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Invariance of Poincaré-Lyapunov polynomials under the group of rotations *

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Abstract

We show that the Poincaré-Lyapunov polynomials at a focus of a family of real polynomial vector fields of degree n on the plane are invariant under the group of rotations. Furthermore, we show that under the multiplicative group $\mathbb{C}^* = \{\rho e^{i\psi}\}$, they are invariant up to a positive factor. These results follow from the weighted-homogeneity of the polynomials that we define in the text.

1 Introduction

Let us consider a real analytic vector field on the plane having a non-degenerate focus at the origin, that is, the Jacobian matrix of the vector field at the focus is not singular. After a linear transformation, we can suppose that the Jacobian matrix at the focus has the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \qquad b \neq 0. \tag{1}$$

Let Σ be a local cross section with one end point at the origin and $U \subseteq \Sigma$, a neighborhood of the origin in Σ. Recall that the displacement function in the neighbourhood of the origin is the Poincaré map $P\colon U \to \Sigma$ minus the Identity. One can show that the displacement function in a neighborhood of the origin has the following form (see [1]):

$$r = (e^{2\pi a/b} - 1)r_0 + u_3 r_0^3 + u_5 r_0^5 + u_7 r_0^7 + \cdots . \tag{2}$$

All the coefficients of the even powers of r_0 are equal to zero. When all the coefficients vanish, the origin is a center. Instead of calculating these coefficients to determine if an equilibrium

point is a center, Poincaré gave in [2] another method which resembles the search for a Lyapunov function to establish the stability of a focus. Let us recall this method.

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Looking at (2), we see that $dr/dr_0 \neq 0$ in a punctured neighborhood of the origin, if $a \neq 0$. Suppose that $a = 0$. If the vector field is linear, the integral curves are circles around the origin: $x^2 + y^2 = k$ (k a constant), or in polar coordinates $r^2 = k$. If the vector field is not linear, it is natural to look for integral curves that are small perturbations of these circles. Using polar coordinates, one tries to find integral curves of the form

$$H(r,\theta) = r^2 + H_3(\theta)r^3 + H_4(\theta)r^4 + \cdots = k\,. \tag{3}$$

If the origin is a center and if $H = k$ is an integral curve, then

$$\frac{dH}{dt} = \frac{\partial H}{\partial r}\dot{r} + \frac{\partial H}{\partial \theta}\dot{\theta} = 0\,.$$

Looking at the coefficients of the powers of r, this equation generates an infinite system of equations with the unknows $H_j(\theta)$ (see section 2). If the origin is not a center, then the equation above cannot be solved.

However, as we will see later on, one can formally solve the equation

$$\frac{dH}{dt} = P_1 r^4 + P_2 r^6 + P_3 r^8 + \cdots\,,$$

where P_j, $j = 1, 2, \ldots$ are constants. The sign of the first non-zero P_j controls the type of stability of the focus. If $P_j > 0$, the focus is unstable; it is stable otherwise. In fact, it is possible to find $H = r^2 + H_3(\theta)r^3 + \cdots + H_{2j+1}(\theta)r^{2j+1}$ such that

$$\frac{dH/dt}{r^{2j+2}}\Big|_{r=0} = P_j\,.$$

H is a Lyapunov function for the focus (see proposition 1 and corollary 2). If all the P_j vanish, it is possible to solve the system and the series in (3) converges in a neighborhood of the origin (see [2]).

There are no standard names for the constants P_j. Some call them focal numbers (or quantities), others call them Lyapunov constants. These names do not match the definitions of Andronov *et al* [1]. According to [1], the j^{th} focal value (or quantity) is the j^{th} derivative of the displacement function r in (2). If the first non-vanishing derivative of r is of order $k = 2j + 1 \geq 3$ ($j \geq 1$), then it is called the k^{th} Lyapunov value. But the P_j are not in general equal to the u_j in (2). Moreover, in the case of a family of vector fields, the P_j are in fact polynomial functions of the parameters (as we will see later on). We adopt the following definition.

Definition P_j is the j^{th} Poincaré-Lyapunov constant. In the case of a family of vector fields, P_j will be called the j^{th} Poincaré-Lyapunov polynomial (associated with this family).

We will study these polynomials for the family of all polynomial vector fields of degree n on the plane. We will prove that they are invariant under the group of rotations $S^1 = \{ \mathrm{e}^{i\psi} \}$ and also invariant under the multiplicative group $\mathbb{C}^* = \{ \rho\mathrm{e}^{i\psi} \}$ modulo a positive factor. Precisely, $\forall j \geq 1$ and for $g = \rho\mathrm{e}^{i\psi} \in \mathbb{C}^*$,

$$P_j(g(a_{rs})) = \rho^{2j} P_j(a_{rs}) \,,$$

where the a_{rs} are the parameters of the family of all polynomial vector fields of degree n on the plane. In this statement, it is important to distinguish a Poincaré-Lyapunov polynomial from the corresponding Poincaré-Lyapunov constant (the value of this polynomial for a certain vector field). Indeed, the statement says that the polynomials are also weighted-homogeneous in a certain sense that we will define in section 3.

2 Poincaré's Method

We suppose that the family of all polynomial vector fields of degree n

has an equilibrium point at the origin with a Jacobian matrix of the form (1) where $a = 0$. We will slightly modify Poincaré's procedure to obtain the main result of this article. Dividing the family by b, it takes the following form in the coordinates $z = x + iy$ and $\bar{z}$:

$$\begin{aligned} \dot{z} &= iz + \sum_{m=2}^{n} \sum_{j+k=m} a_{jk} z^j \bar{z}^k, \\ \dot{\bar{z}} &= -i\bar{z} + \sum_{m=2}^{n} \sum_{j+k=m} \bar{a}_{kj} z^j \bar{z}^k. \end{aligned} \tag{4}$$

Setting $r = \sqrt{z\bar{z}}$ and $\theta = (1/2i)\ln(z/\bar{z})$, we obtain:

$$\begin{aligned} \dot{r} &= \frac{1}{2r}(\dot{z}\bar{z} + z\dot{\bar{z}}) = (1/2)\sum_{m=2}^{n} F_m(\mathrm{e}^{i\theta})r^m \\ \dot{\theta} &= \frac{1}{2r^2}(-i\dot{z}\bar{z} + iz\dot{\bar{z}}) = 1 + (1/2)\sum_{m=2}^{n} G_m(\mathrm{e}^{i\theta})r^{m-1}, \end{aligned} \tag{5}$$

where

$$\begin{aligned} F_m(\mathrm{e}^{i\theta}) &= a_{0m}\mathrm{e}^{-(m+1)i\theta} + \sum_{j+k=m;\ j\neq 0} (a_{jk} + \bar{a}_{(k+1)(j-1)})\mathrm{e}^{(j-k-1)i\theta} \\ &\quad + \bar{a}_{0m}\mathrm{e}^{(m+1)i\theta} \\ G_m(\mathrm{e}^{i\theta}) &= -ia_{0m}\mathrm{e}^{-(m+1)i\theta} + \sum_{j+k=m;\ j\neq 0} (-ia_{jk} + i\bar{a}_{(k+1)(j-1)})\mathrm{e}^{(j-k-1)i\theta} \\ &\quad + i\bar{a}_{0m}\mathrm{e}^{(m+1)i\theta}. \end{aligned} \tag{6}$$

One must find a function

$$H(r, e^{i\theta}) = r^2 + H_3(e^{i\theta})r^3 + H_4(e^{i\theta})r^4 + \cdots$$

such that
$$\frac{dH}{dt} = \frac{\partial H}{\partial r}\dot r + \frac{\partial H}{\partial \theta}\dot\theta = P_1 r^4 + P_2 r^6 + P_3 r^8 + \cdots . \tag{7}$$
We will see, as Poincaré did, that it is in general impossible to find

$H(r, e^{i\theta})$ such that $dH/dt = 0$, except if the origin is a center. In this case, all the constants P_j vanish. We have:

$$\begin{aligned}
&\frac{dH}{dt} = (F_2 + H_3')r^3 + \left(\frac{3}{2}H_3F_2 + F_3 + \frac{1}{2}H_3'G_2 + H_4'\right) r^4 + \cdots \\
&+ \left(\frac{n}{2}H_nF_2 + \cdots + \frac{3}{2}H_3F_{n-1} + F_n + \frac{1}{2}H_3'G_{n-1} + \cdots + \frac{1}{2}H_n'G_2 + H_{n+1}'\right) r^{n+1} \\
&+ \left(\frac{n+1}{2}H_{n+1}F_2 + \cdots + \frac{3}{2}H_3F_n + \frac{1}{2}H_3'G_n + \cdots + \frac{1}{2}H_{n+1}'G_2 + H_{n+2}'\right) r^{n+2} \\
&+ \left(\frac{n+2}{2}H_{n+2}F_2 + \cdots + \frac{4}{2}H_4F_n + \frac{1}{2}H_4'G_n + \cdots + \frac{1}{2}H_{n+2}'G_2 + H_{n+3}'\right) r^{n+3} \\
&+ \cdots
\end{aligned}$$

Notation 1 Let us denote the coefficient of r^k in the previous expression by $L_k(\mathrm{e}^{i\theta}) + H_k'$.

Proposition 1 *Let m be the smallest integer such that $P_m \neq 0$. Then the system of equations $L_k(\mathrm{e}^{i\theta}) + H_k' = 0$ ($3 \le k \le 2m+1$) with the unknowns H_k has a solution. H_k has only powers of $\mathrm{e}^{i\theta}$ of the same parity as k. There is no H_{2m+2} such that $L_{2m+2}(\mathrm{e}^{i\theta}) + H_{2m+2}' = 0$.*

Proof In the sequel, we will say simply powers instead of powers of $\mathrm{e}^{i\theta}$. If we can find H_k', then H_k and H_k' ($k \ge 3$) have the same powers. From (6) we see that F_j and G_j ($j \ge 2$) have (only) powers of the parity opposite to that of j. Since $H_3' = -F_2$,

H_3' and H_3 have odd powers. Up to constants, the terms in L_4 are H_3F_2, F_3 and $H_3'G_2$, where the powers in H_3, F_2, H_3' and G_2 are odd. Then L_4 has even powers. The coefficient of $\mathrm{e}^{0i\theta}$ in L_4 is P_1. If $P_1 = 0$, we can find $H_4(\mathrm{e}^{i\theta})$ such that $L_4(\mathrm{e}^{i\theta}) + H_4' = 0$; in this case H_4 has even powers. If $P_1 \neq 0$, it is impossible to solve the equation.

Let $m \ge 2$. We proceed by induction. Let us suppose that it is possible to solve the equations $L_k(\mathrm{e}^{i\theta}) + H_k' = 0$ up to $k = 2m$ and that the powers in H_k' and H_k have the same parity as k. Up to constants, the terms in L_k are of the form H_rF_s, F_{k-1} and $H_r'G_s$, where $r + s = k + 1$. If $k = 2m + 1$ is odd, then F_{k-1} has odd powers. Since $r + s$ is even, s and r have the same parity and the powers in H_rF_s and $H_r'G_s$ are odd. We conclude that $L_{2m+1}(\mathrm{e}^{i\theta}) + H_{2m+1}' = 0$ has a solution and that H_{2m+1}' and H_{2m+1} have odd powers. Similar arguments show that, when $k = 2m + 2$, F_{k-1}, H_rF_s and $H_r'G_s$ have even powers; then $L_{2m+2}(\mathrm{e}^{i\theta}) + H_{2m+2}' = 0$ has a solution if and only if P_m, the coefficient of $\mathrm{e}^{0i\theta}$ in L_{2m+2}, is zero. If $P_m = 0$, then H_{2m+2}' and H_k have even powers. ∎

Corollary 2 *Let m be the smallest integer such that $P_m \neq 0$. Then the function $r^2 + H_3(\theta)r^3 + \cdots + H_{2m+1}(\theta)r^{2m+1}$, i.e., the solution of the system of equations $L_k(\mathrm{e}^{i\theta}) + H'_k = 0$ $(3 \leq k \leq 2m+1)$, is a Lyapunov function for the focus. If $P_m < 0$, the focus is stable. Otherwise it is unstable.*

To find the Poincaré-Lyapunov polynomials we proceed as follows. Equating dH/dt with the right hand side of (7), we get an infinite set of differential equations with the unknowns H_j $(j \geq 3)$ and P_k $(k \geq 1)$, where P_k is the coefficient of $\mathrm{e}^{0i\theta}$ in L_{2k+2}. If $n = 2k$ is even, the system is:

$$\begin{aligned}
H'_3 &= -F_2 \\
H'_4 &= P_1 - \frac{3}{2}H_3F_2 + F_3 - \frac{1}{2}H'_3G_2 \qquad (8)\\
&\cdots \\
H'_{2k+1} &= -\frac{2k}{2}H_{2k}F_2 - \cdots - \frac{3}{2}H_3F_{2k-1} - F_{2k} - \frac{1}{2}H'_3G_{2k-1} - \cdots - \frac{1}{2}H'_{2k}G_2 \\
H'_{2k+2} &= P_k - \frac{2k+1}{2}H_{2k+1}F_2 - \cdots - \frac{3}{2}H_3F_{2k} \\
&\qquad -\frac{1}{2}H'_3G_{2k} - \cdots - \frac{1}{2}H'_{2k+1}G_2 \\
&\cdots
\end{aligned}$$

If $n = 2k-1$ is odd, the last lines become:

$$\begin{aligned}
H'_{2k+1} &= -\frac{2k}{2}H_{2k}F_2 - \cdots - \frac{3}{2}H_3F_{2k-1} - \frac{1}{2}H'_3G_{2k-1} - \cdots - \frac{1}{2}H'_{2k}G_2 \\
H'_{2k+2} &= P_k - \frac{2k+1}{2}H_{2k+1}F_2 - \cdots - \frac{4}{2}H_4F_{2k-1} \qquad (9)\\
&\qquad -\frac{1}{2}H'_4G_{2k-1} - \cdots - \frac{1}{2}H'_{2k+1}G_2 \\
&\cdots
\end{aligned}$$

Poincaré used the sine and the cosine functions instead of $\mathrm{e}^{i\theta}$.

3 The Main Result

Letting $z = \alpha w$ $(\alpha = \rho\mathrm{e}^{i\psi})$, the vector field
(4) becomes (writing just one equation):

$$\dot{w} = iw + \sum_{m=2}^{n} \sum_{j+k=m} a_{jk}\alpha^{j-1}\bar{\alpha}^k w^j \bar{w}^k.$$

Then we obtain:

Lemma 3 *Under the action of the element $\rho\mathrm{e}^{i\psi}$ of the group $\mathbb{C}^*$, a_{rs} and $\bar{a}_{rs}$, where $r+s=m$, are respectively changed to $a_{rs}\rho^{m-1}\mathrm{e}^{(r-s-1)i\psi}$ and $\bar{a}_{rs}\rho^{m-1}\mathrm{e}^{(s-r+1)i\psi}$.*

Definition Let $c \in \mathbb{C}$ be a constant. If $r+s=m$, the weight of ca_{rs} or $c\bar{a}_{rs}$ with respect to ρ is $m-1$. The respective weights of ca_{rs} and $c\bar{a}_{rs}$ with respect to ψ are $r-s-1$ and $s-r+1$.

Lemma 4 *Let $c \in \mathbb{C}$ be a constant. Each ca_{rs} or $c\bar{a}_{rs}$ in F_m and G_m (see (6)) have a weight with respect of ρ equal to $m-1$. The weight with respect to ψ of each monomial in the coefficient of $\mathrm{e}^{ti\theta}$ is t.*

Proof Because $j+k=m$ $(j,k\geq 0)$, $(k+1)+(j-1)=m$ $(j\neq 0)$ and $0+m=m$, equation (6) implies that the weights with respect to ρ of ca_{jk}, $c\bar{a}_{(k+1)(j-1)}$ and $c\bar{a}_{m0}$ in F_m and G_m are indeed equal to $m-1$. The weight with respect to ψ of ca_{jk} is $j-k-1$, that of $c\bar{a}_{(k+1)(j-1)}$ $(j\neq 0)$, $(j-1)-(k+1)+1=j-k-1$ and that of $c\bar{a}_{0m}$, $m-0+1=m+1$. ∎

Since each monomial in the coefficient of $\mathrm{e}^{si\theta}$ has the same weights, we can, without ambiguity, talk about of the *weights of this coefficient*. The following notation will help to easily determine the weights of the coefficient of $\mathrm{e}^{si\theta}$ in F_m and G_m.

Notation Let us denote the coefficient of $\mathrm{e}^{si\theta}$ in F_m by $c_{[m-1,s]}$. The coefficients of the $\mathrm{e}^{si\theta}$'s in G_m will be denoted in order by

$$-ic_{[m-1,-m-1]}, d_{[m-1,-m+1]}, \dots, d_{[m-1,m-1]}, ic_{[m-1,m+1]}\,.$$

In the particular case of the family of polynomial vector fields of degree 3, one gets:

$$\begin{aligned}
\dot{r} &= \frac{1}{2}\left(c_{[1,-3]}\mathrm{e}^{-3i\theta}+c_{[1,-1]}\mathrm{e}^{-i\theta}+c_{[1,1]}\mathrm{e}^{i\theta}+c_{[1,3]}\mathrm{e}^{3i\theta}\right)r^2\\
&\quad+\frac{1}{2}\left(c_{[2,-4]}\mathrm{e}^{-4i\theta}+c_{[2,-2]}\mathrm{e}^{-2i\theta}+c_{[2,0]}+c_{[2,2]}\mathrm{e}^{2i\theta}+c_{[2,4]}\mathrm{e}^{4i\theta}\right)r^3\\
\dot{\theta} &= 1+\frac{1}{2}\left(-ic_{[1,-3]}\mathrm{e}^{-3i\theta}+d_{[1,-1]}\mathrm{e}^{-i\theta}+d_{[1,1]}\mathrm{e}^{i\theta}+ic_{[1,3]}\mathrm{e}^{3i\theta}\right)r\\
&\quad+\frac{1}{2}\left(-ic_{[2,-4]}\mathrm{e}^{-4i\theta}+d_{[2,-2]}\mathrm{e}^{-2i\theta}+d_{[2,0]}+d_{[2,2]}\mathrm{e}^{2i\theta}+ic_{[2,4]}\mathrm{e}^{4i\theta}\right)r^2.
\end{aligned}$$

Lemma 5 *The following relations are satisfied:*

$$\bar{c}_{[m-1,s]}=c_{[m-1,-s]} \text{ and } \bar{d}_{[m-1,s]}=d_{[m-1,-s]}.$$

Moreover, $c_{[m-1,0]}$ and $d_{[m-1,0]}$ are real.

Proof F_m and G_m are real expressions, since the original family of vector fields is real. Because in (4), $\dot{z}\bar{z}+z\dot{\bar{z}}$ and $-i\dot{z}\bar{z}+iz\dot{\bar{z}}$ are sums of conjugate terms, F_m and G_m are are also sums of conjugate terms. Precisely, the conjugate of the coefficient of $\mathrm{e}^{si\theta}$ is the coefficient of the conjugate of $\mathrm{e}^{si\theta}$. Then $\bar{c}_{[m-1,s]}=c_{[m-1,-s]}$ and $\bar{d}_{[m-1,s]}=d_{[m-1,-s]}$. When $s=0$, the terms $c_{[m-1,0]}$ and $d_{[m-1,0]}$ are self-conjugate, and therefore real. ∎

Definition Let h be a monomial in the unknowns $c[j,s]$ and $d[k,t]$. The weights of h with respect to ρ and ψ are the sums of the respective weights of its unknowns. We will say that a polynomial is weighted-homogeneous of degree (k, r) if all its monomials have the same weights k and r with respect to ρ and ψ respectively.

Proposition 6 *Let Q_t be the coefficient of $\mathrm{e}^{ti\theta}$ in H_s. Then Q_t is weighted-homogeneous of degree $(s-2, t)$. P_k is weighted-homogeneous of degree $(2k, 0)$.*

Proof Let us look at the system of equations (8) or (9). According to lemma 4 and the paragraph following it, the statement is true for all the coefficients Q_t in H_3, since $H_3' = -F_2$. Since $H_s' = -L_s$ (see notation 1), the result follows by induction.

Corollary 7 *P_k is invariant under the group of rotations S^1 and is invariant under the group C^* modulo a positive constant.*

4 Conclusion

We have proved not only that $\forall j \geq 1$ and for $g = \rho \mathrm{e}^{i\psi} \in \mathbb{C}^*$, $P_j(g(a_{rs})) = \rho^{2j} P_j(a_{rs})$, where P_j is a Poincaré-Lyapunov polynomial, but also that P_j is weighted-homogeneous of degree $(2j, 0)$ (according to definition 3).

This result has at least two goals.

New directions of research related to Hilbert's 16^{th} problem which look promising have been given by H. Zoladek in [3] and [4]. One of the questions raised by the Hilbert's 16^{th} problem is about the maximum number of limit cycles that exist in the family of polynomial vector fields of degree less or equal to n. A minor question, but closely related to, is to determine the maximum number of limit cycles near a center-focus. Zoladek proved in [3] that the family of polynomial vector fields of degree less or equal to two has at most 3 limit cycles near a center-focus. In [4], he proved that a family of degree less or equal to three, but without its quadratic part, has at most 5 limit cycles near a center-focus. The proofs follow from his main result that says the ideal generated by the Poincaré-Lyapunov polynomials is a linear combination, with polynomial coefficients in the a_{rs}, of the first Poincaré-Lyapunov polynomials. He utilizes for it the invariance of the Poincaré-Lyapunov polynomials under the group of rotations, but the arguments for proving the invariance, though correct, are rather elliptic. The present article gives a detailed proof.

One knows the importance of the Poincaré-Lyapunov polynomials to determine the stability of an equilibrium point. One could hope to find the Poincaré-Lyapunov polynomials for certain low degree polynomial vector fields. Indeed, using a computer, one could

list all the monomials of P_j, since they must satisfy the (two) homogeneity condition(s). Using the explicit system (8) or (9), one could find the coefficients of the monomials.

Remark The author has received from J.P. Françoise, C. Rousseau and R. Roussarie the main arguments of another proof of the invariance of the Poincaré-Lyapunov polynomials under the group of rotations. They do not have a result on the homogeneity with respect to the weights.

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