)},a_i^{(2)},\dots a_i^{(n-1)}\}$.

Theorem 2.5. *Assuming (H1)–(H4) and that n is even, then all solutions of* (1.5) *are oscillatory.*

Proof. Suppose (1.5) has a non-oscillatory solution $x(t)$. We may assume $x(t)>0$ $(t\ge t_0)$ (the case when $x(t)<0(t\ge t_0)$ can be proved similarly and will not be included here). Lemma 2.3 shows that there exist constants $L\in\{1,3,\dots,n-1\}$ such that (2.1) holds. Moreover, from Lemma 2.4, there exists a $T\ge t_0$ such that $t\ge T$ implies

$$x(t-\tau)\ge M(t-\tau)^{L-1}\ge\frac{M}{2}t^{L-1}=Nt^{L-1},$$

where $N=\frac{M}{2}$. For convenience, let $T=t_0$. Thus

$$x^{(n)}(t)=-p(t)x(t-\tau)\le-Np(t)t^{L-1},\quad(t\ge t_0).$$

Multiplying both sides of the above inequality by t^{n-L-1} and integrating both sides of it from t_k to t, we obtain

$$\int_{t_k}^{t}x^{(n)}(s)s^{n-L-1}ds\le-N\int_{t_k}^{t}p(s)s^{n-2}ds,\quad t\in(t_k,t_{k+1}].$$

Integrating by parts,

$$Q(t)-Q(t_k^+)\le-N\int_{t_k}^{t}p(s)s^{n-2}ds,\tag{2.6}$$

where

$$\begin{aligned}Q(t)=&t^{n-L-1}x^{(n-1)}(t)-(n-L-1)t^{n-L-2}x^{(n-2)}(t)\\&+(n-L-1)(n-L-2)t^{n-L-3}x^{(n-3)}(t)+\dots\\&+(-1)^{n-L-1}(n-L-1)!x^{(L)}(t).\end{aligned}$$

Lemma 2.3 suggests that $Q(t)\ge0$. In view of (2.6), we obtain

$$Q(t_k^+)-N\int_{t_k}^{t_{k+1}}p(s)s^{n-2}ds\ge0.$$

Since

$$Q(t_i^+)=a_i^{(n-1)}t_i^{n-L-1}x^{(n-1)}(t_i)-(n-L-1)a_i^{(n-2)}t_i^{n-L-2}x^{(n-2)}(t_i)$$

$$\begin{aligned}&+(n-L-1)(n-L-2)a_i^{(n-3)}t_i^{n-L-3}x^{(n-3)}(t_i)\\&+\dots+(-1)^{n-L-1}(n-L-1)!a_i^{(L)}x^{(L)}(t_i)\\&\le b_iQ(t_i),\end{aligned}$$

then

$$\begin{aligned}Q(t_1)&\le Q(t_0^+)-N\int_{t_0}^{t_1}p(s)s^{n-2}ds,\\Q(t_2)&<Q(t_1^+)-N\int_{t_1}^{t_2}p(s)s^{n-2}ds\\&\le b_1Q(t_1)-N\int_{t_1}^{t_2}p(s)s^{n-2}ds\\&\le b_1[Q(t_0^+)-N\int_{t_0}^{t_1}p(s)s^{n-2}ds]-N\int_{t_1}^{t_2}p(s)s^{n-2}ds\\&=b_1N[\frac{Q(t_0^+)}{N}-\int_{t_0}^{t_1}p(s)s^{n-2}ds-\frac{1}{b_1}\int_{t_1}^{t_2}p(s)s^{n-2}ds].\end{aligned}$$

Similarly

$$\begin{aligned}&Q(t_3)\\&<Q(t_2^+)-N\int_{t_2}^{t_3}p(s)s^{n-2}ds\\&\le b_2Q(t_2)-N\int_{t_2}^{t_3}p(s)s^{n-2}ds\\&\le b_2b_1N\Big[\frac{Q(t_0^+)}{N}-\int_{t_0}^{t_1}p(s)s^{n-2}ds-\frac{1}{b_1}\int_{t_1}^{t_2}p(s)s^{n-2}ds-\frac{1}{b_1b_2}\int_{t_2}^{t_3}p(s)s^{n-2}ds\Big].\end{aligned}$$

Applying induction, we get

$$\begin{aligned}Q(t_k)\le b_{k-1}\dots b_2b_1N\Big[\frac{Q(t_0^+)}{N}-(\int_{t_0}^{t_1}p(s)s^{n-2}ds+\frac{1}{b_1}\int_{t_1}^{t_2}p(s)s^{n-2}ds\\+\dots+\frac{1}{b_{k-1}\dots b_2b_1}\int_{t_{k-1}}^{t_k}p(s)s^{n-2}ds)\Big].\end{aligned}$$

According to (H4), $Q(t_k)<0$ holds for sufficiently large k. This contradicts that $Q(t_k)\ge 0$. Hence every solution of equation (1.5) is oscillatory. □

Note that in Theorem 2.5, we use a method different from that of [3], and that our result and [3, Theorem 1] are independent. In particular, if $0<\prod_{k=1}^{+\infty}a_k^{(i)}\le\prod_{k=1}^{+\infty}b_k<+\infty, i=0,1,\dots,n-1$, then condition (1.2) turns out to be (1.3). But in this theorem, we only need condition (1.4) to be satisfied.

Theorem 2.6. *Suppose (H1)–(H4) hold, and that n is odd. If*

$$\sum_{k=1}^{\infty}|a_k^{(0)}-1|<+\infty, \tag{2.7}$$

then all the non-oscillatory solutions of (1.5) *tend to zero.*

Proof. Assume that $x(t)$ is a non-oscillatory solution of (1.5). Without loss of generality, we may assume $x(t) > 0$ for $t \geq t_0 - \tau$. By Lemma 2.3, one of the following two statements holds:

(i) $x(t) > 0$, $x'(t) > 0, \dots, x^{(L)}(t) > 0$, $x^{(L+1)}(t) < 0 \dots$, $x^{(n-1)}(t) > 0$, $x^{(n)}(t) \leq 0$, $L \in \{2, 4, \dots, n-1\}$,

(ii) $x(t) > 0$, $x'(t) < 0$, $x''(t) > 0$, $\dots x^{(n-1)}(t) > 0$, $x^{(n)}(t) \leq 0$.

If (i) is true, then employing the conditions (H1)-(H4) and by a similar way of the proof of Theorem 2.5, we find $x(t)$ is oscillatory, which contradicts the assumption that $x(t)$ is a non-oscillatory solution. Thus, only (ii) can be true. Therefore, $x(t)$ is monotonically decreasing in each interval $(t_k, t_{k+1}]$.

We claim that $\lim_{t\to\infty} x(t) = \alpha \geq 0$ exists and is finite. First, we show that

$$\sum_{k=1}^{+\infty} |x(t_k^+) - x(t_k)| < +\infty.$$

It is an easy exercise to prove that condition (2.7) implies

$$\prod_{k=1}^{+\infty} a_k^{(0)} < +\infty. \tag{2.8}$$

Since

$$\begin{aligned} x(t_k^+) &= a_k^{(0)} x(t_k) \leq a_k^{(0)} x(t_{k-1}^+) \\ &= a_k^{(0)} a_{k-1}^{(0)} x(t_{k-1}) \leq \dots \leq a_k^{(0)} a_{k-1}^{(0)} \dots a_1^{(0)} x(t_0^+), \end{aligned}$$

by (2.8), we know that $\{x(t_k^+)\}$ is bounded for $k \in N$. Furthermore, $x(t)$ is non-increasing in every interval $(t_k, t_{k+1}]$, i.e. $x(t) \leq x(t_k^+)$ $t \in (t_k, t_{k+1}]$. Thus, $x(t)$ is bounded on $[t_0, +\infty)$. Now, from (2.7) and the bounded nature of $x(t)$, we find

$$\sum_{k=1}^{+\infty} |x(t_k^+) - x(t_k)| < +\infty.$$

Next, to complete the claim, we let

$$c_l = \sum_{k=1}^{l} [x(t_k^+) - x(t_k)], \quad \lim_{l\to\infty} c_l = c. \tag{2.9}$$

and define a function

$$y(t) = -c_k + x(t), \ t \in (t_k, t_{k+1}], \quad k \in N.$$

We will prove that $y(t)$ is non-increasing and bounded on $[t_0, +\infty)$. From the definition of $y(t)$, we have

$$\begin{aligned} y(t_k) &= -c_{k-1} + x(t_k) = -c_{k-1} - [x(t_k^+) - x(t_k)] + x(t_k^+) \\ &= -c_k + x(t_k^+) \geq -c_k + x(t_{k+1}) = y(t_{k+1}), \quad k \in N. \end{aligned}$$

Now, for any $t_0 < a < b < +\infty$, if there is some k such that $a, b \in (t_k, t_{k+1}]$, then

$$y(a) = -c_k + x(a) \geq -c_k + x(b) = y(b).$$

If there exist $m, k \in N$ such that $0 < m < k$ and $a \in (t_m, t_{m+1}]$, $b \in (t_k, t_{k+1}]$, then from the non-increasing nature of $x(t)$ on $(t_k, t_{k+1}]$ and that $y(t_k) \geq y(t_{k+1})$ we have

$$y(a) = -c_m + x(a) \geq -c_m + x(t_{m+1}) = y(t_{m+1})$$

$$
\begin{aligned}
&\geq y(t_k) = -c_{k-1} + x(t_k) = -c_{k-1} - [x(t_k^+) - x(t_k)] + x(t_k^+) \\
&= -c_k + x(t_k^+) \geq -c_k + x(b) = y(b).
\end{aligned}
$$

Hence, $y(t)$ is non-increasing on $[t_0, +\infty)$. On the other hand, (2.9) and the bounded nature of $x(t)$ imply that $y(t)$ is bounded on $[t_0, +\infty)$. Therefore, the $\lim_{t\to\infty} y(t) = A$ exists and hence the $\lim_{t\to\infty} x(t) = \alpha$ exists and $\alpha = A + c \geq 0$.

Finally, we prove that $\alpha = 0$. If $\alpha > 0$, then there exists $T > t_0$ such that $x(t-\tau) \geq \frac{\alpha}{2}$ for $t \geq T$. Thus

$$
x^{(n)}(t) = -p(t)x(t-\tau) \leq -\frac{\alpha}{2}p(t),
$$

and therefore,

$$
\int_{t_k}^{t} x^{(n)}(s)s^{n-2}ds \leq -\frac{\alpha}{2}\int_{t_k}^{t} p(s)s^{n-2}ds,
$$

where $t \in (t_k, t_{k+1}]$.

The remainder of the proof is similar to that of Theorem 2.5 with $L = 1$ and is omitted. The proof is completed. □

We remark that the Theorem 2.6 is an extension of the result in [8].

For the next theorem we assume

(H5) There exists a sequence $\{t_{k_m}\}_{m=1}^{\infty}$ such that

$$
\int_{t_{k_m}-\tau}^{t_{k_m}} (t_{k_m} - s)^{n-1}p(s)ds > (n-1)!,
$$

Theorem 2.7. *Suppose that (H1)-(H5) hold and that n is odd. Then all solutions of* (1.5) *are oscillatory.*

Proof. Suppose on the contrary that (1.5) has a non-oscillatory solution $x(t)$. Without loss of generality, we may assume that $x(t) > 0$ for $t \geq t_0 - \tau$ and $\{t_{k_m}\} = \{t_k\}$.

Lemma 2.3 suggests that, for sufficiently large t, one of the statements (i) or (ii) in Theorem 2.6 is true. If (i) is true, then one can prove the required conclusion in a similar way as the proof of Theorem 2.5. So only the case (ii) need to be considered.

Since $t_k - t_{k-1} > 3\tau$, for $s \in [t_k - \tau, t_k)$, we have $s - \tau \in (t_{k-1} + \tau, t_k - \tau)$. Using the Taylor formula and (ii), we obtain

$$
\begin{aligned}
x^{(n)}(s) &= -p(s)x(s-\tau) \\
&= -p(s)\big[x(t_k-\tau) + x'(t_k-\tau)(s-t_k) + \frac{1}{2}x''(t_k-\tau)(s-t_k)^2 \\
&\quad + \dots \frac{x^{(n-1)}(t_k-\tau)}{(n-1)!}(s-t_k)^{n-1} + \frac{x^{(n)}(\xi)}{n!}(s-t_k)^n\big] \\
&= -p(s)\big[x(t_k-\tau) - x'(t_k-\tau)(t_k-s) + \frac{1}{2}x''(t_k-\tau)(t_k-s)^2 \dots \\
&\quad + \frac{x^{(n-1)}(t_k-\tau)}{(n-1)!}(t_k-s)^{n-1} - \frac{x^{(n)}(\xi)}{n!}(t_k-s)^n\big] \\
&\leq -p(s)\frac{x^{(n-1)}(t_k-\tau)}{(n-1)!}(t_k-s)^{n-1},
\end{aligned}
$$

where $\xi \in (s-\tau, t_k-\tau)$. Integrating both sides of the above inequality from $t_k-\tau$ to t_k,

$$x^{(n-1)}(t_k) - x^{(n-1)}(t_k-\tau) \le -\frac{x^{(n-1)}(t_k-\tau)}{(n-1)!}\int_{t_k-\tau}^{t_k}(t_k-s)^{n-1}p(s)ds.$$

Since $x^{(n-1)}(t_k) > 0$,

$$-x^{(n-1)}(t_k-\tau) \le -\frac{x^{(n-1)}(t_k-\tau)}{(n-1)!}\int_{t_k-\tau}^{t_k}(t_k-s)^{n-1}p(s)ds,$$

or

$$1 \ge \frac{1}{(n-1)!}\int_{t_k-\tau}^{t_k}(t_k-s)^{n-1}p(s)ds,$$
$$\int_{t_k-\tau}^{t_k}(t_k-s)^{n-1}p(s)ds \le (n-1)!$$

which contradicts (H5). So every solution of (1.5) is oscillatory. □

Corollary 2.8. *Assume that*

(i) *(H1) holds;*
(ii) $a_k^{(i)} \ge 1$, $\prod_{k=1}^{+\infty} b_k < +\infty$, $i = 0,1,2,\dots,n-1$;
(iii) $\int^{+\infty} t^{n-2}p(t)dt = +\infty$.

Then the following statements are true: (a) If n is even, then every solution of (1.5) *is oscillatory. (b) If n is odd, then every non-oscillatory solution of* (1.5) *converges to zero.*

Proof. It is clear that condition $a_k^{(i)} \ge 1$ implies that (H2) and (H3), and that $\prod_{k=1}^{+\infty} b_k < +\infty$ yields that (2.7). (ii) and (iii) yield (H4). Thus, the required conclusion comes out immediately. □

Corollary 2.9. *Assume that (H1), (H2), (H3) hold and there exist an integer $K \ge 0$ and a constant $\alpha \ge 0$ such that $\frac{1}{b_k} \ge (\frac{t_{k+1}}{t_k})^\alpha$ for $k \ge K$, where $b_i = \max\{a_i^{(1.5)}, a_i^{(2)}, \dots a_i^{(n-1)}\}$. If $\int^{+\infty} t^{n-2+\alpha}p(t)dt = +\infty$, then the following statements are true: (a) If n is even,then every solution of* (1.5) *is oscillatory. (b) If n is odd and* (2.7) *is satisfied,then every non-oscillatory solution of* (1.5) *converges to zero.*

Proof. Without loss of generality, assume that $K = 0$. By the assumption of Corollary 2.9, we have

$$\begin{aligned}
\frac{1}{b_1b_2\dots b_k}\int_{t_k}^{t_{k+1}} t^{n-2}p(t)dt &\ge (\frac{t_2}{t_1})^\alpha(\frac{t_3}{t_2})^\alpha\dots(\frac{t_{k+1}}{t_k})^\alpha\int_{t_k}^{t_{k+1}} t^{n-2}p(t)dt \\
&= (\frac{t_{k+1}}{t_1})^\alpha\int_{t_k}^{t_{k+1}} t^{n-2}p(t)dt \\
&\ge \frac{1}{t_1^\alpha}\int_{t_k}^{t_{k+1}} t^{n-2+\alpha}p(t)dt,
\end{aligned}$$

and

$$\int_{t_0}^{t_1} t^{n-2}p(t)dt + \frac{1}{b_1}\int_{t_0}^{t_1} t^{n-2}p(t)dt + \dots + \frac{1}{b_1b_2\dots b_k}\int_{t_k}^{t_{k+1}} t^{n-2}p(t)dt$$

$$\begin{aligned}&\geq \frac{1}{t_1^\alpha}\int_{t_0}^{t_1} t^{n-2+\alpha}p(t)dt + \frac{1}{t_1^\alpha}\int_{t_1}^{t_2} t^{n-2+\alpha}p(t)dt + \dots + \frac{1}{t_1^\alpha}\int_{t_k}^{t_{k+1}} t^{n-2+\alpha}p(t)dt\\ &= \frac{1}{t_1^\alpha}\int_{t_0}^{t_{k+1}} t^{n-2+\alpha}p(t)dt \to +\infty \quad \text{as } k\to+\infty.\end{aligned}$$

This implies that (H4) holds and the required conclusion thus comes from Theorem 2.5 and Theorem 2.6. □

Corollary 2.10. *Assume that (H1) holds and $a_k^{(i)} \geq 1$, $i = 1,2,\dots,n-1$, $k = 1,2,\dots$. Furthermore, there exist an integer $K \geq 0$ and a constant $\alpha < 0$ such that $\frac{1}{b_k} \geq (\frac{t_{k+1}}{t_k})^\alpha$ for $k \geq K$. If*

$$\sum_{k=1}^{+\infty} t_{k+1}^\alpha \int_{t_k}^{t_{k+1}} t^{n-2}p(t)dt = +\infty,$$

then the following statements are true: (a) If n is even, then every solution of (1.5) *is oscillatory. (b) If n is odd and* (2.7) *is satisfied, then every non-oscillatory solution of* (1.5) *converges to zero.*

Proof. We proceed as in the proof of Corollary 2.9 and obtain that

$$\frac{1}{b_1b_2\dots b_k}\int_{t_k}^{t_{k+1}} t^{n-2}p(t)dt \geq (\frac{t_{k+1}}{t_1})^\alpha \int_{t_k}^{t_{k+1}} t^{n-2}p(t)dt.$$

Then

$$\begin{aligned}&\int_{t_0}^{t_1} t^{n-2}p(t)dt + \frac{1}{b_1}\int_{t_1}^{t_2} t^{n-2}p(t)dt + \dots + \frac{1}{b_1b_2\dots b_k}\int_{t_k}^{t_{k+1}} t^{n-2}p(t)dt\\ &\geq \frac{1}{t_1^\alpha}\sum_{k=1}^{+\infty} t_{k+1}^\alpha \int_{t_k}^{t_{k+1}} t^{n-2}p(t)dt = +\infty.\end{aligned}$$

By Theorem 2.5 and Theorem 2.6, the required result follows. □

3. Examples

The following examples illustrate how the results can be applied in practice.

Example 3.1. Consider the impulsive delay differential equation

$$\begin{gathered} x^{(6)}(t) + \frac{2t+1}{t^6}x(t-\frac{1}{5}) = 0, \quad t \neq k, \quad k = 1,2,\dots\\ x^{(i)}(k^+) = \sqrt[2^k]{2}x^{(i)}(k), \quad i = 0,1,\dots,5;\ k = 1,2,\dots \end{gathered} \tag{3.1}$$

where $a_k^{(i)} = \sqrt[2^k]{2} = b_k, i = 0,1,\dots 5, k = 1,2,\dots; p(t) = \frac{2t+1}{t^6}$. One can show easily that $\prod_{k=1}^{+\infty} b_k = 2$ and $\int^{+\infty} t^4 p(t)dt = +\infty$. By Corollary 2.8, every solution of equation (3.1) is oscillatory.

Example 3.2. Consider the impulsive delay differential equation

$$\begin{gathered} x^{(5)}(t) + \frac{1}{t^4}x(t-\frac{1}{4}) = 0, \quad t \neq k,\ k = 1,2,\dots\\ x^{(i)}(k^+) = (1+c^{2^k})x^{(i)}(k), \quad i = 0,1,2,\dots,4;\ k = 1,2,\dots, \end{gathered} \tag{3.2}$$

where c is a constant, $0 < c < 1$. $b_k = a_k^{(i)} = 1 + c^{2^k}$, $i = 0,1,2,3,4$; $k = 1,2,\dots$ and $p(t) = \frac{1}{t^4}$. By simple calculation, we obtain $\lim_{k\to+\infty} b_1b_2\dots b_k =$

$\frac{1}{1-c}$, $\int^{+\infty} t^3 p(t)dt = +\infty$. It follows from Corollary 2.8 that every non-oscillatory solution of equation (3.2) tends to zero.

Example 3.3. Consider the impulsive delay differential equation

$$\begin{gathered} x^{(10)}(t) + (1/t^{\frac{17}{2}})x(t-0.5) = 0, \quad t \neq k^2, k = 1, 2, \dots \\ x^{(i)}(k^+) = ((k+1)/k)x^{(i)}(k), \quad i = 0, 1, \dots, 9;\ k = 1, 2, \dots \end{gathered} \tag{3.3}$$

where $b_k = a_k^{(i)} = \frac{k+1}{k}$, $i = 0, 1, \dots, 9$; $k = 1, 2, \dots$; $p(t) = 1/t^{\frac{17}{2}}$. Let $\alpha = -\frac{1}{2}$, then

$$\frac{1}{b_k} = \frac{k+1}{k} = (\frac{t_{k+1}}{t_k})^\alpha.$$

Furthermore,

$$\sum_{k=1}^{+\infty} t_{k+1}^\alpha \int_{t_k}^{t_{k+1}} t^8 p(t)dt = \frac{1}{2}\sum_{k=1}^{+\infty} \frac{1}{k+1} = +\infty.$$

Thus, by Corollary 2.10, all the solutions of equation (E_3) are oscillatory.

Example 3.4. Consider the impulsive delay differential equation

$$\begin{gathered} x^{(3)}(t) + (2 + \frac{1}{t^2})x(t - \frac{1}{4}) = 0, \quad t \neq k,\ k = 1, 2, \dots \\ x^{(i)}(k^+) = ((k+1)/k)x^{(i)}(k), \\ x^{(0)}(k^+) = x^{(0)}(k),\ i = 1, 2; k = 1, 2, \dots \end{gathered} \tag{3.4}$$

where $a_k^{(0)} = 1$, $b_k = a_k^{(i)} = \frac{k+1}{k}$, $i = 1, 2$, $k = 1, 2, \dots$, $p(t) = 2 + \frac{1}{t^2}$. A simple calculation leads to

$$\sum_{k=1}^{+\infty} \frac{1}{b_1 b_2 \dots b_k} \int_k^{k+1} sp(s)ds = \sum_{k=1}^{+\infty} \frac{1}{k+1} \int_k^{k+1} (2s + \frac{1}{s})ds \geq \sum_{k=1}^{+\infty} \frac{2k+1}{k+1} = +\infty,$$

$$\int_{t-\frac{1}{3}}^{t} (t-s)^2 p(s)ds \geq 2\int_{t-\frac{1}{3}}^{t} (t-s)^2 ds = \frac{4}{3}t^2 - \frac{8}{9} + \frac{2}{27} \to +\infty (t \to +\infty)\,.$$

Thus, (H1)–(H5) hold. By Theorem 2.7, every solution of equation (3.4) is oscillatory.

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