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ENTIRE SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS

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Abstract. We consider existence of entire solutions of a semilinear elliptic equation $\Delta u = k(x)f(u)$ for $x \in \mathbb{R}^n$, $n \geq 3$. Conditions of the existence of entire solutions have been obtained by different authors. We prove a certain optimality of these results and new sufficient conditions for the nonexistence of entire solutions.

1. Introduction

In this paper we study the existence of entire solutions of the semilinear elliptic equation

$$\Delta u = k(x)f(u), \quad x \in \mathbb{R}^n, n \geq 3, \tag{1.1}$$

where $k(x)$ is a nonnegative continuous function in $\mathbb{R}^n$, $f(u)$ is a positive continuous function which is defined either in $\mathbb{R}$ or $\mathbb{R}_+$. We denote here $\mathbb{R}_+ = (0, +\infty)$. By an entire solution of equation (1.1) we mean a function $u \in C^2(\mathbb{R}^n)$ which satisfies (1.1) at every point of $\mathbb{R}^n$. The important particular cases of (1.1) are the equations

$$\Delta u = k(x)u^\sigma,\ \sigma > 1, \quad \Delta u = k(x)\exp(2u). \tag{1.2}$$

The existence and the nonexistence of entire solutions for (1.2) have been investigated by many authors (see, for example, [7] – [11] and the references therein). Equations (1.2) arise in physics and geometry, as stated in [3, 9, 10]. Equation (1.1) has also been studied in papers such as [12, 13, 14], where it is shown the existence of entire solutions. It has also been known [4, 12] that for some classes functions $f(u)$ under the condition

$$\int_0^\infty s\overline{k}(s)\,ds < \infty, \tag{1.3}$$

where $\overline{k}(s) = \sup_{|x|=s} k(x)$, equation (1.1) possesses infinitely many entire solutions if $\operatorname{dom} f = \mathbb{R}$ and infinitely many positive entire solutions if $\operatorname{dom} f = \mathbb{R}_+$. We shall use in this paper the following nonexistence statement of entire solutions of (1.1).

Theorem 1.1. *Let $f(u)$ satisfy the following conditions:*

$$f(u) \text{ is convex}, \tag{1.4}$$

$$\int_1^\infty \Big(\int_0^v f(u)\,du\Big)^{-1/2} dv < \infty, \tag{1.5}$$

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and there exists nonnegative non-increasing continuous function $k_\star(r)$ such that

$$k_\star(|x|) \le k(x), \quad \int_0^{+\infty} s\, k_\star(s)\, ds = +\infty, \tag{1.6}$$

$$\limsup_{r\to+\infty} k_\star(r)\, r^2 > 0. \tag{1.7}$$

Then (1.1) has no entire solutions if $\operatorname{dom} f = \mathbb{R}$ *and has no positive entire solutions if* $\operatorname{dom} f = \mathbb{R}_+$.

Theorem 1.1 is a little more general assertion than [13, Corollary 2.1] and can be easily obtained from that paper.

The main purpose of the present paper is to present new sufficient conditions for nonexistence of entire solutions of (1.1), and to show a certain optimality of (1.3) for the existence of entire solutions of (1.1).

The distribution of this paper is as follows. We show an optimality of the condition (1.3) for the existence of entire solutions of (1.1) for some class functions $f(u)$ in Section 2. In Section 3 we construct example of (1.1) with radially symmetric function $k(x)$ which demonstrates that the condition (1.3) is not necessary for the existence of entire solutions. In Section 4, we give new sufficient conditions for the nonexistence of entire solutions of (1.1). In particular it is shown that Theorem 1.1 is valid without assumption (1.7).

2. Optimality of existence condition

The aim of this section is to show a certain optimality of the condition (1.3) for the existence of entire solutions of (1.1). The similar result for ordinary differential equation of second order with $f(u) = u^\lambda$, $\lambda > 1$, has been obtained in [5] and we shall use here some ideas of that paper.

Theorem 2.1. *Let $f(u)$ satisfy* (1.4), (1.5) *and $\varphi(r)$ be any positive continuous function such that $\varphi(r) \to \infty$ as $r \to \infty$. Then there exist radially symmetric positive continuous function $k(x) = \overline{k}(|x|)$ such that*

$$\int_0^\infty \frac{s\overline{k}(s)}{\varphi(s)}\, ds < \infty, \tag{2.1}$$

and the equation (1.1) has no entire solutions if $\operatorname{dom} f = \mathbb{R}$ *and has no positive entire solutions if* $\operatorname{dom} f = \mathbb{R}_+$.

Proof. Without lose of generality we can suppose that $\varphi(r) \ge 1$ for $r \ge 0$. We shall construct positive locally Hölder continuous function $\overline{\varphi}(r)$ such that

$$\begin{gathered} 1 \le \bar\varphi(r) \le \sqrt{\varphi(r)}, \quad \bar\varphi(r) \text{ does not decrease,} \\ \frac{\bar\varphi(r)}{r} \to 0 \text{ as } r \to \infty \text{ and does not increase for } r \ge R_0, \end{gathered} \tag{2.2}$$

where $R_0 > 0$. Let $r_0 = 0$ and $\varphi_0 = \inf_{r\ge r_0} \sqrt{\varphi(r)} \ge 1$. We put $\varphi_2 = \varphi_0 + 1$ and choose r_1 such that $r_1 \ge \max\{r_0 + 1, \exp(\varphi_0)\}$ and $\inf_{r\ge r_1} \sqrt{\varphi(r)} \ge \varphi_2$. Denote $r_2 = r_1 \exp(1)$. We define $\bar\varphi(r)$ on the interval $[r_0, r_2)$ in the following way

$$\bar\varphi(r) = \begin{cases} \varphi_0, & r \in [r_0, r_1)\,, \\ \varphi_0 + \ln(r/r_1), & r \in [r_1, r_2)\,. \end{cases}$$

Then $\bar\varphi(r_1) = \varphi_0$, $\bar\varphi(r_2) = \varphi_0 + 1 = \varphi_2$. It is easy to see that $\bar\varphi(r) \le \ln r$ for $r \in [r_1, r_2)$. For $k = 2, 3, \dots$ we put $\varphi_{2k} = \varphi_{2k-2} + 1$ and r_{2k-1} choose such that $r_{2k-1} \ge \max\{r_{2k-2} + 1, \exp(\varphi_{2k-2})\}$ and $\inf_{r \ge r_{2k-1}} \sqrt{\varphi(r)} \ge \varphi_{2k}$. Now set $r_{2k} = r_{2k-1}\exp(1)$ and

$$\bar\varphi(r) = \begin{cases} \varphi_{2k-2}, & r \in [r_{2k-2}, r_{2k-1})\,, \\ \varphi_{2k-2} + \ln(r/r_{2k-1}), & r \in [r_{2k-1}, r_{2k})\,. \end{cases}$$

It is not difficult to verify that $\bar\varphi(r_{2k-1}) = \varphi_{2k-2}$, $\bar\varphi(r_{2k}) = \varphi_{2k}$ and $\bar\varphi(r) \le \ln r$ for $r \in [r_{2k-1}, r_{2k})$. Constructed function $\bar\varphi(r)$ is locally Hölder continuous for $r \ge 0$ and satisfies (2.2).

We define now a sequence τ_p, $p = 0, 1, \dots$ as follows:

$$\tau_0 = 0, \quad 1 \le \tau_{p+1} - \tau_p \le \tau_{p+2} - \tau_{p+1}, \quad 2\tau_p \le \tau_{p+1}, \quad (p+1)^2 \le \bar\varphi(\tau_p)$$

and introduce for $r \ge R_0$ the function

$$\overline{k}(r) = \frac{\bar\varphi(r)\,\psi(r)}{r},$$

where $\psi(r)$ is positive locally Hölder continuous function such that

$$\psi(r) = \begin{cases} 1/\delta_p, & r \in [\tau_p, \tau_{p+1} - \delta_p/10), \\ a_p\, r + b_p, & r \in [\tau_{p+1} - \delta_p/10, \tau_{p+1}). \end{cases}$$

Here $p = 0, 1, \dots$, $\delta_p = \tau_{p+1} - \tau_p$, and coefficients a_p and b_p we choose to join points $(\tau_{p+1} - \delta_p/10, 1/\delta_p)$ and $(\tau_{p+1}, 1/\delta_{p+1})$. For $0 \le r < R_0$ we can define $\overline{k}(r)$ in any way to get positive non-increasing locally Hölder continuous function.

Let $R_0 \in [\tau_i, \tau_{i+1})$. Using the definitions of $\bar\varphi(r)$, $\overline{k}(r)$ and $\psi(r)$, we verify the validity of (2.1),

$$\begin{aligned} \int_{\tau_{i+1}}^{\infty} \frac{\overline{k}(s)\,s}{\varphi(s)} ds &= \sum_{p=i+1}^{\infty} \int_{\tau_p}^{\tau_{p+1}} \frac{\bar\varphi(s)\,\psi(s)}{\varphi(s)} ds \\ &\le \sum_{p=i+1}^{\infty} \int_{\tau_p}^{\tau_{p+1}} \frac{\psi(s)\,ds}{\sqrt{\varphi(s)}} \\ &\le \sum_{p=i+1}^{\infty} \frac{1}{\bar\varphi(\tau_p)} \int_{\tau_p}^{\tau_{p+1}} \frac{ds}{\tau_{p+1} - \tau_p} \\ &\le \sum_{p=i+1}^{\infty} \frac{1}{(p+1)^2} < \infty. \end{aligned}$$

Now we show that $\overline{k}(r)$ satisfies (1.6) and (1.7). Indeed, we have

$$\begin{aligned} \int_{R_0}^{\infty} s\overline{k}(s)\,ds &\ge \sum_{p=i+1}^{\infty} \int_{\tau_p}^{\tau_{p+1}} \bar\varphi(s)\,\psi(s)\,ds \\ &\ge \sum_{p=i+1}^{\infty} \int_{\tau_p}^{\tau_{p+1} - \delta_p/10} \frac{\bar\varphi(s)}{\tau_{p+1} - \tau_p} ds \\ &\ge \sum_{p=i+1}^{\infty} \frac{9}{10} \bar\varphi(\tau_p) = \frac{9}{10} \sum_{p=i+1}^{\infty} (p+1)^2 = \infty. \end{aligned}$$

Put $r_p = \dfrac{\tau_{p+1}+\tau_p}{2}$. Then for $p \geq i+1$ we get

$$\begin{aligned}\overline{k}(r_p)\, r_p^2 = \bar{\varphi}(r_p)\,\psi(r_p)\, r_p = \frac{\bar{\varphi}(r_p)\, r_p}{\tau_{p+1}-\tau_p} \geq \frac{\bar{\varphi}(r_p)(\tau_{p+1}+\tau_p)}{2\tau_{p+1}} = \\ = \frac{1}{2}\,\bar{\varphi}(r_p)\big(1+\frac{\tau_p}{\tau_{p+1}}\big) \geq \frac{1}{2}\bar{\varphi}(\tau_p) \geq \frac{1}{2}\,(p+1)^2.\end{aligned}$$

According to Theorem 1.1 the equation (1.1) with function $k(x) = \overline{k}(|x|)$ has no entire solutions if $\operatorname{dom} f = \mathbb{R}$ and has no positive entire solutions if $\operatorname{dom} f = \mathbb{R}_+$. □

3. Counterexample to necessity of (1.3)

The condition (1.3) is not necessary for the existence of entire solutions of the equation (1.1). To show this we give an explicit $k(x) = \overline{k}(|x|)$ which satisfies $\int_0^\infty s\overline{k}(s)\,ds = \infty$ and we construct a solution of (1.1) with this $k(x)$. Note that analogous examples of entire solutions for the equations (1.2) have been constructed in [6]. We modify that construction. Constructed solution will also demonstrate in Section 4 an optimality additional to (1.3) condition for the nonexistence of entire solutions of the equation (1.1).

We suppose that $g(r)$ be any positive nondecreasing continuous function such that $g(r) \to \infty$ and $g(r)/r \to 0$ as $r \to \infty$. Let $\{a_p\}_{p=1}^\infty$ and $\{r_p\}_{p=1}^\infty$ are sequences which have the following properties:

$$\begin{gathered} a_1 = 2\alpha, \quad a_{p+1} = a_p + 2f(\bar{a}_p), \quad f(\bar{a}_p) = \max_{\alpha \leq a \leq a_p} f(a), \\ r_1 > 0, \quad 1 - \Big(\frac{r_p}{r_p + 4(n-2)r_p[g(r_p)]^{-1}}\Big)^{n-2} \leq \frac{1}{2}\,\frac{a_p}{f(\bar{a}_p)}, \\ g(r_p) \geq 4(n-2), \quad r_p + 4(n-2)r_p[g(r_p)]^{-1} < r_{p+1}, \end{gathered} \tag{3.1}$$

where α is some positive constant. We put $\overline{r}_p = r_p + 4(n-2)r_p[g(r_p)]^{-1}$ and denote $\overline{k}(r)$ a smooth function which satisfies the following relations:

$$0 \leq \bar{k}(r) \leq \frac{g(r)}{r^2} \quad \text{for } r_p \leq r < \overline{r}_p,\ p = 1,2,\dots, \tag{3.2}$$

$$\bar{k}(r) = 0 \quad \text{for } 0 \leq r < r_1,\ \overline{r}_p \leq r < r_{p+1},\ p = 1,2,\dots, \tag{3.3}$$

$$\frac{1}{n-2}\int_{r_p}^{r_{p+1}} r\,\bar{k}(r)dr = 1, \quad p = 1,2,\dots. \tag{3.4}$$

It is not difficult to show the existence of $\bar{k}(r)$ with properties (3.2) – (3.4). Indeed

$$\begin{aligned}\int_{r_p}^{\bar{r_p}} \frac{g(r)}{r}\,dr \geq g(r_p)\int_{r_p}^{r_p+\frac{4(n-2)r_p}{g(r_p)}} \frac{dr}{r} = g(r_p)\ln\big(1+\frac{4(n-2)}{g(r_p)}\big) \\ \geq g(r_p)\frac{2(n-2)}{g(r_p)} = 2(n-2).\end{aligned}$$

We used here that $g(r)$ is a nondecreasing function, $g(r_p) \geq 4(n-2)$ and the inequality

$$\ln(1+x) \geq \frac{x}{2}, \quad 0 \leq x \leq 1.$$

Note also that we can choose $\int_{r_p}^{r_{p+1}} r\bar{k}(r)\,dr$ any between 0 and its upper bound $\int_{r_p}^{\bar{r}_p} \frac{g(r)}{r}\,dr$. Let $\bar{w}(r)$ be the piecewise continuous function defined as

$$\bar{w}(r) = \begin{cases} \frac{1}{2}a_1 & \text{for } 0 \le r < r_1, \\ a_p & \text{for } r_p \le r < \bar{r}_p,\ p = 1, 2, \dots, \\ \frac{1}{2}a_{p+1} & \text{for } \bar{r}_p \le r < r_{p+1},\ p = 1, 2, \dots. \end{cases} \tag{3.5}$$

We put

$$T\bar{u} = \alpha + \frac{1}{n-2}\int_0^r \big(1 - \big(\frac{s}{r}\big)^{n-2}\big)\, s\,\bar{k}(s)\, f(\bar{u}(s))\,ds. \tag{3.6}$$

Lemma 3.1. *Let $\bar{u}(r)$ satisfy the inequalities $\alpha \le \bar{u}(r) \le \bar{w}(r)$. Then $T\bar{u}(r) \le \bar{w}(r)$.*

Proof. At first we suppose that $0 \le r \le r_1$. Due to (3.3), (3.5) and (3.6)

$$T\bar{u} = \alpha \le \bar{w}(r).$$

Assume now that $r_p \le r < \bar{r}_p$. Using (3.1) – (3.6) we get

$$\begin{aligned} T\bar{u} &= \frac{1}{2}a_1 + \frac{1}{n-2}\sum_{j=1}^{j=p-1}\int_{r_j}^{\bar{r}_j} \big(1 - (\frac{s}{r})^{n-2}\big) s\,\bar{k}(s)\, f(\bar{u}(s))\,ds \\ &\quad + \frac{1}{n-2}\int_{r_p}^{r} \big(1 - (\frac{s}{r})^{n-2}\big) s\,\bar{k}(s)\, f(\bar{u}(s))\,ds \\ &\le \frac{1}{2}a_1 + \sum_{j=1}^{j=p-1} f(\bar{a}_j) + f(\bar{a}_p)\,\frac{1}{n-2}\Big(1 - (\frac{r_p}{\bar{r}_p})^{n-2}\Big)\int_{r_p}^{\bar{r}_p} s\,\bar{k}(s)\,ds \\ &= \frac{1}{2}a_p + f(\bar{a}_p)\Big(1 - (\frac{r_p}{\bar{r}_p})^{n-2}\Big) \\ &\le \frac{1}{2}a_p + f(\bar{a}_p)\frac{1}{2}\frac{a_p}{f(\bar{a}_p)} = a_p = \bar{w}(r). \end{aligned}$$

For $\bar{r}_p \le r < r_{p+1}$ we have

$$\begin{aligned} T\bar{u} &= \frac{1}{2}a_1 + \frac{1}{n-2}\sum_{j=1}^{j=p}\int_{r_j}^{\bar{r}_j} s\,\bar{k}(s)\, f(\bar{u}(s))\,ds \\ &\le \frac{1}{2}a_1 + \sum_{j=1}^{j=p} f(\bar{a}_j) = \frac{1}{2}a_{p+1} = \bar{w}(r). \end{aligned}$$

□

Now we can prove the main result of this section.

Theorem 3.2. *Let $\bar{k}(r)$be a smooth function satisfying* (3.2) – (3.4)*. Then*

$$\int_0^\infty s\,\bar{k}(s)\,ds = \infty,$$

and the equation (1.1) with $k(x) = \bar{k}(|x|)$ has infinitely many positive entire solutions.

Proof. We consider the problem

$$\begin{gathered}\bar u''(r)+\frac{n-1}{r}\bar u'(r)=\bar k(r)\,f(\bar u(r)),\\ \bar u(0)=\alpha,\quad \bar u'(0)=0,\end{gathered}\tag{3.7}$$

or equivalently the integral equation

$$\bar u(r)=\alpha+\frac{1}{n-2}\int_0^r\big(1-(\frac{s}{r})^{n-2}\big)s\,\bar k(s)\,f(\bar u(s)).\tag{3.8}$$

We shall prove that (3.8) has a solution for each $0<\alpha\le 1$, and therefore the equation (1.1) with $k(x)=\bar k(|x|)$ has infinitely many positive solutions.

Let $C[0,\infty)$ denote the locally convex space of all continuous function on $[0,\infty)$ with the topology of uniform convergence on every compact set of $[0,\infty)$. Let U be the set

$$U=\{\bar u(r)\in C[0,\infty),\ \alpha\le\bar u(r)\le\bar w(r)\text{ for }r\ge 0\}\,,$$

where $0<\alpha\le 1$ and $\bar w(r)$ was defined in (3.5). Clearly, U is a closed convex subset of $C[0,\infty)$. Now we consider the mapping T which was defined in (3.6). It is obvious

$$T\bar u(r)\ge\alpha.$$

Due to Lemma 3.1

$$T\bar u\le\bar w(r).$$

Thus T maps U into itself. It is easy to see that U is continuous. To prove that T is also compact, we just compute

$$0\le(T\bar u)'(r)=\int_0^r\Big(\frac{s}{r}\Big)^{n-1}\bar k(s)\,f(\bar u(s))\,ds\equiv M(r),$$

where $M(r)$ is a bonded function on any segment $[0,R],R>0$. Hence we are able to apply the Schauder-Tychonoff fixed point theorem and conclude that T has a fixed point u in U. This fixed point satisfies (3.8), and so we obtain a solution $u(|x|)$ of (1.1). □

4. Nonexistence of entire solutions

The main purpose of this section is to get new sufficient conditions for nonexistence of entire solutions of (1.1). We introduce an auxiliary function

$$I(\beta)=\int_\beta^\infty\Big(\int_\beta^v f(u)\,du\Big)^{-1/2}dv<\infty,\ \beta>0.$$

Let $\bar u(r)$ denote the mean value of $u(x)$ over the sphere $|x|=r$, that is,

$$\bar u(r)=\frac{1}{\omega_n r^{n-1}}\int_{|x|=r}u(x)\,dS,$$

where ω_n is the surface area of the unit sphere in $\mathbb{R}^n$, dS is the volume element in the surface integral.

We shall use two lemmas which have been proved in [13].

Lemma 4.1. *Let $f(u)$ be convex function and there exists nonnegative continuous function $k_\star(r)$ such that $k_\star(|x|) \le k(x)$. If $u(x)$ is a solution of* (1.1) *then $\bar u(r)$ satisfies the following conditions*

$$\begin{gathered} \bar u''(r) + \frac{n-1}{r}\bar u'(r) \ge k_\star(r)\, f(\bar u(r)), \\ \bar u'(0) = 0, \quad \bar u(0) = u(0). \end{gathered} \tag{4.1}$$

Lemma 4.2. *Let $f(u)$ satisfy* (1.4) *and* (1.5)*. Then function $I(\beta)$ does not increase for sufficiently large values of β and $\lim_{\beta\to\infty} I(\beta) = 0$.*

Now we prove an auxiliary statement which has independent interest.

Theorem 4.3. *Let $f(u)$ satisfy* (1.4), (1.5) *and $k_\star(r)$ be nonnegative continuous function possessing the properties* (1.6) *and*

$$(s/r)^\delta \le \int_{R_0}^{s} t\, k_\star(t)\, dt \Big/ \int_{R_0}^{r} t\, k_\star(t)\, dt \tag{4.2}$$

for $r \ge s \ge R_0^* > R_0$, where δ, R_0^* and R_0 are some positive constants. Then the equation (1.1) has no entire solutions if $\operatorname{dom} f = \mathbb{R}$ *and has no positive entire solutions if* $\operatorname{dom} f = \mathbb{R}_+$.

Proof. Let $u(x)$ be any entire solution of (1.1). Then by Lemma 4.1 $\bar u(r)$ satisfies (4.1) which imply the following integral inequality with $\alpha = u(0)$

$$\bar u(r) \ge \alpha + \frac{1}{n-2}\int_0^r \big(1 - (\frac{s}{r})^{n-2}\big) s k_\star(s) f(\bar u(s))\, ds. \tag{4.3}$$

Moreover, $\bar u(r)$ is nondecreasing and $\bar u(r) \to \infty$ as $r \to \infty$. Since $k_\star(r)$ is nonnegative continuous function then sets $A(R,r) \equiv \{s \in (R,r) : k_\star(s) > 0\}$ and $A(R,\infty) \equiv \{s \in (R,\infty) : k_\star(s) > 0\}$ are union of finite or countable number of intervals. By sets $A(R,r) = \bigcup_i (a_i, b_i)$ and $A(R,\infty) = \bigcup_i (\bar a_i, \bar b_i)$ we introduce the auxiliary sets in the following way $A[R,r) = \bigcup_i [a_i, b_i)$ and $A[R,\infty) = \bigcup_i [\bar a_i, \bar b_i)$.

For $r \in A[R_0,\infty)$, we put

$$h(r) = \int_{A[R_0,r)} s k_\star(s)\, ds. \tag{4.4}$$

By virtue of (1.6) and (4.4) h maps in a one-to-one manner $A[R_0,\infty)$ on $[0,\infty)$. Hence there exists inverse for h function g. We denote

$$t = h(r),\ \tau = h(s),\ \bar u(g(t)) = w(t). \tag{4.5}$$

Due to (1.4), (1.5) function $f(u)$ is increasing for sufficiently large values of u. Therefore $f(\bar u(r))$ is nondecreasing for $r > R_1$ for some $R_1 > 0$. We take R_2 such that $R_2 \ge \max\{R_1, R_0^*\}$, $k_\star(R_2) \ne 0$. Then by (4.3) – (4.5) for $t > h(R_2)$ we get

$$\begin{aligned} w(t) &\ge \alpha + \frac{1}{n-2}\int_{A[R_2,g(t))} \Big(1 - \big(\frac{s}{g(t)}\big)^{n-2}\Big) s\, k_\star(s)\, f(\bar u(s))\, ds \\ &= \alpha + \frac{1}{n-2}\int_{h(R_2)}^{t} \Big(1 - \big(\frac{g(\tau)}{g(t)}\big)^{n-2}\Big) f(w(\tau))\, d\tau. \end{aligned} \tag{4.6}$$

It follows from (4.2) that

$$g(\tau)/g(t) \le (\tau/t)^{1/\delta}. \tag{4.7}$$

From (4.6) and (4.7) we deduce
$$w(t) \geq \alpha + \frac{1}{n-2} \int_{h(R_2)}^{t} \big(1 - (\frac{\tau}{t})^{(n-2)/\delta}\big) f(w(\tau))\, d\tau. \tag{4.8}$$
Let $T > h(R_2)$ and $T \leq \tau \leq t \leq 2T$. Using (4.8) and the inequality
$$1 - (\frac{\tau}{t})^{(n-2)/\delta} \geq (n-2)C(\delta)\frac{t-\tau}{\tau},$$
where $C(\delta) = \min\{1/2, 1/2^{(n-2)/\delta}\}/\delta$, we obtain
$$w(t) \geq \beta + C(\delta) \int_T^t \frac{t-\tau}{\tau} f(w(\tau))\, d\tau.$$
Here we denote
$$\beta = \alpha + \frac{1}{n-2} \int_{h(R_2)}^{T} \big(1 - (\frac{\tau}{t})^{(n-2)/\delta}\big) f(w(\tau))\, d\tau.$$
It is obvious $\beta \to \infty$ as $T \to \infty$. Put
$$z(t) = \beta + C(\delta) \int_T^t \frac{t-\tau}{\tau} f(w(\tau))\, d\tau.$$
Then we have
$$z''(t) = C(\delta)\frac{1}{t} f(w(t)) \geq C(\delta)\frac{1}{t} f(z(t)) \tag{4.9}$$
and $z(T) = \beta, z'(T) = 0$. If we multiply (4.9) by $z'(t)$ and then integrate over $[T, t]$, we get
$$(z'(t))^2 \geq 2C(\delta)\frac{1}{t} \int_\beta^{z(t)} f(u)\, du.$$
Elementary calculations shows that
$$\Big(\int_\beta^{z(t)} f(u)\, du\Big)^{-1/2} z'(t) \geq \sqrt{\frac{2C(\delta)}{t}}.$$
Integrating the above inequality over $[T, t]$, we infer
$$I(\beta) \geq \int_\beta^{z(t)} \Big(\int_\beta^v f(u)\, du\Big)^{-1/2} dv \geq 2\sqrt{2C(\delta)}(\sqrt{t} - \sqrt{T}). \tag{4.10}$$
We put now $t = 2T$ in (4.10) and pass to the limit $T \to \infty$. Then left hand side of (4.10) tends to zero due to Lemma 4.2, on the other hand right hand side of (4.10) tends to infinity. Obtained contradiction proves theorem. □

Corollary 4.4. *Let function $f(u)$ satisfy the conditions* (1.4), (1.5) *and $k_\star(r)$ be nonnegative continuous function possessing the properties* (1.6) *and*
$$k_\star(r) \leq \frac{C}{r^2} \quad \textit{for} \quad r \geq R_3 > 0 \tag{4.11}$$
for some values of R_3 and $C > 0$. Then (1.1) *has no entire solutions if* $\operatorname{dom} f = \mathbb{R}$ *and has no positive entire solutions if* $\operatorname{dom} f = \mathbb{R}_+$.

Proof. We show that (4.2) is valid with $\delta = 1$. Really, it is easy to verify that

$$\frac{d}{dr}\Big(\frac{\int_{R_0}^r t\,k_\star(t)\,dt}{r}\Big) = \frac{r^2\,k_\star(r) - \int_{R_0}^r t\,k_\star(t)\,dt}{r^2} \le \frac{C - \int_{R_0}^r t\,k_\star(t)\,dt}{r^2} < 0$$

for sufficiently large values of r. Now by Theorem 4.3 the conclusion of corollary follows. □

Remark 4.5. We constructed in Section 3 the function $k(x) = \overline{k}(|x|)$ such that $\int_0^\infty s\overline{k}(s)\,ds = \infty$, $\overline{k}(r) \le g(r)/r^2$ for $r \ge r_1 > 0$, where $g(r)$ is any positive nondecreasing continuous function with properties: $g(r) \to \infty$ and $g(r)/r \to 0$ as $r \to \infty$, and the equation (1.1) has infinitely many positive entire solutions. Hence the upper bound in (4.11) is optimal.

Remark 4.6. For the equations (1.2) similar to Theorem 4.3 and Corollary 4.4 statements have been proved in [2] under the additional assumption

$$\int_0^r sk_\star(s)\,ds \text{ is strictly increasing in } [0,\infty).$$

Using Corollary 4.4 and Theorem 1.1 it is not difficult to establish the following assertion.

Corollary 4.7. *Let function $f(u)$ satisfy the conditions* (1.4), (1.5) *and $k_\star(r)$ be nonnegative continuous non-increasing for large values of r function satisfying* (1.6). *Then the equation* (1.1) *has no entire solutions if* $\operatorname{dom} f = \mathbb{R}$ *and has no positive entire solutions if* $\operatorname{dom} f = \mathbb{R}_+$.

Remark 4.8. Corollary 4.7 gives new nonexistence criterion for (1.1) and this statement is more general than any one in [13]. In particular Theorem 1.1 is true without assumption (1.7).

Remark 4.9. All results of this section are valid for more general equation

$$\Delta u = p(x,u)$$

where $p(x,u)$ is nonnegative continuous function satisfying the inequality

$$p(x,u) \ge k(x)f(u).$$

Here the functions $k(x)$ and $f(u)$ possess the same properties as in our statements. In particular the equation (1.1) with function $f(u)$ satisfying the conditions (1.4), (1.5) and function $k(x)$ satisfying the inequality

$$k(x) \ge \{c|x|^2(\ln|x|)(\ln\ln|x|)\dots(\ln\dots\ln|x|)\}^{-1},$$

where $c > 0$ and $|x| \ge r_\star > 0$, has no entire solutions if $\operatorname{dom} f = \mathbb{R}$ and has no positive entire solutions if $\operatorname{dom} f = \mathbb{R}_+$.

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