Electronic Journal of Differential Equations, Vol. 2019 (2019), No. 62, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

ASYMPTOTIC FORMULAS FOR OSCILLATORY BIFURCATION DIAGRAMS OF SEMILINEAR ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. We study the nonlinear eigenvalue problem

$$-u''(t) = \lambda \left(u(t)^p + g(u(t))\right), \quad u(t) > 0, \quad t \in (-1,1),\ u(\pm 1) = 0,$$

where $g(u) = h(u)\sin(u^r)$, p, r are given constants satisfying $p \geq 0$, $0 < r \leq 1$ and $\lambda > 0$ is a parameter. It is known that under suitable conditions on h, λ is parameterized by the maximum norm $\alpha = \|u_\alpha\|_\infty$ of the solution u_λ associated with λ and $\lambda = \lambda(\alpha)$ is a continuous function for $\alpha > 0$. When $p = 1$, $h(u) \equiv 1$ and $r = 1/2$, this equation has been introduced by Chen [4] as a model equation such that there exist infinitely many solutions near $\lambda = \pi^2/4$. We prove that $\lambda(\alpha)$ is an oscillatory bifurcation curve as $\alpha \to \infty$ by showing the asymptotic formula for $\lambda(\alpha)$. It is found that the shapes of bifurcation curves depend on the condition $p > 1$ or $p < 1$. The main tools of the proof are time-map argument and stationary phase method.

1. INTRODUCTION

We consider the nonlinear eigenvalue problems

$$-u''(t) = \lambda\big(u(t)^p + g(u(t))\big), \quad t \in I := (-1,1), \tag{1.1}$$

$$u(t) > 0, \quad t \in I, \tag{1.2}$$

$$u(-1) = u(1) = 0, \tag{1.3}$$

where $g(u) = h(u)\sin(u^r)$ with $h \in C^1[0,\infty)$, p, r are given constants satisfying $p \geq 0$, $0 < r \leq 1$ and $\lambda > 0$ is a parameter. We assume the following conditions :

(A1) $u^p + g(u) > 0$ for $u > 0$.

(A2) Let $0 \leq q < p$, and C_1, C_2 be positive constants. Then for $u \geq 0$,

$$|h(u)| \leq C_1(u+1)^q, \quad |h'(u)| \leq C_2(u+1)^{q-1}. \tag{1.4}$$

(A3) $h(u) \gg u^{q-(r/2)}$ for $u \gg 1$.

A typical example of (1.1) is $p = 2$, $h(u) = \log(1+u^2)$ $(q = 1/4)$ and $r = 1/2$. By (A1), we know from [11] that there exists a unique classical solution pair (λ, u_α) of (1.1)–(1.3) satisfying $\alpha = \|u_\alpha\|_\infty$ for any given $\alpha > 0$. Moreover, λ is a continuous curve $\lambda = \lambda(\alpha)$ for $\alpha > 0$.

The study of global and local structures of bifurcation diagrams have a long history. Many topics considered there have a background in mathematical biology,

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2010 *Mathematics Subject Classification.* 34C23, 34F10.
Key words and phrases. Oscillatory bifurcation; time-map argument; stationary phase method.

Submitted October 31, 2018. Published May 7, 2019.

engineering, and have been investigated intensively by many authors. We refer to [2, 3, 5, 6] and the references therein. In particular, oscillatory phenomena of bifurcation curves seem to be the significant problems to study, since the oscillatory properties of the nonlinear term g give us a lot of interesting shapes of bifurcation curves. We refer to [7, 8, 9, 10, 12, 13, 14, 15] and the references therein.

The equation (1.1)–(1.3) with $p = 1$ and $g(u) = \sin\sqrt{u}$ has been studied in Cheng [4]. This paper was motivated by [1], since this model equation was expected to give an oscillatory bifurcation curve. Precisely, the following result has been proved in [4].

Theorem 1.1 ([4, Theorem 6]). *Let $g(u) = \sin\sqrt{u}$ $(u \ge 0)$. Then for any integer $r \ge 1$, there is $\delta > 0$ such that if $\lambda \in (\pi^2/4 - \delta, \pi^2/4 + \delta)$, then (1.1)–(1.3) has at least r distinct solutions.*

By Theorem 1.1, we expect that $\lambda(\alpha)$ oscillates and intersects the line $\lambda = \pi^2/4$ infinitely many times as $\alpha \to \infty$. Recently, it was shown in [15] that this expectation is valid.

Theorem 1.2 ([15]). *Let $g(u) = \sin\sqrt{u}$. Then as $\alpha \to \infty$,*

$$\lambda(\alpha) = \frac{\pi^2}{4} - \pi^{3/2}\alpha^{-5/4}\sin(\sqrt{\alpha} - \frac{\pi}{4}) + o(\alpha^{-5/4}). \tag{1.5}$$

A rough sketch of $\lambda(\alpha)$ for (1.5) is shown in Figure 1, obtained in [15].

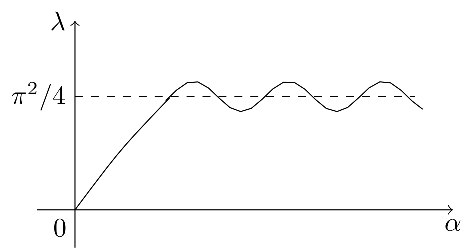

FIGURE 1. Bifurcation curve when $p = 1$, $h \equiv 1$, $r = 1/2$

Motivated by [4, 15], the following asymptotic formulas for $\lambda(\alpha)$ with $p = 1$ and $g(u) = u^q \sin(u^r)$ as $\alpha \to \infty$ when $0 \le q < 1$ and $0 < r \le 1$ were obtained in [16].

Theorem 1.3 ([16]). *Let $p = 1$ and $g(u) = u^q \sin(u^r)$ in (1.1)–(1.3), where $0 \le q < 1$ and $0 < r \le 1$ are fixed constants. Then as $\alpha \to \infty$,*

$$\lambda(\alpha) = \frac{\pi^2}{4} - \frac{\pi^{3/2}}{\sqrt{2r}}\alpha^{q-1-(r/2)}\sin\Big(\alpha^r - \frac{\pi}{4}\Big) + o(\alpha^{q-1-(r/2)}). \tag{1.6}$$

Clearly, if $q = 0$ and $r = 1/2$, then Theorem 1.3 coincides with Theorem 1.2. The proof of Theorem 1.3 was based on the time-map formula and stationary phase method. Unfortunately, however, it seems that only the special cases have been considered in [16]. Moreover, since $h(u)$ is not homogeneous function here, the estimate of the remainder term of $\lambda(\alpha)$ is difficult when we apply the stationary phase method to our case. This phenomenon does not appear for the case $h(u) = u^q$. The purpose of this paper is to overcome this difficulty and generalize the results obtained in [16]. Now we state our main results.

Theorem 1.4. *Assume that* (A1)–(A3) *are satisfied. Then as* $\alpha \to \infty$,

$$\lambda(\alpha) = \frac{p+1}{2}\alpha^{-(p-1)}\Big\{A_p^2 - 2\pi\sqrt{\frac{\pi}{r(p+1)}}A_p h(\alpha)\alpha^{-p-(r/2)}\sin(\alpha^r - \frac{\pi}{4}) + o(h(\alpha)\alpha^{-p-(r/2)})\Big\}, \tag{1.7}$$

where

$$A_p := \int_0^1 \frac{1}{\sqrt{1-s^{p+1}}}ds. \tag{1.8}$$

We see easily that if $p = 1$, $h(u) = u^q$, then Theorem 1.4 coincides with Theorem 1.3. In the proof of Theorem 1.4 the main tools are time-map method and the stationary phase method. The most important part is that, since $f(x)$ in (2.17) and (2.18) in the next section depends on α, we have to be careful about the decay rate of the remainder terms in (2.10). To overcome this difficulty, in the proof, we go back to the starting point of the argument of stationary phase method.

We next establish the asymptotic formulas for $\lambda(\alpha)$ as $\alpha \to 0$ to complete the total shape of $\lambda(\alpha)$. We mention that the conditions (A2) and (A3), which control the growth rate of $h(u)$ as $\alpha \to \infty$ are not necessary.

Theorem 1.5. *Assume that* (A1) *is satisfied and* $h(0) > 0$.

(i) *if* $p < r$, *then as* $\alpha \to 0$,

$$\lambda(\alpha) = \frac{p+1}{2}\alpha^{1-p}\Big\{A_p^2 - \frac{p+1}{r+1}A_p B_{p,r} h(0)\alpha^{r-p} + o(\alpha^{r-p})\Big\}, \tag{1.9}$$

where

$$B_{p,r} := \int_0^1 \frac{1-s^{r+1}}{(1-s^{p+1})^{3/2}}ds. \tag{1.10}$$

(ii) *If* $r < p$, *then as* $\alpha \to 0$,

$$\lambda(\alpha) = \frac{r+1}{2h(0)}\alpha^{1-r}\Big\{A_p^2 - \frac{r+1}{(p+1)h(0)}A_r B_{r,p}\alpha^{p-r} + o(\alpha^{p-r})\Big\}. \tag{1.11}$$

(iii) *If* $1/2 < p = r$ *and* $h'(0) \neq 0$, *then as* $\alpha \to 0$,

$$\lambda(\alpha) = \frac{p+1}{2(1+h(0))}\alpha^{1-p}\Big\{A_p^2 - \frac{(p+1)h'(0)}{(p+2)(1+h(0))}A_p B_{p,p+1}\alpha + o(\alpha)\Big\}. \tag{1.12}$$

We note that in Theorem 1.5 (iii), the case $p = r < 1/2$ can be treated similarly. The proof of Theorem 1.5 depends on the direct calculation by using time-map method and Taylor expansion formula.

Figure 2 shows a rough sketch of $\lambda(\alpha)$ obtained in Theorems 1.4 and 1.5 with $0 < r < 1 < p$ and $0 < r < p < 1$.

2. Proof of Theorem 1.4

In this section, we let $\alpha \gg 1$. Also we denote by C the various positive constants independent of α. For $u \ge 0$, we put

$$G(u) := \int_0^u g(\theta)d\theta = \int_0^u h(\theta)\sin(\theta^r)d\theta. \tag{2.1}$$

It is known that if $(u_\alpha, \lambda(\alpha)) \in C^2(\bar{I}) \times \mathbb{R}_+$ satisfies (1.1)–(1.3), then

$$u_\alpha(t) = u_\alpha(-t), \quad 0 \le t \le 1, \tag{2.2}$$

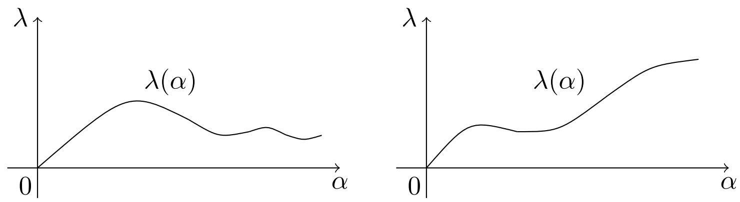

Figure 2. Bifurcation curve when $0 < r < 1 < p$ (left), and when $0 < r < p < 1$ (right)

$$u_\alpha(0) = \max_{-1\le t\le 1} u_\alpha(t) = \alpha, \tag{2.3}$$

$$u_\alpha'(t) > 0, \quad -1 < t < 0. \tag{2.4}$$

By (1.1), we have

$$(u_\alpha''(t) + \lambda\,(u_\alpha(t)^p + g(u_\alpha(t))))\,u_\alpha'(t) = 0.$$

By this and (2.3), putting $t = 0$ we obtain

$$\begin{aligned} &\frac{1}{2}u_\alpha'(t)^2 + \lambda\Big(\frac{1}{p+1}u_\alpha(t)^{p+1} + G(u_\alpha(t))\Big) \\ &= \text{const.} = \lambda\Big(\frac{1}{p+1}\alpha^{p+1} + G(\alpha)\Big). \end{aligned} \tag{2.5}$$

This along with (2.4) implies that for $-1 \le t \le 0$,

$$u_\alpha'(t) = \sqrt{\frac{2\lambda}{p+1}}\sqrt{\alpha^{p+1} - u_\alpha(t)^{p+1} + (p+1)\{(G(\alpha) - G(u_\alpha(t))\}}. \tag{2.6}$$

For $0 \le s \le 1$, by (A2), we have

$$\begin{aligned} |\frac{G(\alpha) - G(\alpha s)}{\alpha^{p+1}(1-s^{p+1})}| &= |\frac{\int_{\alpha s}^{\alpha} g(t)dt}{\alpha^{p+1}(1-s^{p+1})}| \\ &\le C\frac{\alpha^{q+1}(1-s^{q+1}) + \alpha(1-s)}{\alpha^{p+1}(1-s^{p+1})} \\ &\le C(\alpha^{q-p} + \alpha^{-p}) \ll 1. \end{aligned} \tag{2.7}$$

We put $M(s,\alpha) := G(\alpha) - G(\alpha s)$ for $0 \le s \le 1$. By (2.6), (2.7), putting $s := u_\alpha(t)/\alpha$ and using the Taylor expansion, we obtain

$$
\begin{aligned}
&\sqrt{\frac{2\lambda}{p+1}} \\
&= \int_{-1}^{0} \frac{u_\alpha'(t)}{\sqrt{\alpha^{p+1} - u_\alpha(t)^{p+1} + (p+1)(G(\alpha) - G(u_\alpha(t)))}} dt \\
&= \alpha^{-(p-1)/2} \int_0^1 \frac{1}{\sqrt{1 - s^{p+1} + (p+1)M(\alpha,s)/\alpha^{p+1}}} ds \\
&= \alpha^{-(p-1)/2} \int_0^1 \frac{1}{\sqrt{1-s^{p+1}}} \frac{1}{\sqrt{1 + (p+1)M(\alpha,s)/(\alpha^{p+1}(1-s^{p+1}))}} ds \\
&= \alpha^{-(p-1)/2} \int_0^1 \frac{1}{\sqrt{1-s^{p+1}}} \Big\{1 - \frac{(p+1)M(s,\alpha)}{2\alpha^{p+1}(1-s^{p+1})}(1+o(1))\Big\} ds \\
&= \alpha^{-(p-1)/2} \int_0^1 \frac{1}{\sqrt{1-s^{p+1}}} ds - \frac{p+1}{2}\alpha^{-(3p+1)/2} \int_0^1 \frac{M(\alpha,s)}{(1-s^{p+1})^{3/2}} ds \\
&\quad + o(\alpha^{-(3p+1)/2}).
\end{aligned} \tag{2.8}
$$

We put

$$
L(\alpha) := \int_0^1 \frac{M(\alpha,s)}{(1-s^{p+1})^{3/2}} ds. \tag{2.9}
$$

It is evident from (2.8) and (2.9) that the precise asymptotic formula for $L(\alpha)$ as $\alpha \to \infty$ gives us Theorem 1.4. To study the asymptotic properties of $L(\alpha)$, we apply the idea of the stationary phase method to (2.9).

Lemma 2.1. *As $\alpha \to \infty$,*

$$
L(\alpha) = \frac{2}{p+1}\sqrt{\frac{\pi}{r(p+1)}} h(\alpha)\alpha^{(2-r)/2} \sin\big(\alpha^r - \frac{\pi}{4}\big) + o(h(\alpha)\alpha^{(2-r)/2}). \tag{2.10}
$$

Proof. The proof is divided into several steps.

Step 1. We put $s = \sin^{2/(p+1)} x$ in (2.9). Then we have

$$
\begin{aligned}
L(\alpha) &= \frac{2}{p+1} \int_0^{\pi/2} \frac{1}{\cos^2 x} \sin^{(1-p)/(p+1)} x (G(\alpha) - G(\alpha \sin^{2/(p+1)} x)) dx \\
&= \frac{2}{p+1} \int_0^{\pi/2} (\tan x)' \{\sin^{(1-p)/(p+1)} x (G(\alpha) - G(\alpha \sin^{2/(p+1)} x))\} dx \\
&= \frac{2}{p+1} \Big[\tan x \{\sin^{(1-p)/(p+1)} x (G(\alpha) - G(\alpha \sin^{2/(p+1)} x))\}\Big]_0^{\pi/2} \\
&\quad - \frac{2}{p+1} \int_0^{\pi/2} \tan x \Big\{\frac{1-p}{p+1} \sin^{-2p/(p+1)} x \cos x (G(\alpha) - G(\alpha \sin^{2/(p+1)} x)) \\
&\quad - \frac{2}{p+1} \alpha \sin^{2(1-p)/(p+1)} x \cos x g(\alpha \sin^{2/(p+1)} x)\Big\} dx \\
&= \frac{2}{p+1} \big[\tan x \{\sin^{(1-p)/(p+1)} x (G(\alpha) - G(\alpha \sin^{2/(p+1)} x))\}\big]_0^{\pi/2} \\
&\quad + \frac{2(p-1)}{(p+1)^2} \int_0^{\pi/2} \sin^{(1-p)/(p+1)} x (G(\alpha) - G(\alpha \sin^{2/(p+1)} x)) dx
\end{aligned}
$$

$$\begin{aligned}
&+\frac{4}{(p+1)^2}\alpha\int_0^{\pi/2}\sin^{(3-p)/(p+1)}xg(\alpha\sin^{2/(p+1)}x)dx\\
&=: I+\frac{2(p-1)}{(p+1)^2}II+\frac{4\alpha}{(p+1)^2}III.
\end{aligned}\tag{2.11}$$

Step 2. By l'Hôpital's rule, we obtain

$$\begin{aligned}
&\lim_{x\to\pi/2}\frac{G(\alpha)-G(\alpha\sin^{2/(p+1)}x))}{\cos x}\\
&=\lim_{x\to\pi/2}\frac{2}{p+1}\alpha\sin^{-2p/(p+1)}x\cos xg(\alpha\sin^{2/(p+1)}x)=0.
\end{aligned}\tag{2.12}$$

From this, we obtain $I=0$. Now we calculate II. We have

$$II=\int_0^{\pi/2}\sin^{(1-p)/(p+1)}x\Big(\int_{\alpha\sin^{2/(p+1)}x}^{\alpha}h(\theta)\sin(\theta^r)d\theta\Big)dx. \tag{2.13}$$

By putting $\theta=y^{1/r}$ and integration by parts, we obtain

$$\begin{aligned}
&\int_{\alpha\sin^{2/(p+1)}x}^{\alpha}h(\theta)\sin(\theta^r)d\theta\\
&=\frac{1}{r}\int_{\alpha^r\sin^{2r/(p+1)}x}^{\alpha^r}h(y^{1/r})y^{(1-r)/r}\sin y\,dy\\
&=\frac{1}{r}\big[-h(y^{1/r})y^{(1-r)/r}\cos y\big]_{\alpha^r\sin^{(2r)/(p+1)}x}^{\alpha^r}\\
&\quad+\frac{1}{r}\int_{\alpha^r\sin^{2r/(p+1)}x}^{\alpha^r}(h(y^{1/r})y^{(1-r)/r})'\cos y\,dy\\
&=: II_1+II_2.
\end{aligned}\tag{2.14}$$

By (A2) and direct calculation, it is easy to see that

$$|II_1|\le C\alpha^{q+1-r},\quad |II_2|\le C\alpha^{q+1-2r}. \tag{2.15}$$

Step 3. Now by putting $\sin x=\sin^{(p+1)/2}\theta$ and $\theta=\pi/2-y$, we obtain

$$\begin{aligned}
III&=\frac{p+1}{2}\int_0^{\pi/2}\sin\theta\frac{\cos\theta}{\sqrt{1-\sin^{p+1}\theta}}h(\alpha\sin\theta)\sin(\alpha^r\sin^r\theta)d\theta\\
&=\frac{p+1}{2}\int_0^{\pi/2}\cos y\sqrt{\frac{1-\cos^2 y}{1-\cos^{p+1}y}}h(\alpha\cos y)\sin(\alpha^r\cos^r y)dy.
\end{aligned}\tag{2.16}$$

Let $m:=1/r\ge 1$. We put $\cos y=\cos^m v$ and $v=\pi x/2$. Then by (2.16) and direct calculation, we obtain

$$III=\frac{m\pi(p+1)}{4}\int_0^1 f(x)\sin\Big(\alpha^r\cos\Big(\frac{\pi}{2}x\Big)\Big)dx, \tag{2.17}$$

where

$$f(x):=\cos^{2m-1}\big(\frac{\pi}{2}x\big)V(x)h\Big(\alpha\cos^m\big(\frac{\pi}{2}x\big)\Big), \tag{2.18}$$

$$V(x):=\sqrt{\frac{1-\cos^2(\frac{\pi}{2}x)}{1-\cos^{m(p+1)}(\frac{\pi}{2}x)}}. \tag{2.19}$$

Next we use the argument [10, Lemmas 2.24 and 2.25]. We put

$$\mu := \alpha^r, \quad w(x) = \cos\left(\frac{\pi}{2}x\right), \quad x = \phi(t) := \frac{2}{\pi}\cos^{-1}(1-t^2). \tag{2.20}$$

Then we have

$$f(\phi(t)) = (1-t^2)^{2m-1}\sqrt{\frac{1-(1-t^2)^2}{1-(1-t^2)^{m(p+1)}}}h(\alpha(1-t^2)^m), \tag{2.21}$$

$$\phi'(t) = \frac{4}{\pi\sqrt{2-t^2}}, \quad \phi(0)=0, \quad \phi(1)=1. \tag{2.22}$$

Let

$$\tilde{f}(t) := f(\phi(t))\phi'(t), \quad m(t) := \frac{\tilde{f}(t)-\tilde{f}(0)}{t}, \tag{2.23}$$

$$III_1 := \int_0^1 f(x)e^{i\mu w(x)}dx. \tag{2.24}$$

Since $w(x) = 1-t^2$, by change of valuable $x=\phi(t)$, and (2.21)–(2.24), we have

$$\begin{aligned} III_1 &= e^{i\mu}\int_0^1 f(\phi(t))\phi'(t)e^{-i\mu t^2}dt \\ &= e^{i\mu}f(0)\phi'(0)\int_0^1 e^{-i\mu t^2}dt + e^{i\mu}\int_0^1 m(t)te^{-i\mu t^2}dt \\ &= e^{i\mu}f(0)\phi'(0)\Big(\frac{1}{2}\sqrt{\frac{\pi}{\mu}}e^{-i\pi/4} + O(\mu^{-1})\Big) + e^{i\mu}\int_0^1 m(t)te^{-i\mu t^2}dt \\ &=: \frac{1}{2}f(0)\phi'(0)\Big(\sqrt{\frac{\pi}{\mu}}e^{i(\mu-\pi/4)} + O(\mu^{-1})\Big) + e^{i\mu}M_1. \end{aligned} \tag{2.25}$$

Then by direct calculations, we see that $m(t) \in C^1[0,\epsilon]$, and consequently, $m(t) \in C^1[0,1]$. Since $m(t)$ in (2.23) is defined by using f, and f contains α, $m(t)$ also depends on α. So we have to calculate how M_1 depends on $\alpha \gg 1$ precisely. Since $m(0)=0$ and $m(1) = -4h(\alpha)/(\pi\sqrt{m(p+1)})$, by integration by parts and (A2), we have

$$\begin{aligned} M_1 :&= \int_0^1 \frac{-1}{2i\mu}(e^{-i\mu t^2})'m(t)\,dt \\ &= \Big[\frac{-1}{2i\mu}e^{-i\mu t^2}m(t)\Big]_0^1 + \int_0^1 \frac{1}{2i\mu}e^{-i\mu t^2}m'(t)dt \\ &:= O(h(\alpha)\mu^{-1}) + M_2 = O(\alpha^{q-r}) + M_2. \end{aligned} \tag{2.26}$$

We estimate M_2. By (A2), (2.22), (2.23) and direct calculations, for $0 \le t \le 1$. We obtain

$$|m'(t)| \le C\max_{0\le t\le 1}\frac{|\tilde{f}'(t)|}{t} \le C(\max_{0\le t_1\le 1}|h(t_1\alpha)| + \alpha\max_{0\le t_2\le 1}|h'(t_2\alpha)|) \le C\alpha^q. \tag{2.27}$$

We accept (2.27) tentatively, since the argument is long and elementary. For completeness, the proof will be given in the appendix. Since $\mu = \alpha^r$, by this and (2.26), we obtain $M_2 = O(\alpha^{q-r})$. By this, (2.26) and (A3),

$$|M_1| = O(\alpha^{q-r}) \ll h(\alpha)\alpha^{-r/2}. \tag{2.28}$$

Step 4. We have

$$V(0) = \lim_{x\to 0} \sqrt{\frac{1-\cos^2\left(\frac{\pi}{2}x\right)}{1-\cos^{m(p+1)}\left(\frac{\pi}{2}x\right)}} = \sqrt{\frac{2}{m(p+1)}}, \tag{2.29}$$

$$\phi'(0) = \frac{2\sqrt{2}}{\pi}, \quad f(0) = \sqrt{\frac{2}{m(p+1)}}h(\alpha). \tag{2.30}$$

By (2.17), (2.25), (2.26), (2.28)–(2.30), we obtain that as $\alpha \to \infty$,

$$\begin{aligned} III &= \frac{m\pi(p+1)}{4}\mathrm{Im}(III_1) \\ &= \frac{1}{2}\sqrt{m(p+1)\pi}h(\alpha)\alpha^{-r/2}\sin\left(\alpha^r - \frac{\pi}{4}\right) + o(h(\alpha)\alpha^{-r/2}). \end{aligned} \tag{2.31}$$

From this, (2.11) and (2.15), we obtain (2.10). Thus the proof is complete. □

Now Theorem 1.4 follows from (2.8) and Lemma 2.1 immediately.

3. Proof of Theorem 1.5

In this section, we let $0 < \alpha \ll 1$. For $0 < x < \alpha$, we have

$$h(x) = h(0) + h'(0)(1+o(1))x. \tag{3.1}$$

The proof of Theorem 1.5 (ii) is almost the same as that of Theorem 1.5 (i). So we only prove Theorem 1.5 (i) and (iii).

Proof of Theorem 1.5. (i) Let $g(u) = h(u)\sin(u^r)$ and $p < r$. By (3.1) and Taylor expansion, for $0 \le s \le 1$, we have

$$\begin{aligned} M(\alpha, s) &= \int_{\alpha s}^{\alpha} (h(0) + h'(0)(1+o(1))x)\Big(x^r - \frac{1}{6}x^{3r} + O(x^{5r})\Big)dx \\ &= \frac{1}{r+1}h(0)(1+o(1))\alpha^{r+1}(1-s^{r+1}). \end{aligned} \tag{3.2}$$

From this and (2.8), we have

$$\begin{aligned} &\sqrt{\frac{2\lambda}{p+1}} \\ &= \alpha^{-(p-1)/2}\Big[\int_0^1 \frac{1}{\sqrt{1-s^{p+1}}}\Big\{1 - \frac{p+1}{2}\frac{M(\alpha,s)}{\alpha^{p+1}(1-s^{p+1})}\Big\}ds + o(\alpha^{r-p})\Big] \\ &= \alpha^{-(p-1)/2}\{A_p - \frac{p+1}{2}\alpha^{-(p+1)}L_1(\alpha) + o(\alpha^{r-p})\}, \end{aligned} \tag{3.3}$$

where

$$L_1(\alpha) = \frac{1}{r+1}h(0)\alpha^{r+1}\int_0^1 \frac{1-s^{r+1}}{(1-s^{p+1})^{3/2}}ds = \frac{1}{r+1}B_{p,r}h(0)\alpha^{r+1}. \tag{3.4}$$

By this and (3.3), we obtain

$$\lambda = \frac{p+1}{2}\alpha^{-(p-1)}\Big\{A_p^2 - \frac{p+1}{r+1}A_pB_{p,r}h(0)\alpha^{r-p} + o(\alpha^{r-p})\Big\}. \tag{3.5}$$

Thus we obtain (1.9).

(iii) Let $p = r$. Since $h'(0) \neq 0$, by (3.2), we have

$$\begin{aligned} &1 - s^{p+1} + \frac{p+1}{\alpha^{p+1}} M(\alpha, s) \\ &= (1 + h(0))(1 - s^{p+1}) + \frac{p+1}{p+2} h'(0)\alpha(1 - s^{p+2}) + o(\alpha)(1 - s). \end{aligned} \tag{3.6}$$

By this and Taylor expansion, we obtain

$$\begin{aligned} &\sqrt{\frac{2\lambda}{p+1}} \\ &= \alpha^{-(p-1)/2} \int_0^1 \frac{1}{\sqrt{(1 + h(0))(1 - s^{p+1}) + \frac{(p+1)h'(0)}{p+2}(1 - s^{p+2})\alpha + o(\alpha)(1 - s)}} ds \\ &= \frac{\alpha^{-(p-1)/2}}{\sqrt{1 + h(0)}} \int_0^1 \frac{1}{\sqrt{1 - s^{p+1}}\sqrt{1 + \frac{(p+1)h'(0)}{(p+2)(1+h(0))} \frac{1-s^{p+2}}{1-s^{p+1}}\alpha + o(\alpha)}} ds \\ &= \frac{\alpha^{-(p-1)/2}}{\sqrt{1 + h(0)}} \int_0^1 \frac{1}{\sqrt{1 - s^{p+1}}} \Big\{ 1 - \frac{(p+1)h'(0)}{2(p+2)(1 + h(0))} \frac{1 - s^{p+2}}{1 - s^{p+1}} \alpha + o(\alpha) \Big\} ds\,. \end{aligned}$$

From this we obtain (1.12). Thus the proof is complete. $\square$

4. Appendix

In this section, we prove (2.27). For $0 \le t \le 1$. We put

$$W(t) := \sqrt{X(t)}, \quad X(t) := \frac{1 - (1 - t^2)^2}{1 - (1 - t^2)^{m(p+1)}}\,. \tag{4.1}$$

Then by (2.21)–(2.23), we have

$$\tilde{f}(t) = (1 - t^2)^{2m-1} W(t) h(\alpha(1 - t^2)^m) \frac{4}{\pi\sqrt{2 - t^2}}. \tag{4.2}$$

Then

$$\begin{aligned} \tilde{f}'(t) = &-2t(2m - 1)(1 - t^2)^{2m-2} W(t) h(\alpha(1 - t^2)^m) \frac{4}{\pi\sqrt{2 - t^2}} \\ &+ (1 - t^2)^{2m-1} W'(t) h(\alpha(1 - t^2)^m) \frac{4}{\pi\sqrt{2 - t^2}} \\ &- 2mt\alpha(1 - t^2)^{3m-2} W(t) h'(\alpha(1 - t^2)^m) \frac{4}{\pi\sqrt{2 - t^2}} \\ &+ \frac{4}{\pi} t(1 - t^2)^{2m-1} W(t) h(\alpha(1 - t^2)^m)(2 - t^2)^{-3/2}. \end{aligned} \tag{4.3}$$

We have

$$\begin{aligned} &X'(t) \\ &= \Big(\frac{2t^2 - t^4}{1 - (1 - t^2)^{m(p+1)}}\Big)' \\ &= \frac{4(t - t^3)\{1 - (1 - t^2)^{m(p+1)}\} - m(p + 1)(4t^3 - 2t^5)(1 - t^2)^{m(p+1)-1}}{\{1 - (1 - t^2)^{m(p+1)}\}^2}. \end{aligned} \tag{4.4}$$

By Taylor expansion, for $0 \le t \ll 1$, we have

$$X'(t) = \frac{2(mp + m - 2)}{m(p + 1)} t + O(t^3). \tag{4.5}$$

By this, (4.1) and (4.3), for $0 < t \ll 1$, we have

$$\begin{gathered} W'(t) = \frac{1}{2} X(t)^{-1/2} X'(t), \quad X(0) = \frac{2}{m(p+1)}, \\ X'(0) = 0, \quad W'(0) = 0, \quad \tilde{f}'(0) = 0. \end{gathered} \tag{4.6}$$

By this, the mean value theorem and (4.3), for $0 \le t \le 1$, we obtain

$$m'(t) = \frac{\tilde{f}'(t)}{t} - \frac{\tilde{f}(t) - \tilde{f}(0)}{t^2} = \frac{\tilde{f}'(t)}{t} - \eta_t \frac{\tilde{f}'(\eta_t t)}{\eta_t t}, \tag{4.7}$$

where $0 < \eta_t < 1$. By this, (A2), (4.3) and (4.5), we obtain

$$\begin{aligned} |m'(t)| &\le 2 \max_{0 \le s \le t} |\frac{\tilde{f}'(s)}{s}| \le 2 \max_{0 \le t \le 1} |\frac{\tilde{f}'(t)}{t}| \\ &\le C(\max_{0 \le t_1 \le 1} |h(t_1 \alpha)| + \alpha \max_{0 \le t_2 \le 1} |h'(t_2 \alpha)|) \le C\alpha^q. \end{aligned} \tag{4.8}$$

Thus the proof is complete.

Acknowledgments. This work was supported by JSPS KAKENHI Grant Number JP17K05330.

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