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DETERMINATION OF POINT SOURCES IN VIBRATING BEAMS BY BOUNDARY MEASUREMENTS: IDENTIFIABILITY, STABILITY, AND RECONSTRUCTION RESULTS

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Abstract. We consider two inverse problems of determining point sources in vibrating beams by boundary measurements. We show that the boundary observation at one extremity of the domain determines uniquely the sources for an arbitrarily small time of observation. We further establish conditional stability results and give reconstructing schemes.

1. Introduction

Inverse problems of distributed parameter systems are in our days an expanding field. Here we restrict our investigations to the determination of sources using some boundary observations. As usual in such problems the three main steps are the uniqueness (unique solvability of the problem), the stability (small perturbations of the measurements give rise to small perturbations of the sources) and finally the reconstruction (build appropriate processes in order to find a good approximation of the unknowns).

The resolution of such problems using control results of distributed systems (like the wave equation, Petrowsky systems, etc.) has been recently developed, in particular by Yamamoto and coauthors [9, 2, 3, 10]. The main idea is to use some observability estimates and controllability results, using for instance the so-called multiplier method and the Hilbert Uniqueness Method [6], to deduce the uniqueness and the reconstruction process. For the wave equation this method successfully leads to the reconstruction of point sources in 1-dimensional domains by boundary observations in [2, 3, 5, 7]. In higher dimensional domains the same technique leads to the reconstruction of smoother unknown sources using boundary observations [8, 9]. In [3] the authors consider interior pointwise observations for the determination of the point sources in $]0, 1[$. For the standard Petrovsky system (vibrations of beams or plates), pointwise and line observations are treated in a similar spirit in [10].

To our knowledge the determination of point sources by boundary measurements for the beam equation with different boundary conditions has been not yet considered. Therefore, our goal is to answer to this question for two different problems

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by adapting some results from [1, 2, 9, 7]. The main ingredients are the spectral properties of the biharmonic operators, some controllability results [6, 4] and finally appropriate properties of some integral operators [9, 2]. For our first problem since the eigenvalues and eigenvectors of the operator are not explicitly known, our reconstruction process is different from the one in [2] and is more close to the one in [9]. On the contrary for our second system the eigenvalues and eigenvectors of the operator are explicitly known, and therefore our reconstruction process is similar to the one in [2].

The paper is organized as follows: Section 2 is devoted to the first Petrovsky system. In subsection 2.1, we show the wellposedness of the problem, some observability estimates and hidden regularities of the solution. Subsection 2.2 is devoted to the proof of the uniqueness result and is based on the previous observality estimates and some properties of an integral operator between different Sobolev spaces. The conditional stability is deduced in subsection 2.3 and finally the reconstruction is detailed in subsection 2.4. The same questions for the second Petrovsky system are treated in section 3 with the same subdivision into four subsections.

2. The first Petrovsky system

2.1. **Preliminaries.** We consider the initial boundary value problem for a beam equation

$$\begin{gathered} \partial_t^2 u(x,t) + u^{(4)}(x,t) = \lambda(t)a(x) \quad \text{in } Q_T, \\ u(\cdot,0) = 0, \quad \partial_t u(\cdot,0) = 0 \quad \text{in }]0,1[, \\ u(x,t) = u'(x,t) = 0, \quad \text{for } x = 0,1 \text{ and for } t \in]0,T[, \end{gathered} \tag{2.1}$$

where $u^{(4)}(x,t) = \frac{\partial^4 u}{\partial x^4}(x,t)$, $u'(x,t) = \frac{\partial u}{\partial x}(x,t)$, and $Q_T :=]0,1[\times]0,T[$. Above and below $\lambda \in C^1([0,T])$ is a given function satisfying

$$\lambda(0) \neq 0. \tag{2.2}$$

The datum $a \in (H^1(0,1))'$ is assumed to be in the form

$$a(x) = \sum_{k=1}^{K} \alpha_k \delta(x - \xi_k) \tag{2.3}$$

for some positive integer K, some real numbers α_k different from zero and some (different) points ξ_k in $]0,1[$ (enumerated in increasing order), or more precisely

$$\langle a, \phi \rangle = \sum_{k=1}^{K} \alpha_k \phi(\xi_k), \quad \forall \phi \in H^1(0,1).$$

As usual $H^p(0,1)$ is the standard Sobolev space of order $p \in \mathbb{N} := \{0,1,2,\dots\}$ on the interval $]0,1[$.

Our goal is to identify the datum a in the above form (i.e. the location of the point sources ξ_k, the weights α_k and the number K) from boundary measurements, namely the value of $u''(0,t)$, for $0 < t < T$.

To analyse the system (2.1) we introduce the following operator A on the Hilbert space $H = L^2(0,1)$, endowed with the inner product

$$(u,v)_H = \int_0^1 u(x)v(x)\,dx. \tag{2.4}$$

The domain of A is $D(A) = H^4(0,1) \cap H_0^2(0,1)$ and for any $u \in D(A)$ we take $Au = -u^{(4)}$. Remark that A is a negative selfadjoint operator with a compact resolvent since A is the Friedrichs extension of the triple (H, V, a) defined by $V = H_0^2(0,1)$ which is a Hilbert space with the inner product

$$(u,v)_V = \int_0^1 u''(x)v''(x)\,dx, \tag{2.5}$$

and

$$a(u,v) = (u,v)_V. \tag{2.6}$$

The spectrum of this operator A is well known, namely if $\{\lambda_k\}_{k=1}^{\infty}$ denotes the set of eigenvalues of the operator $-A$ in increasing order and repeated according to their multiplicity, then $\lambda_k = \mu_k^2$ where μ_k is a root of $\cosh\sqrt{\mu_k}\cos\sqrt{\mu_k} - 1 = 0$. The eigenvalues have furthermore the asymptotic:

$$C_1 k^4 \le \lambda_k \le C_2 k^4, \quad \forall k = 1, \dots, \infty, \tag{2.7}$$

for some positive constants C_1 and C_2. For future purposes, we need to show that the eigenfunctions are uniformly bounded:

Lemma 2.1. *Let ϕ_k be the eigenfunction of $-A$ associated with λ_k. Then there exists a constant $M > 0$ (independent of k) such that*

$$|\phi_k(x)| \le M, \quad \forall k = 1, \dots, \infty \text{ and } \forall x \in]0,1[.$$

Proof. By simple calculations, we see that the eigenfunctions are

$$\phi_k(x) = C_k[\sin(\sqrt{\mu_k}x) - \sinh(\sqrt{\mu_k}x) - f_k(\cos(\sqrt{\mu_k}x) - \cosh(\sqrt{\mu_k}x))],$$

where

$$f_k = \frac{\sin\sqrt{\mu_k} - \sinh\sqrt{\mu_k}}{\cos\sqrt{\mu_k} - \cosh\sqrt{\mu_k}}$$

and some constant C_k. As $\mu_k \to \infty$ as $k \to \infty$ we readily show that there exists a positive constant C independent of k such that:

$$|f_k - 1| \le C\exp(-\sqrt{\mu_k}). \tag{2.8}$$

This estimate allows to show that there exists a positive constant $C^\star$ independent of k such that

$$|\sin(\sqrt{\mu_k}x) - \sinh(\sqrt{\mu_k}x) - f_k(\cos(\sqrt{\mu_k}x) - \cosh(\sqrt{\mu_k}x))| \le C^\star, \quad \forall x \in [0,1]. \tag{2.9}$$

On the other hand, the constant C_k is chosen such that

$$\int_0^1 |\phi_k(x)|^2 dx = 1.$$

A careful analysis of this constant with respect to μ_k shows that $C_k \to 1$ as $k \to \infty$, which implies the requested estimate. □

We are now ready to prove that our beam equation (2.1) is uniquely solvable and to give regularity of its solution:

Theorem 2.2. *The beam equation* (2.1) *has a unique (weak) solution u satisfying*

$$u \in C([0,T];V) \cap C^1([0,T];H).$$

Proof. We remark that the system (2.1) is equivalently written

$$\begin{gathered} \partial_t^2 u = Au + \lambda(t) a \quad \text{in }]0,T[, \\ u(0) = 0, \quad \partial_t u(0) = 0, \end{gathered} \tag{2.10}$$

where $a \in V'$ is defined by

$$\langle a, \phi \rangle_{V'-V} = \sum_{k=1}^{K} \alpha_k \phi(\xi_k), \quad \forall \phi \in V. \tag{2.11}$$

The solution of this system is given by (using spectral expansions)

$$u(t) = \sum_{k=1}^{\infty} \frac{1}{\mu_k^2} \int_0^t \sin(\mu_k(t-s)) \lambda(s) \, ds \langle a, \phi_k \rangle \phi_k,$$

or equivalently, by integration by parts in the above integral,

$$u(t) = \sum_{k=1}^{\infty} \frac{a_k(t)}{\mu_k^2} \phi_k, \tag{2.12}$$

where a_k is given by

$$a_k(t) = \langle a, \phi_k \rangle (\lambda(t) - \lambda(0) \cos(\mu_k t) - \int_0^t \cos(\mu_k(t-s)) \lambda'(s) \, ds).$$

We now remark that the form of a and Lemma 2.1 allow to conclude the existence of a constant C_1 (depending on T but not on k) such that

$$|a_k(t)| \le C_1, \quad \forall k = 1, \dots, \infty. \tag{2.13}$$

By Parseval's identity we have

$$\|u(t)\|_V^2 \sim \|u(t)\|_{D(A^{1/2})}^2 \sim \sum_{k=1}^{\infty} \frac{|a_k(t)|^2}{\mu_k^2},$$

and consequently we conclude that

$$\|u(t)\|_V^2 \le C_1^2 \sum_{k=1}^{\infty} \frac{1}{\mu_k^2} \le C_2, \forall t \in [0,T],$$

for some positive constant C_2 (depending on T) since (2.7) guarantees the convergence of the series $\sum_{k=1}^{\infty} 1/\mu_k^2$. This implies that the series

$$\sum_{k=1}^{\infty} \frac{a_k(t)}{\mu_k^2} \phi_k$$

is convergent in $L^\infty([0,T]; V)$ and then proves that $u \in C([0,T]; V)$, as limit of elements from $C([0,T]; V)$ (the truncated series).

Similarly by direct calculations we have

$$\|\partial_t u(t)\|_H^2 = \sum_{k=1}^{\infty} \frac{|\partial_t a_k(t)|^2}{\mu_k^4} \le C \sum_{k=1}^{\infty} \frac{1}{\mu_k^2},$$

for some positive constant C (depending on T), and we conclude as before that $u \in C^1([0,T]; H)$. $\square$

Further consider the Petrovsky system

$$\begin{aligned} &\partial_t^2\phi - A\phi = f \quad \text{in }]0,T[, \\ &\phi(0) = \phi_0, \quad \partial_t\phi(0) = \phi_1, \end{aligned} \tag{2.14}$$

where (ϕ_0, ϕ_1) belongs to $V \times H$ and $f \in L^1(]0,T[; H)$. It is well known that this system has a unique solution $\phi \in C([0,T]; V) \cap C^1([0,T]; H)$. Using the direct and inverse estimates of the system (2.14) (see Theorems IV.3.1 and IV.3.3 and Appendix I of [6] and Theorems 2.6 and 6.7 in [4]) and the arguments of Theorem IV.3.6 of [6], we obtain the next (weak) observability estimates.

Lemma 2.3. *For each $a \in V'$ there exists a unique solution v in $C([0,T]; H) \cap C^1([0,T]; V')$ of the equation*

$$\begin{aligned} &\partial_t^2 v - Av = 0 \quad \text{in }]0,T[, \\ &v(0) = 0, \quad \partial_t v(0) = a. \end{aligned} \tag{2.15}$$

Moreover for any $T > 0$, there exist two positive constants C_1 and C_2 depending on T such that

$$C_1\|a\|_{V'} \le \|v''(0,.)\|_{H^{-1}(0,T)} \le C_2\|a\|_{V'}, \tag{2.16}$$

where, as usual, $H^{-1}(0,T)$ is the dual space of $H_0^1(0,T)$.

Let us also give a consequence of the identity with multiplier to the solution u of problem (2.1), namely the hidden regularity of $u''(0,.)$.

Lemma 2.4. *Let $u \in C([0,T]; V) \cap C^1([0,T]; H)$ be the unique solution of (2.1). Then for all $T > 0$, $u''(0,.)$ belongs to $L^2(0,T)$ with the estimate*

$$\|u''(0,.)\|_{L^2(0,T)} \le C(\|u\|_{C([0,T];V)} + \|u\|_{C^1([0,T];H)}), \tag{2.17}$$

for some positive constant C depending on T.

Proof. We set $f(x,t) = \lambda(t)a(x)$ and remark that $f \in L^1((0,T), H^{-1}(0,1))$. We now approximate f by a sequence of more regular data $f_n(x,t) = \lambda(t)a_n(x) \in L^1((0,T), L^2(0,1))$ such that

$$f_n \to f \quad \text{in } L^1((0,T), H^{-1}(0,1)) \text{ as } n \to \infty. \tag{2.18}$$

Namely for n large enough, we take a_n in the form

$$a_n = \sum_{k=1}^{K} \alpha_k \phi_{kn}, \tag{2.19}$$

where $\phi_{kn}(x) = n(\phi(n(x - \xi_k)))$, for all $x \in [0,1]$ with a fixed nonnegative function $\phi \in \mathcal{D}(\mathbb{R})$ with support in $[-1,1]$ and such that $\int_{-1}^{1} \phi(x)dx = 1$.

Let u_n be the solution of (2.1) with datum a_n. Then one has

$$u_n \to u \text{ in } C([0,T]; V) \cap C^1([0,T]; H). \tag{2.20}$$

Now we may apply the identity (IV.3.15) of [6] to u_n with the multiplier q defined by

$$q(x) = (x-1)\eta(x), \quad \forall x \in [0,1],$$

with

$$\eta(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{\xi_1}{3}], \\ 0 & \text{if } x \in [\frac{2\xi_1}{3}, 1]. \end{cases}$$

This choice guarantees that $q\cdot\nu=1$ at $x=0$, $q\cdot\nu=0$ at $x=1$ and that there exists a positive integer $N(\xi_1)$ such that

$$f_n\cdot q\equiv 0,\quad \forall n>N(\xi_1).$$

With this choice the identity (IV.3.15) in [6], for $n>N(\xi_1)$, yields

$$\begin{aligned}\frac{1}{2}\int_0^T |u_n''(0,t)|^2dt &= \int_0^1 \partial_t u_n(x,t)q(x)u_n'(x,t)dx|_0^T\\ &+\frac{1}{2}\int_0^T\int_0^1 q'(|\partial_t u_n'|^2-|u_n''(x,t)|^2)dx\,dt\\ &+2\int_0^1\int_0^T |u_n''(x,t)|^2)dx\,dt.\end{aligned}\tag{2.21}$$

It then remains to estimate the three terms of the above right-hand side. For the first term, Cauchy-Schwarz's inequality gives

$$\begin{aligned}|\int_0^1 \partial_t u_n(x,t)q(x)u_n'(x,t)dx| &\le C\|\partial_t u_n\|_{L^2(0,1)}\|u_n'\|_{L^2(0,1)}\\ &\le C\|u_n\|_{C^1([0,T];L^2(0,1))}\|u_n\|_{C([0,T];H_0^2(0,1))}\\ &\le C\|u_n\|_X^2,\end{aligned}$$

where for short notation we write $\|.\|_X=\|.\|_{C^1([0,T];L^2(0,1))}+\|.\|_{C([0,T];H_0^2(0,1))}$.

On the other hand, we have

$$\begin{aligned}&\int_0^1\int_0^T q'(|\partial_t u_n|^2-|u_n'|^2)dx\,dt\\ &\le CT(\|\partial_t u_n\|^2_{C([0,T];L^2(0,1))}+\|u_n\|^2_{C([0,T];H_0^2(0,1))})\\ &\le 2CT\|u_n\|_X^2,\end{aligned}$$

and similarly

$$\int_0^1\int_0^T |u_n''(x,t)|^2\,dxdt\le T\|u_n\|^2_{C([0,T];H_0^2(0,1))}\le T\|u_n\|_X^2.$$

The three estimates in (2.21) yield

$$\int_0^T |u_n''(0,t)|^2dt\le C(1+T)\|u_n\|_X^2.$$

Passing to the limit in n in that estimate and using (2.20), we conclude that $u''(0,\cdot)$ belongs to $L^2(0,T)$ and obtain the estimate (2.17). $\square$

2.2. **Uniqueness.** We first recall Duhamel's principle (see for instance [9, 2]) which gives the relationship between the solution v of (2.15) and the solution u of (2.10).

Lemma 2.5. *Let $u\in C([0,T];V)\cap C^1([0,T];H)$ be the unique solution of (2.10) with datum a in the form* (2.11) *and let $v\in C([0,T];H)\cap C^1([0,T];V')$ be the unique solution of* (2.15) *with initial speed a. Then*

$$u(t)=(Kv)(t),\forall t\in]0,T[,\tag{2.22}$$

where K is defined by

$$(K\psi)(t)=\int_0^t \lambda(t-s)\psi(s)\,ds,\forall t\in]0,T[,\tag{2.23}$$

and is a bounded operator from $L^2(0,T)$ into itself.

We can now recall the following result proved in [7] (see also [2]).

Lemma 2.6. *If $\lambda \in C^1([0,T])$ satisfies (2.2) then the bounded operator K from $L^2(0,T)$ into itself defined by (2.23) can be extended to a bounded operator from $H_{-1}(0,T)$ onto $L^2(0,T)$ and satisfying*

$$C_1\|K\psi\|_{L^2(0,T)} \le \|\psi\|_{H_{-1}(0,T)} \le C_2\|K\psi\|_{L^2(0,T)}, \forall \psi \in H_{-1}(0,T), \tag{2.24}$$

for some positive constants C_1, C_2.

Here and below the space $H_{-1}(0,T)$ is defined as the dual space of

$$^0H^1(0,T) = \{v \in H^1(0,T) : v(T) = 0\},$$

which is a Hilbert space with the norm

$$\|v\|_{^0H^1(0,T)} = \Big(\int_0^T |\partial_t v(t)|^2 dt\Big)^{1/2}.$$

The above Lemma does not hold in the standard Sobolev space $H^{-1}(0,T)$ but we showed in Lemma 4.3 of [7] that a similar result holds in $H^{-1}(0,T)$ if we replace the operator K by the operator PK, where P is the orthogonal projection (in $L^2(0,T)$) on $\Lambda^\perp$ defined by

$$\Lambda^\perp = \{\eta \in L^2(0,T) : (\lambda, \eta)_{L^2(0,T)} = 0\}.$$

Namely we may state the (see Lemma 4.3 of [7] for the detailed proof).

Lemma 2.7. *If $\lambda \in C^1([0,T])$ satisfies (2.2) then the bounded operator PK from $L^2(0,T)$ into itself can be extended to a bounded operator from $H^{-1}(0,T)$ into $L^2(0,T)$ and satisfying*

$$C_3\|PK\psi\|_{L^2(0,T)} \le \|\psi\|_{H^{-1}(0,T)} \le C_4\|PK\psi\|_{L^2(0,T)}, \quad \forall \psi \in H^{-1}(0,T), \tag{2.25}$$

for some positive constants C_3, C_4.

Corollary 2.8. *Let $u \in C([0,T];V) \cap C^1([0,T];H)$ be the unique solution of (2.10) with datum a in the form* (2.11) *and let $v \in C([0,T];H) \cap C^1([0,T];V')$ be the unique solution of* (2.15) *with initial speed a. Then for all $T > 0$ we have*

$$Pu''(0,\cdot) = PKv''(0,\cdot) \quad \textit{in } L^2(0,T). \tag{2.26}$$

Proof. As in Lemma 2.4 let u_n (resp. v_n) be the solution of (2.10) (resp. (2.15)) with datum $a_n \in V$ (resp. with initial speed a_n) satisfying

$$a_n \to a \text{ in } V'. \tag{2.27}$$

For these solutions their regularity and Lemma 2.5 allow to write

$$u_n''(0,\cdot) = Kv_n''(0,\cdot) \quad \text{in } L^2(0,T).$$

And therefore

$$Pu_n''(0,\cdot) = PKv_n''(0,\cdot) \quad \text{in } L^2(0,T).$$

We conclude by passing to the limit in n and using Lemmas 2.7, 2.3 and 2.4. □

We are now ready to formulate the uniqueness result.

Theorem 2.9. *Fix $T > 0$. Let u^1 (resp. u^2) in $C([0,T];V) \cap C^1([0,T];H)$ be the unique solution of (2.10) with datum a^1 (resp. a^2) in the form*

$$\langle a^l, \phi \rangle_{V'-V} = \sum_{k=1}^{K^l} \alpha_k^l \phi(\xi_k^l), \forall \phi \in V, l = 1, 2,$$

for some positive integers K^l, real numbers α_k^l and points $\xi_k^l \in]0,1[$. If

$$(u^1)''(0,t) = (u^2)''(0,t), \quad \forall t \in (0,T),$$

as elements of $L^2(0,T)$, then $a^1 = a^2$, or equivalently $K^1 = K^2$, $\alpha_k^1 = \alpha_k^2$, $\xi_k^1 = \xi_k^2$.

Proof. We remark that $u = u^1 - u^2$ satisfies (2.10) with datum $a = a^1 - a^2$ which is still in the form (2.11). By the assumption we further have

$$u''(0,\cdot) = 0 \quad \text{in } L^2(0,T).$$

This implies that

$$Pu''(0,\cdot) = 0 \quad \text{in } L^2(0,T).$$

Therefore, by Corollary 2.8 and Lemma 2.7 we get $v''(0,\cdot) = 0$ in $H^{-1}(0,T)$, where v is the unique solution of (2.15) with initial speed a. The application of Lemma 2.3 allows to conclude that $a = 0$. □

2.3. **Stability.** For a fixed positive integer K, we denote

$$\Sigma = \{A = (\alpha_k, \xi_k)_{k=1}^K : \alpha_k \in \mathbb{R} \setminus 0, \xi_k \in]0,1[\}.$$

The above uniqueness result implies that the mapping

$$\eta : \Sigma \to L^2(0,T) : A := (\alpha_k, \xi_k)_{k=1}^K \to u''(0,\cdot),$$

is injective, where u is the unique solution of (2.10) with datum a in the form (2.11). The stability means that the inverse mapping $\eta^{-1} : u''(0,\cdot) \to A$ is continuous once Σ is equipped with the natural distance

$$d(A^1, A^2) = \sum_{k=1}^K (|\alpha_k^1 - \alpha_k^2| + |\xi_k^1 - \xi_k^2|),$$

when $A^l := (\alpha_k^l, \xi_k^l)_{k=1}^K, l = 1, 2$.

We actually will show a slightly weaker result than the continuity of this mapping by only showing that the inverse of the restriction of η to the ball $B(A, \epsilon)$ is Lipschitz continuous for some $\epsilon > 0$ small enough depending on A. Namely we take

$$\epsilon \le \frac{1}{4} \min_{k \ne k'} |\xi_k - \xi_{k'}|, \tag{2.28}$$

$$\epsilon \le \frac{1}{4} \min_k |\xi_k|, \epsilon \le \frac{1}{4} \min_k |1 - \xi_k| \tag{2.29}$$

$$\epsilon \le \frac{1}{2} \min_k |\alpha_k|. \tag{2.30}$$

Under these assumptions we can prove the following conditional stability result.

Theorem 2.10. *Fix $T > 0$ and suppose that $A^2 = (\alpha_k^2, \xi_k^2)_{k=1}^K$ is in $\Sigma \cap B(A, \epsilon)$ with $\epsilon > 0$ satisfying the above constraints. Then there exists a constant C depending on T, $\min_{k \ne k'} |\xi_k - \xi_{k'}|$ and $\min_k |\alpha_k|$ such that*

$$\sum_{k=1}^K (|\alpha_k - \alpha_k^2| + |\xi_k - \xi_k^2|) \le C \|u''(0,\cdot) - (u^2)''(0,\cdot)\|_{L^2(0,T)}. \tag{2.31}$$

Proof. The proof of Theorem 2.9 clearly shows that
$$\|a-a^2\|_{V'} \le C\|u''(0,\cdot)-(u^2)''(0,\cdot)\|_{L^2(0,T)}. \tag{2.32}$$
Therefore it remains to estimate from below the norm of $a-a^2$ in V'. For that purpose we recall that
$$\|a-a^2\|_{V'} = \sup_{\phi\in V,\phi\neq 0} \frac{|<a-a^2,\phi>|}{\|\phi\|_V},$$
and use appropriate test functions ϕ. First we take
$$\phi^{(k)}(x) = \phi_1(\frac{x-\xi_k}{\delta}), \quad \forall x \in]0,1[,$$
where $\delta = \frac{1}{4}\min_{k\neq k'}|\xi_k-\xi_{k'}|$ and ϕ_1 is a fixed function defined by
$$\phi_1(\hat{x}) = \begin{cases} 4(3/2+\hat{x})^2(4\hat{x}-3) & \text{if } -3/2<\hat{x}\le -1,\\ \hat{x} & \text{if } -1<\hat{x}\le 1,\\ 4(-3/2+\hat{x})^2(4\hat{x}-3) & \text{if } 1\le \hat{x}<3/2,\\ 0 & \text{otherwise.}\end{cases}$$
With this choice we have
$$\langle a-a^2,\phi^{(k)}\rangle = \alpha_k\phi^{(k)}(\xi_k) - \alpha_k^2\phi^{(k)}(\xi_k^2) = \alpha_k^2(\phi^{(k)}(\xi_k)-\phi^{(k)}(\xi_k^2)),$$
since $\phi^{(k)}(\xi_k)=0$. By the finite increment theorem and the fact that $|\xi_k-\xi_k^2|<\epsilon$, we then obtain
$$|\langle a-a^2,\phi^{(k)}\rangle| = \frac{|\alpha_k^2|}{\delta}|\xi_k-\xi_k^2|.$$
This estimate yields
$$|\alpha_k^2||\xi_k-\xi_k^2| \le \delta|\langle a-a^2,\phi^{(k)}\rangle| \le \delta\|a-a^2\|_{V'}\|\phi^{(k)}\|_V,$$
and leads to
$$|\alpha_k^2||\xi_k-\xi_k^2| \le \frac{C_1}{\sqrt{\delta}}\|a-a^2\|_{V'}, \tag{2.33}$$
for some positive constant C_1 since one readily checks that $\|\phi^{(k)}\|_V = \frac{C_1}{\delta^{\frac{3}{2}}}$.

From the third assumption on ϵ, we have
$$|\alpha_k^2| \ge m/2,$$
where $m=\min_k|\alpha_k|$. These two estimates finally give
$$|\xi_k-\xi_k^2| \le \frac{2C_1}{m\sqrt{\delta}}\|a-a^2\|_{V'}.$$
Now we take
$$\phi_2^{(k)}(x) = \phi_2(\frac{x-\xi_k}{\delta}), \quad \forall x\in]0,1[,$$
when $\phi_2 \in \mathcal{D}(]-1,1[)$ satisfies $\phi_2(0)=1$. With this choice we have
$$\begin{aligned}\langle a-a^2,\phi_2^{(k)}\rangle &= \alpha_k\phi_2^{(k)}(\xi_k) - \alpha_k^2\phi_2^{(k)}(\xi_k^2)\\ &= (\alpha_k-\alpha_k^2)\phi_2^{(k)}(\xi_k) + \alpha_k^2(\phi_2^{(k)}(\xi_k)-\phi_2^{(k)}(\xi_k^2)),\\ &= (\alpha_k-\alpha_k^2) + \alpha_k^2(\phi_2^{(k)}(\xi_k)-\phi_2^{(k)}(\xi_k^2)).\end{aligned}$$
Therefore by the finite increment theorem we obtain as before
$$|\alpha_k-\alpha_k^2| \le |\langle a-a^2,\phi_2^{(k)}\rangle| + \frac{S}{\delta}|\alpha_k^2||\xi_k-\xi_k^2|,$$

where $S = \max_{-1\leq\hat{x}\leq 1} |\phi_2'(\hat{x})|$ and by estimate (2.33) we get

$$|\alpha_k - \alpha_k^2| \leq |\langle a - a^2, \phi_2^{(k)}\rangle| + \frac{SC_1}{\delta^{\frac{3}{2}}}\|a - a^2\|_{V'}.$$

Since $\|\phi_2^{(k)}\|_V = \frac{C_2}{\delta^{\frac{3}{2}}}$ for some $C_2 > 0$, we obtain

$$|\alpha_k - \alpha_k^2| \leq (\frac{C_2}{\delta^{\frac{3}{2}}} + \frac{SC_1}{\delta^{\frac{3}{2}}})\|a - a^2\|_{V'}.$$

□

In the above theorem if as in [2] we are only interested in the stability of the locations of the point sources, i.e. if we assume that $\alpha_k^2 = \alpha_k$, then we can obtain a more accurate estimate under less assumptions on ϵ, namely we have the

Theorem 2.11. *Fix $T > 0$ and suppose that $A^2 = (\alpha_k, \xi_k^2)_{k=1}^K$ is in $\Sigma \cap B(A, \epsilon)$ with $\epsilon > 0$ satisfying* (2.28) *and* (2.29). *Then there exists a constant C depending on T, $\min_{k\neq k'} |\xi_k - \xi_{k'}|$ and $\min_k |\alpha_k|$ such that*

$$\sum_{k=1}^K |\xi_k - \xi_k^2| \leq C\|u''(0, \cdot) - (u^2)''(0, \cdot)\|_{L^2(0,T)}. \tag{2.34}$$

Proof. It suffices to take

$$\phi^{(k)}(x) = \phi_1(\frac{x - \xi_k}{\delta}) \quad \text{on }]0, 1[,$$

with the same ϕ_1 as before and use the above arguments. □

2.4. **Reconstruction.** For the reconstruction of the point sources from boundary measurements we follow the point of view of [9] which consists in using the following exact controllability result:

Lemma 2.12. *Fix $T > 0$. Then for every $\phi \in V$, there exist a unique control $v \in H_0^1(0, T)$, such that the (weak) solution $\psi \in C([0, T]; H) \cap C^1([0, T]; V')$ of*

$$\begin{gathered}
\partial_t^2\psi(x, t) + \psi^{(4)}(x, t) = 0 \quad \text{in } Q_T, \\
\psi(0, t) = \psi(1, t) = 0, \quad \forall t \in]0, T[, \\
\psi'(0, t) = v, \psi'(1, t) = 0, \quad \forall t \in]0, T[, \\
\psi(\cdot, 0) = \phi, \partial_t\psi(\cdot, 0) = 0 \quad \text{in }]0, 1[,
\end{gathered} \tag{2.35}$$

satisfies

$$\psi(\cdot, T) = \partial_t\psi(\cdot, T) = 0. \tag{2.36}$$

Proof. This lemma is a direct consequence of Lemma 2.3 and of the Hilbert Uniqueness Method of Lions [6, Th.IV.3.4], see also [4]. Note that ψ is only a weak solution of the system (2.35) with the final conditions (2.36) in the sense that ψ is the unique solution of (using the transposition method)

$$\int_{Q_T} \psi f \, dx \, dt = -\langle\partial_t\varphi(0), \phi\rangle_{V'-V} + \langle\varphi''(0), v\angle_{H^{-1}(0,T)-H_0^1(0,T)}, \tag{2.37}$$

for all $f \in L^1(0, T; H)$, $\varphi_0 \in H$, $\varphi_1 \in V'$, where $\varphi \in C([0, T]; H) \cap C^1([0, T]; V')$ is the unique solution of (whose existence follows from Lemma 2.3

$$\begin{gathered}
\partial_t^2\varphi = A\varphi + f \quad \text{in }]0, T[, \\
\varphi(T) = \varphi_0, \partial_t\varphi(T) = \varphi_1.
\end{gathered}$$

□

In view of Lemma 2.12 we can define a bounded linear operator

$$\Pi : V \to H_0^1(0,T) : \phi \to v,$$

where v is the control from the above Lemma driving the system (2.35) to rest at time T.

We further use the adjoint $K^\star_{L^2}$ of the operator K as (bounded) operator from $L^2(0,T)$ into itself and which is given by (see section 6 of [9])

$$(K^\star_{L^2}\eta)(t) = \int_t^T \lambda(s-t)\eta(s)\,ds, \quad 0 < t < T,$$

for all $\eta \in L^2(0,T)$. By the assumption (2.2) we even have (see section 6 of [9])

$$R(K^\star_{L^2}) = {}^0H^1(0,T).$$

Consequently for all $\psi \in {}^0H^1(0,T)$ there exists a unique $\eta \in L^2(0,T)$ solution of $K^\star_{L^2}\eta = \psi$ (since $\ker K^\star_{L^2} = R(K)^\perp = \{0\}$); equivalently, η is solution of the Volterra equation of the first kind

$$\int_t^T \lambda(s-t)\eta(s)\,ds = \psi(t), \quad 0 < t < T.$$

We then define the mapping Φ from ${}^0H^1(0,T)$ into $L^2(0,T)$ by

$$\psi \to \eta := \Phi\psi,$$

when η is solution of the above integral equation. This means that

$$K^\star_{L^2}\Phi = Id \quad \text{ on } {}^0H^1(0,T). \tag{2.38}$$

Now we can formulate our reconstruction result:

Theorem 2.13. *Fix $T > 0$. For all $k = 1, \dots, \infty$ we define $\theta_k = \Phi\Pi\phi_k$. Let $u \in C([0,T];V) \cap C^1([0,T];H)$ be the unique solution of (2.1) with datum a in the form (2.3). Then for $k = 1, \dots, \infty$ we have*

$$\langle a, \phi_k\rangle = (u''(0,\cdot), \theta_k)_{L^2(0,T)}, \tag{2.39}$$

and then a may be reconstructed by

$$a = \sum_{k=1}^{\infty} \langle a, \phi_k\rangle \phi_k = \sum_{k=1}^{\infty} (u''(0,\cdot), \theta_k)_{L^2(0,T)}\phi_k.$$

Proof. Applying the identity (2.37) with $\varphi = v$, where v is the unique solution of (2.15) with initial speed a we have:

$$\langle a, \phi_k\rangle = \langle v''(0,\cdot), \Pi\phi_k\rangle_{H^{-1}(0,T)-H_0^1(0,T)}. \tag{2.40}$$

To conclude we need to show that

$$\langle v''(0,\cdot), \Pi\phi_k\rangle_{H^{-1}(0,T)-H_0^1(0,T)} = (u''(0,\cdot), \theta_k)_{L^2(0,T)}. \tag{2.41}$$

Let us first prove that there exists $h \in H_{-1}(0,T)$ such that

$$u''(0,\cdot) = Kh, \tag{2.42}$$

and satisfies

$$\langle v''(0,\cdot), \chi\rangle_{H^{-1}(0,T)-H_0^1(0,T)} = \langle h, \chi\rangle_{H_{-1}(0,T)-{}^0H^1(0,T)}, \quad \forall \chi \in H_0^1(0,T). \tag{2.43}$$

Indeed the identity (2.42) follows from Lemmas 2.4 and 2.6; moreover using an approximation sequence of a_n as usual, the corresponding u_n and v_n satisfy

$$v_n''(0,\cdot) \to h \text{ in } H_{-1}(0,T), \text{ as } n \to \infty,$$

due to Lemmas 2.4 and 2.6, while by Lemma 2.3 we have

$$v_n''(0,\cdot) \to v''(0,\cdot) \quad \text{in } H^{-1}(0,T), \text{ as } n \to \infty.$$

The identity (2.43) then follows from the two above convergence properties and the continuity of the mapping $Id^\star$ from $H_{-1}(0,T)$ into $H^{-1}(0,T)$ (see [7]). Now by the definition of θ_k and (2.38) we may write

$$K^\star_{L^2}\theta_k = K^\star_{L^2}\Phi\Pi\phi_k = \Pi\phi_k.$$

Therefore, using (2.43) and the above identity, the left-hand side of (2.41) may be transformed as follows

$$\begin{aligned}\langle v''(0,\cdot),\Pi\phi_k\rangle_{H^{-1}(0,T)-H^1_0(0,T)} &= \langle h,\Pi\phi_k\rangle_{H_{-1}(0,T)-{}^0H^1(0,T)}\\ &= \langle h,K^\star_{L^2}\theta_k\rangle_{H_{-1}(0,T)-{}^0H^1(0,T)},\end{aligned}$$

and from the embeddings ${}^0H^1(0,T) \hookrightarrow L^2(0,T) \hookrightarrow H_{-1}(0,T)$, we get

$$\langle h,K^\star_{L^2}\theta_k\rangle_{H_{-1}(0,T)-{}^0H^1(0,T)} = (Kh,\theta_k)_{L^2(0,T)}.$$

This proves (2.41) since the above right-hand side coincides with the right-hand side of (2.41) due to (2.42). □

3. The second Petrovsky system

3.1. **Preliminaries.** We consider the initial boundary value problem for the beam equation with supported boundary conditions:

$$\begin{gathered}\partial_t^2 u(x,t) + u^{(4)}(x,t) = \lambda(t)a(x) \quad \text{in } Q_T,\\ u(\cdot,0)=0, \quad \partial_t u(\cdot,0)=0 \quad \text{in }]0,1[,\\ u(x,t)=u''(x,t)=0, \quad \text{for } x=0,1 \text{ and } \forall t\in]0,T[,\end{gathered} \tag{3.1}$$

where a is in the form (2.3).

As in section 2, our goal is to identify the datum a from boundary measurements, namely from the values of $u'(0,t)$, for $0<t<T$.

To analyse the system (3.1), we define the operator A on the Hilbert space $H=L^2(0,1)$ endowed with the inner product (2.4) as follows:

$$\begin{gathered}D(A) = \{u \in H^4(0,1)\cap H^1_0(0,1) : u''(0)=u''(1)=0\},\\ \forall u \in D(A) : Au = -u^{(4)}.\end{gathered}$$

As before A is a negative selfadjoint operator with a compact resolvent since A is the Friedrichs extension of the triple (H,V,a), where $V=\{u\in H^2(0,1)\cap H^1_0(0,1) : u''(0)=u''(1)=0\}$ equipped with the inner product (2.5) and a is given by (2.6).

Recall that the spectrum $\{\lambda_k\}_{k=1}^\infty$ of $-A$ is given by $\lambda_k = k^4\pi^4$ and the associated eigenfunctions are given by $\phi_k(x) = \sqrt{2}\sin(k\pi x)$ for all $k=1,\dots,\infty$. As in Theorem 2.2, we may prove the following statement.

Theorem 3.1. *The beam equation (3.1) has a unique (weak) solution u satisfying*

$$u \in C([0,T];V)\cap C^1([0,T];H).$$

Proof. The system (3.1) is equivalently written in the form (2.10) and then

$$u(t) = \sum_{k=1}^{\infty} \frac{1}{k^2\pi^2} \int_0^t \sin(k^2\pi^2(t-s))\lambda(s)\,ds\langle a, \phi_k\rangle\phi_k,$$

or equivalently, by integration by parts in the above integral:

$$u(t) = \sum_{k=1}^{\infty} \frac{a_k(t)}{\lambda_k}\phi_k, \tag{3.2}$$

where a_k is here given by

$$a_k(t) = \langle a, \phi_k\rangle(\lambda(t) - \lambda(0)\cos(k^2\pi^2 t) - \int_0^t \cos(k^2\pi^2(t-s))\lambda'(s)\,ds).$$

The remainder of the proof is similar to the one of Theorem 2.2. □

Using the direct and inverse estimates of Theorem 2.10 and 6.11 of [4], we obtain the next (weak) observability estimates.

Lemma 3.2. *For each $a \in V'$ there exists a unique solution v in $C([0,T];H) \cap C^1([0,T];V')$ of*

$$\begin{gathered} \partial_t^2 v - Av = 0 \quad \text{in }]0,T[, \\ v(0) = 0, \quad \partial_t v(0) = a. \end{gathered} \tag{3.3}$$

Moreover for $T > 0$ there exist two positive constants C_1 and C_2 depending on T such that

$$C_1\|a\|_{H^{-1}(0,1)} \le \|v'(0,.)\|_{L^2(0,T)} \le C_2\|a\|_{H^{-1}(0,1)}. \tag{3.4}$$

3.2. **Uniqueness.** As in subsection 2.2, using Lemma 3.2 instead of Lemma 2.3, we obtain the following uniqueness result.

Theorem 3.3. *Fix $T > 0$. Let u^1 (resp. u^2) in $C([0,T];V) \cap C^1([0,T];H)$ be the unique solution of (3.1) with datum a^1 (resp. a^2) in the form*

$$\langle a^l, \phi\rangle_{V'-V} = \sum_{k=1}^{K^l} \alpha_k^l \phi(\xi_k^l), \quad \forall\phi \in V, l = 1, 2,$$

for some positive integers K^l, real numbers α_k^l and points $\xi_k^l \in]0,1[$. If

$$(u^1)'(0,t) = (u^2)'(0,t), \quad \forall t \in (0,T),$$

as elements of $L^2(0,T)$, then $K^1 = K^2$, $\alpha_k^1 = \alpha_k^2$, $\xi_k^1 = \xi_k^2$.

Proof. As before we see that $u = u^1 - u^2$ satisfies (3.1) with datum $a = a^1 - a^2$ and

$$u'(0,\cdot) = 0 \text{ in } L^2(0,T),$$

by the assumption. This implies that $Pu'(0,\cdot) = 0$ in $L^2(0,T)$. Therefore, by Corollary 2.8 and Lemma 2.7 we get

$$v'(0,\cdot) = 0 \quad \text{in } H^{-1}(0,T),$$

and consequently

$$v'(0,\cdot) = 0 \text{ in } L^2(0,T)$$

where v is the unique solution of (3.3) with initial speed a. Lemma 3.2 finally yields $a = 0$. □

3.3. **Stability.** Using the notation from subsection 2.3 and under the same assumptions we have the following conditional stability result.

Theorem 3.4. *Fix $T > 0$. Suppose that $A^2 = \big(\alpha_k^2, \xi_k^2\big)_{k=1}^K$ is in $\Sigma \cap B(A, \epsilon)$ with $\epsilon > 0$ satisfying* (2.28), (2.29) *and (2.30). Then there exists a constant C depending on T, $\min_{k\neq k'} |\xi_k - \xi_{k'}|$ and $\min_k |\alpha_k|$ such that*

$$\sum_{k=1}^K (|\alpha_k - \alpha_k^2| + |\xi_k - \xi_k^2|) \le C(1 + \sqrt{\epsilon})\|u'(0,t) - (u^2)'(0,t)\|_{L^2(0,T)}.$$

Proof. By Theorem 3.3 we have

$$\|a - a^2\|_{H^{-1}(0,1)} \le C\|u'(0,t) - (u^2)'(0,t)\|_{L^2(0,T)}.$$

The conclusion now follows from the next estimate proved in Theorem 5.1 of [7]

$$\sum_{k=1}^K (|\alpha_k - \alpha_k^2| + |\xi_k - \xi_k^2|) \le C(1 + \sqrt{\epsilon})\|a - a^2\|_{H^{-1}(0,1)}.$$

□

If we assume that $\alpha_k^2 = \alpha_k$, then using Theorem 5.2 of [7] we can obtain the following result.

Theorem 3.5. *Fix $T > 0$ and suppose that $A^2 = \big(\alpha_k, \xi_k^2\big)_{k=1}^K$ is in $\Sigma \cap B(A, \epsilon)$ with $\epsilon > 0$ satisfying* (2.28) *and* (2.29). *Then there exists a constant C depending on T, $\min_{k\neq k'} |\xi_k - \xi_{k'}|$ and $\min_k |\alpha_k|$ such that*

$$\sum_{k=1}^K |\xi_k - \xi_k^2| \le C\sqrt{\epsilon}|u'(0,t) - (u^2)'(0,t)\|_{L^2(0,T)}.$$

3.4. **Reconstruction.** For the reconstruction of point sources we could follow the arguments of subsection 2.4 and obtain a reconstruction result similar to Theorem 2.13. We here present an alternative result following the point of view of [2] based on the explicit knowledge of the eigenvalues and the eigenfunctions and some properties of Fourier series. This result seems to be more realistic in the practical point of view than the first one but the prize to pay is that we need boundary observations on a timelength $\frac{1}{\pi}$. For the sake of simplicity, we only consider the case of two point sources, namely

$$(\alpha_1, \xi_1), \quad (\alpha_2, \xi_2), \quad 0 < \xi_1 < \xi_2 < 1.$$

Now we introduce the operator from $L^2(0, \frac{1}{\pi})$ to $L^2(0, \frac{1}{\pi})$ defined by

$$(Lf)(t) = \int_0^t \lambda'(t-s)f(s)ds, \quad 0 \le t \le \frac{1}{\pi}. \tag{3.5}$$

By the assumption (2.2), we see that $-(\lambda(0) + L)^{-1}$ corresponds to the solution of a Volterra equation of second kind and therefore, $-(\lambda(0) + L)^{-1}$ is a bounded operator from $L^2(0, \frac{1}{\pi})$ into itself. We further assume

$$\int_0^{1/\pi} ((\lambda(0) + L)^{-1}\lambda)(t)\, dt \neq 0. \tag{3.6}$$

Henceforth we denote by $(\cdot,\cdot)$, the $L^2(0,\frac{1}{\pi})$-inner product; i.e.,

$$(\phi,\psi)=\int_0^{1/\pi}\phi(t)\psi(t)dt.$$

Moreover, let us set $e_k(t)=\cos(k^2\pi^2t)$, $k\in\mathbb{N}$, and

$$\psi_k=(\lambda(0)+L^\star)^{-1}e_k,\quad k\in\mathbb{N},$$

where $L^\star{:}L^2(0,\frac{1}{\pi})\to L^2(0,\frac{1}{\pi})$ is the adjoint operator of L given by

$$L^\star\psi(t)=\int_t^{1/\pi}\lambda'(s-t)\psi(s)\,ds,\quad 0\le t\le\frac{1}{\pi},$$

and consequently ψ_k is the solution of the Volterra equation of the second kind

$$\lambda(0)\psi_k(t)+\int_t^{1/\pi}\lambda'(s-t)\psi_k(s)\,ds=\cos(k^2\pi^2t),\quad 0\le t\le\frac{1}{\pi}.$$

Remark 3.6. We see directly that the assumption (3.6) is equivalent to

$$(\lambda,\psi_0)\neq 0. \tag{3.7}$$

Now we can state our reconstruction result (compare with [2, Theorem 3]).

Theorem 3.7. *Assume that (3.6) holds. Then for all $k\ge 1$ we have the following identity*

$$\alpha_1\sin(k\pi\xi_1)+\alpha_2\sin(k\pi\xi_2)=k^3\pi^4\Big(\frac{(u'(0,\cdot),\psi_0)}{(\lambda,\psi_0)}(\lambda,\psi_k)-(u'(0,\cdot),\psi_k)\Big). \tag{3.8}$$

In particular, if we assume that $\alpha_1=\alpha_2=1$, then

$$\xi_1=\frac{1}{\pi}\arcsin\theta_1,\quad \xi_2=\frac{1}{\pi}\arcsin\theta_2,$$

with θ_1 and θ_2 being the zeroes of

$$\theta^2-a\theta+\frac{b+4a^3-3a}{12a}=0, \tag{3.9}$$

where

$$a=\pi^4\frac{(u'(0,\cdot),\psi_0)}{(\lambda,\psi_0)}(\lambda,\psi_1)-\pi^4(u'(0,\cdot),\psi_1),$$
$$b=27\pi^4\frac{(u'(0,\cdot),\psi_0)}{(\lambda,\psi_0)}(\lambda,\psi_3)-27\pi^4(u'(0,\cdot),\psi_3).$$

Proof. We remark that (3.2) may be equivalently written

$$\begin{aligned}u(x,t)=&-2\sum_{k=1}^{\infty}\frac{\alpha_1\sin(k\pi\xi_1)+\alpha_2\sin(k\pi\xi_2)}{k^4\pi^4}(\lambda(0)+L)e_k(t)\sin(k\pi x)\\&+2\sum_{k=1}^{\infty}\frac{\alpha_1\sin(k\pi\xi_1)+\alpha_2\sin(k\pi\xi_2)}{k^4\pi^4}\sin(k\pi x)\lambda(t).\end{aligned}$$

Setting

$$g(\xi,x)=2\sum_{k=1}^{\infty}\frac{\sin(k\pi\xi)\sin(k\pi x)}{k^4\pi^4},$$

the above identity may be written as

$$\begin{aligned}-u(x,t) = 2\sum_{k=1}^{\infty} &\frac{\alpha_1 \sin(k\pi\xi_1) + \alpha_2 \sin(k\pi\xi_2)}{k^4\pi^4}(\lambda(0)+L)e_k(t)\sin(k\pi x) \\ &- \lambda(t)(\alpha_1 g(\xi_1,x) + \alpha_2 g(\xi_2,x)).\end{aligned}$$

Differentiating this identity with respect to x, we obtain

$$\begin{aligned}-u'(x,t) = 2\sum_{k=1}^{\infty} &\frac{\alpha_1 \sin(k\pi\xi_1) + \alpha_2 \sin(k\pi\xi_2)}{k^3\pi^3}(\lambda(0)+L)e_k(t)\cos(k\pi x) \\ &- \lambda(t)(\alpha_1 g'(\xi_1,x) + \alpha_2 g'(\xi_2,x)),\end{aligned}$$

so that we can substitute $x = 0$ to get

$$\begin{aligned}-u'(0,t) = \sum_{k=1}^{\infty} &\frac{2}{k^3\pi^3}(\alpha_1 \sin(k\pi\xi_1) + \alpha_2 \sin(k\pi\xi_2))(\lambda(0)+L)e_k(t) \\ &- \lambda(t)(\alpha_1 g'(\xi_1,0) + \alpha_2 g'(\xi_2,0)).\end{aligned} \tag{3.10}$$

On the other hand, we note that

$$((\lambda(0)+L)e_k, (\lambda(0)+L^\star)^{-1}e_j) = \begin{cases} 0 & \text{if } k \neq j,\ k,j \geq 0, \\ \frac{1}{2\pi} & \text{if } k = j. \end{cases}$$

Therefore, in (3.10) taking the $L^2(0, \frac{1}{\pi})$-inner product with $\psi_j = (\lambda(0)+L^\star)^{-1}e_j$, we obtain

$$-(u'(0,\cdot),\psi_0) + (\lambda,\psi_0)(\alpha_1 g'(\xi_1,0) + \alpha_2 g'(\xi_2,0)) = 0, \tag{3.11}$$

$$\begin{aligned}-(u'(0,\cdot),\psi_j) &+ (\lambda,\psi_j)(\alpha_1 g'(\xi_1,0) + \alpha_2 g'(\xi_2,0)) \\ &= \frac{\alpha_1 \sin(j\pi\xi_1) + \alpha_2 \sin(j\pi\xi_2)}{j^3\pi^4}, \forall j \geq 1.\end{aligned} \tag{3.12}$$

The identity (3.11) is equivalent to

$$\alpha_1 g'(\xi_1,0) + \alpha_2 g'(\xi_2,0) = \frac{(u'(0,\cdot),\psi_0)}{(\lambda,\psi_0)},$$

which we combine with (3.12) to obtain (3.8).

Now if we assume that $\alpha_1 = \alpha_2 = 1$, then (3.8) for $k = 1,3$ gives with the notation from the statement of the Theorem:

$$\sin(\pi\xi_1) + \sin(\pi\xi_2) = a,$$
$$\sin(3\pi\xi_1) + \sin(3\pi\xi_2) = b.$$

Using the trigonometric rule $\sin 3\rho = 3\sin\rho - 4\sin^3\rho$ and the above identities we obtain

$$\sin(\pi\xi_1)\sin(\pi\xi_2) = \frac{b + 4a^3 - 3a}{12a}.$$

Consequently the roots θ_1, θ_2 of (3.9) are equal to $\sin(\pi\xi_1)$ and $\sin(\pi\xi_2)$ respectively. □

Remark 3.8. For an arbitrary $T > 0$, the above reconstruction scheme would work if we could find a dual family $(f_k)_{k\in\mathbb{N}}$ to $(e_k)_{k\in\mathbb{N}}$, in the sense that

$$\int_0^T e_k(t)f_l(t)\,dt = \delta_{kl}, \quad \forall k,l \in \mathbb{N}.$$

In that case it would suffice to take

$$\psi_k = (\lambda(0) + L^\star)^{-1} f_k.$$

To our knowledge such a family is not explicitly known except if $T = n/\pi$, for a positive integer n.

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