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# PERIODIC TRAJECTORIES FOR EVOLUTION EQUATIONS IN BANACH SPACES

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Abstract. The existence of periodic solutions for the evolution equation
$$y'(t) + Ay(t) \ni F(t, y(t))$$
is investigated under considerably simple assumptions on $A$ and $F$. Here $X$ is a Banach space, the operator $A$ is $m$-accretive, $-A$ generates a compact semigroup, and $F$ is a Carathéodory mapping. Two examples concerning nonlinear parabolic equations are presented.

## 1. Introduction

Consider the nonlinear evolution equation
$$y'(t) + Ay(t) \ni F(t, y(t)), \quad 0 \le t \le T, \tag{1.1}$$
where $(X, \|\cdot\|)$ is a Banach space, $A : D(A) \subset X \to 2^X$ is $m$-accretive such that $-A$ generates a compact semigroup, $F : [0,T] \times \overline{D(A)} \to X$ is a Carathéodory mapping, i.e., $F(\cdot, x) : [0,T] \to X$ is strongly measurable for every $x \in \overline{D(A)}$ and $F(t, \cdot) : \overline{D(A)} \to X$ is continuous for almost every $t \in [0,T]$.

The aim of this note is to investigate the existence of a mild periodic solution $y \in C([0,T]; X)$, $y(0) = y(T)$ for (1.1) in general Banach spaces. Our main result is the following.

**Theorem 1.1.** *Assume that $A$ is $m$-accretive, $-A$ generates a compact semigroup, $F$ is Carathéodory with*
$$\int_0^T \sup_{x \in \overline{D(A)}, \|x\| \le r} \|F(t,x)\| dt < \infty, \tag{1.2}$$
*for every $r > 0$, and there exist $R > 0$, $b, c \in L^1(0,T)$, $c(t) \ge 0$, $t \in (0,T)$, $c \ne 0$, such that*
$$[x, y - F(t,x)]_+ \ge c(t)\|x\| + b(t), \quad t \in (0,T),\ x \in D(A),\ \|x\| \ge R,\ y \in Ax, \tag{1.3}$$
*where $[x,v]_+ = \lim_{h \downarrow 0} \frac{1}{h}(\|x + hv\| - \|x\|)$, $x, v \in X$. Then (1.1) admits at least one mild $T$-periodic solution.*

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This periodic problem has been intensively studied in the literature [1, 2, 4, 5, 6, 7]. The most common argument used is to apply a fixed point theorem for a suitable Poincaré operator. Generally, Schauder's fixed point theorem produced remarkable results for the general Banach space setting (Vrabie [7], Shioji [4]). For example the main result in Shioji [4] has a statement similar to Theorem 1.1. In addition $X$ is separable, $\overline{D(A)}$ is convex, and condition (1.3) is strengthened to $c(t) = c > 0$, $t \in [0, T]$. The earlier result of Vrabie [7] studies (1.1) under the assumptions that $A - aI$ is $m$-accretive for some $a > 0$ and the growth condition (1.2) is replaced by

$$\limsup_{r\to\infty} \frac{1}{r} \sup\{\|F(t,x)\| : t \geq 0, x \in \overline{D(A)}, \|x\| \leq r\} = m < a.$$

The limiting case $a = 0$ is studied in Hilbert spaces by Cascaval & Vrabie [1] without the growth condition on $F$ but under a supplementary flow-invariant type condition of the form: there exists $r > 0$ such that $\{x \in X;\ \|x\| = r\} \cap D(A)$ is nonempty and

$$\langle x, y - F(t,x)\rangle \geq 0, \quad \text{for every } x \in D(A),\ \|x\| = r,\ y \in Ax,\ 0 \leq t \leq T,$$

where $\langle\cdot,\cdot\rangle$ stands for the inner product of $X$. Also, in [6] the case $m = a \geq 0$ is studied in general Banach spaces under the condition: there exists $r > 0$ such that

$$[x, y - F(t,x)]_+ \geq 0, \quad \text{for } 0 \leq t \leq T,\ x \in D(A),\ y \in Ax,\ \|x\| \geq r.$$

Most of these new assumptions are needed for the invariance of a closed convex ball in the Schauder fixed point theorem argument.

The method we use here relies on the Leray-Schauder topological degree applied for the Green operator and it is neither concerned with the initial value problem for (1.1) nor involves the Poincaré operator. That is why, the convexity of $\overline{D(A)}$ or the strong accretivity of $A$ are no longer needed. Condition (1.3) ensures some "a priori" estimates for the periodic solutions of (1.1) and that the "periodic solution" operator $P_A : L^1(0,T;X) \to C([0,T];X)$, which associates to $g \in L^1(0,T;X)$ all solutions $y \in P_A g$ of the periodic problem $y'(t) + Ay(t) \ni g(t)$, $0 \leq t \leq T$, $y(0) = y(T)$, is compact.

Next section is devoted to preliminaries and main notations. Section 3 contains the proof of our main result. This paper concludes with section 4 where two examples concerning nonlinear evolution equation of parabolic type are presented.

## 2. Preliminaries

For notions such as $m$-accretive operator, mild solution, or compact semigroup the reader is referred to Vrabie [8] and the references therein. We suggest Lloyd [3] for the theory and notations of the topological degree.

Let $(X, \|\cdot\|)$ be a real Banach space and $A : D(A) \subset X \to 2^X$. If $A - \omega I$ is $m$-accretive for some $\omega \in \mathbb{R}$ then for each $(\xi, f) \in \overline{D(A)} \times L^1(0,T;X)$ the initial value problem

$$y'(t) + Ay(t) \ni f(t), \quad 0 \leq t \leq T,\ y(0) = \xi, \tag{2.1}$$

has a unique mild solution denoted by $M(\xi, f)$.

For $y_i = M(\xi_i, f_i)$, where $(\xi_i, f_i) \in \overline{D(A)} \times L^1(0,T;X)$, $i = 1,2$ we have the mild solution inequality

$$e^{\omega t}\|y_1(t) - y_2(t)\| \le e^{\omega s}\|y_1(s) - y_2(s)\| + \int_s^t e^{\omega\tau}\|f_1(\tau) - f_2(\tau)\|d\tau, \tag{2.2}$$

for $0 \le s \le t \le T$.

The norm of a function $f$ in $L^p(0,T;X)$, $1 \le p < \infty$, is denoted by

$$\|f\|_{L^p} = (\int_0^T \|f(t)\|^p dt)^{1/p}.$$

We use the norm $\|y\|_\infty = \sup_{t\in[0,T]} \|y(t)\|$, for $y \in C([0,T];X)$.

A family $\mathcal{G} \subset L^1(0,T;X)$ is called uniformly integrable if for each $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for every measurable set $E$ in $[0,T]$ whose Lebesgue measure is less than $\delta(\varepsilon)$ we have $\int_E \|f(t)\|dt < \varepsilon$, uniformly for $f \in \mathcal{G}$.

Every bounded subset of $L^p(0,T;X)$, $1 < p \le \infty$, is uniformly integrable and every uniformly integrable subset is bounded in $L^1(0,T;X)$. We recall the following result from Vrabie [7, Theorem 2]).

**Theorem 2.1.** *If $A : D(A) \subset X \to 2^X$ is an operator with $A - \omega I$ m-accretive for some $\omega \ge 0$ such that $-A$ generates a compact semigroup, then for each bounded subset $B$ in $\overline{D(A)}$ and each uniformly integrable $\mathcal{G}$ in $L^1(0,T;X)$, the set $M(B\times\mathcal{G})$ of all mild solution of (2.1) corresponding to $(\xi, f) \in B \times \mathcal{G}$ is relatively compact in $C([d,T];X)$ for each $d \in (0,T)$. If, in addition, $B$ is relatively compact in $X$, then $M(B \times \mathcal{G})$ is relatively compact in $C([0,T];X)$.*

Consider the "periodic solution" operator $P_A : L^1(0,T;X) \to 2^{L^1(0,T;X)}$ defined by $(f,y) \in \operatorname{Graph} P_A$ if $y \in C([0,T];X)$, $y(0) = y(T)$, is a mild solution of

$$y'(t) + Ay(t) \ni f(t), \quad 0 \le t \le T.$$

**Theorem 2.2.** *If $\mathcal{G}$ is uniformly integrable in $L^1(0,T;X)$, $A - \omega I$ is m-accretive for some $\omega > 0$, and $-A$ generates a compact semigroup then $P_A(\mathcal{G})$ is relatively compact in $C([0,T];X)$.*

*Proof.* The mild solution inequality and the periodicity offer us the estimate

$$\|y\|_\infty \le \frac{e^{2\omega T}}{e^{\omega T} - 1}\|f\|_{L^1}, \quad \text{for every } y \in P_A f\,, \tag{2.3}$$

where, without loss of generality, we assume that $0 \in A0$. Since $\mathcal{G}$ is bounded, this shows that the set of initial-final data $B = \{y(0);\ y \in P_A(\mathcal{G})\}$ is bounded in $X$. According to Theorem 2.1, this implies that $B$ is relatively compact in $X$ and $P_A(\mathcal{G})$ is relatively compact in $C([0,T];X)$. $\square$

## 3. Proof of Theorem 1.1

In the sequel we assume that all the assumptions in Theorem 1.1 hold. Without loss of generality we may assume that, in (1.3), $R > \|b\|_{L^1}/\|c\|_{L^1}$. Let $K > C := \|b\|_{L^1} + R + 2$ and $\rho \in C^\infty(\mathbb{R})$ such that $0 \le \rho \le 1$, $\rho(u) = 1$ for $|u| \le K$, $\rho(u) = 0$ for $|u| \ge K + 1$.

**Lemma 3.1.** *Let $y \in C([0,T];X)$ be a mild $T$-periodic solution of*

$$y'(t) + Ay(t) \ni \rho(\|y(t)\|)F(t,y(t)), \quad t \in [0,T]. \tag{3.1}$$

*Then $\|y\|_\infty \le C$ or $\|y\|_\infty \ge K$.*

*Proof.* Assume by contradiction that $C < \|y\|_\infty < K$. Therefore, $\rho(\|y(t)\|) = 1$, $0 \le t \le T$. The mild solution definition states that for every $0 < c < T$ and $\varepsilon > 0$ there exist

(i) $0 = t_0 < t_1 < \dots < c \le t_n < T$, $t_k - t_{k-1} \le \varepsilon$ for $k = 1, \dots, n$;

(ii) $f_1, \dots, f_n \in X$ with $\sum_{k=1}^n \int_{t_{k-1}}^{t_k} \|\rho(\|y(t)\|)F(t, y(t)) - f_k\| dt \le \varepsilon$;

(iii) $y_0, \dots, y_n \in X$ satisfying $\frac{y_k - y_{k-1}}{t_k - t_{k-1}} + Ay_k \ni f_k$ for $k = 1, \dots, n$,

such that $\|y(t) - y_k\| \le \varepsilon$ for $t \in [t_{k-1}, t_k)$, $k = 1, \dots, n$.

Suppose that $\inf\limits_{[0,T]} \|y\| > R$. For $\varepsilon < \min\{K - \|y\|_\infty, \inf\limits_{[0,T]} \|y\| - R\}$, we have $R \le \|y_k\| \le K$, $k = 1, \dots, n$. From (1.3) and (iii) we find

$$[y_k, f_k - \frac{y_k - y_{k-1}}{t_k - t_{k-1}} - F(t, y_k)]_+ \ge c(t)\|y_k\| + b(t), \quad \text{a.e. } t \in [0, T]. \tag{3.2}$$

Integrate on $[t_{k-1}, t_k]$ and add from $k = 1$ to $n$, to obtain

$$\begin{aligned} &\|y_n\| + \sum_{k=1}^n \|y_k\| \int_{t_{k-1}}^{t_k} c(t)dt \\ &\le \|y_0\| + \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \|f_k - F(t, y_k)\| + \int_0^{t_n} |b(t)|dt \\ &\le \|y_0\| + \|b\|_{L^1} + \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \|\rho(\|y(t)\|)F(t, y(t)) - f_k\|dt \\ &\quad + \int_0^T \|F(t, y(t)) - F_\varepsilon(t)\|dt, \end{aligned} \tag{3.3}$$

where $F_\varepsilon(t) = F(t, y_k)$ if $t \in [t_{k-1}, t_k)$, $k = 1, \dots, n$, $F_\varepsilon(t) = F(t, y(t))$, $t \in [t_n, T]$. According to (ii), this yields

$$\|y_n\| + R\int_0^{t_n} c(t)dt \le \|y_0\| + \|b\|_{L^1} + \varepsilon + \int_0^T \|F(t, y(t)) - F_\varepsilon(t)\|dt. \tag{3.4}$$

Since $F$ is Carathéodory, from (1.2) and the Lebesgue dominated convergence theorem, we have $\lim\limits_{\varepsilon\to 0} \int_0^T \|F(t, y(t)) - F_\varepsilon(t)\|dt = 0$. Let $\varepsilon \to 0$, $c \to T$ in (3.4). We find $R \le \|b\|_{L^1}/\|c\|_{L^1}$ which is a contradiction. Therefore, $\inf\limits_{[0,T]} \|y\| \le R$. Eventually shifting the time, we may assume without loss of generality, that there exists $t_+ \in [0, T]$, such that $\|y(0)\| = R+1$, $\|y(t_+)\| = \|y\|_\infty$, and $\|y(t)\| \ge R+1$, for every $t \in [0, t_+]$. By the same argument used above we obtain $\|y\|_\infty \le R+1+\|b\|_{L^1} \le C$ which is a contradiction. The proof is complete. □

For $0 \le \lambda \le 1$, define the operators $L_\lambda : C([0, T]; X) \to C([0, T]; X)$,

$$L_\lambda v = P_{A+\omega I}(\lambda\rho(\|v\|)F(\cdot, v) + \lambda\omega v), \quad v \in C([0, T]; X),$$

i.e., $y = L_\lambda v$ is the unique $T$-periodic solution of

$$y'(t) + Ay(t) + \omega y(t) \ni \lambda\rho(\|v(t)\|)F(t, v(t)) + \lambda\omega v(t), \quad 0 \le t \le T. \tag{3.5}$$

where $\omega > 0$ is specified below. Note that $L_0 = 0$ and that Lemma 3.1 contains an "a priori" estimate for the fixed points of $L_1$. Similarly, we provide an "a priori" estimate for the fixed points of $L_\lambda$, $0 < \lambda < 1$.

**Lemma 3.2.** *For $\omega > 0$ big enough, every $T$-periodic solution $y \in C([0,T];X)$, of*

$$y'(t) + Ay(t) + \omega(1-\lambda)y(t) \ni \lambda\rho(\|y(t)\|)F(t,y(t)), \quad 0 \le t \le T,\ 0 < \lambda < 1, \tag{3.6}$$

*satisfies $\|y\|_\infty \le C$ or $\|y\|_\infty \ge K$.*

*Proof.* Consider $a_K(t) = \sup\{\|F(t,x)\|;\ x \in \overline{D(A)}, \|x\| \le K+1\}$, $t \in [0,T]$. According to (1.2) $a_K \in L^1(0,T)$ and $\rho(\|x\|)\|F(t,x)\| \le a_K(t)$, a.e. $t \in [0,T]$, $x \in \overline{D(A)}$.

Assume by contradiction that $C < \|y\|_\infty < K$. Suppose in addition that $\inf_{[0,T]} \|y\| > R$. Reasoning as above we find

$$\omega(1-\lambda)RT + R\|c\|_{L^1} \le (1-\lambda)\|a_K\|_{L^1} + \|b\|_{L^1}, \tag{3.7}$$

which is absurd if $\omega > \|a_K\|_{L^1}/RT$, since $R\|c\|_{L^1} > \|b\|_{L^1}$. Therefore, $\inf_{[0,T]} \|y\| \le R$, and we may assume that there exist $0 \le s_0 < t_0 \le T$ such that $\|y(s_0)\| = R+1$, $\|y(t_0)\| = \|y\|_\infty$, and $\|y(t)\| \ge R+1$, for every $t \in [s_0,t_0]$. Similarly, we obtain

$$\omega(1-\lambda)R(t_0-s_0) + R\int_{s_0}^{t_0} c(t)dt + \|y\|_\infty \le R+1+(1-\lambda)\int_{s_0}^{t_0} a_K(t)dt + \|b\|_{L^1}. \tag{3.8}$$

This leads to

$$\int_{s_0}^{t_0} a_K(t)dt - \omega R(t_0-s_0) \ge C - R - \|b\|_{L^1} - 1 = 1. \tag{3.9}$$

Since $a_K \in L^1(0,T)$ we have $\lim_{|t-s|\to 0} \int_s^t a_K(t)dt = 0$. Relation (3.9) tells us that $t_0 - s_0 \ge \delta_0 > 0$, where $\delta_0$ depends only on $a_K$. If $\omega > \|a_K\|_{L^1}/R\delta_0$, then (3.9) provides us with a contradiction. The proof is complete. $\square$

**Lemma 3.3.** *$H(\lambda) = L_\lambda$, $0 \le \lambda \le 1$, defines a homotopy of compact transformations in $C([0,T];X)$.*

*Proof.* Condition (1.2) ensures the fact that for every $0 \le \lambda \le 1$, the operator $C([0,T];X) \ni v \mapsto \lambda\rho(\|v\|)F(t,v) + \lambda\omega v$ transforms bounded subsets of $C([0,T];X)$ into locally integrable subsets of $L^1(0,T;X)$. According to Theorem 2.2, this implies that $L_\lambda : C([0,T];X) \to C([0,T];X)$ is compact, for every $0 \le \lambda \le 1$. For $0 \le \lambda$, $\mu \le 1$ and $v \in C([0,T];X)$, we have, from the mild solution inequality combined with the periodicity, that

$$\|H(\lambda)v - H(\mu)v\|_\infty \le |\lambda - \mu| e^{2\omega T}/(e^{\omega T} - 1)\|a_K\|_{L^1} \tag{3.10}$$

which shows that $H$ is a homotopy, thereby completing the proof. $\square$

*Proof of Theorem 1.1.* Pick $r_0 \in (C,K)$ and let $B = \{v \in C([0,T];X);\ \|v\|_\infty < r_0\}$, $S = \{v \in C([0,T];X);\ \|v\|_\infty = r_0\}$. Since $C < r_0 < K$ we know from Lemma 3.2 that $0 \notin (I - L_\lambda)(S)$, $0 \le \lambda \le 1$. The invariance of the degree with respect to $H$ gives

$$\deg(I - L_1, B, 0) = \deg(I - L_0, B, 0) = \deg(I, B, 0) = 1 \ne 0, \tag{3.11}$$

i.e., $L_1$ has at least one fixed point in $B$. Since $r_0 < K$, this fixed point of $L_1$ is a $T$-periodic solution of (1.1). $\square$

## 4. Examples

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n \ge 2$, with smooth boundary $\partial\Omega$. First, we study the periodic problem for the nonlinear diffusion equation

$$\begin{gathered}\frac{\partial y}{\partial t} - \Delta\rho(y) = f(t,x,y(t,x)) \quad \text{a.e. } (t,x) \in \mathbb{R}_+ \times \Omega,\\ y = 0 \quad \text{a.e. } (t,x) \in \mathbb{R}_+ \times \partial\Omega,\\ y(t,x) = y(t+T,x) \quad \text{for every } t \ge 0, \text{ and a.e. } x \in \Omega.\end{gathered} \tag{4.1}$$

**Theorem 4.1.** *If $\rho \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$, $\rho(0) = 0$, and there exist $c > 0$, $p > (n-2)/2$ such that $\rho'(r) \ge c|r|^{p-1}$ for every $r \ne 0$, and $f : \mathbb{R}\times\Omega\times\mathbb{R} \to \mathbb{R}$, $f = f(t,x,u)$ is $T$-periodic in $t$, $f(t,x,\cdot)$ is continuous for almost every $(t,x) \in \mathbb{R}\times\Omega$, $f(\cdot,\cdot,u)$ is measurable for every $u \in \mathbb{R}$, and there exist $M > 0$, $a, c \in L^1(0,T)$, $c(t) \ge 0$, $t \in (0,T)$, $c \ne 0$, $b, d \in L^1((0,T)\times\Omega)$ such that*

$$|f(t,x,u)| \le a(t)|u| + b(t,x), \quad (t,x,u) \in [0,T]\times\Omega\times\mathbb{R}, \tag{4.2}$$

*and*

$$\begin{gathered}f(t,x,u)u \le 0, \quad |f(t,x,u)| \ge c(t)|u| + d(t,x),\\ (t,x) \in [0,T]\times\Omega, \quad |u| \ge M.\end{gathered} \tag{4.3}$$

*Then (4.1) has at least one $T$-periodic solution.*

**Remark 4.2.** For the problem above, Shioji [5] considers (4.2) and the condition

$$\limsup_{|u|\to\infty} \operatorname*{ess\,sup}_{(t,x)\in\mathbb{R}\times\Omega} \frac{f(t,x,u)}{u} < 0,$$

which is equivalent to there exist $\delta, M > 0$, $f(t,x,u)/u \le -\delta$ for $(t,x,u) \in \mathbb{R}\times\Omega\times\mathbb{R}$ with $|u| \ge M$. This clearly has the form of (4.3) with $c(t) = \delta$, $t \in [0,T]$, $d = 0$.

*Proof of Theorem 4.1.* The operator $A : D(A) = \{u \in L^1(\Omega);\ \rho(u) \in W_0^{1,1}(\Omega),\ \Delta\rho(u) \in L^1(\Omega)\} \subset L^1(\Omega) \to L^1(\Omega)$ given by $Au := -\Delta\rho(u)$, $u \in D(A)$, is $m$-accretive in $X = L^1(\Omega)$, $\overline{D(A)} = L^1(\Omega)$, and it generates a compact semigroup (Vrabie [8]). Define $F : \mathbb{R}\times L^1(\Omega) \to L^1(\Omega)$ by $F(t,u)(x) := f(t,x,u(x))$, $t \in \mathbb{R}$, $u \in L^1(\Omega)$. ¿From (4.2) $F$ is well defined, Carathéodory, and satisfies (1.2).

Since $A0 = 0$, for every $u \in D(A)$ we have

$$[u, \Delta\rho(u) - F(t,u)]_+ \ge [u, \Delta\rho(u)]_+ - [u, F(t,u)]_+ \ge -[u, F(t,u)]_+. \tag{4.4}$$

Denote by $\{u < 0\} = \{x \in \Omega;\ u(x) < 0\}$, $\{u = 0\} = \{x \in \Omega;\ u(x) = 0\}$, $\{u > 0\} = \{x \in \Omega;\ u(x) > 0\}$, and $|\Omega|$ the measure of $\Omega$. ¿From (4.2) and (4.3), for every $u \in L^1(\Omega)$ we obtain

$$\begin{aligned}&[u, F(t,u)]_+\\ &= \int_{\{u>0\}} f(t,x,u(x))dx - \int_{\{u<0\}} f(t,x,u(x))dx + \int_{\{u=0\}} f(t,x,0)dx\\ &\le 2Ma(t) + 3\int_\Omega b(t,x)dx + \int_{\{u>M\}} f(t,x,u(x))dx - \int_{\{u<-M\}} f(t,x,u(x))dx\\ &\le -c(t)\|u\|_{L^1} + Mc(t)|\Omega| + 2Ma(t) + 3\int_\Omega b(t,x)dx + \int_\Omega |d(t,x)|dx.\end{aligned} \tag{4.5}$$

Relations (4.3) and (4.4) combined prove that (1.3) is fulfilled with $R = 0$. According to Theorem 1.1, (4.1) has at least one $T$-periodic solution. $\square$

Next, we consider the periodic problem for the nonlinear heat equation

$$\begin{gathered} \frac{\partial y}{\partial t} - \Delta_p(y) = f(t,x,y(t,x)) \quad \text{a.e. } (t,x) \in \mathbb{R}_+ \times \Omega, \\ y = 0 \quad \text{a.e. } (t,x) \in \mathbb{R}_+ \times \partial\Omega, \\ y(t,x) = y(t+T,x) \quad \text{for every } t \geq 0, \text{ and a.e. } x \in \Omega, \end{gathered} \tag{4.6}$$

where $\Omega$ is a bounded domain of $\mathbb{R}^n$, with smooth boundary $\partial\Omega$, $p \geq 2$, and

$$\Delta_p y = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(|\frac{\partial y}{\partial x_i}|^{p-2}\frac{\partial y}{\partial x_i}),$$

is the pseudo-Laplace operator.

**Theorem 4.3.** *Suppose that $f : \mathbb{R} \times \Omega \times \mathbb{R} \to \mathbb{R}$, $f = f(t,x,u)$ is $T$-periodic in $t$, $f(t,x,\cdot)$ is continuous for almost every $(t,x) \in \mathbb{R} \times \Omega$, $f(\cdot,\cdot,u)$ is measurable for every $u \in \mathbb{R}$, and there exist $M > 0$, $a,k \in L^1(0,T)$, $a,k \geq 0$, $k \neq 0$, $b,d \in L^1(0,T;L^2(\Omega))$ such that*

$$|f(t,x,u)| \leq a(t)|u| + b(t,x), \quad (t,x,u) \in [0,T] \times \Omega \times \mathbb{R}, \tag{4.7}$$

*and*

$$f(t,x,u)u \leq [c - k(t)]|u|^p + d(t,x)|u|, \quad (t,x) \in [0,T] \times \Omega, \ |u| \geq M, \tag{4.8}$$

*where $c > 0$ is such that*

$$\int_\Omega |\nabla u(x)|^p dx \geq c \int_\Omega |u(x)|^p dx, \quad u \in W_0^{1,p}(\Omega). \tag{4.9}$$

*Then (4.6) has at least one $T$-periodic solution.*

**Remark 4.4.** In [7] the existence of periodic solutions for the nonlinear heat equation governed by the pseudo-Laplace operator is showed under the conditions

$$|f(t,x,u)| \leq a|u| + b, \quad f(t,x,u)u \leq \alpha|u|^p + \beta, \ (t,x,u) \in \mathbb{R} \times \Omega \times \mathbb{R},$$

where $a,b,\alpha,\beta > 0$ and $\alpha < c$. These conditions are particular cases of (4.7), (4.8) with $a(t) = a$, $b(t,x) = b$, $k(t) = c - \alpha$, $d(t,x) = d > 0$, $(t,x) \in \mathbb{R} \times \Omega$, since for $M = \beta/d$ and $|u| \geq M$, $d|u| \geq \beta$.

*Proof of Theorem 4.3.* The operator $A : D(A) = \{u \in W_0^{1,2}(\Omega),\ \Delta_p u \in L^2(\Omega)\} \subset L^2(\Omega) \to L^2(\Omega)$ given by $Au := -\Delta_p u$, $u \in D(A)$, is maximal monotone in $X = L^2(\Omega)$, and it generates a compact semigroup (Vrabie [8]). Define $F : \mathbb{R} \times L^2(\Omega) \to L^2(\Omega)$ by $F(t,u)(x) := f(t,x,u(x))$, $t \in \mathbb{R}$, $u \in L^2(\Omega)$. ¿From (4.7) $F$ is well defined, Carathéodory, and satisfies (1.2). For $u \in D(A)$, according to (4.7), (4.8), we have

$$
\begin{aligned}
&\|u\|_{L^2}[u, Au - F(t,u)]_+ \\
&= \langle u, -\Delta_p u - F(t,u)\rangle_{L^2} \\
&= \int_\Omega |\nabla u(x)|^p dx - \int_\Omega f(t,x,u(x))u(x)dx \\
&\geq c\int_\Omega |u|^p dx - \int_{\{|u|\geq M\}} f(t,x,u(x))u(x)dx - \int_{\{|u|<M\}} f(t,x,u(x))u(x)dx \\
&\geq c\int_{\{|u|<M\}} |u|^p dx + k(t)\int_{\{|u|\geq M\}} |u|^p dx - a(t)\|u\|_{L^2}^2 - \alpha(t)\|u\|_{L^2} \\
&\geq \min\{c, k(t)\}\int_\Omega |u|^p dx - \alpha(t)\|u\|_{L^2},
\end{aligned}
\tag{4.10}
$$

where $\alpha(t) = (\int_\Omega |b(t,x)|^2 dx)^{1/2} + (\int_\Omega |d(t,x)|^2 dx)^{1/2} \in L^1(0,T)$. Using the inequality

$$
\int_\Omega |u|^p dx \geq \|u\|_{L^2}|\Omega|^{(2-p)/2}, \quad u \in L^p(\Omega),
$$

one gets that, for any $R > 0$ and $\|u\|_{L^2} \geq R$,

$$
[u, Au - F(t,u)]_+ \geq c(t)\|u\|_{L^2} - \alpha(t), \tag{4.11}
$$

where $c(t) = \min\{c, k(t)\}R^{p-2}|\Omega|^{(2-p)/2}$, $t \in [0,T]$. Clearly, $c \geq 0$, $c \neq 0$ and all the conditions of Theorem 1.1 are fulfilled. This provides us with a $T$-periodic solution of (4.6). $\square$

## References


[1] R. Caşcaval and I. I. Vrabie; *Existence of periodic solutions for a class of nonlinear evolution equations*, Rev. Mat. Univ. Complutense Madrid **7** (1994), 325-338.

[2] N. Hirano and N. Shioji; *Existence of periodic solutions under saddle point type conditions*, J. Nonlinear Convex Anal. **1** (2000), 115-128.

[3] N. G. Lloyd; *Degree theory*, Cambridge University Press (1978).

[4] N. Shioji; *Periodic solutions for nonlinear evolution equations in Banach spaces*, Funkc. Ekvacioj, Ser. Int. **42** (1999), 157-164.

[5] N. Shioji; *Existence of periodic solutions for nonlinear evolution equations with pseudo monotone operators*, Proc. Am. Math. Soc. **125** (1997), 2921-2929.

[6] M. D. Voisei; *Existence of periodic solutions for nonlinear equations of evolution in Banach spaces*, Math. Sci. Res. Hot-Line **3** (1999), 1-7.

[7] I. I. Vrabie; *Periodic solutions for nonlinear evolution equations in a Banach space*, Proc. Am. Math. Soc. **109**, (1990), 653-661.

[8] I. I. Vrabie; *Compactness methods for nonlinear evolutions. 2nd ed*, Pitman Monographs and Surveys in Pure and Applied Mathematics, Longman (1995).



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