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BLOW-UP CRITERIA AND INSTABILITY OF STANDING WAVES FOR THE INHOMOGENEOUS FRACTIONAL SCHRÖDINGER EQUATION

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Abstract. In this article, we study the blow-up and instability of standing waves for the inhomogeneous fractional Schrödinger equation

$$i\partial_t u - (-\Delta)^s u + |x|^{-b}|u|^p u = 0,$$

where $s \in (\frac{1}{2}, 1)$, $0 < b < \min\{2s, N\}$ and $0 < p < \frac{4s-2b}{N-2s}$. In the L^2-critical and L^2-supercritical cases, i.e., $\frac{4s-2b}{N} \le p < \frac{4s-2b}{N-2s}$, we establish general blow-up criteria for non-radial solutions by using localized virial estimates. Based on these blow-up criteria, we prove the strong instability of standing waves.

1. Introduction

Over the past decade, there has been a great deal of interest in studying the fractional Schrödinger equation

$$i\partial_t u = (-\Delta)^s u + f(u), \tag{1.1}$$

where $0 < s < 1$ and $f(u)$ is the nonlinearity. The fractional differential operator $(-\Delta)^s$ is defined by $(-\Delta)^s u = \mathcal{F}^{-1}[|\xi|^{2s}\mathcal{F}(u)]$, where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform and inverse Fourier transform, respectively. Equation (1.1) was first deduced by Laskin in [24, 25] by extending the Feynman path integral from the Brownian-like to the Lévy-like quantum mechanical paths. The fractional Schrödinger equation also arises in the description of Bonson stars as well as in water wave dynamics (see [16]) and in the continuum limit of discrete models with long-range interactions (see [23]).

In this article, we consider the blow-up criteria and instability of standing waves for the inhomogeneous fractional Schrödinger equation

$$\begin{gathered} i\partial_t u - (-\Delta)^s u + |x|^{-b}|u|^p u = 0, \quad (t,x) \in [0,T^*) \times \mathbb{R}^N, \\ u(0,x) = u_0(x), \end{gathered} \tag{1.2}$$

where $u : [0,T^*) \times \mathbb{R}^N \to \mathbb{C}$ is the complex valued function, $N \ge 1$, $u_0 \in H^s$, $0 < s < 1$, $0 < b < \min\{2s, N\}$, $0 < p < \frac{4s-2b}{N-2s}$.

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This equation enjoys the scaling invariance. That is, if $u(t,x)$ is a solution of (1.2), then

$$u_\lambda(t,x)=\lambda^{\frac{2s-b}{p}}u(\lambda^{2s}t,\lambda x)\quad\text{for all } \lambda>0,$$

is also a solution of (1.2). By simple calculations, we have

$$\|u_\lambda(t)\|_{\dot H^s}=\lambda^{s+\frac{2s-b}{p}-\frac{N}{2}}\|u(\lambda^{2s}t)\|_{\dot H^s}.$$

Thus, the critical Sobolev index is given by

$$s_c:=\frac{N}{2}-\frac{2s-b}{p}. \tag{1.3}$$

When $s_c<0$, equation (1.2) is L^2-subcritical. The smallest power for which blow-up may occur is $p=\frac{4s-2b}{N}$, which is referred to L^2-critical case corresponding to $s_c=0$. When $0<s_c<s$, (1.2) is L^2-supercritical and H^s-subcritical. When $s_c=s$, (1.2) is H^s-critical. In this paper, we are interested in the L^2-critical and L^2-supercritical cases. Therefore, we restrict our attention to the case $0\le s_c<s$. Rewriting this condition in terms of p, we obtain

$$\frac{4s-2b}{N}\le p<\frac{4s-2b}{N-2s}.$$

If one considers initial data in H^s, then the equation enjoys mass and energy conservation laws:

$$M(u(t)):=\|u(t)\|_{L^2}=\|u_0\|_{L^2}, \tag{1.4}$$

and

$$\begin{aligned} E(u(t)):=&\frac{1}{2}\int_{\mathbb{R}^N}|(-\Delta)^{s/2}u(t,x)|^2dx-\frac{1}{p+2}\int_{\mathbb{R}^N}|x|^{-b}|u(t,x)|^{p+2}dx\\ =&E(u_0). \end{aligned} \tag{1.5}$$

Before entering some details of our results, let us recall known blow-up results . For the classical Shrödinger equation, i.e., $s=1$, the Variance-Virial Law holds, that is

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N}|x|^2|u(t,x)|^2dx=2\,\mathrm{Im}\int_{\mathbb{R}^N}\bar u(t,x)x\cdot\nabla u(t,x)dx, \tag{1.6}$$

provided initial data $u_0\in\Sigma:=\{u_0\in H^1 \text{ and } xu_0\in L^2\}$. Combining (1.6) and the virial identity, one can obtain blow-up results for the classical Schrödinger equation with negative energy $E(u_0)<0$ and finite variance, see [2]. Ogawa and Tsutsumi [26] removed the assumption $u_0\in\Sigma$ for the radial symmetry initial data. Applying similar ideas, when $s=1$, Farah [10] and Dinh [7] established the blow-up criteria for equation (1.2) with initial data $u_0\in\Sigma:=\{v\in H^1 \text{ and } xv\in L^2\}$ and radial symmetry initial data. However, when $s<1$, identity (1.6) fails and these arguments cannot work. However, a generalization of the variance for the fractional Schrödinger equation is given by

$$\mathcal{V}^{(s)}[u(t)]:=\int_{\mathbb{R}^N}\bar u(t,x)x\cdot(-\Delta)^{1-s}xu(t,x)dx=\|x(-\Delta)^{\frac{1-s}{2}}u(t)\|_{L^2}^2. \tag{1.7}$$

Let $u(t)$ be the solution of equation $i\partial_t u=(-\Delta)^s u$, a formal calculation yields

$$\frac{1}{2}\frac{d}{dt}\mathcal{V}^{(s)}[u(t)]:=2\,\mathrm{Im}\int_{\mathbb{R}^N}\bar u(t,x)x\cdot\nabla u(t,x)dx. \tag{1.8}$$

Based on this identity, the authors in [3, 4, 33] successfully obtained blow-up results for (1.1) with radial initial data and Hartree-type nonlinearity, i.e., $f(u)=-(|x|^{-\gamma}*$

$|u|^2)u$ with $\gamma \geq 1$. Because it is very hard to control the nontrivial error terms, this method fails to work for the local nonlinearities $f(u) = -|u|^p u$, see [1]. Using the Balakrishman's formula

$$(-\Delta)^s = \frac{\sin \pi s}{\pi} \int_0^\infty m^{s-1} \frac{-\Delta}{-\Delta + m} dm, \tag{1.9}$$

Boulenger, Himmelsbach and Lenzmann [1] established the differential estimate

$$\begin{aligned} &\frac{d}{dt}\Big(\operatorname{Im} \int_{\mathbb{R}^N} \bar{u}(t) \nabla \varphi_R \cdot \nabla u(t) dx \Big) \\ &\leq 4pNE(u_0) - 2\delta \|(-\Delta)^{s/2} u(t)\|_{L^2}^2 + \circ_R(1)(1 + \|(-\Delta)^{s/2} u(t)\|_{L^2}^{p/s+}), \end{aligned}$$

where $\delta = pN - 2s$. Based on this key estimate and a standard comparison ODE argument, they proved the existence of radial blow-up H^s solutions.

For the inhomogeneous fractional Schrödinger equation (1.2), Peng and Zhao in [27] obtained the existence of radial blow-up solutions. In this paper, by using localized virial estimates and the ideas of Du, Wu and Zhang [9], we remove this assumption, and establish general blow-up criteria for non-radial solutions in the L^2-critical and L^2-supercritical cases. The main difficulty is the appearance of the fractional order Laplacian $(-\Delta)^s$ and the singular potential $|x|^{-b}$. When $s = 1$, it easily follows that the time derivative of the virial action

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} \varphi(x) |u(t,x)|^2 dx = 2 \operatorname{Im} \int_{\mathbb{R}^N} \bar{u}(t,x) \nabla \varphi(x) \cdot \nabla u(t,x) dx. \tag{1.10}$$

By applying this identity, Du, Wu and Zhang [9] established an L^2-estimate in the exterior ball. Based on this L^2-estimate and the virial estimates, they established blow-up criteria for the classical Schrödinger equation. In the case $s \in (\frac{1}{2}, 1)$, the identity (1.10) does not hold. However, by using the Balakrishman's formula (1.9) and exploiting the ideas in [1], we can obtain the time derivative of the virial action. We consequently obtain the following general blow-up criteria for non-radial solutions in both L^2-critical and L^2-supercritical cases.

Theorem 1.1. *Let $N \geq 1$, $s \in (\frac{1}{2}, 1)$, $\frac{4s-2b}{N} \leq p < \frac{4s-2b}{N-2s}$, and $u_0 \in H^s$. Assume that $u \in C([0, T^*), H^s)$ is a solution of (1.2). Furthermore, we assume that either $E(u_0) < 0$, or, if $E(u_0) \geq 0$ and*

$$\begin{gathered} E(u_0)^{s_c} \|u_0\|_{L^2}^{2(s-s_c)} < E(Q)^{s_c} \|Q\|_{L^2}^{2(s-s_c)}, \\ \|(-\Delta)^{s/2} u_0\|_{L^2}^{s_c} \|u_0\|_{L^2}^{s-s_c} > \|(-\Delta)^{s/2} Q\|_{L^2}^{s_c} \|Q\|_{L^2}^{s-s_c}, \end{gathered} \tag{1.11}$$

where s_c is defined by (1.3) and Q is the ground state of the elliptic equation

$$(-\Delta)^s Q + Q - |x|^{-b} |Q|^p Q = 0. \tag{1.12}$$

Then one of the following statements holds:

- *$u(t)$ blows up in finite time, i.e. $T^* < +\infty$;*
- *$u(t)$ blows up infinite time, i.e., there exists $(t_n)_{n\geq 1}$ such that $t_n \to +\infty$ and*

$$\lim_{n\to\infty} \|(-\Delta)^{s/2} u(t_n)\|_{L^2} = \infty.$$

Our blow-up criteria also hold for (1.2) with $s = 1$, which to our knowledge is new. When $s = 1$, similar blow-up criteria for (1.2) with radial solutions or initial data $u_0 \in \Sigma := \{v \in H^1 \text{ and } xv \in L^2\}$ have been established in [7, 10]. Here, we remove the assumption of radial solutions and $u_0 \in \Sigma := \{v \in H^1 \text{ and } xv \in L^2\}$.

So our results improve some previous results. Based on blow-up criterion (1.11), we can prove the strong instability of standing waves of (1.2).

Firstly, we introduce some notation. Throughout this paper, we call a standing wave solution of (1.2) of the form $e^{i\omega t}Q_\omega$, where $\omega \in \mathbb{R}$ is a frequency and $Q_\omega \in H^s$ is a nontrivial solution to the elliptic equation

$$(-\Delta)^s Q_\omega + \omega Q_\omega - |x|^{-b}|Q_\omega|^p Q_\omega = 0. \tag{1.13}$$

Let $Q_\omega(x) = \omega^{\frac{2s-b}{2sp}} Q(\omega^{\frac{1}{2s}} x)$ in (1.13), then Q satisfies equation (1.12). In particular, by some basic calculations, we have

$$E(Q_\omega)^{s_c}\|Q_\omega\|_{L^2}^{2(s-s_c)} = E(Q)^{s_c}\|Q\|_{L^2}^{2(s-s_c)}, \tag{1.14}$$

$$\|(-\Delta)^{s/2}Q_\omega\|_{L^2}^{s_c}\|Q_\omega\|_{L^2}^{s-s_c} = \|(-\Delta)^{s/2}Q\|_{L^2}^{s_c}\|Q\|_{L^2}^{s-s_c}. \tag{1.15}$$

In fact, these two quantities are scaling invariant of (1.2).

Definition 1.2. A function $Q \in H^s\backslash\{0\}$ is called a ground state for (1.12) if it is a minimizer of the Weinstein's functional

$$J(v) := \frac{\|v\|_{\dot{H}^s}^{\frac{Np+2b}{2s}} \|v\|_{L^2}^{p+2-\frac{Np+2b}{2s}}}{\int_{\mathbb{R}^N} |x|^{-b}|v(x)|^{p+2}dx}, \tag{1.16}$$

that is,

$$J(Q) = \inf\{J(v) : v \in H^s\backslash\{0\}\}. \tag{1.17}$$

The existence of ground states related to (1.12) has been established in Lemma 2.2. In addition, a direct computation shows that

$$\|Q_\omega\|_{L^2} = \omega^{\frac{4s-2b-Np}{4sp}}\|Q\|_{L^2}, \quad \|Q_\omega\|_{\dot{H}^s} = \omega^{\frac{2sp-Np+4s-2b}{4sp}}\|Q\|_{\dot{H}^s},$$
$$\int_{\mathbb{R}^N} |x|^{-b}|Q_\omega(x)|^{p+2}dx = \omega^{\frac{2sp-Np+4s-2b}{2sp}} \int_{\mathbb{R}^N} |x|^{-b}|Q(x)|^{p+2}dx.$$

These imply that

$$J(Q_\omega) = J(Q).$$

That is, Q_ω is also a minimizer of the Weinstein's functional. Thus, we can define the ground states related to (1.13) as follows: A function $Q_\omega \in H^s\backslash\{0\}$ is called a ground state solution of (1.13) if it is a minimizer of the Weinstein's functional (1.16). We can derive ground states of (1.13) from ground states related to (1.12). This implies the existence of ground states related to (1.13) when $\omega > 0$. In addition, the uniqueness of ground states related to (1.12) is an open problem.

Note also that (1.13) can be written as $S'_\omega(Q_\omega) = 0$, where

$$\begin{aligned} S_\omega(Q) :=& E(Q) + \frac{\omega}{2}\|Q\|_{L^2}^2 \\ =& \frac{1}{2}\|Q\|_{\dot{H}^s}^2 + \frac{\omega}{2}\|Q\|_{L^2}^2 - \frac{1}{p+2}\int_{\mathbb{R}^N} |x|^{-b}|Q(x)|^{p+2}dx, \end{aligned} \tag{1.18}$$

is the action functional. We also define the following functional

$$K(Q) := \partial_\lambda S_\omega(Q^\lambda)|_{\lambda=1} = s\|Q\|_{\dot{H}^s}^2 - \frac{Np+2b}{2p+4}\int_{\mathbb{R}^N} |x|^{-b}|Q(x)|^{p+2}dx, \tag{1.19}$$

where

$$Q^\lambda(x) := \lambda^{N/2}Q(\lambda x). \tag{1.20}$$

To the best of our knowledge, the general method to investigate the strong instability of standing waves for the classical Schrödinger equation is to apply the variational characterization of the ground states as minimizers of the action functional and derive the key estimate $K(u(t)) \le 2(S_\omega(u_0) - S_\omega(Q_\omega))$. Then, it follows from the virial identity that

$$\frac{d^2}{dt^2}\|xu(t)\|_{L^2}^2 = 8K(u(t)) \le 16(S_\omega(u_0) - S_\omega(Q_\omega)),$$

where $K(u(t))$ is defined by (1.19) with $s = 1$. Finally, one can choose the initial data u_0 such that $S_\omega(u_0) - S_\omega(Q_\omega) < 0$. This implies that the solution $u(t)$ of (1.1) with $s = 1$ blows up in finite time. Thus, one can prove the strong instability of ground state standing waves, see [2, 6, 12, 13, 14, 15, 22, 29, 30].

Here, we present a simpler method to study the strong instability of standing waves, which is based on the blow-up criterion (1.11).

Theorem 1.3. *Let $N \ge 1$, $s \in (\frac{1}{2}, 1)$, $\frac{4s-2b}{N} \le p < \frac{4s-2b}{N-2s}$, $\omega > 0$, Q_ω be the ground state related to* (1.13). *Then, the standing wave $u(t,x) = e^{i\omega t}Q_\omega(x)$ is strongly unstable in the following sense: there exists $\{u_{0,n}\} \subset H^s$ such that $u_{0,n} \to Q_\omega$ in H^s as $n \to \infty$ and the corresponding solution u_n of* (1.2) *with initial data $u_{0,n}$ blows up in finite or infinite time for any $n \ge 1$.*

In previous results, to construct blow-up solutions around the ground state solution, one needs to assume that the ground state solution Q_ω is radial or $Q_\omega \in \Sigma := \{v \in H^1 \text{ and } xv \in L^2\}$. Here, we remove these assumptions, so our result greatly improve some previous results.

This article is organized as follows: in Section 2, we recall and prove some lemmas such as the local well-posedness theory of (1.2), Brezis-Lieb's lemma, the sharp Gagliardo-Nirenberg type inequality (2.1) and the localized virial estimate related to (1.2). In section 3, we establish blow-up criteria for (1.2). In section 4, we prove the strong instability of standing waves.

Throughout this article, we use the following notation. $C > 0$ stands for a constant that may be different from line to line when it does not cause any confusion. For any $s \in (0,1)$, the fractional Sobolev space $H^s(\mathbb{R}^N)$ is defined by

$$H^s(\mathbb{R}^N) = \big\{u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N}(1 + |\xi|^{2s})|\hat{u}(\xi)|^2 d\xi < \infty\big\},$$

endowed with the norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \|u\|_{L^2(\mathbb{R}^N)} + \|u\|_{\dot{H}^s(\mathbb{R}^N)},$$

where up to a multiplicative constant,

$$\|u\|_{\dot{H}^s(\mathbb{R}^N)} = \Big\{\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2\alpha}}\,dx\,dy\Big\}^{1/2}$$

is the so-called Gagliardo semi-norm of u. In this paper, we often use the abbreviations $L^r = L^r(\mathbb{R}^N)$, $H^s = H^s(\mathbb{R}^N)$.

2. Preliminary lemmas

In this section, we recall some preliminary results that will be used later. Firstly, let us recall the local theory for the Cauchy problem (1.2). By applying Strichartz's estimates and the contraction mapping argument, Hong and Sire in [21] first studied the local well-posedness for the fractional Schrödinger equation in H^s. Because Strichartz's estimates have a loss of derivatives in the non-radial symmetry case, a weak local well-posedness follows in the energy space compared to the classical

Schrödinger equation, see [5, 21] for more details. In the radial symmetry case, one can remove the loss of derivatives in Strichartz's estimates. But it needs a restriction on the validity of s, namely $\frac{N}{2N-1} \le s < 1$.

For the inhomogeneous Schrödinger equation (1.2) with $s = 1$, Genoud and Stuart [17] first studied the well-posedness by using the argument of Cazenave [2]. By using Strichartz's estimates and the contraction mapping argument, Guzman [19] also established the local well-posedness as well as the small data global well-posedness in Sobolev spaces. By using radial Strichartz's estimates and the contraction mapping argument, we can obtain the following local well-posedness for (1.2) with radial H^s initial data. The proof is standard, see [5, 19, 21]. So we omit it.

Theorem 2.1. *Let $N \ge 2$, $\frac{N}{2N-1} \le s < 1$, $0 < p < \frac{4s-2b}{N-2s}$ and $0 < b < \min\{2s, N\}$. If $u_0 \in H^s$ is radial, then there exists $T = T(\|u_0\|_{H^s})$ such that (1.2) admits a unique solution $u \in C([0,T], H^s)$. Let $[0, T^*)$ be the maximal time interval on which the solution u is well-defined, if $T^* < \infty$, then $\|u(t)\|_{\dot H^s} \to \infty$ as $t \uparrow T^*$. Moreover, for all $0 \le t < T^*$, the solution $u(t)$ satisfies the conservations of mass and energy.*

Next, we recall the following sharp Gagliardo-Nirenberg inequality, which has been established in [27].

Lemma 2.2 ([27])**.** *Let $0 < s < 1$, $0 < p < \frac{4s-2b}{N-2s}$ and $0 < b < \min\{2s, N\}$. Then, for all $u \in H^s$,*

$$\int_{\mathbb{R}^N} |x|^{-b}|u(x)|^{p+2}dx \le C_{\rm opt}\|u\|_{\dot H^s}^{\frac{Np+2b}{2s}} \|u\|_{L^2}^{p+2-\frac{Np+2b}{2s}}, \tag{2.1}$$

where the best constant $C_{\rm opt}$ is given by

$$C_{\rm opt} = \Big(\frac{Np+2b}{2s(p+2)-(Np+2b)}\Big)^{\frac{4s-(Np+2b)}{4s}} \frac{2s(p+2)}{(Np+2b)\|Q\|_{L^2}^{p}},$$

where Q is the ground state of (1.12). Moreover, the following Pohozaev's identities hold

$$\|Q\|_{\dot H^s}^2 = \frac{Np+2b}{2s(p+2)}\int_{\mathbb{R}^N} |x|^{-b}|Q|^{p+2}dx = \frac{Np+2b}{2s(p+2)-(Np+2b)}\|Q\|_{L^2}^2. \tag{2.2}$$

Lemma 2.3 ([1])**.** *Let $N \ge 1$, $\varphi : \mathbb{R}^N \to \mathbb{R}$ and $\nabla\varphi \in W^{1,\infty}(\mathbb{R}^N)$. Then, for all $u \in H^{1/2}$, it follows that*

$$\Big|\int_{\mathbb{R}^N} \overline{u}(x)\nabla\varphi(x)\cdot\nabla u(x)dx\Big| \le C\|\nabla\varphi\|_{W^{1,\infty}}\Big(\||\nabla|^{1/2}u\|_{L^2}^2 + \|u\|_{L^2}\||\nabla|^{1/2}u\|_{L^2}\Big),$$

where $C > 0$ depends only on N.

To study localized virial estimates for (1.2), we introduce an auxiliary function

$$u_m(x) := c_s\frac{1}{-\Delta+m}u(x) = c_s\mathcal{F}^{-1}\Big(\frac{\hat u(\xi)}{|\xi|^2+m}\Big), \quad m > 0, \tag{2.3}$$

where $c_s := \sqrt{\sin(\pi s)/\pi}$.

Lemma 2.4 ([1])**.** *Let $N \ge 1$, $s \in (0,1)$, $\varphi : \mathbb{R}^N \to \mathbb{R}$ and $\Delta\varphi \in W^{2,\infty}(\mathbb{R}^N)$. Then, for all $u \in L^2$,*

$$\Big|\int_0^\infty m^s\int_{\mathbb{R}^N}(\Delta^2\varphi)|u_m|^2\,dx\,dm\Big| \le C\|\Delta^2\varphi\|_{L^\infty}^s\|\Delta\varphi\|_{L^\infty}^{1-s}\|u\|_{L^2}^2,$$

where $C > 0$ depends only on s and N.

Applying the identity

$$\frac{\sin \pi s}{\pi} \int_0^\infty \frac{m^s}{(|\xi|^2 + m)^2} dm = s|\xi|^{2s-2},$$

we deduce form the Plancherel's and Fubini's theorems that

$$\begin{aligned} \int_0^\infty m^s \int_{\mathbb{R}^N} |\nabla u_m|^2 \, dx \, dm &= \int_{\mathbb{R}^N} \Big(\frac{\sin \pi s}{\pi} \int_0^\infty \frac{m^s dm}{(|\xi|^2 + m)^2} \Big) |\xi|^2 |\hat{u}(\xi)|^2 d\xi \\ &= \int_{\mathbb{R}^N} (s|\xi|^{2s-2}) |\xi|^2 |\hat{u}(\xi)|^2 d\xi = s\|(-\Delta)^{s/2} u\|_{L^2}^2, \end{aligned} \tag{2.4}$$

for any $u \in \dot{H}^s$.

Lemma 2.5 ([8]). *Let $N \geq 1, s \in (1/2, 1)$, $\varphi : \mathbb{R}^N \to \mathbb{R}$ and $\nabla \varphi \in W^{1,\infty}$. Then for any $u \in L^2$,*

$$\Big| \int_0^\infty m^s \int_{\mathbb{R}^N} (\Delta \varphi) |u_m|^2 \, dx \, dm \Big| \leq C \|\Delta \varphi\|_{L^\infty}^{2s-1} \|\nabla \varphi\|_{L^\infty}^{2-2s} \|u\|_{L^2}^2,$$

where $C > 0$ depends only on s and N.

Lemma 2.6 ([8]). *Let $N \geq 1, s \in (1/2, 1)$, $\varphi : \mathbb{R}^N \to \mathbb{R}$ and $\nabla \varphi \in W^{1,\infty}$. Then for any $u \in H^{1/2}$,*

$$\Big| \int_0^\infty m^s \int_{\mathbb{R}^N} \overline{u}_m \nabla \varphi \cdot \nabla u_m \, dx \, dm \Big| \leq C \|\nabla \varphi\|_{W^{1,\infty}} \|u\|_{H^{1/2}}^2,$$

where $C > 0$ depends only on N.

Lemma 2.7 (Virial identity). *Let $N \geq 1, s \in (1/2, 1)$ and $\varphi : \mathbb{R}^N \to \mathbb{R}$ be such that $\varphi \in W^{2,\infty}$. Assume that $u \in C([0, T^*), H^s)$ is a solution to (1.2). Then*

$$\begin{aligned} \frac{d}{dt} \mathcal{V}_\varphi[u(t)] = &- i \int_0^\infty m^s \int_{\mathbb{R}^N} (\Delta \varphi) |u_m(t)|^2 \, dx \, dm \\ &- 2i \int_0^\infty m^s \int_{\mathbb{R}^N} \overline{u}_m(t) \nabla \varphi \cdot \nabla u_m(t) \, dx \, dm \end{aligned} \tag{2.5}$$

for any $t \in [0, T^*)$, where

$$\mathcal{V}_\varphi[u(t)] := \int_{\mathbb{R}^N} \varphi(x) |u(t,x)|^2 dx$$

is the localized virial action of u associated to φ and $u_m(t) = c_s(-\Delta + m)^{-1} u(t)$.

Proof. Because the general case follows by an approximation argument, we only prove (2.5) for $u \in C_0^\infty(\mathbb{R}^N)$. Since $u(t)$ satisfies (1.2), it easily follows that

$$\frac{d}{dt} \mathcal{V}_\varphi[u(t)] = \frac{d}{dt} \langle u(t), \varphi u(t) \rangle = i \langle u(t), [(-\Delta)^s, \varphi] u(t) \rangle,$$

where $[X, Y] = XY - YX$ is the commutator of X and Y. To study $[(-\Delta)^s, \varphi]$, we use the fact that for operators $A \geq 0$, B and $m > 0$ any positive real number,

$$\Big[\frac{A}{A+m}, B\Big] = \Big[1 - \frac{m}{A+m}, B\Big] = -m\Big[\frac{1}{A+m}, B\Big] = m \frac{1}{A+m} [A, B] \frac{1}{A+m},$$

see [1]. Using this identity with $A = (-\Delta)^s$ and $B = \varphi$, by the Balakrishman's formula we have

$$[(-\Delta)^s, \varphi] = \frac{\sin \pi s}{\pi} \int_0^\infty m^s \Big[\frac{-\Delta}{-\Delta + m}, \varphi\Big] dm$$

$$= \frac{\sin \pi s}{\pi} \int_0^\infty m^s \frac{1}{-\Delta + m} [-\Delta, \varphi] \frac{1}{-\Delta + m} dm.$$

Thus,

$$\begin{aligned}
&\langle u(t), [(-\Delta)^s, \varphi] u(t) \rangle \\
&= \Big\langle u(t), \Big(\frac{\sin \pi s}{\pi} \int_0^\infty m^s \frac{1}{-\Delta + m} [-\Delta, \varphi] \frac{1}{-\Delta + m} dm \Big) u(t) \Big\rangle \\
&= c_s^2 \int_0^\infty m^s \langle u(t), \frac{1}{-\Delta + m} [-\Delta, \varphi] \frac{1}{-\Delta + m} u(t) \rangle dm \\
&= \int_0^\infty m^s \langle c_s (-\Delta + m)^{-1} u(t), [-\Delta, \varphi] c_s (-\Delta + m)^{-1} u(t) \rangle dm \\
&= \int_0^\infty m^s \int_{\mathbb{R}^N} \overline{u}_m(t) \left(-\Delta \varphi u_m(t) - 2 \nabla \varphi \cdot \nabla u_m(t) \right) dx \, dm \\
&= \int_0^\infty m^s \int_{\mathbb{R}^N} \big((-\Delta \varphi) |u_m(t)|^2 - 2 \overline{u}_m(t) \nabla \varphi \cdot \nabla u_m(t) \big) dx \, dm.
\end{aligned}$$

The proof is complete. □

The following estimate is a direct consequence of Lemmas 2.5, 2.6 and 2.7.

Corollary 2.8. *Let $N \geq 1, s \in (1/2, 1)$ and $\varphi : \mathbb{R}^N \to \mathbb{R}$ be such that $\varphi \in W^{2,\infty}$. Assume that $u \in C([0, T^*), H^s)$ is a solution to (1.2). Then for any $t \in [0, T^*)$,*

$$|\frac{d}{dt} \mathcal{V}_\varphi[u(t)]| \leq C \|\nabla \varphi\|_{W^{1,\infty}} \|u(t)\|_{H^s}^2,$$

for some constant $C > 0$ depending only on s and N.

Now we define the localized Morawetz action of u associated to φ by

$$\mathcal{M}_\varphi[u(t)] := 2 \operatorname{Im} \int_{\mathbb{R}^N} \bar{u}(t, x) \nabla \varphi(x) \cdot \nabla u(t, x) dx. \tag{2.6}$$

By Lemma 2.3, we obtain the bound

$$|\mathcal{M}_\varphi[u(t)]| \leq C \big(\|\nabla \varphi\|_{L^\infty}, \|\Delta \varphi\|_{L^\infty} \big) \|u(t)\|_{H^{1/2}}^2.$$

Hence the quantity $\mathcal{M}_\varphi[u(t)]$ is well-defined, since $u(t) \in H^s$ with some $s > 1/2$ by assumption. By a similar argument as that in [1, Lemma 2.1], we have the following time evolution of $\mathcal{M}_\varphi[u(t)]$.

Lemma 2.9 (Morawetz identity)**.** *Let $N \geq 1, s \in (1/2, 1)$ and $\varphi : \mathbb{R}^N \to \mathbb{R}$ be such that $\nabla \varphi \in W^{3,\infty}$. Assume that $u \in C([0, T^*), H^s)$ is a solution to (1.2). Then for any $t \in [0, T^*)$, it holds that*

$$\begin{aligned}
&\frac{d}{dt} \mathcal{M}_\varphi[u(t)] \\
&= \int_0^\infty m^s \int_{\mathbb{R}^N} \Big\{ 4 \overline{\partial_k u_m(t)} (\partial_{kl}^2 \varphi) \partial_l u_m(t) - (\Delta^2 \varphi) |u_m(t)|^2 \Big\} dx \, dm \\
&\quad - \frac{2p}{p+2} \int_{\mathbb{R}^N} \Delta \varphi |x|^{-b} |u(t, x)|^{p+2} dx \\
&\quad - \frac{4b}{p+2} \int_{\mathbb{R}^N} |x|^{-b-2} x \cdot \nabla \varphi |u(t, x)|^{p+2} dx,
\end{aligned} \tag{2.7}$$

where $u_m(t) = u_m(t, x)$ is defined by (2.3).

Proof. It follows from an integration by parts that

$$
\begin{aligned}
&\langle u(t), [-|x|^{-b}|u(t)|^p, i\Gamma_\varphi]u(t)\rangle \\
&= -\langle u(t), [|x|^{-b}|u(t)|^p, \nabla\varphi\cdot\nabla + \nabla\cdot\nabla\varphi]u(t)\rangle \\
&= 2\int_{\mathbb{R}^N} |x|^{-b}|u(t,x)|^2\nabla\varphi\cdot\nabla|u(t,x)|^p dx + 2\int_{\mathbb{R}^N} |u(t,x)|^{p+2}\nabla\varphi\cdot\nabla|x|^{-b}dx \\
&= -\frac{2p}{p+2}\int_{\mathbb{R}^N} \Delta\varphi|x|^{-b}|u(t,x)|^{p+2}dx - \frac{4b}{p+2}\int_{\mathbb{R}^N} |x|^{-b-2}x\cdot\nabla\varphi|u(t,x)|^{p+2}dx,
\end{aligned}
$$

where we used the identities

$$
\nabla|x|^{-b} = -b|x|^{-b-2}x \quad\text{and}\quad \nabla|u|^{p+2} = \frac{p+2}{p}\nabla|u|^p|u|^2.
$$

Following the method used in [1], we complete the proof. $\square$

3. Blow-up criteria

In this section, we will prove Theorem 1.1. Firstly, we establish the following blow-up criteria for (1.2).

Lemma 3.1. *Let $N \geq 1$, $s \in (\frac{1}{2}, 1)$, $\frac{4s-2b}{N} \leq p < \frac{4s-2b}{N-2s}$. Assume that $u_0 \in H^s$ and $u \in C([0,T^*), H^s)$ is the corresponding solution of (1.2). If there exists $\delta > 0$ such that*

$$
K(u(t)) \leq -\delta \tag{3.1}
$$

for all $t \in [0,T^*)$, then one of the following two statements holds:

- *$u(t)$ blows up in finite time, i.e. $T^* < +\infty$;*
- *$u(t)$ blows up infinite time and there exists $(t_n)_{n\geq 1}$ such that $t_n \to +\infty$ and*

$$
\lim_{n\to\infty} \|(-\Delta)^{s/2}u(t_n)\|_{L^2} = \infty. \tag{3.2}
$$

Proof. If $T^* < +\infty$, then the proof is done. If $T^* = +\infty$, we prove (1.1) by contradiction. If not, the solution $u(t)$ exists globally and there exists $C_0 > 0$ such that

$$
C_0 := \sup_{t\in[0,+\infty)} \|(-\Delta)^{s/2}u(t)\|_{L^2} < \infty. \tag{3.3}
$$

This, together with the conservation of mass, implies that

$$
C_1 := \sup_{t\in[0,+\infty)} \|u(t)\|_{H^s} < \infty. \tag{3.4}
$$

Now, we claim that for every $\eta > 0$, $R > 1$, there exists a constant $C > 0$ independent of R and C_1 such that for any $t \in [0, \frac{\eta R}{CC_1^2}]$,

$$
\int_{|x|\geq R} |u(t,x)|^2 dx \leq \eta + o_R(1). \tag{3.5}
$$

To this end, we define a smooth function $\theta : [0,\infty) \to [0,1]$ that satisfies

$$
\theta(r) = \begin{cases} 0 & \text{if } 0 \leq r \leq 1/2, \\ 1 & \text{if } r \geq 1. \end{cases}
$$

For $R > 1$, we define the radial function

$$
\phi_R(x) = \phi_R(r) := \theta(r/R), \quad r = |x|.
$$

It easily follows that

$$\nabla \phi_R(x) = \frac{x}{rR}\theta'(r/R), \quad \Delta\phi_R(x) = \frac{1}{R^2}\theta''(r/R) + \frac{(N-1)}{rR}\theta'(r/R).$$

In particular, we have

$$\|\nabla\phi_R\|_{W^{1,\infty}} \sim \|\nabla\phi_R\|_{L^\infty} + \|\Delta\phi_R\|_{L^\infty} \le CR^{-1}. \tag{3.6}$$

Now, we can define the localized virial potential

$$\mathcal{V}_{\phi_R}[u(t)] := \int_{\mathbb{R}^N} \phi_R(x)|u(t,x)|^2 dx.$$

We have

$$\begin{aligned} \mathcal{V}_{\phi_R}[u(t)] &= \mathcal{V}_{\phi_R}[u_0] + \int_0^t \frac{d}{d\tau}\mathcal{V}_{\phi_R}[u(\tau)]d\tau \\ &\le \mathcal{V}_{\phi_R}[u_0] + \Big(\sup_{\tau\in[0,t]} \big|\frac{d}{d\tau}\mathcal{V}_{\phi_R}[u(\tau)]\big|\Big)t. \end{aligned}$$

By Corollary 2.8, (3.4) and (3.6), we obtain

$$\sup_{\tau\in[0,t]} \big|\frac{d}{d\tau}\mathcal{V}_{\phi_R}[u(\tau)]\big| \le C\|\nabla\phi_R\|_{W^{1,\infty}} \sup_{\tau\in[0,t]} \|u(\tau)\|_{H^s}^2 \le CC_1^2R^{-1},$$

for some constant $C>0$ independent of R and C_1. We thus obtain

$$\mathcal{V}_{\phi_R}[u(t)] \le \mathcal{V}_{\phi_R}[u_0] + CC_1^2R^{-1}t,$$

for all $t \ge 0$. By the choice of θ and the conservation of mass, we have

$$\mathcal{V}_{\phi_R}[u_0] = \int_{\mathbb{R}^N} \phi_R(x)|u_0(x)|^2dx \le \int_{|x|>R/2} |u_0(x)|^2 dx \to 0,$$

as $R\to\infty$ or $\mathcal{V}_{\phi_R}[u_0] = o_R(1)$. On the other hand, we have

$$\int_{|x|\ge R} |u(t,x)|^2 dx \le \mathcal{V}_{\phi_R}[u(t)].$$

Collecting the above estimates, we can obtain the control on the L^2-norm of the solution outside a large ball, i.e., claim (3.5).

Next, we assume that $\varphi(x)=\varphi(r)$ is radial and satisfies

$$\varphi(r) = \begin{cases} r^2/2 & \text{for } r\le 1, \\ \text{const.} & \text{for } r \ge 10, \end{cases}$$

and $\varphi''(r)\le 1$ for $r\ge 0$. Given $R>0$, we define the rescaled function $\varphi_R : \mathbb{R}^N \to \mathbb{R}$ by

$$\varphi_R(x) := R^2\varphi\big(\frac{x}{R}\big). \tag{3.7}$$

We readily verify the inequalities

$$1-\varphi_R''(r) \ge 0, \quad 1 - \frac{\varphi_R'(r)}{r} \ge 0, \quad N - \Delta\varphi_R(x) \ge 0,$$

for all $r\ge 0$ and all $x\in\mathbb{R}^N$. It is easy to see that

$$\|\nabla^k\varphi_R\|_{L^\infty} \le CR^{2-k}, \quad k=0,\cdots,4,$$

and

$$\operatorname{supp}(\nabla^k\varphi_R) \subset \begin{cases} \{x : |x|\le 10R\} & \text{for } k=1,2, \\ \{x : R\le |x| \le 10R\} & \text{for } k=3,4. \end{cases}$$

By Lemma 2.9, we have

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{M}_{\varphi_R}[u(t)] \\
&= \int_0^\infty m^s \int_{\mathbb{R}^N} \Big\{ 4\overline{\partial_k u_m(t)}(\partial^2_{kl}\varphi_R)\partial_l u_m(t) - (\Delta^2\varphi_R)|u_m(t)|^2 \Big\}\, dx\, dm \\
&\quad - \frac{2p}{p+2}\int_{\mathbb{R}^N} \Delta\varphi_R |x|^{-b}|u(t,x)|^{p+2}dx \\
&\quad - \frac{4b}{p+2}\int_{\mathbb{R}^N} |x|^{-b-2}x\cdot\nabla\varphi_R|u(t,x)|^{p+2}dx
\end{aligned} \tag{3.8}
$$

where $u_m(t) = u_m(t,x)$ is defined in (2.3). Since $\operatorname{supp}(\Delta^2\varphi_R) \subset \{|x| \geq R\}$, by Lemma 2.4, we have

$$
\begin{aligned}
\Big|\int_0^\infty m^s \int_{\mathbb{R}^N} (\Delta^2\varphi_R)|u_m(t)|^2\, dx\, dm\Big| &\leq C\|\Delta^2\varphi_R\|^s_{L^\infty}\|\Delta\varphi_R\|^{1-s}_{L^\infty}\|u(t)\|^2_{L^2(|x|\geq R)} \\
&\leq CR^{-2s}\|u(t)\|^2_{L^2(|x|\geq R)}.
\end{aligned} \tag{3.9}
$$

Since φ_R is radial, we use

$$
\partial^2_{jk} = \Big(\frac{\delta_{jk}}{r} - \frac{x_jx_k}{r^3}\Big)\partial_r + \frac{x_jx_k}{r^2}\partial_r^2
$$

to write

$$
\begin{aligned}
&\int_0^\infty m^s \int_{\mathbb{R}^N} \overline{\partial_k u_m(t)}(\partial^2_{jk}\varphi_R)\partial_l u_m(t)\, dx\, dm \\
&= \int_0^\infty m^s \int_{\mathbb{R}^N} \frac{\varphi_R'}{r}|\nabla u_m(t)|^2\, dx\, dm \\
&\quad + \int_0^\infty m^s \int_{\mathbb{R}^N} \Big(\frac{\varphi_R''}{r^2} - \frac{\varphi_R'}{r^3}\Big)|x\cdot\nabla u_m(t)|^2\, dx\, dm.
\end{aligned}
$$

Using (2.4), we write

$$
\begin{aligned}
&\int_0^\infty m^s \int_{\mathbb{R}^N} \frac{\varphi_R'}{r}|\nabla u_m(t)|^2\, dx\, dm \\
&= s\|(-\Delta)^{s/2}u(t)\|^2_{L^2} + \int_0^\infty m^s \int_{\mathbb{R}^N} \Big(\frac{\varphi_R'}{r} - 1\Big)|\nabla u_m(t)|^2\, dx\, dm.
\end{aligned}
$$

Since $\varphi_R'' \leq 1$, the Cauchy-Schwarz inequality implies

$$
\begin{aligned}
&\int_0^\infty m^s \int_{\mathbb{R}^N} \Big(\frac{\varphi_R'}{r} - 1\Big)|\nabla u_m(t)|^2\, dx\, dm \\
&+ \int_0^\infty m^s \int_{\mathbb{R}^N} \Big(\varphi_R'' - \frac{\varphi_R'}{r}\Big)\frac{|x\cdot\nabla u_m(t)|^2}{r^2}\, dx\, dm \leq 0.
\end{aligned}
$$

Therefore,

$$
4\int_0^\infty m^s \int_{\mathbb{R}^N} \overline{\partial_k u_m(t)}(\partial^2_{jk}\varphi_R)\partial_l u_m(t)\, dx\, dm \leq 4s\|(-\Delta)^{s/2}u(t)\|^2_{L^2}. \tag{3.10}
$$

We next write

$$
\begin{aligned}
&-\frac{2p}{p+2}\int_{\mathbb{R}^N}|x|^{-b}|u(t,x)|^{p+2}\triangle\varphi_R dx\\
&=-\frac{2pN}{p+2}\int_{\mathbb{R}^N}|x|^{-b}|u(t,x)|^{p+2}dx\\
&\quad-\frac{2p}{p+2}\int_{\mathbb{R}^N}|x|^{-b}|u(t,x)|^{p+2}(\triangle\varphi_R-N)dx.
\end{aligned}\tag{3.11}
$$

The second term can be estimated as follows:

$$
\begin{aligned}
&\big|-\frac{2p}{p+2}\int_{\mathbb{R}^N}(\Delta\varphi_R-N)|x|^{-b}|u(t,x)|^{p+2}dx\big|\\
&\le C\int_{|x|\ge R}|x|^{-b}|u(t,x)|^{p+2}dx\\
&\le CR^{-b}\int_{|x|\ge R}|u(t,x)|^{p+2}dx\\
&\le CR^{-b}\|u(t)\|_{L^2(|x|\ge R)}^{p+2-\frac{Np}{2s}}\|u(t)\|_{L^{\frac{2N}{N-2s}}(|x|\ge R)}^{\frac{Np}{2s}}\\
&\le CR^{-b}\|u(t)\|_{L^2(|x|\ge R)}^{p+2-\frac{Np}{2s}}\|u(t)\|_{H^s}^{\frac{Np}{2s}}\\
&\le CC_1^{\frac{Np}{2s}}R^{-b}\|u(t)\|_{L^2(|x|\ge R)}^{p+2-\frac{Np}{2s}}.
\end{aligned}\tag{3.12}
$$

Thus, we have

$$
\begin{aligned}
&-\frac{2p}{p+2}\int_{\mathbb{R}^N}|x|^{-b}|u(t,x)|^{p+2}\triangle\varphi_R dx\\
&\le-\frac{2pN}{p+2}\int_{\mathbb{R}^N}|x|^{-b}|u(t,x)|^{p+2}dx+CC_1^{\frac{Np}{2s}}R^{-b}\|u(t)\|_{L^2(|x|\ge R)}^{p+2-\frac{Np}{2s}}.
\end{aligned}\tag{3.13}
$$

For the last term in (3.8), we have

$$
\begin{aligned}
&-\frac{4b}{p+2}\int_{\mathbb{R}^N}(x\cdot\nabla\varphi_R)|x|^{-b-2}|u(t,x)|^{p+2}dx\\
&=-\frac{4b}{p+2}\int_{\mathbb{R}^N}|x|^{-b}|u(t,x)|^{p+2}dx\\
&\quad-\frac{4b}{p+2}\int_{|x|\ge R}\Big(\frac{x\cdot\nabla\varphi_R(r)}{|x|^2}-1\Big)|x|^{-b}|u(t,x)|^{p+2}dx\,.
\end{aligned}
$$

By the similar method as (3.12), we deduce

$$
\begin{aligned}
&\big|-\frac{4b}{p+2}\int_{|x|\ge R}\Big(\frac{x\cdot\nabla\varphi_R(r)}{|x|^2}-1\Big)|x|^{-b}|u(t,x)|^{p+2}dx\big|\\
&\le CC_1^{\frac{Np}{2s}}R^{-b}\|u(t)\|_{L^2(|x|\ge R)}^{p+2-\frac{Np}{2s}}.
\end{aligned}\tag{3.14}
$$

Collecting (3.9)-(3.14), we obtain

$$
\begin{aligned}
\frac{d}{dt}\mathcal{M}_{\varphi_R}[u(t)]&\le 4s\|(-\Delta)^{s/2}u(t)\|_{L^2}^2-\frac{2Np+4b}{p+2}\int_{\mathbb{R}^N}|x|^{-b}|u(t,x)|^{p+2}dx\\
&\quad+CR^{-2s}\|u(t)\|_{L^2(|x|\ge R)}^2+CC_1^{\frac{Np}{2s}}R^{-b}\|u(t)\|_{L^2(|x|\ge R)}^{p+2-\frac{Np}{2s}}.
\end{aligned}\tag{3.15}
$$

By (3.5), we see that for any $\eta > 0$ and any $R > 1$, there exists $C > 0$ independent of R and C_1 such that for any $t \in [0, T_0]$ with $T_0 = \frac{\eta R}{CC_1^2}$,

$$\begin{aligned}\frac{d}{dt}\mathcal{M}_{\varphi_R}[u(t)] &\le 4K(u(t)) + CR^{-2s}(\eta + o_R(1))^2 + CC_1^{\frac{Np}{2s}} R^{-b}(\eta + o_R(1))^{p+2-\frac{Np}{2s}} \\ &\le -4\delta + CR^{-2s}(\eta^2 + o_R(1)) + CC_1^{\frac{Np}{2s}} R^{-b}(\eta^{p+2-\frac{Np}{2s}} + o_R(1)).\end{aligned}$$

We first choose $\eta > 0$ small enough and $R > 1$ large enough so that

$$\frac{d}{dt}\mathcal{M}_{\varphi_R}[u(t)] \le -\delta < 0, \tag{3.16}$$

for any $t \in [0, T_0]$ with $T_0 = \frac{\eta R}{CC_1^2}$. Note that $\eta > 0$ is fixed, so we can choose $R > 1$ large enough so that T_0 is as large as we want. By (3.16), it follows that

$$\mathcal{M}_{\varphi_R}[u(t)] \le -ct,$$

for all $t \in [t_0, T_0]$ with some sufficiently large $t_0 \in [0, T_0]$. The constant $c > 0$ depends only on δ. On the other hand, we deduce from Lemma 2.3 and the conservation of mass that

$$\begin{aligned}|\mathcal{M}_{\varphi_R}[u(t)]| &\le C(\varphi_R)\Big(\||\nabla|^{1/2}u(t)\|_{L^2}^2 + \|u(t)\|_{L^2}\||\nabla|^{1/2}u(t)\|_{L^2}\Big) \\ &\le C(\varphi_R)\Big(\||\nabla|^{1/2}u(t)\|_{L^2}^2 + \|u(t)\|_{L^2}^2\Big) \\ &\le C(\varphi_R)\Big(\||\nabla|^{1/2}u(t)\|_{L^2}^2 + 1\Big),\end{aligned}$$

for every $t \in [0, +\infty)$. By interpolating between L^2 and $\dot{H}^s$, we obtain

$$ct \le -\mathcal{M}_{\varphi_R}[u(t)] = |\mathcal{M}_{\varphi_R}[u(t)]| \le C(\varphi_R)\left(\|(-\Delta)^{s/2}u(t)\|_{L^2}^{\frac{1}{s}} + 1\right),$$

for any $t \in [t_0, T_0]$. This implies that

$$\|(-\Delta)^{s/2}u(t)\|_{L^2} \ge Ct^s, \tag{3.17}$$

for all $t \in [t_1, T_0]$ with some sufficiently large $t_1 \in [t_0, T_0]$. Taking t close to $T_0 = \frac{\eta R}{CC_1^2}$, we see that $\|(-\Delta)^{s/2}u(t)\|_{L^2} \to \infty$ as $R \to \infty$, which contradicts (3.4). The proof is complete. $\square$

Applying Lemma 3.1, we can prove blow-up criteria for (1.2).

Proof of Theorem 1.1. We only check that (3.1) holds under the assumptions of this Theorem. In the L^2-critical case, i.e., $s_c = 0$. The blow-up condition (1.11) implies that $\|u_0\|_{L^2} < \|Q\|_{L^2}$ and $\|u_0\|_{L^2} > \|Q\|_{L^2}$, which is impossible. Thus, for $s_c = 0$ the only admissible condition is $E(u_0) < 0$. It follows from the conservation of energy and $p = \frac{4s-2b}{N}$ that

$$\begin{aligned}K(u(t)) =& s\|u(t)\|_{\dot{H}^s}^2 - \frac{Np+2b}{2p+4}\int_{\mathbb{R}^N} |x|^{-b}|u(t,x)|^{p+2}dx \\ =& 2sE(u(t)) + \frac{4s-Np-2b}{2p+4}\int_{\mathbb{R}^N} |x|^{-b}|u(t,x)|^{p+2}dx \\ =& 2sE(u_0),\end{aligned}$$

for all $t \in [0, T^*)$. Hence, when $E(u_0) < 0$, (3.1) follows with $\delta = -2sE(u_0)$.

Next, we consider the case $E(u_0) > 0$. The assumption (1.11) implies

$$\begin{gathered} E(u_0)\|u_0\|_{L^2}^{2\sigma} < E(Q)\|Q\|_{L^2}^{2\sigma}, \\ \|(-\Delta)^{s/2}u_0\|_{L^2}\|u_0\|_{L^2}^{\sigma} > \|(-\Delta)^{s/2}Q\|_{L^2}\|Q\|_{L^2}^{\sigma}, \end{gathered} \tag{3.18}$$

where

$$\sigma := \frac{s-s_c}{s_c} = \frac{2sp-Np+4s-2b}{Np+2b-4s}.$$

We notice that the sharp constant in Gagliardo-Nirenberg inequality (2.1) can be written as

$$C_{\rm opt} = \frac{\int_{\mathbb{R}^N}|x|^{-b}|Q(x)|^{p+2}dx}{\|Q\|_{\dot H^s}^{\frac{Np+2b}{2s}}\|Q\|_{L^2}^{p+2-\frac{Np+2b}{2s}}}, \tag{3.19}$$

which, by (2.2), can be rewritten as

$$C_{\rm opt} = \frac{2s(p+2)}{Np+2b}\frac{1}{(\|Q\|_{\dot H^s}\|Q\|_{L^2}^{\sigma})^{\frac{Np+2b-4s}{2s}}}. \tag{3.20}$$

It easily follows that

$$E(Q)\|Q\|_{L^2}^{2\sigma} = \frac{Np+2b-4s}{2(Np+2b)}(\|Q\|_{\dot H^s}\|Q\|_{L^2}^{\sigma})^2. \tag{3.21}$$

Multiplying both sides of $E(u(t))$ by $\|u(t)\|_{L^2}^{2\sigma}$, we deduce from the sharp Gagliardo-Nirenberg inequality (2.1) that

$$\begin{aligned} E(u(t))\|u(t)\|_{L^2}^{2\sigma} &= \frac{1}{2}\|u(t)\|_{\dot H^s}^2\|u(t)\|_{L^2}^{2\sigma} - \frac{1}{p+2}\int_{\mathbb{R}^N}|x|^{-b}|u(t,x)|^{p+2}dx\|u(t)\|_{L^2}^{2\sigma} \\ &\geq \frac{1}{2}(\|u(t)\|_{\dot H^s}\|u(t)\|_{L^2}^{\sigma})^2 - \frac{C_{\rm opt}}{p+2}(\|u(t)\|_{\dot H^s}\|u(t)\|_{L^2}^{\sigma})^{\frac{Np+2b}{2s}} \\ &= f(\|u(t)\|_{\dot H^s}\|u(t)\|_{L^2}^{\sigma}), \end{aligned}$$

where $f(x) := \frac{1}{2}x^2 - \frac{C_{\rm opt}}{p+2}x^{\frac{Np+2b}{2s}}$. It is easy to see that f is increasing on $(0, x_0)$ and decreasing on (x_0, ∞), where

$$x_0 = \Big(\frac{2sp+4s}{C_{\rm opt}(Np+2b)}\Big)^{\frac{2s}{Np+2b-4s}} = \|Q\|_{\dot H^s}\|Q\|_{L^2}^{\sigma},$$

where the last equality follows from (3.20). It follows from (3.20) and (3.21) that

$$f(\|Q\|_{\dot H^s}\|Q\|_{L^2}^{\sigma}) = E(Q)\|Q\|_{L^2}^{2\sigma}.$$

Thus the conservation of mass and energy together with the first condition in (1.11) imply

$$\begin{aligned} f(\|u(t)\|_{\dot H^s}\|u(t)\|_{L^2}^{\sigma}) &\leq E(u(t))\|u(t)\|_{L^2}^{2\sigma} = E(u_0)\|u_0\|_{L^2}^{2\sigma} \\ &< E(Q)\|Q\|_{L^2}^{2\sigma} = f(\|Q\|_{\dot H^s}\|Q\|_{L^2}^{\sigma}), \end{aligned}$$

for all $t \in [0, T^*)$. Using the second condition (1.11), the continuity argument shows that

$$\|u(t)\|_{\dot H^s}\|u(t)\|_{L^2}^{\sigma} > \|Q\|_{\dot H^s}\|Q\|_{L^2}^{\sigma} \tag{3.22}$$

for any $t \in [0, T^*)$. On the other hand, since $E(u_0)\|u_0\|_{L^2}^{2\sigma} < E(Q)\|Q\|_{L^2}^{2\sigma}$, we pick $\eta > 0$ small enough so that

$$E(u_0)\|u_0\|_{L^2}^{2\sigma} \leq (1-\eta)E(Q)\|Q\|_{L^2}^{2\sigma}.$$

Thus, by the conservation of energy, (3.21) and (3.22), we have

$$
\begin{aligned}
K(u(t))\|u(t)\|_{L^2}^{2\sigma} &= \frac{Np+2b}{2}E(u(t))\|u(t)\|_{L^2}^{2\sigma} - \frac{Np+2b-4s}{4}\|u(t)\|_{\dot H^s}^2\|u(t)\|_{L^2}^{2\sigma} \\
&= \frac{Np+2b}{2}E(u_0)\|u_0\|_{L^2}^{2\sigma} - \frac{Np+2b-4s}{4}(\|u(t)\|_{\dot H^s}\|u(t)\|_{L^2}^{\sigma})^2 \\
&\le \frac{Np+2b}{2}(1-\eta)E(Q)\|Q\|_{L^2}^{2\sigma} - \frac{Np+2b-4s}{4}(\|Q\|_{\dot H^s}\|Q\|_{L^2}^{\sigma})^2 \\
&= -\eta\frac{Np+2b}{2}E(Q)\|Q\|_{L^2}^{2\sigma},
\end{aligned}
$$

for all $t \in [0, T^*)$. This implies (3.1) with $\delta = \eta\frac{Np+2b}{2}E(Q)\|Q\|_{L^2}^{2\sigma}$. Thus, the solution $u(t)$ of (1.2) blows up in finite or infinite time. This completes the proof. □

4. Strong instability

In this section, we apply the blow-up criteria in Theorem 1.1 to prove Theorem 1.3.

Proof of Theorem 1.3. We divide the proof into two cases: (1) $p = \frac{4s-2b}{N}$ and (2) $\frac{4s-2b}{N} < p < \frac{4s-2b}{N-2s}$.

Case (1) $p = \frac{4s-2b}{N}$. Firstly, we deduce from Pohozaev's identities (2.2) that $E(Q_\omega) = 0$, where Q_ω is the ground state solution of (1.13). Thus, if we can construct initial data $u_{0,n}$ such that $E(u_{0,n}) < 0$ and $u_{0,n} \to Q_\omega$ in H^s, as $n \to \infty$, then the corresponding solution u_n blows up in finite or infinite time by applying Theorem 1.1. This implies that the standing wave $u(t,x) = e^{i\omega t}Q_\omega(x)$ is unstable.

Let $\{c_n\} \subseteq \mathbb{C}$ be such that $|c_n| > 1$ and $\lim_{n\to\infty}|c_n| = 1$, and $\{\lambda_n\} \subseteq \mathbb{R}^+$ be such that $\lim_{n\to\infty}\lambda_n = 1$. We take the initial data

$$
u_{0,n}(x) := c_n\lambda_n^{N/2}Q_\omega(\lambda_n x).
$$

Then, we have

$$
\begin{gathered}
\lim_{n\to\infty}\|u_{0,n}\|_{L^2} = \lim_{n\to\infty}|c_n|\|Q_\omega\|_{L^2} = \|Q_\omega\|_{L^2}, \\
\lim_{n\to\infty}\|u_{0,n}\|_{\dot H^s} = \lim_{n\to\infty}|c_n|\lambda_n^s\|Q_\omega\|_{\dot H^s} = \|Q_\omega\|_{\dot H^s}.
\end{gathered}
$$

Thus, from Brezis-Lieb's lemma we deduce that $u_{0,n} \to Q_\omega$ in H^s as $n \to \infty$.

On the other hand, from Pohozaev's identities (2.2) we deduce that

$$
\begin{aligned}
E(u_{0,n}) &= \frac{1}{2}\|u_{0,n}\|_{\dot H^s}^2 - \frac{1}{p+2}\int_{\mathbb{R}^N}|x|^{-b}|u_{0,n}(x)|^{p+2}dx \\
&= \frac{|c_n|^2\lambda_n^{2s}}{2}\|Q_\omega\|_{\dot H^s}^2 - \frac{|c_n|^{p+2}\lambda_n^{b+\frac{Np}{2}}}{p+2}\int_{\mathbb{R}^N}|x|^{-b}|Q_\omega(x)|^{p+2}dx \\
&= \frac{(|c_n|^2 - |c_n|^{p+2})\lambda_n^{2s}}{2}\|Q_\omega\|_{\dot H^s}^2 < 0.
\end{aligned}
$$

Applying Theorem 1.1, the solution u_n of (1.2) with initial data $u_{0,n}$ blows up in finite time.

Case (2) $\frac{4s-2b}{N} < p < \frac{4s-2b}{N-2s}$. Let Q_ω be the ground state related to (1.13), a direct computation shows

$$S_\omega(Q_\omega^\lambda) = \frac{1}{2}\lambda^{2s}\|Q_\omega\|_{\dot{H}^s}^2 + \frac{\omega}{2}\|Q_\omega\|_{L^2}^2 - \frac{\lambda^{\frac{Np}{2}+b}}{p+2}\int_{\mathbb{R}^N}|x|^{-b}|Q_\omega(x)|^{p+2}dx,$$

and

$$\begin{aligned}\partial_\lambda S_\omega(Q_\omega^\lambda) &= s\lambda^{2s-1}\|Q_\omega\|_{\dot{H}^s}^2 - \frac{(Np+2b)\lambda^{\frac{Np}{2}+b-1}}{2p+4}\int_{\mathbb{R}^N}|x|^{-b}|Q_\omega(x)|^{p+2}dx \\ &= \frac{K(Q_\omega^\lambda)}{\lambda}.\end{aligned}$$

It is easy to see that the equation $\partial_\lambda S_\omega(Q_\omega^\lambda) = 0$ has a unique non-zero solution,

$$\Big(\frac{s(2p+4)\|Q_\omega\|_{\dot{H}^s}^2}{(Np+2b)\int_{\mathbb{R}^N}|x|^{-b}|Q_\omega(x)|^{p+2}dx}\Big)^{\frac{2}{Np+2b-4s}} = 1.$$

The last inequality comes from the fact that $K(Q_\omega) = 0$, which follows from Pohozaev's identities (2.2). We thus obtain

$$\partial_\lambda S_\omega(Q_\omega^\lambda)\begin{cases} > 0 & \text{if } \lambda \in (0,1), \\ < 0 & \text{if } \lambda \in (1,\infty).\end{cases}$$

This implies that $S_\omega(Q_\omega^\lambda) < S_\omega(Q_\omega)$ for any $\lambda > 0$ and $\lambda \neq 1$. This, together with $\|Q_\omega^\lambda\|_{L^2} = \|Q_\omega\|_{L^2}$, implies that for any $\lambda > 1$,

$$E(Q_\omega^\lambda) < E(Q_\omega). \tag{4.1}$$

Let $\lambda_n > 1$ such that $\lim_{n\to\infty}\lambda_n = 1$. We take the initial data

$$u_{0,n}(x) = Q_\omega^{\lambda_n}(x) = \lambda_n^{N/2}Q_\omega(\lambda_n x).$$

By Brezis-Lieb's lemma, we have $u_{0,n} \to Q_\omega$ in H^s as $n \to \infty$. We deduce from (4.1) that

$$E(u_{0,n}) < E(Q_\omega),$$

and

$$\|(-\Delta)^{s/2}u_{0,n}\|_{L^2} = \lambda_n^s\|(-\Delta)^{s/2}Q_\omega\|_{L^2} > \|(-\Delta)^{s/2}Q_\omega\|_{L^2}.$$

Thus, by $\|u_{0,n}\|_{L^2} = \|Q_\omega\|_{L^2}$, (1.14) and (1.15), we have

$$E(u_{0,n})^{s_c}\|u_{0,n}\|_{L^2}^{2(s-s_c)} < E(Q_\omega)^{s_c}\|Q_\omega\|_{L^2}^{2(s-s_c)} = E(Q)^{s_c}\|Q\|_{L^2}^{2(s-s_c)},$$

and

$$\|(-\Delta)^{s/2}u_{0,n}\|_{L^2}^{s_c}\|u_{0,n}\|_{L^2}^{s-s_c} > \|(-\Delta)^{s/2}Q_\omega\|_{L^2}^{s_c}\|Q_\omega\|_{L^2}^{s-s_c} = \|(-\Delta)^{s/2}Q\|_{L^2}^{s_c}\|Q\|_{L^2}^{s-s_c},$$

where $s_c = \frac{N}{2} - \frac{2s-b}{p}$. Applying Theorem 1.1, the solution u_n of (1.2) and initial data $u_{0,n}$ blows up in finite time. This completes the proof. □

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