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ASYMPTOTIC REPRESENTATION OF SOLUTIONS TO THE DIRICHLET PROBLEM FOR ELLIPTIC SYSTEMS WITH DISCONTINUOUS COEFFICIENTS NEAR THE BOUNDARY

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Abstract. We consider variational solutions to the Dirichlet problem for elliptic systems of arbitrary order. It is assumed that the coefficients of the principal part of the system have small, in an integral sense, local oscillations near a boundary point and other coefficients may have singularities at this point. We obtain an asymptotic representation for these solutions and derive sharp estimates for them which explicitly contain information on the coefficients.

1. Introduction

Let $B_+(\delta) = \mathbb{R}^n_+ \cap B(\delta)$, where $\mathbb{R}^n_+ = \{x = (x', x_n) : x_n > 0\}$ and $B(\delta)$ is the ball with the center at the origin and with the radius $\delta > 0$. We consider solutions to the Dirichlet problem

$$\mathcal{L}(x, D_x)u = 0 \quad \text{on } B_+(\delta), \tag{1.1}$$

$$\partial^k_{x_n} u\big|_{x_n=0} = 0 \quad \text{for } k = 0, 1, \dots, m-1,\ |x'| < \delta \tag{1.2}$$

for the differential operator

$$\mathcal{L}(x, D_x)u = L(D_x)u - N(x, D_x)u\,, \tag{1.3}$$

where $D_x = -i\partial_x$ and $L(D_x)$ is a strongly elliptic differential operator with constant $d \times d$-matrix coefficients. The operator

$$N(x, D_x)u = \sum_{|\alpha|,|\beta|\le m} D^\alpha_x \big(N_{\alpha\beta}(x) D^\beta_x u\big) \tag{1.4}$$

will be treated as a perturbation operator and we shall characterize it by the function

$$\kappa(x) = \sum_{|\alpha|,|\beta|\le m} x_n^{2m-|\alpha+\beta|} |N_{\alpha\beta}(x)|. \tag{1.5}$$

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The function κ is assumed to be bounded and the function

$$r \to \int_{r/e<|y|<r} \kappa(y)|y|^{-n} dy \tag{1.6}$$

is small. Other conditions on the operator $\mathcal{L}$ are its ellipticity and a local estimate outside the origin; see (H1)–(H3) in Section 2.1. Under these assumptions, we prove an asymptotic representation for solution u to problem (1.1), (1.2), see Theorems 2.3 and 2.4. Conditions (H1)–(H3) are satisfied for an important particular case $\kappa(y) \le \omega_0$, where ω_0 is a sufficiently small constant, see Remark 2.2. Let us formulate here our results under this assumption.

We are interested in variational solutions to problem (1.1), (1.2). Our goal is to describe the structure of these solutions near the origin and derive explicit, sharp estimates for them.

In order to formulate the main result we introduce some notation. Let $\mathcal{Q}(y)$ be the matrix $\{\mathcal{Q}_{kj}(y)\}_{k,j=1}^d$ with

$$\mathcal{Q}_{kj}(y) = m! \sum_{|\alpha|,|\beta|\le m} (N_{\alpha\beta}(y) D_y^\alpha y_n^m e_k, D_y^\beta E_j(y)), \tag{1.7}$$

where $(\cdot,\cdot)$ is the standard inner product in $\mathbb{C}^d$, E_j is the Poisson kernels of the adjoint operator $L^*(D_x)$ defined in Section 2.2 and e_k is the d-vector with kth component 1 and all other components 0. Let

$$\mathbf{R}(\rho) = \frac{1}{2}\,\rho^n \int_{S_+^{n-1}} \big(\mathcal{Q}(\xi) + \mathcal{Q}^*(\xi)\big) d\theta, \tag{1.8}$$

where $\rho = |\xi|$, $\theta = \xi/|\xi|$, $d\theta$ is a standard measure on the unit sphere S^{n-1} and S_+^{n-1} is the upper hemisphere. Our main result is the following asymptotic formula for solution to (1.1), (1.2) subject to a certain mild growth condition at the origin:

$$u(x) \sim C \exp\Big(\int_r^\delta \big(\big(\mathbf{R}(\rho)\mathbf{q}(\rho), \mathbf{q}(\rho)\big) + \Upsilon_1(\rho)\big)\frac{d\rho}{\rho}\Big) x_n^m\,\mathbf{q}, \tag{1.9}$$

where C is a constant, $\mathbf{q}$ is a vector function subject to $|\mathbf{q}(\rho)| = 1$ for all ρ. The functions $\Upsilon_1(r)$ and $r\partial_r \mathbf{q}(r)$ are small in the sense of (2.27). Moreover,

$$\int_r^\delta \Upsilon_1(\rho)\frac{d\rho}{\rho} \le c \int_{r<|y|<\delta, y_n>0} \kappa^2(y)|y|^{-n} dy + c\omega_0^2,$$

where c_1 and c_2 are two constants independent of δ. For more explicit formulation of relation (1.9) as well as for the corresponding relations for derivatives of u we refer to Theorems 2.4 and 2.3.

A direct consequence of (1.9) is the asymptotic formula $u(x) \sim \mathbf{c} x_n^m$ for solution u to (1.1), (1.2) proved in Corollary 2.9, under the assumption that

$$\int_{B_+(\delta)} \kappa(x)|x|^{-n} dx < \infty.$$

We note that actually this result is proved without smallness assumption on the function κ.

Another application is the following. It is well known that any variational solution belongs to the Sobolev space $(W^{m,p})^d$ with sufficiently large p, depending on $\operatorname{ess\,sup} \kappa$ (see [1] and [2] for second order equations). We prove the following estimate for solutions to problem (1.1), (1.2) from $(W^{m,2}(B_+(\delta))^d$:

$$|\nabla_k u(x)| \le CJ(u,\delta)|x|^{m-k} \exp\Big(\int_{|x|<|y|<\delta, y_n>0} (\lambda_+(y) + c\kappa^2(y)|y|^{-n})dy\Big) \quad (1.10)$$

for $|x| < \delta/2$ and $k = 0, 1, \dots, m-1$. Here $\nabla_k u$ is the vector $\{\partial_x^\alpha u\}_{|\alpha|=k}$, $\lambda_+(y)$ is the maximal eigenvalue of the matrix $\Re\mathcal{Q}(y)$. The constants C and c in (1.10) depend only on the operator $L(D_x)$ and n, and

$$J(u,\delta) = \delta^{-n/2}\Big(\int_{|y|<\delta} |\nabla_m u|^2 dy\Big)^{1/2}.$$

Estimate (1.10) follows directly from Remark 2, Corollary 2.8 with $p > n$ and the Sobolev imbedded theorem.

The case of a scalar operator $\mathcal{L}(x, D_x)$ was treated in [4] under the smallness assumption of the function κ. But even in the scalar case this work improves asymptotic formulae from [4] in the following way. In the exponent in (1.10) and in similar formulae in Corollaries 2.5–2.8 we have a remainder term of the form $\kappa^2(y)$, whereas in [4] instead of this term we have $(\operatorname{ess\,sup}_{|y|/e<|\xi|<|y|} \kappa(\xi))^2$. The importance of this improvement will be demonstrated in the forthcoming paper on estimates of solutions to Dirichlet boundary value problem for elliptic systems in convex domains.

Let us describe the idea of the proof. We use the same reduction to the first order evolution system as in [4]–[8] and transfer the study of behavior of solutions near the boundary point to the study of behavior of solutions to the evolution system at infinity. The next step is a reduction of the infinite dimensional system to a finite system of ordinary differential equations perturbed by nonlocal integro-differential operator, which was used in [4]-[8]. The new feature here is a more refine study of this finite dimensional system, which is performed in Section 4.3–4.6.

2. Preliminaries and formulation of main results

2.1. **Assumptions and some functional spaces.** In parallel to (1.1), (1.2) we shall consider the Dirichlet problem

$$\mathcal{L}(x, D_x)u = f(x) \quad \text{in } \mathbb{R}^n_+, \quad (2.1)$$

$$\partial^k_{x_n} u\big|_{x_n=0} = 0 \quad \text{for } k = 0, 1, \dots, m-1 \quad \text{on } \mathbb{R}^{n-1} \setminus \mathcal{O}. \quad (2.2)$$

We suppose that

$$\mathcal{L}(x, D_x)u = \sum_{|\alpha|,|\beta|\le m} D_x^\alpha\big(\mathcal{L}_{\alpha\beta}(x) D_x^\beta u\big)\,, \quad (2.3)$$

where the coefficients $\mathcal{L}_{\alpha\beta}$ are measurable complex valued $d \times d$-matrix functions on $\mathbb{R}^n_+$. We write the operator $L(D_x)$ in (1.3) as

$$L(D_x) = \sum_{|\alpha|=|\beta|=m} L_{\alpha\beta} D_x^{\alpha+\beta} \quad (2.4)$$

and assume that the matrix $\Re L(\xi)$ is positively definite for all $\xi \in \mathbb{R}^n \setminus \mathcal{O}$. According to (1.3) and (2.4)

$$N_{\alpha\beta} = \begin{cases} L_{\alpha\beta} - \mathcal{L}_{\alpha\beta} & \text{if } |\alpha| = |\beta| = m \\ -\mathcal{L}_{\alpha\beta} & \text{if } |\alpha| + |\beta| < 2m. \end{cases} \tag{2.5}$$

We consider solutions u from the space $(\mathring{W}^{m,p}_{\rm loc}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O}))^d$, where $\mathring{W}^{m,p}_{\rm loc}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$, $1 < p < \infty$, denotes the space of functions w defined on $\mathbb{R}^n_+$ such that $\eta w \in \mathring{W}^{m,p}(\mathbb{R}^n_+)$ for all smooth functions η with compact support in $\overline{\mathbb{R}^n} \setminus \mathcal{O}$. We introduce a family of seminorms in $\mathring{W}^{m,p}_{\rm loc}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$ by

$$\mathfrak{M}^m_p(w; K_{ar,br}) = \Big(\sum_{k=0}^m \int_{K_{ar,br}} |\nabla_k w(x)|^p |x|^{pk-n} dx \Big)^{1/p}, \quad r > 0, \tag{2.6}$$

where $K_{\rho,r} = \{x \in \mathbb{R}^n_+ : \rho < |x| < r\}$, a and b are positive constants, $a < b$ and $\nabla_k w$ is the vector $\{\partial^\alpha_x w\}_{|\alpha|=k}$. Here and elsewhere $|\cdot|$ denotes the euclidian norm, the only exception is the use of $|\cdot|$ for multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$, in this case $|\alpha| = \alpha_1 + \dots + \alpha_n$. Due to (2.2) the seminorm $\mathfrak{M}^m_p(w; K_{ar,br})$ is equivalent to the seminorm

$$\Big(\int_{K_{ar,br}} |\nabla_m w(x)|^p |x|^{pm-n} dx \Big)^{1/p}.$$

We say that a function v belongs to the space $\mathring{W}^{m,q}_{\rm comp}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$, $pq = p + q$, if $v \in \mathring{W}^{m,q}_{\rm loc}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$ and v has a compact support in $\overline{\mathbb{R}^n_+} \setminus \mathcal{O}$. By $W^{-m,p}_{\rm loc}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$ we denote the dual of $\mathring{W}^{m,q}_{\rm comp}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$ with respect to the inner product in $L^2(\mathbb{R}^n_+)$. We supply $W^{-m,p}_{\rm loc}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$ with the seminorms

$$\mathfrak{M}^{-m}_p(f; K_{ar,br}) = r^{-n} \sup \Big| \int_{\mathbb{R}^n_+} f\, \overline{v}\, dx \Big|, \tag{2.7}$$

where the supremum is taken over all functions $v \in \mathring{W}^{m,q}_{\rm comp}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$ supported in $ar \le |x| \le br$ and such that $\mathfrak{M}^m_p(v; K_{ar,br}) \le 1$.

In what follows we use the same notations for the norms of scalar and vector functions.

We require that the right-hand side f in (2.1) belongs to $\big(W^{-m,p}_{\rm loc}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})\big)^d$ and consider a solution u of (2.1) in the space $\big(\mathring{W}^{m,p}_{\rm loc}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})\big)^d$. This solution satisfies

$$\int_{\mathbb{R}^n_+} \sum_{|\alpha|,|\beta| \le m} \big(\mathcal{L}_{\alpha\beta}(x) D^\beta_x u(x), D^\alpha_x v(x)\big) dx = \int_{\mathbb{R}^n_+} \big(f, v\big) dx \tag{2.8}$$

for all $v \in \big(\mathring{W}^{m,p'}_{\rm comp}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})\big)^d$, $p' = p/(p-1)$. Here and elsewhere $\big(\cdot,\cdot\big)$ is the standard inner product in $\mathbb{C}^d$ and the integral on the right is understood in the distribution sense.

Let us formulate three assumptions on the operator $\mathcal{L}$.

(H1) (*Ellipticity of* $\mathcal{L}$) We suppose that the function κ, given by (1.5), is bounded by a constant γ^{-1} and that

$$\sum_{|\alpha|,|\beta| \le m} \Re \int_{\mathbb{R}^n_+} \big(\mathcal{L}_{\alpha\beta}(x) D^\beta_x u, D^\alpha_x u\big) dx \ge \gamma \sum_{|\alpha| = m} \int_{\mathbb{R}^n_+} \big(D^\alpha_x u, D^\alpha_x u\big) dx \tag{2.9}$$

for all $u \in (C^\infty_0(\mathbb{R}^n_+))^d$.

(H2) (*A local estimate*) We suppose that for certain p and p_1, $2 \le p \le p_1$, the following local estimate is valid: if $u \in \mathring{W}^{m,p}_{\rm loc}(K)$ solves problem (2.1), (2.2) with $f \in W^{-m,p_1}_{\rm loc}(K)$, then $u \in \mathring{W}^{m,p_1}_{\rm loc}(K)$ and

$$\mathfrak{M}^m_{p_1}(u; K_{r/a,r}) \le b_0\big(r^{2m}\mathfrak{M}^{-m}_{p_1}(f; K_{r/a^2,ar}) + \mathfrak{M}^m_p(u; K_{r/a^2,ar})\big), \tag{2.10}$$

where b_0 is a constant independent of r, u and f, but it can depend on $a > 1$. We shall suppose that $b_0 \ge 1$ and that

$$p_1 \le \begin{cases} np/(n-p) & \text{if } 2p < n \\ 2p & \text{if } 2p \ge n. \end{cases} \tag{2.11}$$

Below we use the notation

$$\kappa_s(r) = \Big(\int_{r/e<|y|<r} \kappa^s(y)|y|^{-n}dy\Big)^{1/s} \tag{2.12}$$

for $1 \le s \le \infty$. If $s = \infty$ then

$$\kappa_\infty(r) = \operatorname*{ess\,sup}_{x\in K_{r/e,r}} \kappa(x) = \operatorname*{ess\,sup}_{x\in K_{r/e,r}} \sum_{|\alpha|,|\beta|\le m} x_n^{2m-|\alpha+\beta|}|N_{\alpha\beta}(x)|\,. \tag{2.13}$$

Clearly,

$$\kappa_s(r) \le \kappa_1^{1/s}(r)\kappa_\infty^{1-1/s}(r) \quad \text{for } r > 0. \tag{2.14}$$

(H3) (*Smallness of N*) We shall require that

$$b_0\kappa_{\frac{p_1p}{p_1-p}}(r) \le \omega_0, \tag{2.15}$$

where p_1 and p are the same as in (H2) and ω_0 is a small constant depending on m, n, p, γ and on the unperturbed operator L.

Remark 2.1. We note that in the case $p_1 > p$, (H3) follows from boundedness of κ and smallness of κ_1, because of (2.14). From (2.11) it follows that $p_1 \le 2p$ and hence $p_1 \le p_1p/(p_1-p)$. This together with (2.15) implies, in particular, that

$$b_0\kappa_1(r) + b_0\kappa_{p_1}(r) \le c\omega_0 \tag{2.16}$$

with c depending on n. Assumption (2.11) implies also that $p_1 \le pn/(n-p)$ if $p < n$. This we need in order to handle commutators, see the proof of Theorem 2.4.

Remark 2.2. If

$$\kappa_\infty(r) \le \omega_0\,, \tag{2.17}$$

where ω_0 is a sufficiently small constant, then clearly (H1) is satisfied. Condition (H2) is valid with $p_1 = p$ and $b_0 = 1$. With this choice p and p_1 relation (2.17) implies (2.15) possibly with another small constant ω_0. Moreover, for every $p \ge 2$ one can choose ω_0 being sufficiently small such that if $u \in \mathring{W}^{m,2}_{\rm loc}(K)$ solves problem (2.1), (2.2) with $f \in W^{-m,p}_{\rm loc}(K)$, then $u \in \mathring{W}^{m,p}_{\rm loc}(K)$ and

$$\mathfrak{M}^m_p(u; K_{r/a,r}) \le c\big(r^{2m}\mathfrak{M}^{-m}_p(f; K_{r/a^2,ar}) + \mathfrak{M}^m_2(u; K_{r/a^2,ar})\big), \tag{2.18}$$

with a constant c depending on n, p, a and the operator L.

By the classical Hardy inequality

$$\mathfrak{M}^{-m}_p((\mathcal{L}-L)(u); K_{r/e,r}) \le c\,\kappa_\infty(r)\mathfrak{M}^m_p(u; K_{r/e,r})\,, \tag{2.19}$$

where c depends only on n, m and p. Therefore, the boundedness of the function κ implies that the operator $\mathcal{L}(x, D_x)$ maps continuously $(\mathring{W}^{m,p}_{\rm loc}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O}))^d$ into $(W^{-m,p}_{\rm loc}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O}))^d$.

2.2. **Main results.** Let $L^*_{\alpha\beta}$ denote the matrix adjoint to $L_{\alpha\beta}$ and also let E_j be the Poisson kernel corresponding to the operator $L(D_x)$, i.e. it satisfies

$$\sum_{|\alpha|=|\beta|=m} L^*_{\alpha\beta} D_x^{\alpha+\beta} E_j(x) = 0 \quad \text{in } \mathbb{R}^n_+, \tag{2.20}$$

it is a positive homogeneous of degree $m-n$ vector valued function and subject to the following Dirichlet conditions on the hyperplane $x_n = 0$:

$$\partial^j_{x_n} E_j = 0 \quad \text{for} \quad 0 \le j \le m-2\,, \quad \text{and} \quad \partial^{m-1}_{x_n} E_j = (L^*_0)^{-1} e_j \delta(x')\,, \tag{2.21}$$

where δ is the Dirac function, e_j is the column vector with j-th component equals 1 and all other components zero, and L_0 is the coefficient before $D^{2m}_{x_n}$ in (2.4).

We shall use the notation

$$\Omega(r) = b_0 \Big(\int_{K_{r/e,r}} \kappa^{p_1 p/(p_1-p)}(y)|y|^{-n} dy \Big)^{(p_1-p)/p_1 p}, \tag{2.22}$$

where b_0, p_1 and p are the same as in (H2) and (H3). By (2.15) $\Omega(r) \le \omega_0$.

In what follows by c and $\mathcal{C}$ (sometimes enumerated) we denote different positive constants which depend only on m, n, p, γ (the constant in (H1)) and the coefficients $L_{\alpha,\beta}$.

Theorem 2.3. *Let (H1)–(H3) be fulfilled and let $\delta > 0$. Let also $Z \in (\mathring{W}^{m,p}_{\rm loc}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O}))^d$ be a solution of $\mathcal{L}(x, D_x)Z = 0$ on $\mathbb{R}^n_+ \setminus \mathcal{O}$ subject to*

$$\mathfrak{M}^m_p(Z; K_{r/e,r}) = o\Big(r^{m-n} \exp\Big(-\mathcal{C} \int_r^\delta \Omega(\rho) \frac{d\rho}{\rho} \Big)\Big) \tag{2.23}$$

as $r \to 0$ and

$$\mathfrak{M}^m_p(Z; K_{r/e,r}) = o\Big(r^{m+1} \exp\Big(-\mathcal{C} \int_\delta^r \Omega(\rho) \frac{d\rho}{\rho} \Big)\Big) \tag{2.24}$$

as $r \to \infty$. Then $Z \in (\mathring{W}^{m,p_1}_{\rm loc}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O}))^d$ and

$$(r\partial_r)^k Z(x) = J_Z \exp\Big(\int_r^\delta \Upsilon(\rho) \frac{d\rho}{\rho} \Big) \big(x_n^m m^k \mathbf{q}(r) + r^m v_k(x) \big) \tag{2.25}$$

for $k = 0, 1, \dots, m$. Here $r = |x|$ and

$$\Upsilon(r) = \big(\mathbf{R}(r)\mathbf{q}(r), \mathbf{q}(r) \big) + \Upsilon_1(r)\,, \tag{2.26}$$

where $\mathbf{R}(r)$ is given by (1.8), the vector function $\mathbf{q}$ and the scalar function Υ_1 are measurable and satisfy $|\mathbf{q}(r)| = 1$ and

$$\int_{r/e}^r |\partial_\rho \mathbf{q}(\rho)| d\rho \le c(\kappa_1(r) + \chi(r)), \quad \int_{r/e}^r |\Upsilon_1(\rho)| \frac{d\rho}{\rho} \le c\chi(r) \tag{2.27}$$

for all $r > 0$ with

$$\begin{aligned} \chi(r) = b_0 \kappa_{p_1'}(r) \Big(& r^{-n} \int_0^r e^{\mathcal{C} \int_\rho^r \Omega(s) \frac{ds}{s}} \kappa_{p_1}(\rho) \rho^{n-1} d\rho \\ & + r \int_r^e e^{\mathcal{C} \int_r^\rho \Omega(s) \frac{ds}{s}} \kappa_{p_1}(\rho) \rho^{-2} d\rho \Big), \end{aligned} \tag{2.28}$$

where $p_1' = p_1/(p_1 - 1)$. The constant J_Z in (2.25) admits the estimates

$$c_1 \mathfrak{M}^0_2(Z; K_{\delta/e,\delta}) \le |J_Z| \delta^m \le c_2 \mathfrak{M}^0_2(Z; K_{\delta/e,\delta})\,. \tag{2.29}$$

The functions v_k belong to $L^{p_1}_{\rm loc}((0,\infty);(\mathring{W}^{m-k,p_1}(S^{n-1}_+))^d)$ and satisfy

$$\begin{aligned} &\Big(\int_{r/e}^{r}\big(\|v_k(\rho,\cdot)\|^{p_1}_{W^{m-k,p_1}(S^{n-1}_+)}+\|\rho\partial_\rho v_k(\rho,\cdot)\|^{p_1}_{W^{m-k-1,p_1}(S^{n-1}_+)}\big)\frac{d\rho}{\rho}\Big)^{1/p_1}\\ &\le cb_0\Big(r^{-n}\int_0^r e^{\mathcal{C}\int_\rho^r\Omega(s)\frac{ds}{s}}\kappa_{p_1}(\rho)\rho^{n-1}d\rho+r\int_r^e e^{\mathcal{C}\int_r^\rho\Omega(s)\frac{ds}{s}}\kappa_{p_1}(\rho)\rho^{-2}d\rho\Big) \end{aligned} \tag{2.30}$$

where $k=0,\dots,m-1$ and $\mathring{W}^{m-k,p_1}(S^{n-1}_+)$ is the completion of $C_0^\infty(S^{n-1}_+)$ in the norm of the Sobolev space $W^{m-k,p_1}(S^{n-1}_+)$. In the case $k=m$ estimate (2.30) *holds without the second norm in the left-hand side.*

The dimension of the space of such solutions Z is equal to d.

We note that by (2.16) the left-hand side of (2.30) is small. Let us formulate a local version of the above theorem.

Theorem 2.4. *Assume that (H1)–(H3) are fulfilled. Let $u\in(\mathring{W}^{m,p}_{\rm loc}(\overline{\mathbb{R}^n_+}\setminus\mathcal{O}))^d$ be a solution of $\mathcal{L}(x,D_x)u=0$ on B^+_δ, $\delta>0$, subject to*

$$\mathfrak{M}^m_p(u;K_{r/e,r})=o\Big(r^{m-n}\exp\Big(-\mathcal{C}\int_r^\delta\Omega(\rho)\frac{d\rho}{\rho}\Big)\Big) \tag{2.31}$$

as $r\to0$. Then

$$u=Z+w\,, \tag{2.32}$$

where Z is a special solution from Theorem 2.3, which admits the asymptotic representation (2.25) *with*

$$|J_Z|\le cb_0\delta^{-m}\mathfrak{M}^m_{p_1}(u;K_{\delta/4,\delta}) \tag{2.33}$$

and

$$\mathfrak{M}^m_{p_1}(w;K_{r/e,r})\le cb_0\Big(\frac{r}{\delta}\Big)^{m+1}e^{\mathcal{C}\int_r^\delta\Omega(s)\frac{ds}{s}}\mathfrak{M}^m_{p_1}(u;K_{\delta/4,\delta}) \tag{2.34}$$

for $r<\delta$.

The proofs of these theorems are presented in Sections 3–5.

2.3. **Corollaries of the main results.** In this section we present several corollaries of Theorems 2.3 and 2.4 concerning the case when (2.17) is satisfied with sufficiently small ω_0.

Corollary 2.5. *Let $p\ge2$. There exists $\omega_0>0$ depending on n, p and L such that if* (2.17) *is satisfied then the following assertion is valid. If $Z\in(\mathring{W}^{m,2}_{\rm loc}(\overline{\mathbb{R}^n_+}\setminus\mathcal{O}))^d$ is a solution of $\mathcal{L}(x,D_x)Z=0$ on $\mathbb{R}^n_+\setminus\mathcal{O}$ subject to* (2.23) *and* (2.24) *with p replaced by 2, then $Z\in(\mathring{W}^{m,p}_{\rm loc}(\overline{\mathbb{R}^n_+}\setminus\mathcal{O}))^d$ and for every $\delta>0$ representation* (2.25) *holds for $k=0,1,\dots,m$, where Υ is given by* (2.26) *with the same $\mathbf{R}$ and $\mathbf{q}$. Moreover, estimates* (2.27) *are fulfilled with*

$$\begin{aligned} \chi(r)=\kappa_2(r)\Big(&r^{-n}\int_0^r e^{\mathcal{C}\int_\rho^r\Omega(s)\frac{ds}{s}}\kappa_2(\rho)\rho^{n-1}d\rho\\ &+r\int_r^e e^{\mathcal{C}\int_r^\rho\Omega(s)\frac{ds}{s}}\kappa_2(\rho)\rho^{-2}d\rho\Big). \end{aligned} \tag{2.35}$$

The coefficient J_Z satisfies (2.29) *and the remainder term v_k is subject to* (2.30) *with p_1 replaced by p.*

The dimension of the space of such solutions Z is equal to d.

Corollary 2.6. *Let $p \geq 2$ and $\delta > 0$. There exists $\omega_0 > 0$ depending on n, p and L such that if* (2.17) *is satisfied for $r < \delta$ then the following assertion is valid. Let $u \in (\mathring{W}^{m,2}_{\rm loc}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O}))^d$ be a solution of $\mathcal{L}(x, D_x)u = 0$ on B^+_δ subject to* (2.31) *with p replaced by 2. Then $u \in (\mathring{W}^{m,p}_{\rm loc}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O}))^d$ and satisfies* (2.32)*, where Z is a special solution from* Corollary 2.5*, which admits the asymptotic representation* (2.25) *with*

$$|J_Z| \leq c\delta^{-m}\mathfrak{M}^m_2(u; K_{\delta/16,\delta}), \tag{2.36}$$

$$\mathfrak{M}^m_p(w; K_{r/e,r}) \leq c\Big(\frac{r}{\delta}\Big)^{m+1} e^{\mathcal{C}\int_r^\delta \Omega(s)\frac{ds}{s}}\mathfrak{M}^m_2(u; K_{\delta/16,\delta}) \tag{2.37}$$

for $r < \delta/2$.

The next two corollaries give a rougher but more explicit description of solutions to $\mathcal{L}u = 0$. We denote by $\Upsilon_-(\rho)$ and $\Upsilon_+(\rho)$ the minimal and maximal eigenvalue of the matrix $\mathbf{R}(\rho)$.

Corollary 2.7. *Let* (2.17) *be fulfilled with sufficiently small constant ω_0 depending on n, p and L. Let also $Z \in (\mathring{W}^{m,2}_{\rm loc}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O}))^d$ be a solution of $\mathcal{L}(x, D_x)Z = 0$ on $\mathbb{R}^n_+ \setminus \mathcal{O}$ subject to* (2.23) *and* (2.24) *with p replaced by 2. Then $Z \in (\mathring{W}^{m,p}_{\rm loc}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O}))^d$ and for every $\delta > 0$*

$$\begin{aligned}
&C_1 J(Z)\Big(\frac{r}{\delta}\Big)^m \exp\Big(\int_r^\delta (\Upsilon_-(\rho) - c\nu(\rho))\frac{d\rho}{\rho}\Big)\\
&\leq \mathfrak{M}^m_p(Z; K_{r/e,r})\\
&\leq C_2 J(Z)\Big(\frac{r}{\delta}\Big)^m \exp\Big(\int_r^\delta (\Upsilon_+(\rho) + c\nu(\rho))\frac{d\rho}{\rho}\Big)
\end{aligned} \tag{2.38}$$

for $r < \delta$. Here

$$\nu(\rho) = \int_{S^{n-1}_+} \kappa^2(\xi)d\theta\,, \quad \rho = |\xi|, \quad \theta = \xi/|\xi|, \tag{2.39}$$

and $J(Z) = \mathfrak{M}^0_2(Z; K_{\delta/e,\delta})$. The dimension of the space of such solutions Z is equal to d.

Corollary 2.8. *Let* (2.17) *be fulfilled with sufficiently small constant ω_0 depending on n, p and L. Let $u \in (\mathring{W}^{m,2}_{\rm loc}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O}))^d$ be a solution of $\mathcal{L}(x, D_x)u = 0$ on B^+_δ, $\delta > 0$, subject to* (2.31) *with p replaced by 2. Then $u \in (\mathring{W}^{m,p}_{\rm loc}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O}))^d$ and*

$$\mathfrak{M}^m_p(u; K_{r/e,r}) \leq CJ_m(u)\Big(\frac{r}{\delta}\Big)^m \exp\Big(\int_r^\delta (\Upsilon_+(\rho) + c\nu(\rho))\frac{d\rho}{\rho}\Big) \tag{2.40}$$

for $r < \delta/2$. Here

$$J_m(u) \leq c\mathfrak{M}^m_2(u; K_{\delta/16,\delta})\,. \tag{2.41}$$

The following consequence of Corollary 2.6 treats the case when u has the same asymptotics as in the constant coefficient case.

Corollary 2.9. *Let (H1) be valid and let*

$$\int_{B_+(\delta)} \kappa(x)|x|^{-n}dx < \infty\,. \tag{2.42}$$

Then there exists $p_1 > 2$, depending on L, m, n and γ such that if $u \in (\mathring{W}^{m,2}(\mathbb{R}^n_+))^d$ be a solution of $\mathcal{L}(x, D_x)u = 0$ on $B^+_{2\delta}$, $\delta > 0$, then $u \in (\mathring{W}^{m,p_1}(B^+_\delta))^d$ and

$$u(x) = \mathbf{c}x_n^m + v(x)\,,$$

where $\mathbf{c}$ *is a constant vector and* v *satisfies the relation*

$$\mathfrak{M}^m_{p_1}(v; K_{r/e,r}) = o(r^m) \tag{2.43}$$

as $r \to 0$.

The proofs of these corollaries can be found in Section 5.

2.4. **Solvability results for the Dirichlet problem in $\mathbb{R}^n_+$.** The next statement for $d = 1$ and $a = e$ is proved in [4, Proposition 1]. The proof for arbitrary d and $a > 1$ is the same since the arguments there do not use the facts $d = 1$ and $a = e$.

Proposition 2.10. (i) *Let* $f \in (W^{-m,q}_{\mathrm{loc}}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O}))^d$, $q \in (1, \infty)$, *be subject to*

$$\int_0^1 \rho^m \mathfrak{M}^{-m}_q(f; K_{\rho/a,\rho}) \frac{d\rho}{\rho} + \int_1^\infty \rho^{m-1} \mathfrak{M}^{-m}_q(f; K_{\rho/a,\rho}) \frac{d\rho}{\rho} < \infty\,, \tag{2.44}$$

where $a > 1$. *Then the system*

$$L(D_x)u = f \quad \text{in } \mathbb{R}^n_+ \tag{2.45}$$

has a solution $u \in (\mathring{W}^{m,q}_{\mathrm{loc}}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O}))^d$ *satisfying*

$$\begin{aligned}\mathfrak{M}^m_q(u; K_{r/a,r}) \le c\Big(&\int_0^r r^m \rho^m \mathfrak{M}^{-m}_q(f; K_{\rho/a,\rho}) \frac{d\rho}{\rho} \\ &+ \int_r^\infty r^{m+1} \rho^{m-1} \mathfrak{M}^{-m}_q(f; K_{\rho/a,\rho}) \frac{d\rho}{\rho}\Big).\end{aligned} \tag{2.46}$$

Estimate (2.46) *implies*

$$\mathfrak{M}^m_q(u; K_{r/a,r}) = \begin{cases} o(r^m) & \text{if } r \to 0 \\ o(r^{m+1}) & \text{if } r \to \infty. \end{cases} \tag{2.47}$$

Solution $u \in (\mathring{W}^{m,q}_{\mathrm{loc}}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O}))^d$ *of equation* (2.45) *subject to* (2.47) *is unique.*

(ii) *Let* $f \in (W^{-m,q}_{\mathrm{loc}}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O}))^d$ *be subject to*

$$\int_0^1 \rho^{m+n} \mathfrak{M}^{-m}_q(f; K_{\rho/a,\rho}) \frac{d\rho}{\rho} + \int_1^\infty \rho^m \mathfrak{M}^{-m}_q(f : K_{\rho/a,\rho}) \frac{d\rho}{\rho} < \infty\,. \tag{2.48}$$

Then system (2.45) *has a solution* $u \in (\mathring{W}^{m,q}_{\mathrm{loc}}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O}))^d$ *satisfying*

$$\begin{aligned}\mathfrak{M}^m_q(u; K_{r/a,r}) \le c\Big(&\int_0^r r^{m-n} \rho^{m+n} \mathfrak{M}^{-m}_q(f; K_{\rho/a,\rho}) \frac{d\rho}{\rho} \\ &+ \int_r^\infty r^m \rho^m \mathfrak{M}^{-m}_q(f; K_{\rho/a,\rho}) \frac{d\rho}{\rho}\Big).\end{aligned} \tag{2.49}$$

Estimate (2.49) *implies*

$$\mathfrak{M}^m_q(u; K_{r/a,r}) = \begin{cases} o(r^{m-n}) & \text{if } r \to 0 \\ o(r^m) & \text{if } r \to \infty. \end{cases} \tag{2.50}$$

The solution $u \in (\mathring{W}^{m,q}_{\mathrm{loc}}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O}))^d$ *of equation* (2.45) *subject to* (2.50) *is unique.*

The next proposition contains a solvability result for problem (2.1), (2.2).

Proposition 2.11. *Let (H1)–(H3) be fulfilled and let* $f \in (W^{-m,p_1}_{\rm loc}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O}))^d$ *be subject to*

$$\begin{aligned} &\int_0^1 \rho^{m+n} e^{\mathcal{C}\int_\rho^1 \Omega(y)\frac{dy}{y}} \mathfrak{M}^{-m}_{p_1}(f; K_{\rho/e,\rho}) \frac{d\rho}{\rho} \\ &+ \int_1^\infty \rho^m e^{\mathcal{C}\int_1^\rho \Omega(y)\frac{dy}{y}} \mathfrak{M}^{-m}_{p_1}(f : K_{\rho/e,\rho}) \frac{d\rho}{\rho} < \infty\,. \end{aligned} \tag{2.51}$$

Then there exists a solution $u \in (\mathring{W}^{m,p_1}_{\rm loc}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O}))^d$ *of*

$$\mathcal{L}(x, D_x)u = f \quad \text{in } \mathbb{R}^n_+ \tag{2.52}$$

satisfying

$$\begin{aligned} \mathfrak{M}^m_{p_1}(u; K_{r/e,r}) \le c b_0 \Big(&\int_0^r r^{m-n} \rho^{m+n} e^{\mathcal{C}\int_\rho^r \Omega(y)\frac{dy}{y}} \mathfrak{M}^{-m}_{p_1}(f; K_{\rho/e,\rho}) \frac{d\rho}{\rho} \\ &+ \int_r^\infty r^m \rho^m e^{\mathcal{C}\int_r^\rho \Omega(y)\frac{dy}{y}} \mathfrak{M}^{-m}_{p_1}(f; K_{\rho/e,\rho}) \frac{d\rho}{\rho} \Big). \end{aligned} \tag{2.53}$$

Estimate (2.53) *implies*

$$\mathfrak{M}^m_p(u; K_{r/e,r}) = \begin{cases} o(r^{m-n} e^{-\mathcal{C}\int_r^1 \Omega(s)\frac{ds}{s}}) & \text{if } r \to 0 \\ o(r^m e^{-\mathcal{C}\int_1^r \Omega(s)\frac{ds}{s}}) & \text{if } r \to \infty. \end{cases} \tag{2.54}$$

The solution $u \in (\mathring{W}^{m,p}_{\rm loc}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O}))^d$ *of problem* (2.52) *subject to* (2.54) *is unique.*

Proof. (1) *Solvability in* $(\mathring{W}^{m,2}(\mathbb{R}^n_+))^d$. Using Lax-Milgram Theorem together with (H1) we obtain unique solvability of problem (2.52) in the space $(\mathring{W}^{m,2}(\mathbb{R}^n_+))^d$ for every $f \in (W^{-m,2}(\mathbb{R}^n_+))^d$.

(2) *Solvability in* $(\mathring{W}^{m,p_1}_{\rm loc}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O}))^d$. Since $f \in (W^{-m,p_1}_{\rm loc}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O}))^d$ can be approximated by functions from $(W^{-m,2}(\mathbb{R}^n_+))^d$ with compact supports in the norm defined by the left-hand side in (2.51), for establishing the existence result together with estimate (2.53) it suffices to prove (2.53) for solutions from (1).

We start with estimating the norm $\mathfrak{M}^{-m}_p(Nu; K_{r/a,r})$. By Hölder and Hardy inequalities, we have

$$\begin{aligned} &\Big| \int_{K_{r/a,r}} \sum_{|\alpha|,|\beta| \le m} \big(N_{\alpha\beta}(x) D^\beta_x u, D^\alpha_x v\big) dx \Big| \\ &\le C \Big(\int_{K_{r/a,r}} \kappa(x)^s dx \Big)^{1/s} \Big(\int_{K_{r/a,r}} |\nabla_m u|^{p_1} dx \Big)^{1/p_1} \Big(\int_{K_{r/a,r}} |\nabla_m v|^{p'} dx \Big)^{1/p'}, \end{aligned}$$

where $p' = p/(p-1)$ and $s = p_1 p/(p_1 - p)$. This leads to

$$r^{2m} \mathfrak{M}^{-m}_p(Nu; K_{r/a,r}) \le C \kappa_{s,a}(r) \mathfrak{M}^m_{p_1}(u; K_{r/a,r}) \tag{2.55}$$

with $\kappa_{s,a} = \|\kappa\|_{L^s(K_{r/a,r})}$. We write equation (2.52) in the form $Lu = f_1$ with $f_1 = f + Nu$. One can check that the function f_1 satisfies (2.48). Applying Proposition 2.10(ii) with $q = p$, we obtain that

$$\begin{aligned} \mathfrak{M}^m_p(u; K_{r/a,r}) \le c \Big(&\int_0^r r^{m-n} \rho^{m+n} \mathfrak{M}^{-m}_p(f + Nu; K_{\rho/a,\rho}) \frac{d\rho}{\rho} \\ &+ \int_r^\infty r^m \rho^m \mathfrak{M}^{-m}_p(f + Nu; K_{\rho/a,\rho}) \frac{d\rho}{\rho} \Big). \end{aligned}$$

Now, using this estimate with a close to 1 together with (2.55) and (2.10) we arrive at

$$\begin{aligned}
&\mathfrak{M}^m_{p_1}(u;K_{r/e,r})\\
&\le cb_0\Big(\int_0^r\Big(\frac{r}{\rho}\Big)^{m-n}\big(\rho^{2m}\mathfrak{M}^{-m}_{p_1}(f;K_{\rho/e,\rho})+\kappa_s(\rho)\mathfrak{M}^m_{p_1}(u;K_{\rho/e,\rho})\big)\frac{d\rho}{\rho}\\
&\quad+\int_r^\infty\Big(\frac{r}{\rho}\Big)^{m}\big(\rho^{2m}\mathfrak{M}^{-m}_{p_1}(f;K_{\rho/e,\rho})+\kappa_s(\rho)\mathfrak{M}^m_{p_1}(u;K_{\rho/e,\rho})\big)\frac{d\rho}{\rho}\Big).
\end{aligned}$$

Iterating this estimate we obtain

$$\mathfrak{M}^m_{p_1}(u;K_{r/e,r})\le cb_0\int_0^\infty g_s(r,\rho)\rho^{2m}\mathfrak{M}^{-m}_{p_1}(f;K_{\rho/e,\rho})\frac{d\rho}{\rho},\tag{2.56}$$

where

$$\begin{aligned}
g(e^{-t},e^{-\tau})=\mu(t-\tau)+\sum_{k=1}^\infty(cb_0)^k\int_{\mathbb{R}^k}\mu(t-\tau_1)\kappa_s(e^{-\tau_1})\mu(\tau_1-\tau_2)\\
\dots\kappa_s(e^{-\tau_k})\mu(\tau_k-\tau)d\tau_1\dots d\tau_k.
\end{aligned}$$

Here $\mu(t)=e^{-mt}/n$ if $t\ge 0$ and $\mu(t)=e^{(n-m)t}/n$ if $t<0$. Since $(m-n-\partial_t)(\partial_t+m)\mu(t)=\delta(t)$, it can be checked that the function $\mu_s(t,\tau)=g(e^{-t},e^{-\tau})$ satisfies

$$\big((m-n-\partial_t)(\partial_t+m)-cb_0\kappa_s(e^{-t})\big)\mu_s(t,\tau)=\delta(t-\tau).$$

Using [3, Proposition 6.3.1], we obtain

$$\begin{gathered}
\mu_s(t,\tau)\le C\exp\Big(-m(t-\tau)+cb_0\int_\tau^t\kappa_s(y)dy\Big)\quad\text{if } t\ge\tau\,,\\
\mu_s(t,\tau)\le C\exp\Big((n-m)(t-\tau)+cb_0\int_t^\tau\kappa_s(y)dy\Big)\quad\text{if } t<\tau\,.
\end{gathered}$$

These estimates together with (2.56) give (2.53) with the norm $\mathfrak{M}^m_p(u;K_{r/e,r})$ in the left-hand side. The last norm can be replaced by $\mathfrak{M}^m_{p_1}(u;K_{r/e,r})$ by using (2.10).

(3) *Uniqueness.* Let $\mathcal{L}u=0$, $u\in(\mathring{W}^{m,p}_{\rm loc}(\overline{\mathbb{R}^n_+}\setminus\mathcal{O}))^d$ and let u be subject to (2.54). By (2.10) one can replace p by p_1 here. Let R be a large positive number and let $\eta_R(r)$ be smooth function equals 1 for $R^{-1}\le r\le R$ and 0 for $r\le(Re)^{-1}$ and for $r\ge Re$. We can suppose that $|\partial_r^k\eta_R(r)|\le c_kr^{-k}$ with c_k independent of R. Since $\eta_Ru\in(\mathring{W}^{m,2}(\mathbb{R}^n_+))^d$, we can apply uniqueness result from (1) and obtain from (2) that

$$\begin{aligned}
\mathfrak{M}^m_{p_1}(\eta_Ru;K_{r/e,r})\le cb_0\Big(&\int_0^r r^{m-n}\rho^{m+n}e^{\mathcal{C}\int_\rho^r\Omega(y)\frac{dy}{y}}\mathfrak{M}^{-m}_{p_1}(\mathcal{L}(\eta_Ru);K_{\rho/e,\rho})\frac{d\rho}{\rho}\\
&+\int_r^\infty r^m\rho^m e^{\mathcal{C}\int_r^\rho\Omega(y)\frac{dy}{y}}\mathfrak{M}^{-m}_{p_1}(\mathcal{L}(\eta_Ru);K_{\rho/e,\rho})\frac{d\rho}{\rho}\Big),
\end{aligned}$$

which implies

$$\begin{aligned}
\mathfrak{M}^m_{p_1}(u;K_{r/e,r})\le cb_0\Big(&r^{m-n}R^{m-n}e^{\mathcal{C}\int_{1/R}^r\Omega(y)\frac{dy}{y}}\mathfrak{M}^m_{p_1}(u);K_{1/(Re),1/R})\\
&+r^mR^{-m}e^{\mathcal{C}\int_r^R\Omega(y)\frac{dy}{y}}\mathfrak{M}^m_{p_1}(u;K_{R,Re})\Big)
\end{aligned}$$

for $eR^{-1}\le r\le R$. By (2.54) the right-hand side tends to 0 as $R\to 0$. Therefore $u=0$. $\square$

3. First order system associated with (2.1), (2.2)

3.1. **Reduction of problem (2.1), (2.2) to the Dirichlet problem in a cylinder.** We shall use the variables

$$t = -\log|x| \quad \text{and} \quad \theta = x/|x|. \tag{3.1}$$

The mapping $x \to (\theta, t)$ transforms $\mathbb{R}^n_+$ onto the cylinder $\Pi = S^{n-1}_+ \times \mathbb{R}$.

The images of $\mathring{W}^{m,p}_{\rm loc}(\overline{\mathbb{R}^n_+}\backslash\mathcal{O})$ and $W^{-m,p}_{\rm loc}(\overline{\mathbb{R}^n_+}\backslash\mathcal{O})$ under mapping (3.1) we denote by $\mathring{W}^{m,p}_{\rm loc}(\Pi)$ and $W^{-m,p}_{\rm loc}(\Pi)$. These spaces can be defined independently as follows. The space $\mathring{W}^{m,p}_{\rm loc}(\Pi)$ consists of functions whose derivatives up to order m belong to $L^p(D)$ for every compact subset D of $\overline{\Pi}$ and whose derivatives up to order $m-1$ vanish on $\partial\Pi$. The seminorm $\mathfrak{M}^m_p(u; K_{e^{-a-t},e^{-t}})$ in $\mathring{W}^{m,p}_{\rm loc}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O})$ is equivalent to the seminorm

$$\|u\|_{W^{m,p}(\Pi_{t,t+a})}\ , \quad t \in \mathbb{R}\ ,$$

where

$$\Pi_{t,t+a} = \{(\theta,\tau) \in \Pi : \tau \in (t, t+a)\}.$$

If $a = 1$ than we shall use also the notation Π_t for $\Pi_{t,t+1}$. The space $W^{-m,p}_{\rm loc}(\Pi)$ consists of the distributions f on Π such that the seminorm

$$\|f\|_{W^{-m,p}(\Pi_t)} = \sup\Big|\int_{\Pi_t} f\,\overline{v} d\tau d\theta\Big| \tag{3.2}$$

is finite for every $t \in \mathbb{R}$. The supremum in (3.2) is taken over all $v \in \mathring{W}^{m,p'}_{\rm loc}(\Pi)$, $p' = p/(p-1)$, supported in $\overline{\Pi_t}$ and subject to $\|v\|_{W^{m,p'}(\Pi_t)} \le 1$. The seminorm (3.2) is equivalent to $\mathfrak{M}^{-m}_p(f; K_{e^{-1-t},e^{-t}})$.

In the variables (θ, t) the operator L takes the form

$$L(D_x) = e^{2mt}\mathbf{A}(\theta, D_\theta, D_t)\ , \tag{3.3}$$

where $\mathbf{A}$ is an elliptic partial differential operator of order $2m$ on Π with smooth matrix coefficients. We introduce the operator $\mathbb{N}$ by

$$L(D_x) - \mathcal{L}(x, D_x) = e^{2mt}\mathbb{N}(\theta, t, D_\theta, D_t). \tag{3.4}$$

Now problem (2.1), (2.2) can be written as

$$\begin{gathered} \mathbf{A}(\theta, D_\theta, D_t)u = \mathbb{N}(\theta, t, D_\theta, D_t)u + e^{-2mt}f \quad \text{on } \Pi \\ u \in (\mathring{W}^{p,m}_{\rm loc}(\Pi))^N\ , \end{gathered} \tag{3.5}$$

where $f \in (W^{-m,p}_{\rm loc}(\Pi))^d$. By (2.19), the operator $\mathbb{N}$ satisfies

$$\|\mathbb{N}\|_{\mathring{W}^{m,p}(\Pi_t)\to W^{-m,p}(\Pi_t)} \le c\,\kappa_\infty(e^{-t}) \le c\gamma$$

with the same γ as in (H1). We put

$$\omega(t) = \Omega(e^{-t})\,, \tag{3.6}$$

where Ω is given by (2.22). Clearly, $\omega(t) \le \omega_0$, where ω_0 is the same as in (H3). By the change of variables (3.1) we can formulate Proposition 2.11 as follows

Proposition 3.1. *Let (H1)–(H3) be fulfilled and let $f \in (W^{-m,p_1}_{\rm loc}(\Pi))^d$ be subject to*

$$\begin{aligned}&\int_0^\infty e^{-(m+n)\tau+\mathcal{C}\int_0^\tau \omega(s)ds}\|f\|_{W^{-m,p_1}(\Pi_\tau)}d\tau\\&+\int_{-\infty}^0 e^{-m\tau+\mathcal{C}\int_\tau^0 \omega(s)ds}\|f\|_{W^{-m,p_1}(\Pi_\tau)}d\tau<\infty\,.\end{aligned}\tag{3.7}$$

Then problem (3.5) *has a solution $u \in (\mathring{W}^{m,p_1}_{\rm loc}(\Pi))^d$ satisfying the estimate*

$$\begin{aligned}\|u\|_{W^{m,p_1}(\Pi_t)}\le c\Big(&\int_t^\infty e^{(n-m)t-(m+n)\tau+\mathcal{C}\int_t^\tau \omega(s)ds}\|f\|_{W^{-m,p_1}(\Pi_\tau)}d\tau\\&+\int_{-\infty}^t e^{-m(t+\tau)+\mathcal{C}\int_\tau^t \omega(s)ds}\|f\|_{W^{-m,p_1}(\Pi_\tau)}d\tau\Big).\end{aligned}\tag{3.8}$$

Estimate (3.8) *implies*

$$\|u\|_{W^{m,p}(\Pi_t)}=\begin{cases}o(e^{(n-m)t-\mathcal{C}\int_0^t \omega(s)ds}) & \text{if } t\to+\infty\\ o(e^{-mt-\mathcal{C}\int_t^0 \omega(s)ds}) & \text{if } t\to-\infty.\end{cases}\tag{3.9}$$

The solution $u \in (\mathring{W}^{m,p}_{\rm loc}(\Pi))^d$ of problem (3.5) *subject to* (3.9) *is unique.*

Let $W^{-m,p}(S^{n-1}_+)$ denote the dual of $\mathring{W}^{m,q}(S^{n-1}_+)$, $q=p/(p-1)$, with respect to the inner product in $L^2(S^{n-1}_+)$. We introduce the operator pencil

$$\mathcal{A}(\lambda):(\mathring{W}^{m,p}(S^{n-1}_+))^d\to(W^{-m,p}(S^{n-1}_+))^d\tag{3.10}$$

by

$$\mathcal{A}(\lambda)U(\theta)=r^{i\lambda+2m}L(D_x)r^{-i\lambda}U(\theta)=\mathbf{A}(\theta,D_\theta,\lambda)U(\theta).\tag{3.11}$$

The following properties of $\mathcal{A}$ and its adjoint are standard and their proofs can be found, for example in [6, Section 10.3]. The operator (3.10) is Fredholm for all $\lambda\in\mathbb{C}$ and its spectrum consists of eigenvalues with finite geometric multiplicities. These eigenvalues are

$$i(m+k)\quad\text{and}\quad i(m-n-k)\quad\text{for } k=0,1,\dots,\tag{3.12}$$

and there are no generalized eigenvectors. The eigenvectors corresponding to the eigenvalue im are $c|x|^{-m}x_n^m=c\theta_n^m$, where $c\in\mathbb{C}^d$.

We introduce the operator pencil $\mathcal{A}^*(\lambda)$ defined on $\mathring{W}^{m,p}(S^{n-1}_+)$ by the formula

$$\mathcal{A}^*(\lambda)U(\theta)=r^{i\lambda+2m}L^*(D_x)r^{-i\lambda}U(\theta)\,.$$

This pencil has the same eigenvalues as the pencil $\mathcal{A}(\lambda)$. Eigenvector corresponding to the eigenvalue $i(m-n)$ are linear combinations of $|x|^{n-m}E_j(x)=E_j(\theta)$, where E_j are defined in Section 2.2. Moreover, the following biorthogonality condition holds:

$$\int_{\mathbb{R}^n_+}(L\big(D_x)(e_k\zeta x_n^m),E_j(x))dx=m!\delta_j^k\,,\tag{3.13}$$

where ζ is a smooth function equal to 1 in a neighborhood of the origin and zero for large $|x|$. This relation can be checked by integration by parts.

Using the definitions of the above pencils and Green's formula for L and $\overline{L}$ one can show that

$$(\mathcal{A}(\lambda))^*=\mathcal{A}^*(\overline{\lambda}+(2m-n)i)\,,\tag{3.14}$$

where $*$ in the left-hand side denotes passage to the adjoint operator in $(L^2(S_+^{n-1}))^d$. This implies, in particular,

$$(\mathcal{A}(im))^* E_j(\theta) = 0 \quad \text{for } j = 1, \dots, d. \tag{3.15}$$

3.2. **Reduction of problem (3.5) to a first order system in t.** To reduce problem (3.5) to a first order system, first we represent the right-hand side $f \in W_{\mathrm{loc}}^{-m,p}(\Pi)$ as

$$f = e^{2mt} \sum_{j=0}^{m} D_t^{m-j} f_j \,, \tag{3.16}$$

where $f_j \in L^p_{\mathrm{loc}}(\mathbb{R}; W^{-j,p}(S_+^{n-1}))$. This representation can be chosen to satisfy

$$c_1 \mathfrak{M}_p^{-m}(f; K_{e^{-1-t}, e^{-t}}) \le e^{2mt} \sum_{j=0}^{m} \|f_j\|_{W^{-j,p}(\Pi_t)} \le c_2 \mathfrak{M}_p^{-m}(f; K_{e^{-2-t}, e^{1-t}}) \,,$$

where c_1 and c_2 are constants depending only on n, m and p (see [4, Lemma 1]).

Next, we represent the operators $r^{|\alpha|} D_x^\alpha$ and $r^{2m} D_x^\alpha (r^{-2m+|\alpha|} \,\cdot\,)$ as polynomials with respect to $-rD_r$ we obtain

$$r^{|\alpha|} D_x^\alpha u = \sum_{l=0}^{|\alpha|} Q_{\alpha l}(\theta, D_\theta)(-rD_r)^l u \,,$$
$$r^{2m} D_x^\alpha (r^{-2m+|\alpha|} u) = \sum_{l=0}^{|\alpha|} P_{\alpha l}(\theta, D_\theta)(-rD_r)^l u \,,$$

where $Q_{\alpha l}(\theta, D_\theta)$ and $P_{\alpha l}(\theta, D_\theta)$ are differential operators of order $|\alpha| - l$ with smooth coefficients. Furthermore, integrating by parts in

$$\int_{\mathbb{R}^n_+} D_x^\alpha (r^{-2m+|\alpha|} u) r^{2m-n} \overline{v} dx,$$

we obtain

$$\sum_{l=0}^{|\alpha|} Q_{\alpha l}(-rD_r + i(2m-n))^l = \sum_{l=0}^{|\alpha|} P^*_{\alpha l}(-rD_r)^l \,, \tag{3.17}$$

where $P^*_{\alpha l}$ is the differential operator on S^{n-1} adjoint to $P_{\alpha l}$. Now we write $\mathbf{A}$ in the form

$$\mathbf{A}(\theta, D_\theta, D_t) = \sum_{j=0}^{m} D_t^{m-j} \mathcal{A}_j(D_t) \,,$$

where

$$\mathcal{A}_j(D_t) = \sum_{k=0}^{m} A_{jk} D_t^{m-k}$$

with

$$A_{jk} = \sum_{|\alpha|=|\beta|=m} P_{\alpha, m-j}(\theta, D_\theta) L_{\alpha\beta} Q_{\beta, m-k}(\theta, D_\theta) \,.$$

It is clear that

$$A_{jk} : (\mathring{W}^{k,p}(S_+^{n-1}))^d \to (W^{-j,p}(S_+^{n-1}))^d \tag{3.18}$$

are differential operators of order $\le j+k$ on S_+^{n-1} with smooth matrix coefficients. We also write

$$\mathbb{N}(\theta, t, D_\theta, D_t)u = \sum_{j=0}^{m} D_t^{m-j}\big(\mathcal{N}_j(t, D_t)u\big)\,, \tag{3.19}$$

where

$$\mathcal{N}_j(t, D_t) = \sum_{k=0}^{m} \mathcal{N}_{jk}(t) D_t^{m-k} \tag{3.20}$$

with

$$\mathcal{N}_{jk} = \sum_{m-j\le|\alpha|\le m}\ \sum_{m-k\le|\beta|\le m} P_{\alpha,m-j} e^{-(2m-|\alpha|-|\beta|)t} N_{\alpha\beta} Q_{\beta,m-k}\,, \tag{3.21}$$

where $N_{\alpha\beta}$ is defined by (2.5). By (3.21) the operators

$$\mathcal{N}_{jk}(t)\ :\ (\mathring{W}^{k,p}(S_+^{n-1}))^d \to (W^{-j,p}(S_+^{n-1}))^d$$

are continuous. By (3.21) and (3.17), for almost all $r>0$

$$\begin{aligned}
&\int_{S_+^{n-1}} \sum_{j=0}^{m} \Big(\mathcal{N}_j(D_t)u, D_t^{m-j}(e^{(2m-n)t}v)\Big) d\theta \\
&= \int_{S_+^{n-1}} \sum_{j,k\le m}\ \sum_{m-j\le|\alpha|\le m}\ \sum_{m-k\le|\beta|\le m} \Big(N_{\alpha\beta} Q_{\beta,m-k} D_t^{m-k} u, \\
&\quad e^{-(2m-|\alpha|-|\beta|)t} P^*_{\alpha,m-j} D_t^{m-j}(e^{(2m-n)t}v)\Big) d\theta \\
&= r^n \int_{S_+^{n-1}} \sum_{|\alpha|,|\beta|\le m} \Big(N_{\alpha\beta}(x) D_x^\beta u, D_x^\alpha v\Big) d\theta\,,
\end{aligned} \tag{3.22}$$

where u and v are in $(\mathring{W}^{m,p}_{\rm loc}(\overline{\mathbb{R}^n_+ \setminus \mathcal{O}}))^d$.

Using the operators $\mathcal{A}_j(D_t)$ and $\mathcal{N}_j(t, D_t)$, and (3.16) we write problem (3.5) in the form

$$\sum_{j=0}^{m} D_t^{m-j} \mathcal{A}_j(D_t) u(t) = \sum_{j=0}^{m} D_t^{m-j}\big(\mathcal{N}_j(t, D_t)u + f_j(t)\big) \quad \text{on } \mathbb{R}, \tag{3.23}$$

where we consider u and f_j as functions on $\mathbb{R}$ taking values in function spaces $(\mathring{W}^{m,p}(S_+^{n-1}))^d$ and $(W^{-j,p}(S_+^{n-1}))^d$ respectively. By (2.13) and (3.21)

$$\|\mathcal{N}_{jk}(t)\|_{(\mathring{W}^{k,p}(S_+^{n-1}))^d \to (W^{-j,p}(S_+^{n-1}))^d} \le\ c\,\kappa_\infty(e^{-t})\ . \tag{3.24}$$

Therefore, $\mathcal{N}_j$ acts from $(\mathring{W}^{m,p}_{\rm loc}(\Pi))^d$ to $(L^p_{\rm loc}(\mathbb{R}; W^{-j,p}(S_+^{n-1})))^d$. The local estimate (H2) can be reformulated now as follows

(H2a) Let p and p_1 be the same as in (H2) and $u \in (\mathring{W}^{m,p}_{\rm loc}(\Pi))^d$ satisfies (3.23) with $f_j \in (L^{p_1}_{\rm loc}(\mathbb{R}; W^{-j,p_1}(S_+^{n-1})))^d$, then $u \in (\mathring{W}^{m,p_1}_{\rm loc}(\Pi))^d$ and

$$\begin{aligned}
&\|u\|_{W^{m,p_1}(\Pi_{t,t+a})} \\
&\le c b_0 \Big(\sum_{j=0}^{m} \|f_j\|_{L^{p_1}(t-a,t+2a; W^{-j,p_1}(S_+^{n-1}))} + \|u\|_{W^{m,p}(\Pi_{t-a,t+2a})}\Big),
\end{aligned} \tag{3.25}$$

where c may depend on $a>0$.

Let $\mathcal{U} = \mathrm{col}(\mathcal{U}_1, \dots, \mathcal{U}_{2m})$, where

$$\mathcal{U}_k = D_t^{k-1} u, \quad k = 1, \dots, m, \tag{3.26}$$

$$\mathcal{U}_{m+1} = \mathcal{A}_0(D_t)u - \mathcal{N}_0(t, D_t)u - f_0 \,, \tag{3.27}$$

$$\mathcal{U}_{m+j} = D_t \mathcal{U}_{m+j-1} + \mathcal{A}_{j-1}(D_t)u - \mathcal{N}_{j-1}(t, D_t)u - f_{j-1} \tag{3.28}$$

for $j = 2, \dots, m$. With this notation (3.23) takes the form

$$D_t \mathcal{U}_{2m} + \mathcal{A}_m(D_t)u - \mathcal{N}_m(t, D_t)u - f_m = 0. \tag{3.29}$$

Using (3.26) we write (3.27) as

$$(A_{00} - \mathcal{N}_{00}(t)) D_t^m u = \mathcal{U}_{m+1} - \sum_{k=0}^{m-1} (A_{0,m-k} - \mathcal{N}_{0,m-k}(t)) \mathcal{U}_{k+1} + f_0 \,. \tag{3.30}$$

Since $Q_{\alpha,|\alpha|} = P_{\alpha,|\alpha|} = \theta^\alpha$, we have

$$A_{00} - \mathcal{N}_{00} = L(\theta) - \sum_{|\alpha|=|\beta|=m} N_{\alpha\beta}(e^{-t}\theta)\theta^{\alpha+\beta},$$

and by (H1) the matrix $A_{00} - \mathcal{N}_{00}$ is invertible, and the norm of the inverse matrix is bounded by a constant times γ^{-1}. Thus equation (3.30) is uniquely solvable with respect to $D_t^m u$, and

$$D_t^m u = \mathcal{S}(t)\hat{\mathcal{U}} \,, \tag{3.31}$$

where

$$\mathcal{S}(t)\hat{\mathcal{U}} = (A_{00} - \mathcal{N}_{00}(t))^{-1} \Big(\mathcal{U}_{m+1} - \sum_{k=0}^{m-1} (A_{0,m-k} - \mathcal{N}_{0,m-k}(t)) \mathcal{U}_{k+1} + f_0 \Big) \tag{3.32}$$

and $\hat{\mathcal{U}} = (\mathcal{U}_1, \dots, \mathcal{U}_m, \mathcal{U}_{m+1})$. If we introduce the following two operators

$$\mathcal{S}_0 \hat{\mathcal{U}} = A_{00}^{-1} \Big(\mathcal{U}_{m+1} - \sum_{k=0}^{m-1} A_{0,m-k} \mathcal{U}_{k+1} \Big) \tag{3.33}$$

$$\mathcal{S}'(t)\hat{\mathcal{U}} = \mathcal{S}(t)\hat{\mathcal{U}} - (A_{00} - \mathcal{N}_{00}(t))^{-1} f_0 \,, \tag{3.34}$$

then

$$\mathcal{S}'(t)\hat{\mathcal{U}} - \mathcal{S}_0(t)\hat{\mathcal{U}} = A_{00}^{-1} \Big(\sum_{k=0}^{m-1} \mathcal{N}_{0,m-k}(t) \mathcal{U}_{k+1} + \mathcal{N}_{00}(t) \mathcal{S}'(t)\hat{\mathcal{U}} \Big). \tag{3.35}$$

One verifies directly that

$$\|\mathcal{S}'(t)\hat{\mathcal{U}}\|_{L^q(S_+^{n-1})} \le C \sum_{j=0}^{m} \|\mathcal{U}_{j+1}\|_{W^{m-j,q}(S_+^{n-1})} \tag{3.36}$$

and

$$\|\mathcal{S}_0(t)\hat{\mathcal{U}}\|_{L^q(S_+^{n-1})} \le C \sum_{j=0}^{m} \|\mathcal{U}_{j+1}\|_{W^{m-j,q}(S_+^{n-1})}$$

for $q \in (1, \infty)$. From (3.26) it follows that

$$D_t \mathcal{U}_k = \mathcal{U}_{k+1} \quad \text{for } k = 1, \dots, m-1. \tag{3.37}$$

By (3.31) we have

$$D_t \mathcal{U}_m = \mathcal{S}(t)\hat{\mathcal{U}}. \tag{3.38}$$

Using (3.31), we write (3.28) as

$$\begin{aligned} D_t\mathcal{U}_{m+j} = {} & \mathcal{U}_{m+j+1} - \sum_{k=0}^{m-1} (A_{j,m-k} - \mathcal{N}_{j,m-k}(t))\mathcal{U}_{k+1} \\ & - (A_{j0} - \mathcal{N}_{j0}(t))\mathcal{S}(t)\hat{\mathcal{U}} + f_j \end{aligned} \tag{3.39}$$

for $j = 1, \dots, m-1$ and (3.29) takes the form

$$D_t\mathcal{U}_{2m} + \sum_{k=0}^{m-1} (A_{m,m-k} - \mathcal{N}_{m,m-k}(t))\mathcal{U}_{k+1} + (A_{m0} - \mathcal{N}_{m0}(t))\mathcal{S}(t)\hat{\mathcal{U}} - f_m = 0. \tag{3.40}$$

Relations (3.37)–(3.40) can be written as the first order evolution system

$$(\mathcal{I}D_t + \mathfrak{A})\mathcal{U}(t) - \mathfrak{N}(t)\widehat{\mathcal{U}}(t) = \mathcal{F}(t) \quad \text{on } \mathbb{R}, \tag{3.41}$$

where

$$\mathcal{F}(t) = \operatorname{col}(0, \dots, 0, \mathcal{F}_m(t), \mathcal{F}_{m+1}(t), \dots, \mathcal{F}_{2m}(t)) \tag{3.42}$$

with

$$\mathcal{F}_m(t) = (A_{00} - \mathcal{N}_{00}(t))^{-1} f_0(t), \tag{3.43}$$

$$\mathcal{F}_{m+j}(t) = f_j(t) - (A_{j0} - \mathcal{N}_{j0}(t))(A_{00} - \mathcal{N}_{00}(t))^{-1} f_0(t), \quad j = 1, \dots, m\,. \tag{3.44}$$

The operator $\mathfrak{N}$ is given by

$$\mathfrak{N}(t)\widehat{\mathcal{U}} = \operatorname{col}(0, \dots, 0, \mathfrak{N}_m(t)\mathcal{U}, \mathfrak{N}_{m+1}(t)\mathcal{U}, \dots, \mathfrak{N}_{2m}(t)\mathcal{U}), \tag{3.45}$$

where

$$\mathfrak{N}_m(t)\widehat{\mathcal{U}} = A_{00}^{-1}\Big(\sum_{k=0}^{m-1} \mathcal{N}_{0,m-k}(t)\mathcal{U}_{k+1} + \mathcal{N}_{00}(t)\mathcal{S}'(t)\hat{\mathcal{U}}\Big) \tag{3.46}$$

and

$$\begin{aligned} \mathfrak{N}_{m+j}(t)\widehat{\mathcal{U}} = {} & \sum_{k=0}^{m-1} \mathcal{N}_{j,m-k}(t)\mathcal{U}_{k+1} + \mathcal{N}_{j0}(t)\mathcal{S}'(t)\hat{\mathcal{U}} \\ & - A_{j0}A_{00}^{-1}\Big(\sum_{k=0}^{m-1} \mathcal{N}_{0,m-k}(t)\mathcal{U}_{k+1} + \mathcal{N}_{00}(t)\mathcal{S}'(t)\hat{\mathcal{U}}\Big) \end{aligned} \tag{3.47}$$

for $j = 1, \dots, m$. In (3.41), by $\mathcal{I}$, we denote the identity operator. We also use the operator matrix

$$\mathfrak{A} = -\mathcal{J} + \mathfrak{L} \tag{3.48}$$

with $\mathcal{J} = \{\mathcal{J}_{jk}\}_{j,k=1}^{2m}$ given by

$$\begin{array}{cc} & \begin{array}{ccccccc} & & & & (m+1) & & \end{array} \\ (m) & \begin{pmatrix} 0 & I & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & I & \cdots & \vdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \vdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & A_{00}^{-1} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & I \\ 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix} \end{array}$$

and with $\mathfrak{L} = \{\mathfrak{L}_{jk}\}_{j,k=1}^{2m}$ equal to

$$
\begin{pmatrix}
0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \cdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 \\
A_{00}^{-1}A_{0,m} & \cdots & A_{00}^{-1}A_{0,1} & 0 & \cdots & 0 \\
A_{1,m} - A_{1,0}A_{00}^{-1}A_{0,m} & \cdots & A_{1,1} - A_{1,0}A_{00}^{-1}A_{0,1} & A_{10}A_{00}^{-1} & \cdots & 0 \\
\vdots & \cdots & \vdots & \cdots & \ddots & \vdots \\
A_{m,m} - A_{m,0}A_{00}^{-1}A_{0,m} & \cdots & A_{m,1} - A_{m,0}A_{00}^{-1}A_{0,1} & A_{m0}A_{00}^{-1} & \cdots & 0
\end{pmatrix}
$$

We put

$$\mathcal{T} = B_m \times \cdots \times B_1 \times B_0 \times (B_{-m})^{m-1}, \tag{3.49}$$

$$\mathcal{R} = B_{m-1} \times \cdots \times B_1 \times B_0 \times (B_{-m})^{m}\,, \tag{3.50}$$

where

$$
\begin{gathered}
B_j = (\mathring{W}^{j,p}(S_+^{n-1}))^d \quad \text{for } j = 1, \dots, m, \quad B_0 = (L^p(S_+^{n-1}))^d\,, \\
B_{-j} = (W^{-j,p}(S_+^{n-1}))^d \quad \text{for } j = 1, \dots, m.
\end{gathered} \tag{3.51}
$$

By (3.18) the operator $\mathfrak{A} : \mathcal{T} \to \mathcal{R}$ is continuous. Sometimes we shall write B_j^p, $\mathcal{T}_p$ and $\mathcal{R}_p$ in order to mark the dependence of these spaces on p.

3.3. **Spectral properties of the pencil $\lambda\mathcal{I} + \mathfrak{A}$.** We introduce the operator matrix $\mathcal{E}(\lambda) = \{\mathcal{E}_{pq}(\lambda)\}_{p,q=1}^{2m}$ as

$$
\begin{array}{rc}
 & \qquad\qquad\qquad (m) \\
(m+1) & \begin{pmatrix}
e_1(\lambda) & e_2(\lambda) & \cdots & e_m(\lambda) & \cdots & e_{2m-1}(\lambda) & e_{2m}(\lambda) \\
-I & 0 & \cdots & 0 & \cdots & 0 & 0 \\
0 & -I & \cdots & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & \cdots & -A_{00} & \cdots & 0 & 0 \\
\vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & -I & 0
\end{pmatrix}
\end{array} \tag{3.52}
$$

where

$$
\begin{gathered}
e_{2m-j}(\lambda) = \lambda^j, \quad j = 0, \dots, m-1, \\
e_m = \sum_{j=0}^{m} \lambda^{m-j} A_{j0}, \\
e_{m-k}(\lambda) = \sum_{s=0}^{k} \sum_{j=0}^{m} \lambda^{k+m-s-j} A_{js}, \quad k = 1, \dots, m-1.
\end{gathered}
$$

Let also

$$J(\lambda)=\begin{pmatrix} I & 0 & \dots & 0 & 0\\ -\lambda & I & \dots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \dots & I & 0\\ 0 & 0 & \dots & -\lambda & I \end{pmatrix},\ \mathcal{M}=\begin{pmatrix} 0 & 0 & \dots & 0 & 0\\ A_{10}A_{00}^{-1} & 0 & \dots & 0 & 0\\ A_{20}A_{00}^{-1} & 0 & \dots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots\\ A_{m-1,0}A_{00}^{-1} & 0 & \dots & 0 & 0 \end{pmatrix}$$

be two $m\times m$-matrices. One can check directly that $\mathcal{E}^{-1}(\lambda)$ is given by

$$\begin{array}{cc} & (m+1) \\ (m) & \begin{pmatrix} 0 & -I & \cdots & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots\\ 0 & 0 & \cdots & -A_{00}^{-1} & \cdots & 0 & 0\\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & \cdots & 0 & -I\\ I & e_1(\lambda) & \cdots & e_m(\lambda) & \cdots & e_{2m-2}(\lambda) & e_{2m-1}(\lambda) \end{pmatrix} \end{array}$$

and that

$$J(\lambda)^{-1}=\begin{pmatrix} I & 0 & 0 & \dots & 0 & 0\\ \lambda & I & 0 & \dots & \vdots & \vdots\\ \lambda^2 & \lambda & I & \dots & \vdots & \vdots\\ \vdots & \vdots & \vdots & \dots & I & 0\\ \lambda^{m-1} & \lambda^{m-2} & \lambda^{m-3} & \dots & \lambda & I \end{pmatrix}. \tag{3.53}$$

The following assertion is proved in [4]

Proposition 3.2. *For all $\lambda\in\mathbb{C}$*

$$\mathcal{E}(\lambda)(\lambda\mathcal{I}+\mathfrak{A})=\operatorname{diag}(\mathcal{A}(\lambda),I,\dots,I)\begin{pmatrix} J(\lambda) & 0\\ -\mathcal{B}(\lambda) & J(\lambda)-\mathcal{M} \end{pmatrix} \tag{3.54}$$

where the $m\times m$-matrix $\mathcal{B}(\lambda)$ is defined by

$$\begin{aligned}\mathcal{B}(\lambda)=&\begin{pmatrix} A_{0m} & \dots & A_{02} & A_{01}\\ A_{1,m} & \dots & A_{12} & A_{11}\\ \vdots & \dots & \vdots & \vdots\\ A_{m-1,m} & \dots & A_{m-1,2} & A_{m-1,1} \end{pmatrix}\\ &-\begin{pmatrix} 0 & \dots & 0 & -\lambda A_{00}\\ A_{10}A_{00}^{-1}A_{0,m} & \dots & A_{10}A_{00}^{-1}A_{02} & A_{10}A_{00}^{-1}A_{01}\\ \vdots & \dots & \vdots & \vdots\\ A_{m-1,0}A_{00}^{-1}A_{0,m} & \dots & A_{m-1,0}A_{00}^{-1}A_{02} & A_{m-1,0}A_{00}^{-1}A_{01} \end{pmatrix}\end{aligned}$$

Moreover,

$$\begin{pmatrix} J(\lambda) & 0\\ -\mathcal{B}(\lambda) & J(\lambda)-\mathcal{M} \end{pmatrix}^{-1}=\begin{pmatrix} J^{-1}(\lambda) & 0\\ Q(\lambda) & J^{-1}(\lambda)(I+\mathcal{M}) \end{pmatrix}, \tag{3.55}$$

where the elements of the matrix $Q(\lambda) = \{Q_{jk}(\lambda)\}_{j,k=1}^m$ *are given by*

$$Q_{jk}(\lambda) = \sum_{l=0}^{j-1} \sum_{q=k-1}^{m} \lambda^{q+l+1-k} A_{j-l-1,m-q}. \tag{3.56}$$

The relation (3.54) allows one to establish the following correspondence between $\mathcal{A}(\lambda)$ and the linear pencil $\lambda\mathcal{I} + \mathfrak{A}$.

Proposition 3.3. (see [4]) (i) *The operator*

$$\lambda\mathcal{I} + \mathfrak{A} : \mathcal{T} \to \mathcal{R} \tag{3.57}$$

is Fredholm for all $\lambda \in \mathbb{C}$.

(ii) *The spectra of the operator* $\mathfrak{A}$ *and the pencil* $\mathcal{A}(\lambda)$ *coincide and consist of eigenvalues of the same multiplicity.*

We put

$$\phi_j(\theta) = \theta_n^m\, e_j \quad \text{and} \quad \psi_j(\theta) = (m!)^{-1} E_j(\theta) \quad j = 1, \dots, d,$$

where e_j and E_j are defined in the beginning of Section 2.2. By (3.13) and (3.11)

$$\int_{\Pi} (\mathcal{A}(D_t)(\eta(t) e^{-mt}\phi_k(\theta))\, , e^{mt}\psi_j(\theta)) d\theta dt = \delta_k^j\, , \tag{3.58}$$

where η is a smooth function equal to 1 for large positive t and 0 for large negative t. The equality (3.58) can be written as

$$\int_{S_+^{n-1}} (\mathcal{A}'(im)\phi_k(\theta)\, , \psi_j(\theta)) d\theta = i\delta_k^j\, . \tag{3.59}$$

We introduce the vector functions

$$\Phi_j = \operatorname{col}(\Phi_{jl})_{l=1}^{2m} = \begin{pmatrix} J^{-1}(im) & 0 \\ Q(im) & J^{-1}(im)(I + \mathcal{M}) \end{pmatrix} \operatorname{col}(\phi_j, 0, \dots, 0). \tag{3.60}$$

Owing to (3.54) and (3.55) we obtain

$$(im\mathcal{I} + \mathfrak{A})\Phi_j = 0\, . \tag{3.61}$$

Using (3.53) and the definitions of the matrices $\mathcal{M}$ and $\mathcal{B}$ we get

$$\Phi_{jl} = (im)^{l-1}\phi_j, \quad l = 1, \dots, m, \tag{3.62}$$

$$\Phi_{j,m+l} = \sum_{p=0}^{l-1} \sum_{q=0}^{m} A_{l-p-1,m-q} (im)^{p+q} \phi_j \tag{3.63}$$

for $l = 1, \dots, m$.

We introduce the vector $\Psi_j = \operatorname{col}(\Psi_{jl})_{l=1}^{2m}$, by

$$\Psi_j = \mathcal{E}^*(-im) \operatorname{col}(\psi_j, 0, \dots, 0) \tag{3.64}$$

where $\mathcal{E}^*(\lambda)$ is the adjoint of $\mathcal{E}(\overline{\lambda})$. Since ψ_j is the eigenfunction of the pencil $(\mathcal{A}(\lambda))^*$ corresponding to the eigenvalue $\lambda = im$ (see (3.15)), it follows from (3.54) that

$$(-im\mathcal{I} + \mathfrak{A}^*)\Psi_j = 0. \tag{3.65}$$

By (3.52)

$$\Psi_{jl} = \sum_{p=0}^{m-l} \sum_{q=0}^{m} A_{qp}^* (-im)^{2m-l-q-p} \psi_j$$

for $l = 1, \dots, m-1$,

$$\Psi_{jm} = \sum_{q=0}^{m} A^*_{q0}(-im)^{m-q}\psi_j \,, \tag{3.66}$$

$$\Psi_{j,m+l} = (-im)^{m-l}\psi_j \tag{3.67}$$

for $l = 1, \dots, m$. Clearly, $\Phi_j \in \mathcal{T}, \quad \Psi_j \in \mathcal{R}^*$, where

$$\begin{aligned}\mathcal{R}^* = {}& (W^{1-m,q}(S^{n-1}_+))^d \times \dots \times (W^{-1,q}(S^{n-1}_+))^d \\ & \times (L^q(S^{n-1}_+))^d \times ((\mathring{W}^{m,q}(S^{n-1}_+))^d)^m \,.\end{aligned}$$

Proposition 3.4. *The biorthogonality condition*

$$\langle \Phi_k, \Psi_j\rangle = i\delta^j_k \tag{3.68}$$

is valid, where

$$\langle \Phi_k, \Psi_j\rangle = \sum_{s=1}^{2m}\int_{S^{n-1}_+}(\Phi_{ks}, \Psi_{js})d\theta \,.$$

Proof. We put

$$\Phi_{k\lambda} = \begin{pmatrix} J^{-1}(\lambda) & 0 \\ Q(\lambda) & J^{-1}(\lambda)(I+\mathcal{M})\end{pmatrix}\operatorname{col}(\phi_k, 0, \dots, 0)$$

and $\Psi_{j\lambda} = \mathcal{E}^*(\overline{\lambda})\operatorname{col}(\psi_j, 0, \dots, 0)$. Then by (3.54) and (3.55)

$$\langle (\lambda\mathcal{I} + \mathfrak{A})\Phi_{k\lambda}, \Psi_{j\lambda}\rangle = \int_{S^{n-1}_+}(\mathcal{A}(\lambda)\phi_k, \psi_j)d\theta \,.$$

Differentiating this equality with respect to λ, setting $\lambda = im$ and using (3.61), (3.65) together with (3.59) we obtain (3.68). □

We introduce the spectral projector $\mathcal{P}$ corresponding to the eigenvalue $\lambda = im$:

$$\mathcal{P}\mathcal{F} = -i\sum_{q=1}^{d}\langle \mathcal{F}, \Psi_q\rangle\Phi_q \,. \tag{3.69}$$

This operator maps $\mathcal{R}$ into $\mathcal{T}$. Using (3.61) we obtain

$$\mathfrak{A}\mathcal{P} = -im\mathcal{P} \,. \tag{3.70}$$

3.4. **Equivalence of equation (3.23) and system (3.41).** Here we collect definitions of some spaces which are used in the sequel. Let $\mathcal{T}$ and $\mathcal{R}$ be the spaces defined by (3.49) and (3.50). We introduce the space $\mathbb{T}(a,b)$ of vector functions $\mathcal{U} = \operatorname{col}(\mathcal{U}_j)^{2m}_{j=1}$ defined on (a,b), taking values in $\mathcal{T}$ and supplied with the norm

$$\|\mathcal{U}\|_{\mathbb{T}(a,b)} = \Big(\int_a^b \big(\|\mathcal{U}(\tau)\|^p_{\mathcal{T}} + \|D_\tau\mathcal{U}(\tau)\|^p_{\mathcal{R}}\big)d\tau\Big)^{1/p}.$$

This definition is equivalent to

$$\mathbb{T}(a,b) = \big\{\mathcal{U} : \mathcal{U} \in L_p(a,b;\mathcal{T}), D_t\mathcal{U} \in L_p(a,b;\mathcal{R})\big\}. \tag{3.71}$$

Here p is the same number as in the definition of the spaces B_k.

By $\mathbb{T}'(a,b)$ we denote the space of vector-functions

$$\mathcal{U}' = (\mathcal{U}_1, \dots, \mathcal{U}_m) \,,$$

with values in $B_m \times \cdots \times B_1$ which is endowed with the norm

$$\|\mathcal{U}'\|_{\mathbb{T}'(a,b)} = \Big(\sum_{j=1}^{m} \int_a^b \big(\|\mathcal{U}_j(\tau)\|^p_{B_{m-j+1}} + \|D_\tau \mathcal{U}_j(\tau)\|^p_{B_{m-j}} \big) d\tau \Big)^{1/p},$$

where B_j is defined by (3.51). Also let $\hat{\mathbb{T}}(a,b)$ be the product $\mathbb{T}'(a,b) \times L_p(a,b;B_0)$, which consists of the vector functions

$$\hat{\mathcal{U}} = (\mathcal{U}', \mathcal{U}_{m+1})$$

with the norm

$$\|\hat{\mathcal{U}}\|_{\hat{\mathbb{T}}(a,b)} = (\|\mathcal{U}'\|^p_{\mathbb{T}'(a,b)} + \|\mathcal{U}_{m+1}\|^p_{L_p(a,b;B_0)})^{1/p}.$$

Furthermore, we use the spaces $\mathbb{T}'_{\rm loc}(\mathbb{R})$ and $\hat{\mathbb{T}}_{\rm loc}(\mathbb{R})$ endowed with the seminorms $\|\mathcal{U}'\|_{\mathbb{T}'(t,t+1)}$ and $\|\hat{\mathcal{U}}\|_{\hat{\mathbb{T}}(t,t+1)}$, $t \in \mathbb{R}$.

If $u \in \mathring{W}^m_{p,\rm loc}(\Pi)$ then by setting $\mathcal{U}_j = D_t^{j-1} u$ we see that

$$\mathcal{U}' \in \mathbb{T}'_{\rm loc}(\mathbb{R}) \quad \text{and} \quad \hat{\mathcal{U}} \in \hat{\mathbb{T}}_{\rm loc}(\mathbb{R})$$

and

$$2^{-1/p} \|\hat{\mathcal{U}}\|_{\hat{\mathbb{T}}(t,t+1)} \le \|u\|_{W^m_p(t,t+1;\{B_k\}_{k=0}^m)} \le \|\mathcal{U}'\|_{\mathbb{T}'(t,t+1)}\,. \tag{3.72}$$

Let $W_0^{m,p}(\Pi_{a,b})$ be the closure of the space of smooth functions u defined on $\Pi_{a,b}$ and equal zero in a neighborhood of $\partial\Pi \cap \overline{\Pi}_{a,b}$. By $\mathbf{S}(a,b)$ we denote the space of all vector functions $\mathcal{U}(t)$ represented in the form

$$\mathcal{U}(t) = \text{col}\ \big(u(t), \dots, D_t^{m-1} u(t), u_{m+1}(t), \dots, u_{2m}(t) \big) \tag{3.73}$$

where $u \in W_0^{m,p}(\Pi_{a,b})$,

$$u_{m+1} \in L_p(a,b;B_0), \quad D_t u_{m+1} \in L_p(a,b;B_{-m}) \tag{3.74}$$

$$u_{m+j}, D_t u_{m+j} \in L_p(a,b;B_{-m}), \quad j = 2, \dots, m. \tag{3.75}$$

We equip the space $\mathbf{S}(a,b)$ with the norm

$$\begin{aligned} \|\mathcal{U}\|_{\mathbf{S}(a,b)} = \Big(& \|u\|^p_{W^{m,p}(\Pi_{a,b})} + \|u_{m+1}\|^p_{L_p(a,b;B_0)} + \|D_t u_{m+1}\|^p_{L_p(a,b;B_{-m})} \\ & + \sum_{j=2}^{m} (\|u_{m+j}\|^p_{L_p(a,b;B_{-m})} + \|D_t u_{m+j}\|^p_{L_p(a,b;B_{-m})}) \Big)^{1/p} \end{aligned}$$

Clearly, $\mathbf{S}(a,b) \subset \mathbb{T}(a,b)$ and

$$\|\mathcal{U}\|_{\mathbb{T}(a,b)} \le c \|\mathcal{U}\|_{\mathbf{S}(a,b)} \tag{3.76}$$

for all $\mathcal{U} \in \mathbf{S}(a,b)$. Furthermore, if $m = 1$, then $\mathbf{S}(a,b) = \mathbb{T}(a,b)$ and the norms are equivalent.

We use the notation $\mathbb{T}_{\rm loc}(\mathbb{R})$ for the space of vector functions defined on $\mathbb{R}$ whose restrictions to an arbitrary finite interval (a,b) belong to $\mathbb{T}(a,b)$. In the same way, the space $\mathbf{S}_{\rm loc}(\mathbb{R})$ is defined.

Lemma 3.5. (i) *If $u \in (\mathring{W}^{m,p}_{\rm loc}(\overline{\Pi}))^d$ is a solution of* (3.23), *then the vector function $\mathcal{U}$ given by* (3.26)–(3.28) *belongs to $\mathbf{S}_{\rm loc}(\mathbb{R})$ and satisfies* (3.41).

(ii) *If $\mathcal{U} \in \mathbb{T}_{\rm loc}(\mathbb{R})$ is a solution of* (3.41), *then $\mathcal{U} \in \mathbf{S}_{\rm loc}(\mathbb{R})$ and the vector function $u = \mathcal{U}_1$ belongs to $(\mathring{W}^{m,p}_{\rm loc}(\overline{\Pi}))^d$ satisfies* (3.23).

For the proof of the above lemma see [4].

Sometimes, to emphasize the dependence of the spaces $\mathbb{T}$ and $\mathbf{S}$ on p we shall write $\mathbb{T}_p$ and $\mathbf{S}_p$ respectively.

Let us show that for solutions of (3.41) the following local estimate is valid

(H2b) Let p and p_1 be the same as in (H2) and (H2a) and let $\mathcal{U} \in \mathbb{T}_{p,\mathrm{loc}}(\mathbb{R})$ be a solution of (3.41) with $\mathcal{F}_{m+j} \in (L^{p_1}_{\mathrm{loc}}(\mathbb{R}; W^{-j,p_1}(S^{n-1}_+)))^d$, then $\mathcal{U} \in \mathbb{T}_{p_1,\mathrm{loc}}(\mathbb{R})$ and

$$\|\hat{\mathcal{U}}\|_{\hat{\mathbb{T}}_{p_1}(t,t+a)} \le cb_0\big(\sum_{j=0}^{m} \|\mathcal{F}_{m+j}\|_{L_{p_1}(t-a,t+2a;W^{-j,p_1}(S^{n-1}_+))} + \|\hat{\mathcal{U}}\|_{\hat{\mathbb{T}}_p(t-a,t+2a)}\big).$$

Let $\mathcal{U} \in \mathbb{T}_{p,\mathrm{loc}}(\mathbb{R})$ be a solution of (3.41) with $\mathcal{F}_{m+j} \in (L_{p_1,\mathrm{loc}}(\mathbb{R}; W^{-j,p_1}(S^{n-1}_+)))^d$. Then by Lemma 3.5(ii) the function $u = \mathcal{U}_1$ is a solution of (3.23). Clearly, the functions f_j in (3.43) and (3.44) belong to $(L_{p_1,\mathrm{loc}}(\mathbb{R}; W^{-j,p_1}(S^{n-1}_+)))^d$. By (H2a) $u \in (\mathring{W}^{m,p_1}_{\mathrm{loc}}(\overline{\mathbb{R}^n_+}))^d$ and (3.25) holds. This together with (3.43) and (3.44) gives (H2b).

4. Description of solutions to the homogeneous system (3.41)

Our goal here is to describe all solutions $\mathcal{U} \in \mathbb{T}_{p,\mathrm{loc}}(\mathbb{R})$ to equation

$$(\mathcal{I}D_t + \mathfrak{A})\mathcal{U}(t) - \mathfrak{N}(t)\widehat{\mathcal{U}}(t) = \mathcal{O} \quad \text{on } \mathbb{R}, \tag{4.1}$$

subject to

$$\|\mathcal{U}\|_{\mathbb{T}_p(t,t+1)} = \begin{cases} o\big(e^{(n-m)t-c_0\int_0^t \omega(s)ds}\big) & \text{as } t \to +\infty \\ o\big(e^{-(m+1)t-c_0\int_t^0 \omega(s)ds}\big) & \text{as } t \to -\infty, \end{cases} \tag{4.2}$$

where ω is given by (3.6) and c_0 is a sufficiently large constant. The main theorem is contained in Section 4.6.

4.1. Spaces $\mathbb{X}$ and $\mathbb{Y}$.

Here we add some new function spaces to spaces $\mathbb{T}$, $\mathbb{T}'$, $\hat{\mathbb{T}}$ and $\mathbf{S}$. By $\mathbb{X}(a,b)$ we denote the space of all vector functions

$$\mathcal{U}(t) = (\mathcal{I} - \mathcal{P})\mathcal{V}(t) \tag{4.3}$$

with $\mathcal{V} \in \mathbf{S}(a,b)$. We define the space $\mathbb{X}_{\mathrm{loc}}(\mathbb{R})$ of all vector functions on $\mathbb{R}$ which are represented in the form (4.3) with a certain $\mathcal{V} \in \mathbf{S}_{\mathrm{loc}}(\mathbb{R})$. Clearly, $\mathbb{X}_{\mathrm{loc}}(\mathbb{R}) \subset \mathbb{T}_{\mathrm{loc}}(\mathbb{R})$ and we shall use seminorms $\|\cdot\|_{\mathbb{T}(t,t+1)}$ in $\mathbb{X}_{\mathrm{loc}}(\mathbb{R})$. If $m = 1$ then $\mathbb{X}(a,b)$ is a closed subspace in $\mathbb{T}(a,b)$ consisting of functions $\mathbf{v} \in \mathbb{T}(a,b)$ satisfying $(\mathcal{I} - \mathcal{P})\mathbf{v}(t) = \mathbf{v}(t)$ almost for all $t \in (a,b)$. For the case $m \ge 2$ we prove the following

Lemma 4.1. *Let $m \ge 2$. Then*
(i) *if $(\mathcal{I} - \mathcal{P})\mathcal{U} = 0$ with $\mathcal{U} \in \mathbf{S}(a,b)$ then*

$$\mathcal{U} = e^{-mt} \sum_{j=1}^{d} c_j \Phi_j \tag{4.4}$$

with some constants c_j;
(ii) *if $\mathbf{v} \in \mathbb{X}(a,b)$ then there exists $\mathcal{U} \in \mathbf{S}(a,b)$ such that $\mathbf{v} = (\mathcal{I} - \mathcal{P})\mathcal{U}$ and*

$$\|\mathcal{U}\|_{\mathbb{T}(a,b)} \le c\|\mathbf{v}\|_{\mathbb{T}(a,b)} \tag{4.5}$$

with constant c depending only on $b - a$, n, m, p and $\mathcal{P}$.

Proof. (i) The equality $\mathcal{U}=\mathcal{P}\mathcal{U}$ implies

$$\mathcal{U}(t)=\sum_{k=1}^{d}h_k(t)\Phi_k.$$

Since $\mathcal{U}_2(t)=D_t\mathcal{U}_1(t)$ and $\Phi_{k1}=\phi_k$, $\Phi_{k2}=im\phi_k$ we have that

$$\sum_{k=1}^{d}D_th_k(t)\phi_k=im\sum_{k=1}^{d}h_k(t)\phi_k.$$

Using linear independence of the functions ϕ_k we obtain that $D_th_k(t)=imh_k(t)$ or $h_k(t)=c_ke^{-mt}$.

(ii) We introduce the factor space $\mathbb{T}_0=\mathbb{T}(a,b)/K$, where K is the subspace of elements of the form (4.4). The norm is defined by

$$\|\mathcal{U}\|_{\mathbb{T}_0}=\min_{\mathcal{V}\in K}\|\mathcal{U}+\mathcal{V}\|_{\mathbb{T}_0}.$$

Clearly the minimum is attained for a certain $\mathcal{V}$. Suppose that the assertion (ii) is not valid. Then there exist functions $\mathcal{U}_j$ such that $\|\mathcal{U}_j\|_{\mathbb{T}(a,b)}=\|\mathcal{U}_j\|_{\mathbb{T}_0}=1$ and $\|(\mathcal{I}-\mathcal{P})\mathcal{U}_j\|_{\mathbb{T}(a,b)}\to 0$ as $j\to\infty$. We write

$$\mathcal{P}\mathcal{U}_j(t)=\sum_{k=1}^{d}h_k^{(j)}(t)\Phi_k.$$

Using

$$\|\mathcal{U}_{j1}-\sum_{k=1}^{d}h_k^{(j)}(t)\phi_k\|_{L_p(a,b;B_m)}\to 0,$$
$$\|\mathcal{U}_{j2}-im\sum_{k=1}^{d}h_k^{(j)}(t)\phi_k\|_{L_p(a,b;B_{m-1})}\to 0,$$
$$\|D_t\mathcal{U}_{j1}-\sum_{k=1}^{d}D_th_k^{(j)}(t)\phi_k\|_{L_p(a,b;B_{m-1})}\to 0$$

together with $D_t\mathcal{U}_{j1}=\mathcal{U}_{j2}$, we obtain that

$$\|D_th_k^{(j)}-imh_k^{(j)}\|_{L^p(a,b)}\to 0\quad\text{as } j\to\infty.$$

Putting $f_k^{(j)}=D_th_k^{(j)}-imh_k^{(j)}$, we obtain

$$h_k^{(j)}(t)=c_k^{(j)}e^{-mt}+F_k^{(j)}(t)\quad\text{with}\quad F_k^{(j)}(t)=\int_a^t e^{-m(t-\tau)}f_k^{(j)}(\tau)d\tau,$$

where $c_k^{(j)}$ are constants. Clearly, $F_k^{(j)}\to 0$ in $W^{1,p}(a,b)$. If we introduce

$$\mathcal{U}_j'=\mathcal{U}_j-e^{-mt}\sum_{k=1}^{d}c_k^{(j)}\Phi_k\,,$$

then $\|\mathcal{U}_j'\|_{\mathbb{T}(a,b)}\geq 1$ and $\|\mathcal{P}\mathcal{U}_j'\|_{\mathbb{T}(a,b)}\to 0$ as $j\to 0$. Since $(\mathcal{I}-\mathcal{P})\mathcal{U}_j=(\mathcal{I}-\mathcal{P})\mathcal{U}_j'$ we have also $\|(\mathcal{I}-\mathcal{P})\mathcal{U}_j\|_{\mathbb{T}(a,b)}\to 0$. This implies that $\|\mathcal{U}_j'\|_{\mathbb{T}(a,b)}\to 0$ as $j\to 0$. This contradiction proves (ii). $\square$

Corollary 4.2. *The space* $\mathbb{X}_{\mathrm{loc}}(\mathbb{R})$ *is closed in* $\mathbb{T}_{\mathrm{loc}}(\mathbb{R})$.

Proof. For $m = 1$ this is obvious. Let $m \geq 2$ and let $\mathbf{v}_j \in \mathbb{X}_{\rm loc}(\mathbb{R})$ and $\mathbf{v}_j \to \mathbf{v}$ in $\mathbb{T}_{\rm loc}(\mathbb{R})$. We put $\delta_k = (k, k+3/2)$. Then $\mathbf{v}_j \to \mathbf{v}$ in $\mathbb{T}_{\rm loc}(\delta_k)$ for each $k \in \mathbb{Z}$. By Lemma 4.1(ii) there exists $\mathcal{U}_j^{(k)} \in \mathbb{T}_{\rm loc}(\delta_k)$ such that $(\mathcal{I} - \mathcal{P})\mathcal{U}_j^{(k)} = \mathbf{v}_j$ on δ_k and estimate (4.5) holds for the interval δ_k. Therefore the sequence $\{\mathcal{U}_j^{(k)}\}$ has the limit $\mathcal{U}^{(k)}$ in $\mathbb{T}_{\rm loc}(\delta_k)$ and $(\mathcal{I} - \mathcal{P})\mathcal{U}^{(k)} = \mathbf{v}$. By Lemma 4.1(i)

$$\mathcal{U}_{k+1} - \mathcal{U}_k = e^{-mt} \sum_{j=1}^{d} c_j^{(k)} \Phi_j$$

with some constants $c_j^{(k)}$. This implies that there exists $\mathcal{U} \in \mathbb{T}_{\rm loc}(\mathbb{R})$ such that $(\mathcal{I} - \mathcal{P})\mathcal{U} = \mathbf{v}$ on $\mathbb{R}$ and $\mathcal{U} - \mathcal{U}_j$ has the same form as the right-hand side of (4.4) on each δ_k. Therefore, $\mathbf{v} \in \mathbb{X}_{\rm loc}(\mathbb{R})$. □

We shall also use the space $\mathbb{Y}_{\rm loc}(\mathbb{R})$ of vector functions

$$\mathcal{F}(t) = \operatorname{col}\big(0, \dots, 0, \mathcal{F}_m(t), \mathcal{F}_{m+1}(t), \dots, \mathcal{F}_{2m}\big) \tag{4.6}$$

with some $\mathcal{F}_{m+j} \in L_{p,\rm loc}(\mathbb{R}; B_{-j})$, $j = 0, \dots, m$. We equip this space with the seminorms

$$\|\mathcal{F}\|_{\mathbb{Y}(t,t+1)} = \Big(\sum_{j=0}^{m} \|\mathcal{F}_{m+j}\|^p_{L_p(t,t+1;B_{-j})} \Big)^{1/p}.$$

We put

$$\widehat{\mathcal{T}} = B_m \times B_{m-1} \times \dots \times B_1 \times B_0\,, \tag{4.7}$$

$$\varkappa_s(t) = \kappa_s(e^{-t}). \tag{4.8}$$

We shall use also the notation $\mathbb{X}_p$, $\mathbb{Y}_p$, B_k^p and $\mathcal{T}_p$ parallel to $\mathbb{X}$, $\mathbb{Y}$, B_k and $\mathcal{T}$ in order to indicate their dependence on p.

Let us prove the following estimate

Lemma 4.3. *Let $q \geq p$, $\delta > 0$ and let $\widehat{\mathcal{U}} \in L_q(t, t+\delta; \hat{\mathcal{T}}_q)$. Then*

$$\|\mathfrak{N}\widehat{\mathcal{U}}\|_{\mathbb{Y}_p(t,t+\delta)} \leq c\varkappa_{s,\delta}(t) \|\widehat{\mathcal{U}}\|_{L_q(t,t+\delta;\hat{\mathcal{T}}_q)}, \tag{4.9}$$

where

$$\varkappa_{s,\delta}(t) = \Big(\int_{K_{e^{-\delta-t},e^{-t}}} \kappa^s(x)|x|^{-n} dx \Big)^{1/s} \tag{4.10}$$

and $s = qp/(q-p)$.

Proof. Using definitions (3.46) and (3.34) of the operators $\mathfrak{N}_m$ and $\mathcal{S}'$, we have

$$\|\mathfrak{N}_m\widehat{\mathcal{U}}\|_{L_p(t,t+\delta;B_0^p)} \leq c\Big(\sum_{k=0}^{m-1} \|\mathcal{N}_{0,m-k}\mathcal{U}_{k+1}\|_{L_p(t,t+\delta;B_0^p)} + \|\mathcal{N}_{00}\mathcal{S}'\widehat{\mathcal{U}}\|_{L_p(t,t+\delta;B_0^p)} \Big). \tag{4.11}$$

By (3.36) and Hölder's inequality, we get

$$\|\mathcal{N}_{00}\mathcal{S}'\widehat{\mathcal{U}}\|_{L_p(t,t+\delta;B_0^p)} \leq c\varkappa_{s,\delta}(t)\|\widehat{\mathcal{U}}\|_{L_q(t,t+\delta;\hat{\mathcal{T}}_q)}.$$

By (3.21) the sum in the right-hand side in (4.11) is estimated by

$$c \sum_{k=0}^{m-1} \sum_{|\alpha|=m} \sum_{k \leq |\beta| \leq m} e^{(|\beta|-m)t} \|N_{\alpha\beta} Q_{\beta k} \mathcal{U}_{k+1}\|_{L_p(t,t+\delta;B_0^p)}.$$

Using Hardy's and Hölder's inequalities we estimate this sum by the right-hand side in (4.9). Thus the norm of $\mathfrak{N}_m\widehat{\mathcal{U}}$ is estimated. The corresponding norms of $\mathfrak{N}_{m+j}\widehat{\mathcal{U}}$, $j = 1, \dots, m$, are estimated analogously. $\square$

4.2. **Spectral splitting of system (3.41).** Let

$$\mathbf{u}(t) = \mathcal{P}\mathcal{U}(t), \quad \mathbf{v}(t) = (\mathcal{I} - \mathcal{P})\mathcal{U}(t). \tag{4.12}$$

Then

$$\mathcal{U}(t) = \mathbf{u}(t) + \mathbf{v}(t)\,. \tag{4.13}$$

Also, let $\hat{\mathbf{u}} = \mathrm{col}(\mathbf{u}_1, \dots, \mathbf{u}_{m+1})$ and $\hat{\mathbf{v}} = \mathrm{col}(\mathbf{v}_1, \dots, \mathbf{v}_{m+1})$. Applying $\mathcal{P}$ to equation (3.41) and using (3.70) we arrive at

$$(D_t - im)\mathbf{u} - \mathcal{P}\mathfrak{N}(t)(\hat{\mathbf{u}} + \hat{\mathbf{v}}) = \mathcal{P}\mathcal{F} \quad \text{on } \mathbb{R}. \tag{4.14}$$

Applying $\mathcal{I} - \mathcal{P}$ to (3.41) we obtain

$$(\mathcal{I}D_t + \mathfrak{A})\mathbf{v} - (\mathcal{I} - \mathcal{P})\mathfrak{N}(t)\hat{\mathbf{v}} = (\mathcal{I} - \mathcal{P})(\mathcal{F} + \mathfrak{N}(t)\hat{\mathbf{u}}) \quad \text{on } \mathbb{R}. \tag{4.15}$$

Thus we have split system (3.41) into the finite-dimensional system (4.14) and the infinite-dimensional system (4.15). Clearly, $\mathcal{U} \in \mathbb{T}_{p,\mathrm{loc}}(\mathbb{R})$ implies that $\mathbf{u}$ and $D_t\mathbf{u}$ belong to $L_{p,\mathrm{loc}}(\mathbb{R}; \mathcal{T}_q)$ for all $q \geq p$.

The next proposition shows the equivalence of (3.41) and the split system (4.14), (4.15).

Proposition 4.4. (i) *Let $\mathcal{U} \in \mathbb{T}_{\mathrm{loc}}(\mathbb{R})$ be a solution of* (3.41). *Then $\mathcal{U} \in \mathbf{S}_{\mathrm{loc}}(\mathbb{R})$ and the pair $\mathbf{u}, \mathbf{v}$ given by* (4.12) *satisfy systems* (4.14), (4.15).

(ii) *Let $\mathbf{u}$ and $\mathbf{v}$ belong to $\mathbb{T}_{\mathrm{loc}}(\mathbb{R})$, satisfy* (4.14), (4.15) *and be subject to $\mathbf{u}(t) = \mathcal{P}\mathbf{u}(t)$ and $\mathbf{v}(t) = (\mathcal{I} - \mathcal{P})\mathbf{v}(t)$ on $\mathbb{R}$. Then the function* (4.13) *satisfies system* (3.41).

The proof of the above proposition is obvious. This proposition, combined with Lemma 3.5, ensures the equivalence of equation (3.23) and the split system (4.14), (4.15).

4.3. **The infinite-dimensional part of the split system.** We start with the case $\mathfrak{N} = 0$, i.e. we consider the system

$$(\mathcal{I}D_t + \mathfrak{A})\mathbf{v} = (\mathcal{I} - \mathcal{P})F \quad \text{on } \mathbb{R}. \tag{4.16}$$

We put

$$\mu(t) = \begin{cases} e^{-(m+1)t} & \text{for } t \geq 0 \\ e^{(n-m)t} & \text{for } t < 0. \end{cases} \tag{4.17}$$

The following result is proved in [KM2, Lemma 8].

Lemma 4.5. (i) (Existence) *Let $F \in \mathbb{Y}_{q,\mathrm{loc}}(\mathbb{R})$, $q \in (1, \infty)$ and $\delta > 0$. Suppose that*

$$\int_{\mathbb{R}} \mu(-\tau)\|F\|_{\mathbb{Y}_q(\tau,\tau+\delta)} d\tau < \infty\,. \tag{4.18}$$

Then (4.16) *has a solution $\mathbf{v} \in \mathbb{X}_{q,\mathrm{loc}}(\mathbb{R})$ satisfying*

$$\|\mathbf{v}\|_{\mathbb{T}_q(t,t+\delta)} \leq c \int_{\mathbb{R}} \mu(t - \tau)\|F\|_{\mathbb{Y}_q(\tau,\tau+\delta)} d\tau, \tag{4.19}$$

where c is a constant independent of F.

(ii) (Uniqueness) *Let* $\mathbf{v} \in \mathbb{T}_{q,\mathrm{loc}}(\mathbb{R})$ *satisfy* (4.16) *with* $F = 0$ *and* $\mathcal{P}\mathbf{v}(t) = 0$ *for almost every* $t \in \mathbb{R}$. *Also let*

$$\|\mathbf{v}\|_{\mathbb{T}_q(t,t+\delta)} = \begin{cases} o(e^{(n-m)t}) & \text{if } t \to +\infty \\ o(e^{-(m+1)t}) & \text{if } t \to -\infty \end{cases} \tag{4.20}$$

be valid. Then $\mathbf{v} = 0$.

Now, we study the system

$$(\mathcal{I}D_t + \mathfrak{A})\mathbf{v} - (\mathcal{I} - \mathcal{P})\mathfrak{N}(t)\hat{\mathbf{v}} = (\mathcal{I} - \mathcal{P})F \quad \text{on } \mathbb{R}. \tag{4.21}$$

We introduce the function

$$\mu_\omega(t,\tau) = \begin{cases} \exp\big(-(m+1)(t-\tau) + c_0\int_\tau^t \omega(s)ds\big) & \text{for } t \ge \tau \\ \exp\big((n-m)(t-\tau) + c_0\int_t^\tau \omega(s)ds\big) & \text{for } t < \tau, \end{cases} \tag{4.22}$$

where c_0 is a sufficiently large positive constant depending on n, m, p, γ and L.

Proposition 4.6. *Let assumptions (H1)–(H3) be fulfilled and let* p *and* p_1 *be the same as in (H2). Then the following assertions are valid:*

(i) *Let* F *belong to* $\mathbb{Y}_{p_1,\mathrm{loc}}(\mathbb{R})$ *and let*

$$\int_{\mathbb{R}} \mu_\omega(0,\tau)\|F\|_{\mathbb{Y}_{p_1}(\tau,\tau+1)}d\tau < \infty\,. \tag{4.23}$$

Then system (4.21) *has a solution* $\mathbf{v} \in \mathbb{X}_{p_1,\mathrm{loc}}(\mathbb{R})$ *satisfying*

$$\|\mathbf{v}\|_{\mathbb{T}_p(t,t+1)} \le c\int_{\mathbb{R}} \mu_\omega(t,\tau)\|F\|_{\mathbb{Y}_{p_1}(\tau,\tau+1)}d\tau\,, \tag{4.24}$$

$$\|\hat{\mathbf{v}}\|_{\hat{\mathbb{T}}_{p_1}(t,t+1)} \le cb_0\int_{\mathbb{R}} \mu_\omega(t,\tau)\|F\|_{\mathbb{Y}_{p_1}(\tau,\tau+1)}d\tau\,. \tag{4.25}$$

(ii) *The solution* $\mathbf{v} \in \mathbb{X}_{p,\mathrm{loc}}(\mathbb{R})$ *to* (4.21) *subject to*

$$\|\mathbf{v}\|_{\mathbb{T}_p(t,t+1)} = \begin{cases} o\Big(e^{(n-m)t - c_0\int_0^t \omega(\tau)d\tau}\Big) & \text{as } t \to +\infty \\ o\Big(e^{-(m+1)t - c_0\int_t^0 \omega(\tau)d\tau}\Big) & \text{as } t \to -\infty \end{cases} \tag{4.26}$$

is unique. (We note that (4.23) *together with* (4.24) *imply* (4.26).*)*

Proof. (1). *Solvability in* $\mathbb{X}_2(\mathbb{R})$. We introduce the space $\mathbb{T}_2(\mathbb{R})$, which consists of vector functions $\mathcal{U} \in \mathbb{T}_{2,\mathrm{loc}}(\mathbb{R})$ with finite norm

$$\|\mathcal{U}\|_{\mathbb{T}_2(\mathbb{R})} = \Big(\int_{\mathbb{R}} e^{(2m-n)t}\|\mathcal{U}\|^2_{\mathbb{T}_2(t,t+1))}dt\Big)^{1/2}.$$

The space $\mathbf{S}_2(\mathbb{R})$ contains vector functions (3.73) with finite norm

$$\|\mathcal{U}\|_{\mathbf{S}_2(\mathbb{R})} = \Big(\int_{\mathbb{R}} e^{(2m-n)t}\|\mathcal{U}\|^2_{\mathbf{S}_2(t,t+1))}dt\Big)^{1/2}.$$

The space $\mathbb{X}_2(\mathbb{R})$ consists of $\mathbf{v}$ represented as $(\mathcal{I} - \mathcal{P})\mathcal{U}$ with $\mathcal{U} \in \mathbf{S}_2(\mathbb{R})$. Let also $\mathbb{Y}_2(\mathbb{R})$ consists of vector functions $\mathcal{F}$ from $\mathbb{Y}_{2,\mathrm{loc}}(\mathbb{R})$ with finite norm

$$\|\mathcal{F}\|_{\mathbb{Y}_2(\mathbb{R})} = \Big(\int_{\mathbb{R}} e^{(2m-n)t}\|\mathcal{F}\|^2_{\mathbb{Y}_2(t,t+1))}dt\Big)^{1/2}.$$

Consider first problem (3.41) with $\mathcal{F} \in \mathbb{Y}_2(\mathbb{R})$. Using the solvability result for (2.52) from (1) in the proof of Proposition 2.11 and connection of problems (2.52) and (3.41) established in Section 3 we obtain that for every $\mathcal{F} \in \mathbb{Y}_2(\mathbb{R})$ there exists the unique $\mathcal{U} \in \mathbb{T}_2(\mathbb{R})$ solving (3.41) and it satisfies

$$\|\mathcal{U}\|_{\mathbb{T}_2(\mathbb{R})} \le c\|\mathcal{F}\|_{\mathbb{Y}_2(\mathbb{R})}.$$

Therefore the function $\mathbf{v} = (\mathcal{I} - \mathcal{P})\mathcal{U}$ belongs to $\mathbb{X}_2(\mathbb{R})$ solves (4.21) with $F = \mathcal{F}$.

(2). *Local estimate for* $\mathbf{v}$. Let $\mathbf{v} = (\mathcal{I} - \mathcal{P})\mathcal{U}$ with $\mathcal{U} \in \mathbf{S}_{p,\mathrm{loc}}(\mathbb{R})$ satisfy (4.21) with $F \in \mathbb{Y}_{p_1,\mathrm{loc}}(\mathbb{R})$ and let $m \ge 2$. According to Lemma 4.1(ii) we can suppose that $\mathcal{U}$ is subject to (4.5) with $a = t - \delta$ and $b = t + 2\delta$, where t and δ are fixed. We write equation (4.21) as

$$(\mathcal{I}D_t + \mathfrak{A})\mathcal{U} - \mathfrak{N}(t)\hat{\mathbf{v}} = F + \mathcal{P}G, \tag{4.27}$$

where

$$G = (\mathcal{I}D_t + \mathfrak{A})\mathcal{U} - \mathfrak{N}(t)\hat{\mathbf{v}} - F.$$

System (4.27) consists of $2m$ equations. Since $\mathcal{U} \in \mathbf{S}_{p,\mathrm{loc}}(\mathbb{R})$ and the first components of $\mathfrak{N}(t)\hat{\mathbf{v}}$ and F are zero, we have $(\mathcal{P}G)_1 = 0$. But

$$(\mathcal{P}G)(t) = \sum_{j=1}^{d} h_j(t)\Phi_j$$

and the vector functions $(\Phi_j)_1 = \phi_j$ are linear independent, which implies $h_j = 0$ and hence $\mathcal{P}G = 0$. Thus system (4.27) becomes

$$(\mathcal{I}D_t + \mathfrak{A})\mathcal{U} - \mathfrak{N}(t)\hat{\mathcal{U}} = F - \mathfrak{N}(t)\widehat{\mathcal{P}\mathcal{U}}. \tag{4.28}$$

Since $\mathcal{U} \in \mathbf{S}_{p,\mathrm{loc}}(\mathbb{R}) \subset \mathbb{T}_{p,\mathrm{loc}}(\mathbb{R})$ it follows that $\mathcal{P}\mathcal{U}$ and $\partial_t\mathcal{P}\mathcal{U}$ belong to $L_{p,\mathrm{loc}}(\mathbb{R};\mathcal{T}_q)$ for all $q \ge p$ and

$$\|\mathcal{P}\mathcal{U}\|_{L_p(t,t+\delta;\mathcal{T}_q)} + \|\partial_\tau\mathcal{P}\mathcal{U}\|_{L_p(t,t+\delta;\mathcal{T}_q)} \le c\|\mathcal{U}\|_{\mathbb{T}_p(t,t+\delta)}.$$

This implies

$$\|\mathcal{P}\mathcal{U}\|_{L_q(t,t+\delta;\mathcal{T}_q)} \le c\|\mathcal{U}\|_{\mathbb{T}_p(t,t+\delta)}. \tag{4.29}$$

Now applying (H2b) to (4.28) and using the last inequality together with Lemma 4.3, we obtain that $\mathcal{U} \in \mathbb{T}_{p_1,\mathrm{loc}}(\mathbb{R})$ and

$$\|\hat{\mathcal{U}}\|_{\hat{\mathbb{T}}_{p_1}(t,t+\delta)} \le cb_0\big(\sum_{j=0}^{m} \|F_{m+j}\|_{L_{p_1}(t-\delta,t+2\delta;W^{-j,p_1}(S_+^{n-1}))} + \|\hat{\mathcal{U}}\|_{\hat{\mathbb{T}}_p(t-\delta,t+2\delta)}\big).$$

Now using (4.5) we arrive at

$$\|\hat{\mathbf{v}}\|_{\hat{\mathbb{T}}_{p_1}(t,t+\delta)} \le cb_0\big(\|F\|_{\mathbb{Y}_{p_1}(t-\delta,t+2\delta)} + \|\mathbf{v}\|_{\mathbb{T}_p(t-\delta,t+2\delta)}\big). \tag{4.30}$$

When $m = 1$, a direct application of (H2b) to the system $(\mathcal{I}D_t + \mathfrak{A})\mathbf{v} - \mathfrak{N}(t)\hat{\mathbf{v}} = F - \mathcal{P}\mathfrak{N}(t)\hat{\mathbf{v}}$ gives

$$\begin{aligned}\|\hat{\mathbf{v}}\|_{\hat{\mathbb{T}}_{p_1}(t,t+\delta)} \le cb_0\big(&\sum_{j=0}^{1} \|(F - \mathcal{P}\mathfrak{N}(t)\hat{\mathbf{v}})_{1+j}\|_{L_{p_1}(t-\delta,t+2\delta;W^{-j,p_1}(S_+^{n-1}))} \\ &+ \|\mathbf{v}\|_{\mathbb{T}_p(t-\delta,t+2\delta)}\big).\end{aligned}$$

This together with (4.29) implies (4.30) for $m = 1$. The local estimate (4.30) together with (4.9), with $q = p_1$, gives

$$\|\mathfrak{N}\hat{\mathbf{v}}\|_{\mathbb{Y}_p(t,t+\delta)} \le c\big(\omega_0\|F\|_{\mathbb{Y}_{p_1}(t-\delta,t+2\delta)} + b_0\varkappa_{s,\delta}(t)\|\mathbf{v}\|_{\mathbb{T}_p(t-\delta,t+2\delta)}\big), \tag{4.31}$$

where $s = p_1p/(p_1 - p)$ and $\varkappa_{s,\delta}$ is given by (4.10).

(3). *Existence of solution.* One can verify that every vector function F from $\mathbb{Y}_{p_1,\mathrm{loc}}(\mathbb{R})$ subject to (4.23) can be approximated by vector functions from $\mathbb{Y}_2(\mathbb{R})$ with compact support in the norm defined by the left-hand side in (4.23). Therefore it suffices to prove estimate (4.24) and (4.25) for solutions from (1). We write system (4.21) in the form

$$(\mathcal{I}D_t + \mathfrak{A})\mathbf{v} = (\mathcal{I} - \mathcal{P})(F + \mathfrak{N}(t)\hat{\mathbf{v}}).$$

Using $\mathbf{v} \in \mathbb{X}_2(\mathbb{R})$ one can check that the right-hand side satisfies (4.18). Applying to this equation Lemma 4.5 with $q = p$ and using (4.31), we arrive at

$$\|\mathbf{v}\|_{\mathbb{T}_p(t,t+\delta)} \le c \int_{\mathbb{R}} \mu(t-\tau)\big(\|F\|_{\mathbb{Y}_{p_1}(t-\delta,t+2\delta)} + b_0\varkappa_{s,\delta}(t)\|\mathbf{v}\|_{\mathbb{T}_p(t-\delta,t+2\delta)}\big)d\tau.$$

Taking here δ sufficiently small we derive the estimate

$$\|\mathbf{v}\|_{\mathbb{T}_p(t,t+1)} \le c \int_{\mathbb{R}} \mu(t-\tau)\big(\|F\|_{\mathbb{Y}_{p_1}(t,t+1)} + b_0\varkappa_s(t)\|\mathbf{v}\|_{\mathbb{T}_p(t,t+1)}\big)d\tau.$$

Iterating this inequality we obtain

$$\|\mathbf{v}\|_{\mathbb{T}_p(t,t+1)} \le c \int_{\mathbb{R}} g_\omega(t,\tau)\|F\|_{\mathbb{Y}_{p_1}(t,t+1)}d\tau, \tag{4.32}$$

where

$$g_\omega(t,\tau) = \mu(t-\tau) + \sum_{k=1}^{\infty}(cb_0)^k \int_{\mathbb{R}^k} \mu(t-\tau_1)\varkappa_s(\tau_1)\mu(\tau_1-\tau_2)\dots\varkappa_s(\tau_k)\mu(\tau_k-\tau)d\tau_1\dots d\tau_k.$$

Since $(m-n-\partial_t)(\partial_t+m+1)\mu(t) = (n+1)\delta(t)$, we can check that

$$\big((m-n-\partial_t)(\partial_t+m+1) - (n+1)cb_0\kappa_s(e^{-t})\big)g_\omega(t,\tau) = (n+1)\delta(t-\tau).$$

Using [3, Proposition 6.3.1], we obtain

$$g_\omega(t,\tau) \le C\mu_\omega(t,\tau).$$

This together with (4.32) leads to (4.24). Estimate (4.24) together with the local estimate (4.30) gives (4.25).

(4) *Uniqueness.* First we observe that we can start in (3) from a solution $\mathbf{v} \in \mathbb{X}_{p,\mathrm{loc}}(\mathbb{R})$ subject to a certain growth restrictions at $\pm\infty$, for example $\mathbf{v}$ has a compact support with respect to t, and reasoning as above we will arrive at estimates (4.24) and (4.25) for such $\mathbf{v}$. This leads to uniqueness in the class of functions with compact support. The uniqueness in the class of functions subject to (4.26) is proved in the same way as in Proposition 2.11. □

4.4. **The finite dimensional system.** By Proposition 4.6 we can introduce the operator $\mathfrak{M} : F \to \hat{\mathbf{v}}$ which is defined on $F \in \mathbb{Y}_{p_1,\mathrm{loc}}(\mathbb{R})$ subject to (4.23) and $\mathfrak{M}(F) = \hat{\mathbf{v}}$ where $\mathbf{v}$ is the solution of (4.21) from Proposition 4.6. By (4.24) we have

$$\|\mathfrak{M}(F)\|_{\hat{\mathbb{T}}_{p_1}(t,t+1)} \le cb_0 \int_{\mathbb{R}} \mu_\omega(t,\tau)\|F\|_{\mathbb{Y}_{p_1}(\tau,\tau+1)}d\tau\ . \tag{4.33}$$

Using the operator $\mathfrak{M}$, we write (4.14) in the form

$$(D_t - im)\mathbf{u} - \mathcal{P}\mathfrak{N}(\hat{\mathbf{u}} + \mathfrak{M}(\mathfrak{N}\hat{\mathbf{u}}))(t) = \mathcal{P}(\mathcal{F} + \mathfrak{N}\mathfrak{M}(\mathcal{F}))(t) \quad \text{on } \mathbb{R}.$$

We rewrite this system as

$$(D_t - im)\mathbf{u} - \mathcal{P}\mathfrak{N}^{(0)}\hat{\mathbf{u}} - \mathcal{P}\mathcal{K}(\hat{\mathbf{u}}) = \mathcal{P}(\mathcal{F} + \mathfrak{N}\mathfrak{M}(\mathcal{F}))(t) \quad \text{on } \mathbb{R}, \tag{4.34}$$

where

$$\mathcal{K}[\hat{\mathbf{u}}](t) = (\mathfrak{N}(t) - \mathfrak{N}^{(0)}(t))\hat{\mathbf{u}} + \mathfrak{N}(t)\mathfrak{M}(\mathfrak{N}\hat{\mathbf{u}}) \,, \tag{4.35}$$

$$\mathfrak{N}^{(0)}(t)\hat{\mathcal{U}} = \mathrm{col}(0, \dots, 0, \mathfrak{N}_m^{(0)}(t)\hat{\mathcal{U}}, \mathfrak{N}_{m+1}^{(0)}(t)\hat{\mathcal{U}}, \dots, \mathfrak{N}_{2m}^{(0)}(t)\hat{\mathcal{U}}) \tag{4.36}$$

with

$$\mathfrak{N}_m^{(0)}(t)\hat{\mathcal{U}} = A_{00}^{-1}\Big(\sum_{k=0}^{m-1} \mathcal{N}_{0,m-k}(t)\mathcal{U}_{k+1} + \mathcal{N}_{00}(t)\mathcal{S}_0(t)\hat{\mathcal{U}}\Big) \tag{4.37}$$

and

$$\begin{aligned} \mathfrak{N}_{m+j}^{(0)}(t)\hat{\mathcal{U}} &= \sum_{k=0}^{m-1} \mathcal{N}_{j,m-k}(t)\mathcal{U}_{k+1} + \mathcal{N}_{j0}(t)\mathcal{S}_0(t)\hat{\mathcal{U}} \\ &\quad - A_{j0}A_{00}^{-1}\Big(\sum_{k=0}^{m-1} \mathcal{N}_{0,m-k}(t)\mathcal{U}_{k+1} + \mathcal{N}_{00}(t)\mathcal{S}_0(t)\hat{\mathcal{U}}\Big) \end{aligned} \tag{4.38}$$

for $j = 1, \dots, m$. Here $\mathcal{S}_0$ is given by (3.33).

The vector function $\mathbf{u}$ can be represented as

$$\mathbf{u}(t) = e^{-mt} \sum_{j=1}^{d} h_j(t)\Phi_j \,, \tag{4.39}$$

where Φ_j is given by (3.62) and (3.63). Inserting (4.39) into (4.34), multiplying then (4.58) by vectors (3.64) and using (3.68) and (3.69), we obtain the system for the vector function $\mathbf{h} = \mathrm{col}(h_1, \dots, h_d)$

$$\partial_t \mathbf{h}(t) - \mathcal{R}(t)\mathbf{h}(t) - (\mathcal{M}\mathbf{h})(t) = \mathbf{g} \,, \tag{4.40}$$

where

$$(\mathcal{R}(t)\mathbf{h}(t))_k = \sum_{j=1}^{N} \mathcal{R}_{kj}(t)h_j(t)$$

with

$$\mathcal{R}_{kj}(t) = \langle \mathfrak{N}^{(0)}(t)\Phi_j, \Psi_k\rangle \quad \text{and} \quad (\mathcal{M}\mathbf{h})(t) = ((\mathcal{M}\mathbf{h})_1(t), \dots, \mathcal{M}\mathbf{h})_d(t)),$$

where

$$(\mathcal{M}\mathbf{h})_k(t) = e^{mt}\langle \mathcal{K}(e^{-m\tau} \sum_{j=1}^{d} h_j\hat{\Phi}_j)(t), \Psi_k\rangle \,. \tag{4.41}$$

The right-hand side $\mathbf{g}(t) = (g_1(t), \dots, g_d(t))$ is defined by

$$g_k(t) = e^{mt}\langle \mathcal{F}(t) + \mathfrak{N}\mathfrak{M}(\mathcal{F})(t), \Psi_k\rangle \,. \tag{4.42}$$

Using (3.62), (3.63) and (3.33) we obtain that $\mathcal{S}_0\Phi_j = (im)^m\varphi_j$. Therefore,

$$\mathfrak{N}_m^{(0)}(t)\Phi_j = A_{00}^{-1}\mathcal{N}_0(t, im)\varphi_j \,,$$

$$\mathfrak{N}_{m+q}^{(0)}(t)\Phi_j = \mathcal{N}_q(t, im)\varphi_j - A_{q0}A_{00}^{-1}\mathcal{N}_0(t, im)\varphi_j$$

for $q = 1, \dots, m$, where we have used notation (3.20) and formulae (4.37), (4.38). This together with (3.66) and (3.67) gives

$$\mathcal{R}_{kj}(t) = \sum_{q=0}^{m} \int_{S_+^{n-1}} (\mathcal{N}_q(t, im)\varphi_j, (-im)^{m-q}\psi_k)d\theta \,. \tag{4.43}$$

Furthermore, $\mathbf{u} \in L_{q,\mathrm{loc}}(\mathbb{R}; \mathcal{T}_q)$ if and only if $\mathbf{h} \in (L^q_{\mathrm{loc}}(\mathbb{R}))^d$ and

$$C_1 \|\mathbf{h}\|_{L^q(t,t+1)} \le e^{mt} \|\mathbf{u}\|_{L_q(t,t+1;\mathcal{T}_q)} \le C_2 \|\mathbf{h}\|_{L^q(t,t+1)} \tag{4.44}$$

with constants independent of t.

To derive estimates for $\mathcal{M}$ we need some formulae. Using definition (3.34) of the operator $\mathcal{S}'$ together with (3.62) and (3.63) we obtain

$$\mathcal{S}'(t)\hat{\Phi}_j = (im)^m \varphi_j + (A_{00} - \mathcal{N}_{00})^{-1} \mathcal{N}_0(t, im)\varphi_j \,. \tag{4.45}$$

Therefore,

$$(\mathfrak{N}_m(t) - \mathfrak{N}_m^{(0)}(t))\hat{\Phi}_j = A_{00}^{-1} \mathcal{N}_{00}(t)(A_{00} - \mathcal{N}_{00})^{-1} \mathcal{N}_0(t, im)\varphi_j$$

and

$$\begin{aligned}(\mathfrak{N}_{m+q}(t) - \mathfrak{N}_{m+q}^{(0)}(t))\hat{\Phi}_j &= \mathcal{N}_{q0}(t)(A_{00} - \mathcal{N}_{00})^{-1} \mathcal{N}_0(t, im)\varphi_j \\ &\quad - A_{q0} A_{00}^{-1} \mathcal{N}_{00}(A_{00} - \mathcal{N}_{00})^{-1} \mathcal{N}_0(t, im)\varphi_j\end{aligned}$$

for $q = 1, \dots, m$. These relations together with (3.66) and (3.67) give

$$\begin{aligned}&\langle (\mathfrak{N}(t) - \mathfrak{N}^{(0)}(t))\hat{\Phi}_j, \Psi_k \rangle \\ &= \int_{S_+^{n-1}} ((A_{00} - \mathcal{N}_{00})^{-1} \mathcal{N}_0(t, im)\varphi_j, \sum_{q=0}^{m} \mathcal{N}_{q0}^*(t)(-im)^{m-q}\psi_k) d\theta \,.\end{aligned} \tag{4.46}$$

We also need the formulae

$$\mathfrak{N}_m \Phi_j = (A_{00} - \mathcal{N}_{00})^{-1} \mathcal{N}_0(t, im)\varphi_j \,, \tag{4.47}$$

$$\mathfrak{N}_{m+q} \Phi_j = \mathcal{N}_q(t, im)\varphi_j - (A_{q0} - \mathcal{N}_{q0})(A_{00} - \mathcal{N}_{00})^{-1} \mathcal{N}_0(t, im)\varphi_j \tag{4.48}$$

for $q = 1, \dots, m$, which can be checked directly by using the definitions of the operator $\mathfrak{N}$, the vector function Φ_j and (4.45).

Using again definitions of the operator $\mathfrak{N}$ and the vector functions Ψ_k one verifies that

$$\begin{aligned}\langle \mathfrak{N}(t)\hat{\mathcal{U}}, \Psi_k \rangle &= \sum_{s=0}^{m} \sum_{q=0}^{m-1} \int_{S_+^{n-1}} (\mathcal{N}_{s,m-q} \mathcal{U}_{q+1}, (-im)^{m-s}\psi_k) d\theta \\ &\quad + \sum_{s=0}^{m} \int_{S_+^{n-1}} (\mathcal{N}_{s0} \mathcal{S}' \hat{\mathcal{U}}, (-im)^{m-s}\psi_k) d\theta.\end{aligned} \tag{4.49}$$

Now we estimate the norm of the operator $\mathcal{M}$. We use the notation

$$\tilde{\mu}_\omega(t, \tau) = \mu_\omega(t, \tau) e^{m(t-\tau)} \,. \tag{4.50}$$

Using (4.33) and the definition of $\mathcal{M}$ one can derive the following estimate

$$\|\mathcal{M}(\mathbf{h})\|_{L^{p_1}(t,t+1)} \le c \int_{\mathbb{R}} \tilde{\mu}_\omega(t, \tau) \|\mathbf{h}\|_{L^{p_1}(\tau,\tau+1)} d\tau \,, \tag{4.51}$$

which is valid for $\mathbf{h}$ subject to

$$\int_{\mathbb{R}} \tilde{\mu}_\omega(0, \tau) \|\mathbf{h}\|_{L^{p_1}(\tau,\tau+1)} d\tau < \infty \,.$$

In what follows we shall need also another estimate for $\mathcal{M}$.

Lemma 4.7. *For all* $\mathbf{h} \in (L^\infty_{\mathrm{loc}}(\mathbb{R}))^d$ *subject to*

$$\int_{\mathbb{R}} \tilde{\mu}_\omega(0,\tau)\varkappa_{p_1}(\tau)\|\mathbf{h}\|_{L^\infty(\tau,\tau+1)}d\tau < \infty \tag{4.52}$$

the following estimate holds:

$$\|\mathcal{M}(\mathbf{h})\|_{L^1(t,t+1)} \le cb_0\varkappa_{p_1'}(t)\int_{\mathbb{R}} \tilde{\mu}_\omega(t,\tau)\varkappa_{p_1}(\tau)\|\mathbf{h}\|_{L^\infty(\tau,\tau+1)}d\tau\,, \tag{4.53}$$

where $p_1' = p_1/(p_1-1)$ *and* $\varkappa_s$ *is defined by* (4.8) *and* (2.12).

Proof. We start with proving the estimates

$$\|\mathfrak{N}_{m+k}(t)\hat{\Phi}_j\|_{W^{-k,s}(S^{n-1}_+)} \le c\overline{\varkappa}_s(t)\,, \tag{4.54}$$

$$\Big|\langle\mathfrak{N}(t)\hat{\mathcal{U}},\Psi_k\rangle\Big| \le c\overline{\varkappa}_{s'}(t)\sum_{j=1}^{m+1}\|\mathcal{U}_j(t)\|_{W^{m-j+1,s}(S^{n-1}_+)}\,, \tag{4.55}$$

where $s\in(1,\infty)$, $1/s' = 1-1/s$,

$$\overline{\varkappa}_s(\tau) = \sum_{|\alpha|,|\beta|\le m}\Big(\int_{S^{n-1}_+}((e^{-t}\theta_n)^{2m-|\alpha|-|\beta|}|N_{\alpha\beta}(e^{-\tau}\theta)|)^s d\theta\Big)^{1/s} \tag{4.56}$$

and the constant c depends on n, s and coefficients $L_{\alpha\beta}$. By (4.47) and (4.48)

$$\|\mathfrak{N}_{m+k}(t)\hat{\Phi}_j\|_{W^{-k,s}(S^{n-1}_+)} \le c\big(\|\mathcal{N}_0(t,im)\varphi_j\|_{L^s(S^{n-1}_+)} + \|\mathcal{N}_k(t,im)\varphi_j\|_{W^{-k,s}(S^{n-1}_+)}\big)\,.$$

Using (3.20), (3.21) and that $\varphi_j(\theta) = \theta_n^m e_j$, we can estimate the right-hand side by

$$\begin{aligned}
&c\sum_{|\alpha|\le m}\sum_{|\beta|\le m} e^{-(2m-|\alpha|-|\beta|)t}\|N_{\alpha\beta}(e^{-t}\theta)\theta_n^{m-|\beta|}\|_{W^{|\alpha|-m,s}(S^{n-1}_+))}\\
&\le c\sum_{|\alpha|\le m}\sum_{|\beta|\le m} e^{-(2m-|\alpha|-|\beta|)t}\|N_{\alpha\beta}(e^{-t}\theta)\theta_n^{2m-|\beta|-|\alpha|}\|_{L^s(S^{n-1}_+)}\,,
\end{aligned}$$

which is estimated by $c\overline{\varkappa}_s$. Inequality (4.54) is proved.

Using (4.49) and (3.35) together with (3.21) and observing that ψ_k is equal to ϕ_k times a smooth function, we arrive at

$$\begin{aligned}
\Big|\langle\mathfrak{N}(t)\hat{\mathcal{U}},\Psi_k\rangle\Big| \le c\sum_{q=0}^{m}\sum_{|\alpha|\le m}\sum_{q\le|\beta|\le m} &e^{-(2m-|\alpha|-|\beta|)t}\\
&\times\|N^*_{\alpha\beta}\theta_n^{2m-|\alpha|-|\beta|}\|_{L^{s'}(S^{n-1}_+)}\|\theta_n^{|\beta|-m}Q_{\beta,q}\mathcal{U}_{q+1}\|_{L^s(S^{n-1}_+)}\,.
\end{aligned}$$

Using the Hardy inequality for estimating the last factor we arrive at (4.55).

We represent $\mathcal{M}$ as the sum $\mathcal{M}_1+\mathcal{M}_2$, where

$$\mathcal{M}_1(\mathbf{h}) = \sum_{j=1}^{d}(\langle\mathfrak{N}(t)-\mathfrak{N}^{(0)}(t))\hat{\Phi}_j,\Psi_k\rangle h_j(t)\,,$$

$$\mathcal{M}_2(\mathbf{h})_k = \langle\mathfrak{N}(t)\mathfrak{M}(\mathfrak{N}\sum_{j=1}^{d}h_j(t)\hat{\Phi}_j),\Psi_k\rangle\,.$$

From (4.46) and (3.20), (3.21) it follows that

$$|\mathcal{M}_1(\mathbf{h})| \le c \sum_{|\alpha|=m} \sum_{m-k\le|\beta|} e^{-(2m-|\alpha|-|\beta|)t} \|N_{\alpha\beta}\theta_n^{2m-|\beta|-k}\|_{L^{p_1'}(S_+^{n-1})} \times \sum_{|\beta|=m} \sum_{m-j\le|\alpha|} e^{-(2m-|\alpha|-|\beta|)t} \|N^*_{\alpha\beta}\theta_n^{2m-|\alpha|-j}\|_{L^{p_1}(S_+^{n-1})} |\mathbf{h}(t)|\,,$$

which implies

$$|\mathcal{M}_1(\mathbf{h})(t)| \le c\overline{\varkappa}_{p_1'}(t)\overline{\varkappa}_{p_1}(t)|\mathbf{h}(t)|\,. \tag{4.57}$$

Furthermore, by (4.55)

$$|\mathcal{M}_2(\mathbf{h})(t)| \le C\overline{\varkappa}_{p_1'}(t) \sum_{j=1}^{m+1} \sum_{k=1}^{d} \|\mathfrak{M}(\mathfrak{N}h_k(t)\hat{\Phi}_k)_j\|_{W^{m-j+1,p_1}(S_+^{n-1})}\,.$$

Now, using (4.33) together with (4.54) with $s = p_1$ and (4.44), we obtain

$$\|\mathcal{M}_2(\mathbf{h})\|_{L^1(t,t+1)} \le Cb_0\|\overline{\varkappa}_{p_1'}\|_{L^{p_1'}(t,t+1)} \int_{\mathbb{R}} \tilde{\mu}_\omega(t,\tau)\varkappa_{p_1}(\tau)\|\mathbf{h}\|_{L^\infty(\tau,\tau+1)}d\tau\,.$$

From (4.57) it follows the same estimate for $\|\mathcal{M}_1(\mathbf{h})\|_{L^1(t,t+1)}$. These two estimates give (4.53). The proof is complete. □

4.5. **Homogeneous equation** (4.34)**.** Here we shall study the homogeneous equation (4.40), i.e.

$$\partial_t\mathbf{h}(t) - \mathcal{R}(t)\mathbf{h}(t) - (\mathcal{M}\mathbf{h})(t) = 0 \quad \text{on } \mathbb{R}. \tag{4.58}$$

We start with a uniqueness result.

Lemma 4.8. *If* $\mathbf{h} \in (W^{1,1}_{\rm loc}(\mathbb{R}))^d$ *is a solution of* (4.58) *subject to*

$$|\mathbf{h}(t)| = \begin{cases} o\Big(e^{nt-c_1\int_0^t \omega(s)ds}\Big) & \text{as } t\to+\infty \\ o\Big(e^{-t-c_1\int_t^0 \omega(s)ds}\Big) & \text{as } t\to-\infty, \end{cases} \tag{4.59}$$

with sufficient large c_1 *and* $\mathbf{h}(t_0) = 0$ *for some* t_0 *then* $\mathbf{h}(t) = 0$ *for all* $t \in \mathbb{R}$*.*

Proof. Without loss of generality we can assume that $t_0 = 0$. By (4.59) the function $\mathbf{h}$ satisfies (4.52). Integrating (4.58) from 0 to t and using (4.53) together with the inequality

$$\int_a^b |f(t)|dt \le \int_{a-1}^b \int_t^{t+1} |f(\tau)|d\tau dt,$$

we arrive at

$$|\mathbf{h}(t)| \le c\int_{-1}^t \varkappa_1(\tau)\|\mathbf{h}\|_{L^\infty(\tau,\tau+1)}d\tau + cb_0\int_{-1}^t \varkappa_{p_1'}(\tau)\int_{\mathbb{R}}\tilde{\mu}_\omega(\tau,s)\varkappa_{p_1}(s)\|\mathbf{h}\|_{L^\infty(s,s+1)}ds$$

if $t \le 0$ and

$$|\mathbf{h}(t)| \le c\int_{t-1}^0 \varkappa_1(\tau)\|\mathbf{h}\|_{L^\infty(\tau,\tau+1)}d\tau + cb_0\int_{t-1}^0 \varkappa_{p_1'}(\tau)\int_{\mathbb{R}}\tilde{\mu}_\omega(\tau,s)\varkappa_{p_1}(s)\|\mathbf{h}\|_{L^\infty(s,s+1)}ds$$

if $t < 0$. Now repeating the proof of [4, Lemma 12] we obtain $\mathbf{h} = 0$. □

Lemma 4.9. *For each* $\mathbf{a} \in \mathbb{C}^d \setminus O$ *equation* (4.58) *has a solution* $\mathbf{h} \in (W^{1,p}_{\rm loc}(\mathbb{R}))^d$ *which has the form*

$$\mathbf{h}(t) = |\mathbf{a}| \exp \int_0^t \Lambda(\tau) d\tau \; \Theta(t) , \tag{4.60}$$

where $|\Theta(t)| = 1$ *for all* $t \in \mathbb{R}$, $\Theta(0) = \mathbf{a}/|\mathbf{a}|$, *and*

$$\Lambda(t) = \Re\big(\mathcal{R}(t)\Theta(t), \Theta(t)\big) + \Lambda_1(t) \tag{4.61}$$

where Λ_1 *satisfies the estimate*

$$\|\Lambda_1\|_{L^1(t,t+1)} \le cb_0 \varkappa_{p_1'}(t) \int_{\mathbb{R}} \tilde{\mu}_\omega(t,\tau) \varkappa_{p_1}(\tau) d\tau , \tag{4.62}$$

where p_1' *and* p_1 *are the same as in* Lemma 4.7. *Furthermore,*

$$\|\Theta'\|_{L^1(t,t+1)} \le C \wp(t) , \tag{4.63}$$

where $\Theta'(t) = d\Theta(t)/dt$ *and*

$$\wp(t) = \varkappa_1(t) + b_0 \varkappa_{p_1'}(t) \int_{\mathbb{R}} \tilde{\mu}_\omega(t,\tau) \varkappa_{p_1}(\tau) d\tau . \tag{4.64}$$

Proof. It suffices to prove the assertion for vectors $\mathbf{a}$ with $|\mathbf{a}| = 1$. Let us first prove the existence of a solution $\mathbf{h}$ subject to the estimate

$$|\mathbf{h}(t)| \le C \exp c_1 \Big| \int_0^t \wp(\tau) d\tau \Big| \tag{4.65}$$

with some positive constants c_1 and C. In order to construct a solution we shall use the following iterative procedure: $\mathbf{h}_0(t) = \mathbf{a}$, and

$$\mathbf{h}_{k+1} = \mathbf{a} + \int_0^t (\mathcal{R}(\tau)\mathbf{h}_k(\tau) + (\mathcal{M}\mathbf{h}_k)(\tau)) d\tau$$

for $k = 0, 1, \dots$ We introduce the Banach space $B_\wp$ which consists of measurable vector functions $\mathbf{h} = (h_1, \dots, h_d)$ on $\mathbb{R}$ with the norm

$$\|\mathbf{h}\|_{B_\wp} = \sup_{t \in \mathbb{R}} \Big(\exp\Big(-c_1 \Big| \int_0^t \wp(\tau) d\tau \Big| \Big) \|\mathbf{h}\|_{L^\infty(t,t+1)} \Big) ,$$

where the constant c_1 will be chosen later. Let us show that the sequence $\{\mathbf{h}_k\}_{k=0}^\infty$ is convergent in $B_\wp$ if c_1 is sufficiently large. Using (4.43) we obtain $|\mathcal{R}(t)| \le c\overline{\varkappa}_1(t)$, where $\overline{\varkappa}_s$ is defined by (4.56). By the last estimate and by (4.53)

$$\|\mathbf{h}_{k+1} - \mathbf{h}_k\|_{L^\infty(t,t+1)} \le c_2 \int_{-1}^t \wp(\tau) \|\mathbf{h}_{k+1} - \mathbf{h}_k\|_{L^\infty(\tau,\tau+1)} d\tau \tag{4.66}$$

for $t \ge 0$ and

$$\|\mathbf{h}_{k+1} - \mathbf{h}_k\|_{L^\infty(t,t+1)} \le c_2 \int_{t-1}^0 \wp(\tau) \|\mathbf{h}_{k+1} - \mathbf{h}_k\|_{L^\infty(\tau,\tau+1)} d\tau$$

for $t \le 0$. Let $t \ge 0$. Then (4.66) implies

$$\|\mathbf{h}_{k+1} - \mathbf{h}_k\|_{B_\wp} \le \frac{c_2}{c_1} \|\mathbf{h}_k - \mathbf{h}_{k-1}\|_{B_\wp} . \tag{4.67}$$

Analogously, for $t \le 0$ we have

$$\|\mathbf{h}_{k+1} - \mathbf{h}_k\|_{B_\wp} \le \frac{c_2}{c_1} \sup_{t \le 0} e^{c_1 \int_{t-1}^t \wp(\tau) d\tau} \|\mathbf{h}_k - \mathbf{h}_{k-1}\|_{B_\wp} . \tag{4.68}$$

By (2.16) and $b_0 \geq 1$, the function $\wp$ does not exceed ω_0 times a constant. Therefore, we can choose c_1 sufficiently large and ω_0 sufficiently small so that the constants in the right-head sides in (4.67) and (4.68) are less than 1. This guarantees the convergence of $\{\mathbf{h}_k\}_{k=0}^{\infty}$ to $\mathbf{h} \in B_\wp$. Clearly this vector function satisfies the equation

$$\mathbf{h} = \mathbf{a} + \int_0^t (\mathcal{R}(\tau)\mathbf{h}(\tau) + (\mathcal{M}\mathbf{h})(\tau))d\tau\,,$$

which is equivalent to (4.58).

We define $q(t) = |\mathbf{h}(t)|$ and $\Theta(t) = \mathbf{h}(t)/q(t)$. We note that by Lemma 4.8 the function q is positive for all t. Multiplying equation (4.58) by $\Theta(t)$ and taking the real part we obtain

$$\frac{dq}{dt}(t) - a(t)q(t) - (\mathcal{M}_s q)(t) = 0\,, \tag{4.69}$$

where

$$a(t) = \Re(\mathcal{R}(t)\Theta(t), \Theta(t)) \quad \text{and} \quad (\mathcal{M}_s q)(t) = \Re(\mathcal{M}(\Theta q)(t), \Theta(t))\,. \tag{4.70}$$

¿From (4.53) it follows that

$$\|\mathcal{M}_s q\|_{L^1(t,t+1)} \leq c b_0 \varkappa_{p_1'}(t) \int_{\mathbb{R}} \tilde{\mu}_\omega(t,\tau)\varkappa_{p_1}(\tau)\|q\|_{L^\infty(t,t+1)} d\tau\,. \tag{4.71}$$

Let us show that

$$q(t) = \exp \int_0^t \Lambda(\tau)d\tau\,, \tag{4.72}$$

where Λ is a measurable function satisfying estimate (4.62). We shall consider (4.69) as an equation with respect to q only, supposing Θ be fixed. Making substitution $q(t) = \exp\big(\int_0^t a(\tau)d\tau\big)\, z(t)$ we arrive at the equation

$$\frac{dz}{dt}(t) - (\mathcal{M}_2 z)(t) = 0\,, \tag{4.73}$$

where $(\mathcal{M}_2 z)(t) = (\mathcal{M}_s)_{\tau\to t}(\exp\big(\int_t^\tau a(\tau)d\tau\big) z(\tau))$. One can check directly that the operator $\mathcal{M}_2$ also satisfies the estimate (4.71), possibly with another constant c_0 in definition (4.22) of the function μ. Equation (4.73) has the form (173) in [4], but the operator $\mathcal{M}_2$ is estimated in different norms. Therefore the representation $z(t) = \exp\big(\int_0^t \Lambda_1(\tau)d\tau\big)$ with Λ_1 subject to (4.62) follows actually from [4, Lemma 13] if one makes there evident changes caused by the only available L^1-estimate for $\mathcal{M}_2$.

It remains to prove (4.63). Expressing $\mathbf{h}'$ from (4.58) and using (4.60) and (4.53) for estimating the second and the third terms in the left-hand side of (4.58) we arrive at

$$\|\mathbf{h}'\|_{L^1(t,t+1)} \leq |a| C \wp(t) \exp \int_0^t \Lambda(\tau)d\tau\,,$$

From representation (4.60) and this estimate we derive (4.63). The proof is complete. $\square$

4.6. **Solutions to the homogeneous system** (3.41)**.** Using (4.41), (4.70) and (4.35) we can represent the operator $\mathcal{M}_s$ in (4.69) as

$$\mathcal{M}_s(q)(t) = \Re\langle(\mathfrak{N}(t) - \mathfrak{N}^{(0)}(t))\sum_{j=1}^{d}\Theta_j\hat{\Phi}_j, \sum_{k=1}^{d}\Theta_k\Psi_k\rangle q + e^{mt}\Re\langle\mathfrak{N}(t)\hat{\mathbf{v}}, \sum_{k=1}^{d}\Theta_k\Psi_k\rangle\,, \tag{4.74}$$

where the vector function $\mathbf{v}$ satisfies (4.15) with $\mathcal{F} = 0$ and $\hat{\mathbf{u}} = e^{-mt}\sum_{j=1}^{d}\Theta_j\hat{\Phi}_j q$.

Let $\mathbf{a} \in \mathbb{C}$ and $|\mathbf{a}| = 1$. We denote by $\mathbf{h}$ the unique solution of (4.58) having the form (4.60). Then $q(t) := |h(t)| = \exp\int_0^t\Lambda(\tau)d\tau$ and this function satisfies (4.69). We represent the vector function $\mathbf{v}$ as

$$\mathbf{v} = \exp\Big(-mt + \int_0^t\Lambda(\tau)d\tau\Big)\,\mathbf{V}(t)\,. \tag{4.75}$$

Inserting these q and $\mathbf{v}$ into (4.58) and using (4.74) we arrive at

$$\Lambda(t) - a(t) - b(t,\Lambda) = 0\,, \tag{4.76}$$

where a is given by (4.70) and

$$b(t,\Lambda) = \Re\langle(\mathfrak{N}(t) - \mathfrak{N}^{(0)}(t))\sum_{j=1}^{d}\Theta_j\hat{\Phi}_j, \sum_{k=1}^{d}\Theta_k\hat{\Psi}_k\rangle + \Re\langle\mathfrak{N}(t)\hat{\mathbf{V}}, \sum_{k=1}^{d}\Theta_k\hat{\Psi}_k\rangle \tag{4.77}$$

with $\mathbf{V}$ satisfying

$$(\mathcal{I}D_t + \mathfrak{A} + im - i\Lambda)\mathbf{V} - (\mathcal{I}-\mathcal{P})\mathfrak{N}\hat{\mathbf{V}} = \sum_{j=1}^{d}(\mathcal{I}-\mathcal{P})\Theta_j\mathfrak{N}\widehat{\Phi_j} \quad \text{on } \mathbb{R}. \tag{4.78}$$

Using estimate (4.24) for the function $\mathbf{v}$ and observing that $\mathbf{v}$ satisfies (4.21) with

$$F = \exp\Big(-mt + \int_0^t\Lambda(\tau)d\tau\Big)\sum_{j=1}^{d}\Theta_j\mathfrak{N}\widehat{\Phi_j},$$

we arrive at the following estimate for $\mathbf{V}$:

$$\|\hat{\mathbf{V}}\|_{\hat{\mathbb{T}}_{p_1}(t,t+1)} + \|\mathbf{V}\|_{\mathbb{T}_p(t,t+1)} \le cb_0\sum_{j=1}^{d}\int_{\mathbb{R}}\tilde{\mu}_\omega(t,\tau)\|\mathfrak{N}\hat{\Phi}_j\|_{\mathbb{Y}_{p_1}(\tau,\tau+1)}d\tau\,.$$

This together with (4.54) gives the estimate

$$\|\hat{\mathbf{V}}\|_{\hat{\mathbb{T}}_{p_1}(t,t+1)} + \|\mathbf{V}\|_{\mathbb{T}_p(t,t+1)} \le cb_0\int_{\mathbb{R}}\mu_\omega(t,\tau)\varkappa_{p_1}(\tau)d\tau\,, \tag{4.79}$$

where $\tilde{\mu}_\omega$ is given by (4.50) and (4.22) possibly with a larger constant c_0. Here we used the inequalities $|\Lambda(\tau)| \le c\wp(\tau) \le c\omega(\tau)$.

Now we are in position to formulate and prove an assertion about solutions to (4.1) subject to (4.2).

Lemma 4.10. *Let $\mathcal{U} \in \mathbb{T}_{p,\mathrm{loc}}(\mathbb{R})$ be a solution to system* (4.1) *subject to* (4.2)*. Then*

$$\mathcal{U}(t) = c\exp\Big(-mt + \int_0^t\Lambda(\tau)d\tau\Big)\Big(\sum_{j=1}^{d}\Theta_j\Phi_j + \mathbf{V}\Big)\,, \tag{4.80}$$

where c is a constant, the function $\Lambda(t)$ admits representation (4.61)*, where Λ_1 satisfies* (4.62)*, the vector function Θ subject to $|\Theta(t)| = 1$ and* (4.63)*, and the function $\mathbf{V} \in \mathbb{T}_{p,\mathrm{loc}}(\mathbb{R})$ satisfies equation* (4.78) *and estimate* (4.79)*.*

Proof. By (4.13) and (4.39) we obtain

$$\mathcal{U} = e^{-mt}\sum_{j=1}^{d} h_j(t)\Phi_j + \mathbf{v}\,. \tag{4.81}$$

Using (4.60) together with (4.75) we arrive at the representation (4.80) with $\mathbf{V}$ solving equation (4.78) and satisfying estimate (4.79). The proof is complete. □

Let us denote by $\Lambda_+(t)$ and $\Lambda_-(t)$ the largest and the least eigenvalue of the matrix $\Re\mathcal{R}$. We finish this section by the following two-side estimates for Λ.

Lemma 4.11. (i) *The function* Λ *satisfies the estimates*

$$\Lambda_-(t) - c_1\wp_0(t) \le \Lambda(t) \le \Lambda_+(t) + c_2\wp_0(t)\,, \tag{4.82}$$

where

$$\wp_0(t) = b_0\varkappa_{p_1'}(t)\int_{\mathbb{R}} \tilde{\mu}_\omega(t,\tau)\varkappa_{p_1}(\tau)d\tau\,. \tag{4.83}$$

(ii) *Furthermore, if* (2.17) *is fulfilled with a sufficiently small* ω_0 *depending on* m*,* n*,* p*,* γ *and* L *then*

$$\int_a^b \Lambda(\tau)d\tau \le \int_a^b \Lambda_+(\tau)d\tau + c\int_a^b \varkappa_2^2(\tau)d\tau + c\omega_0^2 \tag{4.84}$$

and

$$\int_a^b \Lambda(\tau)d\tau \ge \int_a^b \Lambda_-(\tau)d\tau - c\int_a^b \varkappa_2^2(\tau)d\tau - c\omega_0^2 \tag{4.85}$$

for $a < b$.

Proof. (i) The inequalities (4.82) are a direct consequence of (4.61), (4.62) and the definition of $\Lambda_\pm$.

(ii) Let

$$\wp_2(t) = \varkappa_2(t)\int_{\mathbb{R}} \tilde{\mu}_\omega(t,\tau)\varkappa_2(\tau)d\tau\,.$$

We have

$$\begin{aligned}\int_a^b \wp_2(\tau)d\tau \le c\Big(&\int_a^b \varkappa_2(\tau)\int_a^b \tilde{\mu}_\omega(\tau,s)\varkappa_2(s)ds\,d\tau \\ &+ \int_a^b \varkappa_2(\tau)\Big(\int_{-\infty}^a + \int_b^\infty\Big)\tilde{\mu}_\omega(\tau,s)\varkappa_2(s)ds\,d\tau\Big).\end{aligned} \tag{4.86}$$

The first double integral in the right-hand side is estimated by

$$c\int_a^b \varkappa_2^2(\tau)d\tau.$$

Since $\tilde{\mu}_\omega(\tau,s) \le c\tilde{\mu}_\omega(\tau,z)\tilde{\mu}_\omega(z,s)$ if $\tau \le z \le s$ or $\tau \ge z \ge s$ and since $\varkappa(\tau) \le c\omega_0$, we estimate the second term in (4.86) by

$$c\omega_0\int_a^b (\tilde{\mu}_\omega(\tau,a) + \tilde{\mu}_\omega(\tau,b))\varkappa_2(\tau)d\tau,$$

which is less than $c\omega_0^2$. Therefore

$$\int_a^b \wp_2(\tau)d\tau \le c\Big(\int_a^b \varkappa_2^2(\tau)d\tau + c\omega_0^2\Big) \tag{4.87}$$

This along with (4.82) proves (4.84) and (4.85). □

Remark 4.12. An analog of estimate (4.87) for the function (4.83) is

$$\int_a^b \wp_0(\tau)d\tau \le cb_0 \,\|\varkappa_{p_1'}\|_{L^{p_1'}(a,b)}\|\varkappa_{p_1}\|_{L^{p_1}(a,b)} + c\omega_0^2, \tag{4.88}$$

where ω_0 is the same constant as in (H3) and (2.16). The proof is similar to that of (4.87).

4.7. **A solvability result for equation** (2.52)**.** In what follows we need the following analog of Proposition 2.10(i) for equation (2.52)

Proposition 4.13. *Let (H1)–(H3) be fulfilled and let* $f \in (W^{-m,p_1}_{\rm loc}(\overline{\mathbb{R}^n_+}\setminus\mathcal{O}))^d$ *be subject to*

$$\begin{aligned}&\int_0^1 \rho^m e^{\mathcal{C}\int_\rho^1 \Omega(y)\frac{dy}{y}}\mathfrak{M}^{-m}_{p_1}(f;K_{\rho/e,\rho})\frac{d\rho}{\rho}\\ &+\int_1^\infty \rho^{m-1}e^{\mathcal{C}\int_1^\rho \Omega(y)\frac{dy}{y}}\mathfrak{M}^{-m}_{p_1}(f:K_{\rho/e,\rho})\frac{d\rho}{\rho}<\infty\ ,\end{aligned} \tag{4.89}$$

with sufficiently large $\mathcal{C}$*. Then there exists a solution* $u \in (\mathring{W}^{m,p_1}_{\rm loc}(\overline{\mathbb{R}^n_+}\setminus\mathcal{O}))^d$ *of* (2.52) *satisfying*

$$\begin{aligned}\mathfrak{M}^m_{p_1}(u;K_{r/e,r}) \le cb_0\Big(&\int_0^r r^m\rho^m e^{\mathcal{C}\int_\rho^r \Omega(y)\frac{dy}{y}}\mathfrak{M}^{-m}_{p_1}(f;K_{\rho/e,\rho})\frac{d\rho}{\rho}\\ &+\int_r^\infty r^{m+1}\rho^{m-1}e^{\mathcal{C}\int_r^\rho \Omega(y)\frac{dy}{y}}\mathfrak{M}^{-m}_{p_1}(f;K_{\rho/e,\rho})\frac{d\rho}{\rho}\Big).\end{aligned} \tag{4.90}$$

Estimate (4.90) *implies*

$$\mathfrak{M}^m_p(u;K_{r/e,r}) = \begin{cases} o\big(r^m e^{-\mathcal{C}\int_r^1 \Omega(s)\frac{ds}{s}}\big) & \text{if } r\to 0\\ o\big(r^{m+1}e^{-\mathcal{C}\int_1^r \Omega(s)\frac{ds}{s}}\big) & \text{if } r\to\infty.\end{cases} \tag{4.91}$$

The solution $u \in (\mathring{W}^{m,p}_{\rm loc}(\overline{\mathbb{R}^n_+}\setminus\mathcal{O}))^d$ *of problem* (2.52) *subject to* (4.91) *is unique.*

Proof. Using the reduction to the first order system (3.41) from Section 3.2, we arrive at (3.41) for the vector function (3.26)–(3.28) with the right hand side given by (3.42)–(3.44). Inequality (4.89) implies

$$\int_{\mathbb{R}} e^{-m\tau}g_\omega(0,\tau)\|\mathcal{F}\|_{\mathbb{Y}_{p_1}(\tau,\tau+1)}d\tau<\infty,$$

where

$$g_\omega(t,\tau) = ce^{-(t-s)+C\int_s^t \omega(z)dz}\ \text{ for } t\ge s \text{ and }\ g_\omega(t,s)=ce^{C\int_t^s\omega(z)dz}\ \text{ for } t<s.$$

Relation (4.91) takes the form

$$\|\mathcal{U}\|_{\mathbb{T}_p(t,t+1)} = \begin{cases} o\Big(e^{(-m)t-C\int_0^t\omega(s)ds}\Big) & \text{as } t\to+\infty\\ o\Big(e^{-(m+1)t-c_0\int_t^0\omega(s)ds}\Big) & \text{as } t\to-\infty.\end{cases} \tag{4.92}$$

Furthermore, we shall use the spectral splitting

$$\mathcal{U} = e^{-mt}\sum_{j=1}^d h_j(t)\Phi_j + \mathbf{v}(t),$$

where $\mathbf{h}=(h_1,\dots,h_d)$ satisfies (4.40) and $\mathbf{v}$ solves (4.15).

(1). *Solvability of equation (4.40).* We solve equation (4.40) by using the iterative procedure

$$\partial_t \mathbf{h}_{k+1}(t) - \mathcal{R}(t)\mathbf{h}_{k+1}(t) = (\mathcal{M}\mathbf{h}_k)(t) + \mathbf{g} \tag{4.93}$$

for $k = 0, 1, \dots$ and $\mathbf{h}_0 = 0$. Here $\mathbf{g} = (g_1, \dots, g_d)$ is defined by (4.42). The ordinary differential equations $\partial_t \mathbf{h}(t) - \mathcal{R}(t)\mathbf{h}(t) = \mathbf{f}(t)$ will be solved as

$$\mathbf{h}(t) = -\int_t^\infty G(\tau, t)\mathbf{f}(\tau)d\tau,$$

where $G(t,\tau)$ is the Green matrix, which satisfies the estimate

$$|G(\tau, t)| \le ce^{\int_t^\tau |\mathcal{R}(s)|ds}$$

Using the estimate $\int_t^{t+1} |\mathcal{R}(s)|ds \le c_1\omega(t)$ and (4.53), we obtain

$$\begin{aligned}
&\|\mathbf{h}_{k+1} - \mathbf{h}_k\|_{L^\infty(t,t+1)} \\
&\le c\int_{t-1}^\infty e^{c_1\int_t^\tau \omega(s)ds}\varkappa_{p_1'}(\tau)\int_{\mathbb{R}} \tilde{\mu}_\omega(\tau, s)\varkappa_{p_1}\|\mathbf{h}_k - \mathbf{h}_{k-1}\|_{L^\infty(s,s+1)}ds\, d\tau,
\end{aligned}$$

which implies

$$\|\mathbf{h}_{k+1} - \mathbf{h}_k\|_{L^\infty(t,t+1)} \le c\omega_0\int_{\mathbb{R}} g_1(t,s)\omega(s)\|\mathbf{h}_k - \mathbf{h}_{k-1}\|_{L^\infty(s,s+1)}ds,$$

where

$$g_1(t,s) = e^{-(t-s)+c_0\int_s^t \omega(z)dz}$$

for $t \ge s$ and

$$g_1(t,s) = e^{c_1\int_t^s \omega(z)dz}$$

for $t < s$. We put

$$g_2(t,\tau) = \sum_{j=0}^\infty (c\omega_0)^j \int_{\mathbb{R}^j} g_1(t,\tau_1)\omega(\tau_1)\dots g_1(\tau_j,\tau)d\tau_1\dots d\tau_j.$$

Then

$$\sum_{k=0}^\infty \|\mathbf{h}_{k+1} - \mathbf{h}_k\|_{L^\infty(t,t+1)} \le c\int_{\mathbb{R}} g_2(\tau, s)\|\mathbf{h}_1\|_{L^\infty(s,s+1)}ds$$

Furthermore, one can check that g_2 is estimated by the Green function of the second order operator $-\partial_t(\partial_t + 1) - c_2\omega(t)$ and therefore $g_2(t,\tau) \le cg_\omega(t,\tau)$. This leads to convergence of $\{\mathbf{h}_k\}$ in $L^\infty_{\rm loc}$ and to the estimate for the function $\mathbf{h}$:

$$\|\mathbf{h}\|_{L^\infty(t,t+1)} \le c\int_{\mathbb{R}} g_\omega(\tau, s)\|\mathcal{F}\|_{\mathbb{Y}_{p_1}(s,s+1)}ds\,. \tag{4.94}$$

By (4.51) we obtain also that

$$\|\mathbf{h}'\|_{L^{p_1}(t,t+1)} \le c\int_{\mathbb{R}} g_\omega(\tau, s)\|\mathcal{F}\|_{\mathbb{Y}_{p_1}(s,s+1)}ds\,. \tag{4.95}$$

(2). *Solvability of (4.15).* Using (4.94) and (4.95) together with estimate (4.25) we arrive at

$$\|\hat{\mathbf{v}}\|_{\hat{\mathbb{T}}_{p_1}(t,t+1)} \le cb_0\int_{\mathbb{R}} g_\omega(\tau, s)\|\mathcal{F}\|_{\mathbb{Y}_{p_1}(s,s+1)}ds. \tag{4.96}$$

Estimates (4.94), (4.95) together with (4.96) lead to (4.90). Uniqueness result follows from the uniqueness results from Proposition 4.6 and Lemma 4.8. $\square$

5. Proofs of the main results and their corollaries

5.1. **Solutions to the homogeneous equation** (3.23)**.** Here we describe solutions to the homogeneous equation (3.23).

Lemma 5.1. *Let $u \in \mathring{W}^{m,p}_{\rm loc}(\Pi)$ be a solution to equation* (3.23) *with $f_j = 0$, $j = 0, \dots, m$, subject to*

$$\|u\|_{W^{m,p}(\Pi_t)} = \begin{cases} o\Big(e^{(n-m)t - c_0 \int_0^t \omega(s)ds}\Big) & \text{as } t \to +\infty \\ o\Big(e^{-(m+1)t - c_0 \int_t^0 \omega(s)ds}\Big) & \text{as } t \to -\infty. \end{cases} \tag{5.1}$$

Then $u \in \mathring{W}^{m,p_1}_{\rm loc}(\Pi)$ and

$$D_t^k u(t) = J(u) \exp\Big(-mt + \int_0^t \Lambda(\tau)d\tau\Big)\Big(\sum_{j=1}^d \Theta_j \phi_j (im)^k + \mathbf{w}_k\Big) \tag{5.2}$$

for $k = 0, 1, \dots, m$, where Λ and Θ are the same functions as in Lemma 4.10*, and the function $\mathbf{w}_k$ belongs to $L^{p_1}_{\rm loc}(\mathbb{R}; \mathring{W}^{m-k,p_1}(S^{n-1}_+))$ and satisfies the estimate*

$$\begin{aligned} &\|\mathbf{w}_k\|_{L^{p_1}(t,t+1;W^{m-k,p_1}(S^{n-1}_+))} + \|\partial_t \mathbf{w}_k\|_{L^{p_1}(t,t+1;W^{m-k-1,p_1}(S^{n-1}_+))} \\ &\le C b_0 \int_{\mathbb{R}} \mu_\omega(t,\tau) \varkappa_{p_1}(\tau) d\tau \end{aligned} \tag{5.3}$$

for $k = 0, 1, \dots, m-1$, where $\varkappa_s$ is given by (4.8)*. The remainder $\mathbf{w}_m$ satisfies also* (5.3) *with $k = m$ but without the second term in the left-hand side. The constant $J(u)$ in* (5.2) *admits the estimates*

$$c_1 \|u\|_{L^2(\Pi_{t_0})} \le |J(u)| \exp\Big(-mt_0 + \int_0^{t_0} \Lambda(\tau)d\tau\Big) \le c_2 \|u\|_{L^2(\Pi_{t_0})} \tag{5.4}$$

for every real number t_0. The dimension of the space of such solutions is equal to d.

Proof. Using the reduction of (3.23) to the first order system (3.41), described in Section 3.2, we arrive at (4.1) for the vector function $\mathcal{U}$ defined by (3.26)–(3.28). Moreover, $\mathcal{U}$ satisfies (4.2) because of (5.1). By Lemma 4.10 the vector function $\mathcal{U}$ admits representation (4.80). Using (4.80), (3.62) and (3.26) we obtain (5.2) for $k = 1, \dots, m-1$ with $\mathbf{w}_k = \mathbf{V}_k$ and estimate (5.3) follows from (4.79). By (3.31), (3.35) and (4.80) we obtain

$$D_t^m u = c \exp\Big(-mt + \int_0^t \Lambda(\tau)d\tau\Big)\Big(\sum_{j=1}^d \Theta_j \mathcal{S}'(t)\hat{\Phi}_j + \mathcal{S}'(t)\hat{\mathbf{V}}\Big).$$

Using (3.32) and the definition of Φ_j we get

$$\mathcal{S}'(t)\hat{\Phi}_j = (im)^m \phi_j + (A_{00} - \mathcal{N}_{00}(t))^{-1} \mathcal{N}_0(t, im)\phi_j\,,$$

which gives (5.2) with $k = m$ and

$$\mathbf{w}_m(t) = \mathcal{S}'(t)\hat{\mathbf{V}} + \sum_{j=1}^d \Theta_j (A_{00} - \mathcal{N}_{00}(t))^{-1} \mathcal{N}_0(t, im)\phi_j\,. \tag{5.5}$$

By (3.36) and (4.79)

$$\|\mathcal{S}'\hat{\mathbf{V}}\|_{L^{p_1}(\Pi_t)} \le c b_0 \int_{\mathbb{R}} \tilde{\mu}_\omega(t,\tau) \varkappa_{p_1}(\tau) d\tau\,.$$

Using (3.20), (3.21) and that $\phi_j = \theta_n^m e_j$ we obtain

$$\begin{aligned}&\| \sum_{j=1}^{d} \Theta_j (A_{00} - \mathcal{N}_{00}(t))^{-1} \mathcal{N}_0(t, im)\phi_j \|_{L^{p_1}(\Pi_t)} \\ &\le c \sum_{|\alpha|=m} \sum_{|\beta|\le m} \| N_{\alpha\beta} \theta_n^{m-|\beta|} \|_{L^{p_1}(\Pi_t)} \le c\varkappa_{p_1}\,.\end{aligned}$$

Using the last two estimates and (5.5) we arrive at (5.3) for $k = m$ without the second term in the left-hand side.

To obtain estimate (5.4) we calculate the $L^2(\Pi_{t_0})$ norms of the left-hand and right-hand sides in (5.2) with $k = 0$. We have

$$\begin{aligned}&|J(u)| \exp\Big(- mt_0 + \int_0^{t_0} \Lambda(\tau)d\tau\Big)\Big(\| \sum_{j=1}^{d} \Theta_j\phi_j \|_{L^2(\Pi_{t_0})} - \|\mathbf{w}_0\|_{L^2(\Pi_{t_0})}\Big) \\ &\le c\|u\|_{L^2(\Pi_{t_0})}\,.\end{aligned}$$

One can check that

$$\| \sum_{j=1}^{d} \Theta_j\phi_j \|_{L^2(\Pi_{t_0})} \ge c_1.$$

Furthermore,

$$\|\mathbf{w}_0\|_{L^2(\Pi_{t_0})} \le C\omega_0 \int_{\mathbb{R}} \mu_\omega(t,\tau)d\tau \le c_2\omega_0$$

because of (5.3) and (2.16). Therefore,

$$|J(u)| \exp\Big(- mt_0 + \int_0^{t_0} \Lambda(\tau)d\tau\Big)(c_1 - c_2\omega_0) \le c\|u\|_{L^2(\Pi_{t_0})},$$

which implies the left-hand side estimate in (5.4) provided ω_0 is sufficiently small. The right-hand side estimate in (5.4) is proved analogously. □

Corollary 5.2. *Let $u \in \mathring{W}^{m,p}_{\rm loc}(\Pi)$ be a solution to equation* (3.23) *subject to* (5.1)*. Then for every t_0*

$$c_1\|u\|_{L^2(\Pi_{t_0})} \le \|u\|_{W^{m,p}(\Pi_t)} \exp\Big(m(t - t_0) - \int_{t_0}^{t} \Lambda(\tau)d\tau\Big) \le c_2\|u\|_{L^2(\Pi_{t_0})}. \quad (5.6)$$

Proof. By (5.2) the norm $\|u\|_{W^{m,p}(\Pi_t)}$ is estimated from below by

$$|J(u)| \exp\Big(- mt + \int_0^{t} \Lambda(\tau)d\tau\Big)\Big(\|\phi\|_{W^{m,p}(\Pi_t)} - \sum_{k=0}^{m} \|\mathbf{w}_k\|_{L^p(t,t+1;W^{m-k,p}(S_+^{n-1}))}\Big),$$

where $\phi(\theta) = \theta_n^m$. Since $\|\phi_j\|_{W^{m,p}(\Pi_t)} \ge c_1$ and

$$\sum_{k=0}^{m} \|\mathbf{w}_k\|_{L^p(t,t+1;W^{m-k,p}(S_+^{n-1}))} \le c_2\omega_0,$$

which follows from (5.3) and (2.16), we arrive at the left-hand side inequality in (5.6) by using (5.4). The right-hand inequality in (5.6) is proved analogously.

5.2. **Proof of Theorem 2.3.** Let $u \in (\mathring{W}^{m,p}_{\rm loc}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O}))^d$ be a solution of the equation $\mathcal{L}(x, D_x)u = 0$ on $\mathbb{R}^n_+ \setminus \mathcal{O}$ subject to (2.23) and (2.24). In the variables t and θ, defined by (3.1), the function u belongs to $(\mathring{W}^{m,p}_{\rm loc}(\Pi))^d$, satisfies (3.23) with $f_j = 0$, $j = 0, \dots, m$, and is subject to (5.1) because of (2.23) and (2.24). Therefore, by Lemma 5.1 u admits representation (5.2), which implies (2.25) with $\delta = e^{-t_0}$,

$$\Upsilon(\rho) = \Lambda\big(\log \rho^{-1}\big), \quad \mathbf{q}(r) = \Theta\big(\log r^{-1}\big), \quad J_Z = J(u) \exp\Big(\int_0^{t_0} \Lambda(\tau) d\tau\Big)$$

and

$$v_k(x) = v_k(r\theta) = \mathbf{w}_k\big(\theta, \log r^{-1}\big).$$

Using definition (4.43) of the matrix $\mathcal{R}$ together with the definitions of the vector functions φ_j and ψ_k we have

$$\mathcal{R}_{kj}(t) = m! \sum_{q=0}^{m} \int_{S^{n-1}_+} \big(\mathcal{N}_q(t, D_t)(e^{-mt}\theta_n^m e_j), D_t^{m-q}(e^{(2m-n)t} E_k(e^{-t}\theta))\big) d\theta.$$

By (3.22)

$$\mathcal{R}_{kj}(\log r^{-1}) = m! r^n \int_{S^{n-1}_+} \sum_{|\alpha|,|\beta| \le m} \big(N_{\alpha\beta}(x) D_x^\beta(x_n^m e_j), D_x^\alpha E_k(x)\big) d\theta.$$

Therefore, $\mathbf{R}(r) = \Re \mathcal{R}(\log r^{-1})$ and representation (2.26) follows from (4.61) if we take $\Upsilon_1(\rho) = \Lambda_1(\log \rho^{-1})$. Since

$$\chi(r) = \wp_0(\log r^{-1}), \tag{5.7}$$

where $\wp_0$ is introduced by (4.83), estimate (2.27) follows from (4.63) and (4.62), and (2.30) and (2.29) follow from (5.3) and (5.4) respectively. The proof is complete.

5.3. **Proof of Theorem 2.4.** We introduce a smooth function $\eta_\delta = \eta_\delta(r)$, which is equal to 1 in a neighborhood of $(0, \delta/2]$ and equal to 0 for $r > 2\delta/3$. We can choose η_δ such that $|d^k \eta_\delta(r)/dr^k| \le C_k \delta^{-k}$. Let ζ_δ be another smooth function, which is equal to 1 in a neighborhood of $[\delta/2, 2\delta/3]$ and is zero outside the interval $(\delta/3, 3\delta/4)$. We can suppose that $|d^k \zeta_\delta(r)/dr^k| \le C_k \delta^{-k}$. We set $u_\delta = \eta_\delta u$. Then u_δ satisfies

$$\begin{aligned} \mathcal{L}(x, D_x) u_\delta = &\sum_{|\alpha|,|\beta| \le m} D_x^\alpha \big(\mathcal{L}_{\alpha\beta}(x)(D_x^\beta \eta_\delta \zeta_\delta u - \eta_\delta D_x^\beta \zeta_\delta u)\big) \\ &+ \sum_{|\alpha|,|\beta| \le m} (D_x^\alpha \eta_\delta - \eta_\delta D_x^\alpha)\big(\mathcal{L}_{\alpha\beta}(x) D_x^\beta \zeta_\delta u\big). \end{aligned} \tag{5.8}$$

If we denote the functional corresponding to the left-hand side in (5.8) by f then it is supported by $\delta/2 \le |x| \le 2\delta/3$ and, by Sobolev imbedding theorem and by (2.11),

$$\mathfrak{M}^{-m}_{p_2}(f; K_{r/e,r}) \le c\delta^{-2m} \mathfrak{M}^m_p(\zeta_\delta u; K_{r/e,r}),$$

where $p_1 = \min(2n/(n-2), p)$ if $n > 2$ and $p_1 = p$ if $n = 2$. We can verify that all requirements of Proposition T4.2zzZ are fulfilled and therefore there exists a solution $v \in (\mathring{W}^{m,p_1}_{\rm loc}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O}))^d$ of problem (2.1), (2.2) satisfying estimate (4.90), which takes, in our case, the form

$$\mathfrak{M}^m_{p_1}(v; K_{r/e,r}) \le c b_0 \Big(\frac{r}{\delta}\Big)^{m+1} e^{\mathcal{C} \int_r^\delta \Omega(s) \frac{ds}{s}} \mathfrak{M}^m_{p_1}(u; K_{\delta/4,\delta}) \tag{5.9}$$

for $r \le \delta$ and

$$\mathfrak{M}^m_{p_1}(v; K_{r/e,r}) \le c b_0 \Big(\frac{r}{\delta}\Big)^m e^{\mathcal{C}\int_\delta^r \Omega(s)\frac{ds}{s}} \mathfrak{M}^m_{p_1}(u; K_{\delta/4,\delta}) \tag{5.10}$$

for $r > \delta$. The function $Z = u_\delta - v$ satisfies all conditions of Theorem 2.3 and hence, admits representation (2.25). Thus we arrive at representation (2.32) with $w = v + (1 - \eta_\delta)u$ and estimate (2.34) follows from (5.9).

In order to prove (2.33) we observe that

$$\mathfrak{M}^0_2(Z; K_{\delta/e,\delta}) \le \mathfrak{M}^0_2(u_\delta; K_{\delta/e,\delta}) + \mathfrak{M}^0_2(v; K_{\delta/e,\delta})$$

and using (5.9) we obtain

$$\mathfrak{M}^0_2(Z; K_{\delta/e,\delta}) \le c b_0 \mathfrak{M}^m_p(u; K_{\delta/4,\delta}).$$

Using this estimate together with the right-hand inequality in (2.29), we arrive at (2.33). This completes the proof of Theorem 2.4. □

5.4. **Proof of Corollaries 2.5 and 2.6.**

Proof of Corollaries 2.5. As it was noted in Remark 2.2 under assumption (2.17) all conditions **H1**–**H3** are satisfied with $p_1 = p$ as well as with $p_1 = p = 2$ and $b_0 = 1$ provided ω_0 is sufficiently small. Inclusion $Z \in (\mathring{W}^{m,2}_{\rm loc}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O}))^d$ together with (2.18) implies $Z \in (\mathring{W}^{m,p}_{\rm loc}(\overline{\mathbb{R}^n_+} \setminus \mathcal{O}))^d$ as well as (2.23) and (2.24). Thus we can apply Theorem 2.3. Choosing in (2.28) $p_1 = p = 2$ we arrive at (2.27) with χ given by (2.35). The required estimate for v_k follows from (2.30) if we take there $p_1 = p$. □

Corollaries 2.6 is proved similarly. The only new element here is that first we obtained (2.36) and (2.37) with $\mathfrak{M}^m_{p_1}(u; K_{\delta/8,\delta/2})$ instead of $\mathfrak{M}^m_2(u; K_{\delta/16,\delta})$ but using the local estimate for u we arrive at the required estimates.

5.5. **Proof of Corollary 2.7.** To prove this assertion we use Corollary 2.5. In our case $p_1 = p$, $b_0 = 1$ and $\kappa_p(r) \le c\omega_0$. Therefore from the asymptotic representation (2.25) we derive the estimates

$$\begin{aligned} c_1 |J_Z| r^m \exp\Big(\int_r^1 \Upsilon(\rho)\frac{d\rho}{\rho}\Big) &\le \mathfrak{M}^m_p(Z; K_{r/e,r}) \\ &\le c_2 |J_Z| r^m \exp\Big(\int_r^1 \Upsilon(\rho)\frac{d\rho}{\rho}\Big). \end{aligned}$$

Using (2.29) for estimating the constant J_Z we obtain

$$\begin{aligned} &C_1 J(Z)\Big(\frac{r}{\delta}\Big)^m \exp\Big(\int_r^\delta \Upsilon(\rho)\frac{d\rho}{\rho}\Big) \\ &\le \mathfrak{M}^m_p(Z; K_{r/e,r}) \\ &\le C_2 J(Z)\Big(\frac{r}{\delta}\Big)^m \exp\Big(\int_r^\delta \Upsilon(\rho)\frac{d\rho}{\rho}\Big). \end{aligned} \tag{5.11}$$

Since $\Upsilon(\rho) = \Lambda\big(\log \rho^{-1}\big)$ and $\Upsilon_\pm(\rho) = \Lambda_\pm\big(\log \rho^{-1}\big)$, estimates (2.38) follows from (4.84), (4.85) and (5.11).

5.6. **Proof of Corollary 2.8.** Here we apply Corollary 2.6 for proving this assertion. Since

$$\frac{r}{\delta}\exp\Big(\mathcal{C}\int_r^\delta \Omega(s)\frac{ds}{s}\Big) \le c\exp\Big(\int_r^\delta (\Upsilon_+(\rho)+c\nu(\rho))\frac{d\rho}{\rho}\Big),$$

estimate (2.37) for the remainder w in (2.32) implies

$$\mathfrak{M}_p^m(w;K_{r/e,r}) \le c\,\mathfrak{M}_2^m(u;K_{\delta/16,\delta})\Big(\frac{r}{\delta}\Big)^m\exp\Big(\int_r^\delta (\Upsilon_+(\rho)+c\nu(\rho))\frac{d\rho}{\rho}\Big). \tag{5.12}$$

Using the right-hand inequality in (2.38) and this estimate, we obtain

$$\begin{aligned}\mathfrak{M}_p^m(u;K_{r/e,r}) &\le c\big(\mathfrak{M}_2^m(u;K_{\delta/16,\delta})+\mathfrak{M}_2^0(Z;K_{\delta/e,\delta})\big)\Big(\frac{r}{\delta}\Big)^m\\ &\quad\times\exp\Big(\int_r^\delta (\Upsilon_+(\rho)+c\nu(\rho))\frac{d\rho}{\rho}\Big).\end{aligned}\tag{5.13}$$

Since

$$\mathfrak{M}_2^0(Z;K_{\delta/e,\delta}) \le \mathfrak{M}_2^0(u;K_{\delta/e,\delta})+\mathfrak{M}_2^0(w;K_{\delta/e,\delta}),$$

we apply (5.12) to estimate the last term and obtain

$$\mathfrak{M}_2^0(w;K_{\delta/e,\delta}) \le c\mathfrak{M}_2^m(u;K_{\delta/16,\delta}).$$

This along with (5.13) proves (2.40).

5.7. **Proof of Corollary2.9.** (1) *Existence of p_1 and validity of (H2).* Let us show first that there exists $p_1>2$, depending on m, n, γ and L such that the following local estimate is valid: if $u\in \mathring{W}^{m,2}_{\rm loc}(K)$ solves problem (2.1), (2.2) with $f\in W^{-m,p_1}_{\rm loc}(K)$, then $u\in \mathring{W}^{m,p_1}_{\rm loc}(K)$ and

$$\mathfrak{M}_{p_1}^m(u;K_{r/e,r}) \le b_0\big(r^{2m}\mathfrak{M}_{p_1}^{-m}(f;K_{r/e^2,er})+\mathfrak{M}_2^m(u;K_{r/e^2,er})\big), \tag{5.14}$$

where b_0 is a constant depending on m, n, γ and L. We note that this is (H2) condition with $p=2$.

Indeed, consider the operator

$$\mathcal{L}\,:\,(\mathring{W}^{m,p}(\mathbb{R}^n_+))^d \to (\mathring{W}^{-m,p}(\mathbb{R}^n_+))^d \quad\text{with } p\in[q',q], \tag{5.15}$$

where $q>2$ and $q'=q/(q-1)$. The norm of this operator is bounded by a constant depending on the constants γ, m, n and q. By (2.9) this operator has inverse for $p=2$ with the norm which depends on the same constants. Using Shneiberg result [11] (see also [6] and references there), we conclude that there exist constants $p_1>2$ and C depending on γ, m, n and q, such that the operator (5.15) is invertible for $p\in[p_1',p_1]$ and the norms of inverse operators are bounded by C.

Let $\eta=\eta(\tau)$ be a smooth function on $(0,\infty)$ such that $\eta(\tau)=1$ for $\tau\in[e^{-1},1]$ and $\eta(\tau)=0$ outside $[e^{-2},e]$. Let also $\eta_r(\tau)=\eta(\tau/r)$. Then $\mathcal{L}(\eta_r u)=\eta_r f+(\mathcal{L}\eta_r-\eta_r\mathcal{L})u$. Using that the operator (5.15) is isomorphism for $p_1\in[p_2',p_2]$ we obtain

$$\|\eta_r u\|_{\mathring{W}^{m,p_1}(\mathbb{R}^n_+)} \le c(\|\eta_r f\|_{W^{-m,p_1}(\mathbb{R}^n_+)}+\|(\mathcal{L}\eta_r-\eta_r\mathcal{L})u\|_{W^{-m,p_1}(\mathbb{R}^n_+)}).$$

This estimate together with Sobolev's imbedding theorem implies (5.14).

(2) *Validity of (H3).* From (2.42) it follows that $\kappa_1(r)\to 0$ as $r\to 0$. This together with boundedness of κ implies (H3) because of (2.14).

(3) *Application of Theorem 2.4.* Since (H1)–(H3) are valid we can apply to solution u Theorem 2.4 and we obtain the asymptotic representation (2.32) with Z satisfying (2.25) and w subject to (2.34). The first inequality in (2.27) implies that

the vector function $\mathbf{q}(r)$ has a limit as $r \to 0$ because of (2.42), which we denote by $\mathbf{q}_0$. Let us show that the integral

$$\int_r^\delta \Upsilon(\rho)\frac{d\rho}{\rho}$$

has a limit as $r \to 0$. Using the definition (1.8) of $\mathbf{R}(\rho)$, we check that

$$|(\mathbf{R}(\rho)\mathbf{q}(\rho), \mathbf{q}(\rho))| \le c\int_{S_+^{n-1}} \kappa(y)d\theta \tag{5.16}$$

where $\rho = |y|$ and $\theta = y/|y|$. From (4.88) and the definition (2.35) of the function χ, it follows

$$\begin{aligned}\int_0^\delta |\chi(\rho)|\frac{d\rho}{\rho} &\le c\Big(\int_0^\delta \kappa_{p_1'}^{p_1'}(\rho)\frac{d\rho}{\rho}\Big)^{1/p_1'}\Big(\int_0^\delta \kappa_{p_1}^{p_1}(\rho)\frac{d\rho}{\rho}\Big)^{1/p_1} + c\omega_0^2 \\ &\le C\int_0^\delta \kappa_1(\rho)\frac{d\rho}{\rho} + c\omega_0^2\,.\end{aligned} \tag{5.17}$$

Therefore,

$$\int_0^\delta |\Upsilon(\rho)|\frac{d\rho}{\rho} < \infty$$

because of (2.42), and consequently,

$$\int_r^\delta \Upsilon(\rho)\frac{d\rho}{\rho} = C - \int_0^r \Upsilon(\rho)\frac{d\rho}{\rho}\,, \tag{5.18}$$

where the last integral is absolutely convergent and, hence, is $o(1)$ as $r \to 0$. This leads to

$$\exp\Big(\int_r^\delta \Upsilon(\rho)\frac{d\rho}{\rho}\Big) = C_1 + o(1) \quad \text{as } r \to 0. \tag{5.19}$$

We put $v = u - \mathbf{c}x_n^m$, where $\mathbf{c} = J_Z C_1 \mathbf{q}_0$ and J_Z is the constant in (2.25). Due to (2.34) in order to show that v satisfies (2.43) it suffices to show that $Z - \mathbf{c}x_n^m$ satisfies (2.43). By (2.25) we have

$$(r\partial_r)^k(Z - \mathbf{c}x_n^m) = J_Z m^k x_n^m\Big(\exp\Big(\int_r^\delta \Upsilon(\rho)\frac{d\rho}{\rho}\Big)\mathbf{q}(r) - C_1\mathbf{q}_0\Big) + J_Z r^m v_k(x)\,.$$

By (2.30), $\mathbf{q}(r) \to \mathbf{q}_0$ as $r \to 0$ and by (5.19) this implies (2.43).

Now the result follows from Corollary 2.4 and from the asymptotic representation (2.25) for the first term in the right-hand side in (2.32).

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