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A nonexistence result for a system of quasilinear degenerate elliptic inequalities in a half-space *

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Abstract

We show that a system of quasilinear degenerate elliptic inequalities does not have non-trivial solutions for a certain range of parameters in the system. The proof relies on a suitable choice of the test function in the weak formulation of the inequalities.

1 Introduction

For $N \geq 2$, let $\Omega = \mathbb{R}^N_+ = \{(x', x_N) : x' \in \mathbb{R}^{N-1}, x_N > 0\}$ and $\partial\Omega$ its boundary. On this domain, we consider the system

$$\begin{aligned} -|x|^\alpha \Delta u &\geq |v|^p, \\ -|x|^\beta \Delta v &\geq |u|^q, \end{aligned} \tag{1.1}$$

which can be viewed as the elliptic part of a system of wave equations where the velocity in each equation vanishes near $x = 0$. This accounts for the effect of a medium that is dense near $x = 0$.

Definition The couple (u, v) is called a solution of (1.1), if

$$u \in L^1(\partial\Omega) \cap L^q_{\rm loc}(\Omega, |x|^{-\beta}\, dx), \quad v \in L^1(\partial\Omega) \cap L^p_{\rm loc}(\Omega, |x|^{-\alpha}\, dx),$$
$$\partial u/\partial\nu, \ \ \partial v/\partial\nu \in L^1_{\rm loc}(\partial\Omega)\,.$$

and for every positive regular function ψ,

$$\begin{aligned} -\int_\Omega u\Delta\psi - \int_{\partial\Omega} \frac{\partial u}{\partial\nu}\,\psi + \int_{\partial\Omega} \frac{\partial\psi}{\partial\nu}\,u &\geq \int_\Omega |v|^p|x|^{-\alpha}\psi, \\ -\int_\Omega v\Delta\psi - \int_{\partial\Omega} \frac{\partial v}{\partial\nu}\,\psi + \int_{\partial\Omega} \frac{\partial\psi}{\partial\nu}\,v &\geq \int_\Omega |u|^q|x|^{-\beta}\psi. \end{aligned}$$

NOTATION. We let $L^m_{\rm loc}(\Omega, |x|^{-\delta}\, dx)$ be the set of all functions $f : \Omega \to \mathbb{R}$ such that for every compact set $K \subseteq \Omega$, $\int_K |f|^m\, |x|^{-\delta}\, dx < \infty$.

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Before we present our results, let us dwell a moment on some previous interesting articles.

In their celebrated article, Brézis and Cabré [1] considered the problem

$$-|x|^2\Delta u \geq u^2, \quad x \in D,$$
$$u = 0, \quad \text{on } \partial D,$$

where D is a smooth bounded domain of $\mathbb{R}^N$ containing 0. They proved that it admits as a weak solution only the trivial solution. Moreover, they gave nonexistence results of weak positive solutions for general equations of the form

$$-\Delta u = a(x)g(u) + b(x), \quad x \in D,$$

under some assumptions on $a(x)$ and $b(x)$, with g a continuous function on $\mathbb{R}$, nondecreasing on $\mathbb{R}^+$, such that $\int_1^\infty (1/g(s))ds < \infty$.

On the other hand, Esteban and Giacomoni in [3] studied the structure of the set of solutions to the problem

$$-|x|^2\Delta u = \lambda u + g(u), \quad x \in B = \{x \in \mathbb{R}^N : |x| < 1\},$$
$$u \geq 0 \quad \text{in} \quad B,$$
$$u = 0, \quad \text{on } \partial B.$$

Concerning equations posed in a half-space, Chipot, Chlebík, Fila and Shafrir [2] considered the problem

$$-\Delta u = au^p, \quad \text{on } \Omega = \mathbb{R}^N_+,$$
$$-\frac{\partial u}{\partial x_N} = u^q, \quad \text{on } \partial\Omega,$$

where $a \geq 0$ and $p, q > 1$. They proved the existence of positive solutions, for

$$p \geq \frac{N+2}{N-2} \quad \text{and} \quad q \geq \frac{N}{N-2},$$

and obtained nonexistence results for $a > 0$ when one of the following requirements is satisfied:

(i) $p \leq \frac{N+2}{N-2}$ and $q \leq \frac{N}{N-2}$ with at least one strict inequality,

(ii) $p < \frac{N}{N-2}$,

(iii) $q < \frac{N}{N-1}$.

Concerning our results, they can be summarized as follows: In section 2, we show that (1.1) cannot admit nontrivial solutions (u, v) for some range of p and q whenever

$$\int_{\{x_N=0\}} (u + v)\, dx' > 0\,.$$

However in section 3, we treat the particular case of positive solutions to (1.1) and obtain different results under conditions different from those of section 2. This is due to the methods employed. Furthermore, in each section, nonexistence results are extended to systems of $m \geq 2$ inequalities.

2 Nonexistence via Young's inequality

Theorem 2.1 *Assume* $p > 1$, $q > 1$, $\alpha \leq 2$, $\beta \leq 2$, *and that*

$$N \leq \min\Big(\frac{p+1-\alpha}{p-1}, \frac{q+1-\beta}{q-1}\Big).$$

Then, there exist no nontrivial solutions (u, v) *of the problem (1.1) such that*

$$(u+v)|_{x_N=0} \in L^1(\mathbb{R}^{N-1}), \quad \int_{\{x_N=0\}} (u+v)\,dx' > 0. \tag{2.1}$$

Remark 2.2 Observe that in the usual case where $\alpha = \beta = 0$, we have nonexistence for $N \geq 2$ and

$$1 < p \leq \min\Big(q, \frac{N+1}{N-1}\Big) \quad \text{or} \quad 1 < q \leq \min\Big(p, \frac{N+1}{N-1}\Big).$$

Remark 2.3 For $\alpha = \beta$ and $1 < p = q$, there is no solution if

$$p \leq \frac{N+1-\alpha}{N-1} \quad \Longleftrightarrow \quad N \leq \frac{p+1-\alpha}{p-1}.$$

Proof of Theorem 2.1 The proof is divided into two steps. First, we construct a suitable test function and make some estimations. Then, we introduce a re-scaling technique as in [4, 5].

The proof is by contradiction. For, suppose that problem (1.1) admits a nontrivial solution (u, v) such that

$$\int_{\{x_N=0\}} (u+v)\,dx' > 0.$$

Let φ be a positive test function in $\mathcal{C}^2(\Omega)$, φ decreasing, and $\varphi(x) = \varphi_0^\lambda(|x|/R)$, where $(R > 0)$ and

$$\varphi_0(\xi) = \begin{cases} 1 & \text{if } 0 \leq \xi \leq 1 \\ 0 & \text{if } \xi \geq 2. \end{cases}$$

The parameter λ will be specified later.

Let $\psi(x) = x_N\varphi(x) \geq 0$. Then

$$\frac{\partial\psi}{\partial x_N} = \varphi(x) + x_N\frac{\partial\varphi}{\partial x_N}, \quad \nabla_{x'}\psi = x_N\nabla_{x'}\varphi,$$

$$\Delta\psi = \frac{\partial^2\psi}{\partial x_N^2} + \sum_{i=1}^{N-1}\frac{\partial^2\psi}{\partial x_i^2} = 2\frac{\partial\varphi}{\partial x_N} + x_N\frac{\partial^2\varphi}{\partial x_N^2} + x_N\sum_{i=1}^{N-1}\frac{\partial^2\varphi}{\partial x_i^2}.$$

Since $\int_{\{x_N=0\}}(\partial u/\partial\nu)\psi = 0$ and $\partial\psi/\partial\nu = -\partial\psi/\partial x_N$, from the above definition we obtain

$$-\int_\Omega \Delta\psi u - \int_{\{x_N=0\}}\frac{\partial\psi}{\partial x_N}u \geq \int_\Omega |v|^p\,|x|^{-\alpha}\psi.$$

Since $\partial\psi/\partial x_N(x',0)=\varphi$, we have

$$\int_\Omega |v|^p\,|x|^{-\alpha}\,\psi \le -\int_\Omega \Delta\psi\,u - \int_{\{x_N=0\}} \varphi\,u.$$

Then it follows that

$$\int_\Omega |v|^p\,|x|^{-\alpha}\psi + \int_{\{x_N=0\}} \varphi u \le \int_\Omega |\Delta\psi||u|. \tag{2.2}$$

We have also

$$\int_\Omega |u|^q\,|x|^{-\beta}\psi + \int_{\{x_N=0\}} \varphi v \le \int_\Omega |\Delta\psi||v|. \tag{2.3}$$

Now, using (2.2), (2.3) and Young's inequality, we obtain

$$\int_\Omega |u|^q|x|^{-\beta}\psi + \int_{\{x_N=0\}} \varphi v \le \varepsilon\int_\Omega |v|^p|x|^{-\alpha}\psi + C(\varepsilon)\int_\Omega |\Delta\psi|^{p'}\psi^{1-p'}|x|^{\alpha(p'-1)},$$

with $p+p'=pp'$, and

$$\int_\Omega |v|^p|x|^{-\alpha}\psi + \int_{\{x_N=0\}} \varphi u \le \varepsilon\int_\Omega |u|^q|x|^{-\beta}\psi + C(\varepsilon)\int_\Omega |\Delta\psi|^{q'}\psi^{1-q'}|x|^{\beta(q'-1)},$$

with $q+q'=qq'$. Therefore,

$$\begin{aligned}(1-\varepsilon)\int_\Omega |u|^q|x|^{-\beta}\psi + (1-\varepsilon)\int_\Omega |v|^p|x|^{-\alpha}\psi + \int_{\{x_N=0\}} (u+v)\varphi \\ \le C(\varepsilon)\int_\Omega |\Delta\psi|^{p'}\psi^{1-p'}|x|^{\alpha(p'-1)} + C(\varepsilon)\int_\Omega |\Delta\psi|^{q'}\psi^{1-q'}|x|^{\beta(q'-1)}.\end{aligned}$$

Hence for $0<\varepsilon<1$, there exists $C>0$ such that

$$\begin{aligned}\int_\Omega |u|^q|x|^{-\beta}\psi + \int_\Omega |v|^p|x|^{-\alpha}\psi + \int_{\{x_N=0\}} (u+v)\varphi \\ \le C\Big(\int_\Omega |\Delta\psi|^{p'}\psi^{1-p'}|x|^{\alpha(p'-1)} + \int_\Omega |\Delta\psi|^{q'}\psi^{1-q'}|x|^{\beta(q'-1)}\Big).\end{aligned} \tag{2.4}$$

At this stage, we introduce the scaled variables:

$$\eta=(\eta_1,\cdots,\eta_N)=R^{-1}x=(R^{-1}x_1,R^{-1}x_2,\cdots,R^{-1}x_N).$$

We have

$$\Delta\psi = R^{-1}\Big(2\frac{\partial\varphi_0^\lambda}{\partial\eta_N} + \eta_N\Delta\varphi_0^\lambda\Big) =: R^{-1}A(\eta).$$

It is clear that the support of $\partial\varphi_0^\lambda/\partial\eta_N$ and the support of $\Delta\varphi_0^\lambda$ are subsets of $\mathcal{C}:=\{\eta\in\mathbb{R} : 1\le|\eta|\le 2\}$.

The relation (2.4) is then written

$$\begin{aligned}\int_\Omega |u|^q|x|^{-\beta}\psi + \int_\Omega |v|^p|x|^{-\alpha}\psi + \int_{\{x_N=0\}}(u+v)\varphi \\ \le C_1R^{N+p'(\alpha-2)+(1-\alpha)} + C_2R^{N+q'(\beta-2)+(1-\beta)}.\end{aligned}$$

where for $\lambda \gg 1$,

$$\begin{aligned}\int_{\mathcal{C}} \frac{|A(\eta)|^{p'}|\eta|^{\alpha(p'-1)}}{|\eta_N|^{p'-1}\varphi_0^{\lambda(p'-1)}(\eta)}\,d\eta \le C_1 < \infty,\\ \int_{\mathcal{C}} \frac{|A(\eta)|^{q'}|\eta|^{\beta(q'-1)}}{|\eta_N|^{q'-1}\varphi_0^{\lambda(q'-1)}(\eta)}\,d\eta \le C_2 < \infty.\end{aligned}$$

Since condition (2.1) implies $\int_{\{x_N=0\}}(u+v)\varphi \ge 0$ for R large enough, it follows that

$$\int_\Omega |u|^q|x|^{-\beta}\psi + \int_\Omega |v|^p|x|^{-\alpha}\psi + \int_{\{x_N=0\}}(u+v)\varphi \le \tilde{C}R^{N+\gamma_1}, \tag{2.5}$$

where $\gamma_1 = \max\Big((\alpha-2)p' + 1 - \alpha, (\beta-2)q' + 1 - \beta\Big)$. It is easy to see that

$$N + \gamma_1 \le 0 \quad\iff\quad N \le \min\Big(\frac{p+1-\alpha}{p-1}, \frac{q+1-\beta}{q-1}\Big).$$

For $N + \gamma_1 < 0$, we let $R \to \infty$ in (2.5) to obtain

$$\int_\Omega |u|^q|x|^{-\beta} + \int_\Omega |v|^p|x|^{-\alpha} = 0$$

which implies $u = v = 0$. This is a contradiction.

For $N + \gamma_1 = 0$, we deduce from (2.5) that

$$\int_\Omega |u|^q|x|^{-\beta}\psi < \infty, \quad \int_\Omega |v|^p|x|^{-\alpha}\psi < \infty$$

since condition (2.1) implies $\int_{\{x_N=0\}}(u+v)\varphi \ge 0$ for large R. It follows that

$$\lim_{R\to\infty}\int_{\{R\le|x|\le 2R\}} |u|^q|x|^{-\beta}\psi = \lim_{R\to\infty}\int_{\{R\le|x|\le 2R\}} |v|^p|x|^{-\alpha}\psi = 0.$$

Now, we use Hölder's inequality in the right-hand side of (2.2) and (2.3) and a scaling argument as in (2.5) to obtain

$$\begin{aligned}&\int_{\{x_N=0\}} v\varphi + \int_\Omega |v|^p|x|^{-\alpha}\,\psi\\ &\le \Big(\int_\Omega |u|^q|x|^{-\beta}\,\psi\Big)^{1/q}\Big(\int_\Omega |\Delta\psi|^{q'}|x|^{\beta(q'-1)}\psi^{1-q'}\Big)^{1/q'}\\ &\le \Big(\int_{\operatorname{supp}\Delta\psi} |u|^q|x|^{-\beta}\psi\Big)^{1/q}\Big(C_2R^{N+q'(\beta-2)+(1-\beta)}\Big)^{1/q'},\end{aligned} \tag{2.6}$$

and

$$\begin{aligned} \int_{\{x_N=0\}} u\varphi + \int_\Omega |u|^q|x|^{-\beta}\,\psi \\ &\le \Big(\int_\Omega |v|^p|x|^{-\alpha}\,\psi\Big)^{1/p}\Big(\int_\Omega |\Delta\psi|^{p'}|x|^{\alpha(p'-1)}\psi^{1-p'}\Big)^{1/p'} \\ &\le \Big(\int_{\operatorname{supp}\Delta\psi} |v|^p|x|^{-\alpha}\,\psi\Big)^{1/p}\Big(C_1R^{N+p'(\alpha-2)+(1-\alpha)}\Big)^{1/p'}. \end{aligned} \tag{2.7}$$

Since $\operatorname{supp}\psi \subset \{R \le |x| \le 2R\}$, then for $N + p'(\alpha-2) + (1-\alpha) = 0$ or $N + q'(\beta-2) + (1-\beta) = 0$, we let $R \to \infty$ in (2.6) and (2.7) to obtain, as before,

$$\int_\Omega |u|^q|x|^{-\beta} + \int_\Omega |v|^p|x|^{-\alpha} \le 0 \quad \Longrightarrow \quad u = v = 0.$$

This completes the proof of Theorem 2.1. $\square$

Without difficulties, we can extend the results to the system of m inequalities

$$\begin{gathered} -|x|^{\alpha_i}\Delta u_i \ge |u_{i+1}|^{p_i}, \quad x \in \Omega, \quad 1 \le i \le m, \\ u_{m+1} = u_1. \end{gathered} \tag{2.8}$$

Theorem 2.4 *Let $p_i > 1$. If p_i and α_i are such that*

$$2 \le N \le \min_{1\le i\le m}\Big(\frac{p_i + 1 - \alpha_i}{p_i - 1}\Big),$$

then problem (2.8) does not admit nontrivial solutions $(u_1, u_2, \dots, u_m)$ satisfying $\sum_{i=1}^m u_i|_{x_N=0} \in L^1(\mathbb{R}^{N-1})$, $\quad \int_{\{x_N=0\}} \sum_{i=1}^m u_i\,dx' > 0$.

3 Nonexistence of positive solution via Hölder's inequality

Theorem 3.1 *Suppose $p > 1$, $q > 1$, and α, β satisfy*

$$1 < N \le \max\Big(\frac{pq + 1 - \beta + (2-\alpha)q}{pq - 1}, \frac{pq + 1 - \alpha + (2-\beta)p}{pq - 1}\Big).$$

Then system (1.1) does not admit nontrivial positive solutions.

Proof of Theorem 3.1 This proof is done by contradiction. Suppose that (1.1) admits a nontrivial solution (u, v) such that $u \ge 0$ and $v \ge 0$.

Let ψ be the same test function as in the proof of Theorem 2.1. Then, relations (2.2) and (2.3) become, respectively,

$$\int_\Omega v^p\,|x|^{-\alpha}\psi \le \int_\Omega |\Delta\psi|\,u, \tag{3.1}$$

$$\int_\Omega u^q\,|x|^{-\beta}\psi \le \int_\Omega |\Delta\psi|\,v. \tag{3.2}$$

Now, using Hölder's inequality in the right-hand side of the above inequalities, we have

$$\int_\Omega u^q\,|x|^{-\beta}\psi \le \Big(\int_\Omega v^p|x|^{-\alpha}\psi\Big)^{1/p}\Big(\int_\Omega |\Delta\psi|^{p'}\psi^{1-p'}|x|^{\alpha(p'-1)}\Big)^{1/p'},$$
$$\int_\Omega v^p|x|^{-\alpha}\psi \le \Big(\int_\Omega u^q|x|^{-\beta}\psi\Big)^{1/q}\Big(\int_\Omega |\Delta\psi|^{q'}\psi^{1-q'}|x|^{\beta(q'-1)}\Big)^{1/q'},$$

where $p' = p/(p-1)$ and $q' = q/(q-1)$. Therefore,

$$\begin{aligned}&\Big(\int_\Omega u^q|x|^{-\beta}\psi\Big)^{1-1/pq}\\ &\le \Big(\int_\Omega |\Delta\psi|^{q'}\psi^{1-q'}|x|^{\beta(q'-1)}\Big)^{1/pq'}\Big(\int_\Omega |\Delta\psi|^{p'}\psi^{1-p'}|x|^{\alpha(p'-1)}\Big)^{1/p'},\end{aligned}$$

and

$$\begin{aligned}&\Big(\int_\Omega v^p|x|^{-\alpha}\psi\Big)^{1-1/pq}\\ &\le \Big(\int_\Omega |\Delta\psi|^{p'}\psi^{1-p'}|x|^{\alpha(p'-1)}\Big)^{1/qp'}\Big(\int_\Omega |\Delta\psi|^{q'}\psi^{1-q'}|x|^{\alpha(q'-1)}\Big)^{1/q'}.\end{aligned}$$

Using the change of variable $x = R\eta$ and choosing λ as in the proof of Theorem 2.1, it follows that

$$\Big(\int_\Omega u^q|x|^{-\beta}\psi\Big)^{1-1/pq} \le C_1R^{\lambda_1}, \tag{3.3}$$

where

$$\lambda_1 = \frac{1}{pq}\Big\{N(pq-1) - pq - 1 + \beta + (2-\alpha)q\Big\},$$

and

$$\Big(\int_\Omega v^p|x|^{-\alpha}\psi\Big)^{1-1/pq} \le C_2R^{\lambda_2}, \tag{3.4}$$

where

$$\lambda_2 = \frac{1}{pq}\Big\{N(pq-1) - pq - 1 + \alpha + (2-\beta)p\Big\}.$$

Note that

$$\lambda_1 \le 0 \quad\Longleftrightarrow\quad N \le \frac{pq+1-\beta+(2-\alpha)q}{pq-1},$$

and

$$\lambda_2 \le 0 \quad\Longleftrightarrow\quad N \le \frac{pq+1-\alpha+(2-\beta)p}{pq-1}.$$

For $\lambda_1 < 0$ and $\lambda_2 < 0$, letting $R \to \infty$ in (3.3) and (3.4), we deduce $u = 0$ and $v = 0$, respectively. This is a contradiction.

For $\lambda_1 = 0$ or $\lambda_2 = 0$, we can use the same argument developed in the last part of the proof of Theorem 2.1 and show that $u = 0$ or $v = 0$, when $R \to \infty$.

Observe that, thanks to (3.1) and (3.2), when $u = 0$ then $v = 0$ and vice versa. $\square$

To obtain a generalization of Theorem 3.1 to the case of m inequalities, we first analyze a system with three inequalities:

$$\begin{aligned} -|x|^{\alpha}\Delta u &\ge v^p, \quad x \in \Omega,\\ -|x|^{\beta}\Delta v &\ge w^q, \quad x \in \Omega,\\ -|x|^{\gamma}\Delta w &\ge u^r, \quad x \in \Omega. \end{aligned} \tag{3.5}$$

Definition The vector (u, v, w) is called a solution of (3.5), if

$$\begin{gathered} u \in L^1_{\rm loc}(\partial\Omega) \cap L^r_{\rm loc}(\Omega, |x|^{-\gamma}\,dx),\\ v \in L^1_{\rm loc}(\partial\Omega) \cap L^p_{\rm loc}(\Omega, |x|^{-\alpha}\,dx),\\ w \in L^1_{\rm loc}(\partial\Omega) \cap L^q_{\rm loc}(\Omega, |x|^{-\beta}\,dx), \end{gathered}$$

and for any positive regular function ψ we have

$$\begin{gathered} -\int_\Omega u\Delta\psi - \int_{\partial\Omega}\frac{\partial u}{\partial\nu}\,\psi + \int_{\partial\Omega}\frac{\partial\psi}{\partial\nu}\,u \ge \int_\Omega v^p|x|^{-\alpha}\psi,\\ \int_\Omega v\Delta\psi - \int_{\partial\Omega}\frac{\partial v}{\partial\nu}\,\psi + \int_{\partial\Omega}\frac{\partial\psi}{\partial\nu}\,v \ge \int_\Omega w^q|x|^{-\beta}\psi,\\ \int_\Omega v\Delta\psi - \int_{\partial\Omega}\frac{\partial w}{\partial\nu}\,\psi + \int_{\partial\Omega}\frac{\partial\psi}{\partial\nu}\,w \ge \int_\Omega u^r|x|^{-\gamma}\psi. \end{gathered}$$

Let ψ be defined as in the proof of Theorem 2.1. Then, using Hölder's inequality, we have

$$\begin{gathered} \int_\Omega v^p|x|^{-\alpha}\psi \le \Big(\int_\Omega u^r|x|^{-\gamma}\psi\Big)^{1/r}\Big(\int_\Omega |\Delta\psi|^{r'}\psi^{1-r'}|x|^{\gamma(r'-1)}\Big)^{1/r'},\\ \int_\Omega w^q|x|^{-\beta}\psi \le \Big(\int_\Omega v^p|x|^{-\alpha}\psi\Big)^{1/p}\Big(\int_\Omega |\Delta\psi|^{p'}\psi^{1-p'}|x|^{\alpha(p'-1)}\Big)^{1/p'}\\ \int_\Omega u^r|x|^{-\gamma}\psi \le \Big(\int_\Omega w^q|x|^{-\beta}\psi\Big)^{1/q}\Big(\int_\Omega |\Delta\psi|^{q'}\psi^{1-q'}|x|^{\beta(q'-1)}\Big)^{1/q'}. \end{gathered}$$

Put

$$I_1 = \Big(\int_\Omega |\Delta\psi|^{r'}\psi^{1-r'}|x|^{\gamma(r'-1)}\Big)^{1/r'}, \quad I_2 = \Big(\int_\Omega |\Delta\psi|^{p'}\psi^{1-p'}|x|^{\alpha(p'-1)}\Big)^{1/p'}$$
$$I_3 = \Big(\int_\Omega |\Delta\psi|^{q'}\psi^{1-q'}|x|^{\beta(q'-1)}\Big)^{1/q'}.$$

Then, we have

$$\Big(\int_\Omega v^p|x|^{-\alpha}\psi\Big)^{(pqr-1)/p} \le I_1^{qr}\,I_2\,I_3^{q}, \quad \Big(\int_\Omega w^q|x|^{-\beta}\psi\Big)^{(pqr-1)/q} \le I_1^{r}\,I_2^{pr}\,I_3,$$
$$\Big(\int_\Omega u^r|x|^{-\gamma}\psi\Big)^{(pqr-1)/q} \le I_1\,I_2^{p}\,I_3^{pq}.$$

The same change of variables used in the proof of Theorem 2.1 gives

$$I_1 = R^{\lambda_1}\Big(\int_{\mathcal{C}} \frac{|A(\eta)|^{r'}|\eta|^{\gamma(r'-1)}}{|\eta_N|^{r'-1}\varphi_0^{\lambda(r'-1)}(\eta)}\,d\eta\Big)^{1/r'}$$

where $\lambda_1 = (N+1-\gamma)/r' + \gamma - 2$. For a suitable choice of λ, we have

$$\int_{\mathcal{C}} \frac{|A(\eta)|^{r'}|\eta|^{\gamma(r'-1)}}{|\eta_N|^{r'-1}\varphi_0^{\lambda(r'-1)}(\eta)}\,d\eta < \infty.$$

Therefore, there exists a constant $C_1 > 0$ such that

$$I_1 \le C_1 R^{\lambda_1}.$$

Analogously we have

$$\begin{aligned}
I_2 &\le C_2 R^{\lambda_2}, \quad \text{with} \quad \lambda_2 = \frac{N+1-\alpha}{p'} + \alpha - 2,\\
I_3 &\le C_3 R^{\lambda_3}, \quad \text{with} \quad \lambda_3 = \frac{N+1-\beta}{r'} + \beta - 2.
\end{aligned}$$

It follows that

$$\begin{aligned}
\Big(\int_\Omega v^p|x|^{-\alpha}\psi\Big)^{(pqr-1)/p} &\le \tilde{C}_1 R^{\lambda_1 qr + \lambda_2 + \lambda_3 q} =: \tilde{C}_1 R^{\sigma_1},\\
\Big(\int_\Omega w^q|x|^{-\beta}\psi\Big)^{(pqr-1)/q} &\le \tilde{C}_2 R^{\lambda_1 r + \lambda_2 pr + \lambda_3} =: \tilde{C}_2 R^{\sigma_2},\\
\Big(\int_\Omega u^r|x|^{-\gamma}\psi\Big)^{(pqr-1)/q} &\le \tilde{C}_3 R^{\lambda_1 + \lambda_2 p + \lambda_3 pq} =: \tilde{C}_3 R^{\sigma_3}.
\end{aligned}$$

Note that

$$\begin{aligned}
\sigma_1 \le 0 \quad &\Longleftrightarrow N \le \frac{pqr + (2-\gamma)pq + (2-\beta)p + 1 - \alpha}{pqr-1} = 1 + X_1,\\
\sigma_2 \le 0 \quad &\Longleftrightarrow N \le \frac{pqr + (2-\alpha)qr + (2-\gamma)q + 1 - \beta}{pqr-1} = 1 + X_2,\\
\sigma_3 \le 0 \quad &\Longleftrightarrow N \le \frac{pqr + (2-\beta)pr + (2-\alpha)r + 1 - \gamma}{pqr-1} = 1 + X_3,
\end{aligned}$$

where

$$\begin{aligned}
X_1 &= \frac{(2-\gamma)pq + (2-\beta)p + (2-\alpha)}{pqr-1},\\
X_2 &= \frac{(2-\alpha)qr + (2-\gamma)q + (2-\beta)}{pqr-1},\\
X_3 &= \frac{(2-\beta)pr + (2-\alpha)r + (2-\gamma)}{pqr-1}
\end{aligned}$$

are solutions of the linear system

$$\begin{pmatrix} 1 & -p & 0 \\ 0 & 1 & -q \\ -r & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} \alpha - 2 \\ \beta - 2 \\ \gamma - 2 \end{pmatrix}. \tag{3.6}$$

We have the following nonexistence result.

Theorem 3.2 *Let $(X_1, X_2, X_3)^T$ be the solution of (3.6). Then, if $N \le X_1 + 1$, or $N \le X_2 + 1$, or $N \le X_3 + 1$, system (3.5) cannot admit nontrivial weak solutions (u, v, w) such that $u \ge 0$, $v \ge 0$ and $w \ge 0$.*

Now, we are able to announce the nonexistence result of positive solutions for the system (2.8).

Theorem 3.3 *Suppose $p_i > 1$ for $1 \le i \le m$. Let $(X_1, X_2, \dots, X_m)^T$ be the solution of the linear system*

$$\begin{pmatrix} 1 & -p_1 & 0 & 0 & 0 \\ 0 & 1 & -p_2 & 0 & 0 \\ \vdots & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 1 & -p_{m-1} \\ -p_m & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{m-1} \\ X_m \end{pmatrix} = \begin{pmatrix} \alpha_1 - 2 \\ \alpha_2 - 2 \\ \vdots \\ \alpha_{m-1} - 2 \\ \alpha_m - 2 \end{pmatrix}.$$

Then, if $N \le 1 + \max(X_1, X_2, ..., X_m)$, system (2.8) cannot admit a nontrivial positive solution.

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