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Quantitative, uniqueness, and vortex degree estimates for solutions of the Ginzburg-Landau equation *

Igor Kukavica

Abstract

In this paper, we provide a sharp upper bound for the maximal order of vanishing for non-minimizing solutions of the Ginzburg-Landau equation

$$\Delta u = -\frac{1}{\epsilon^2}(1-|u|^2)u$$

which improves our previous result [12]. An application of this result is a sharp upper bound for the degree of any vortex. We treat Dirichlet (homogeneous and non-homogeneous) as well as Neumann boundary conditions.

1 Introduction

In this paper, we provide vortex degree estimates for solutions of the Ginzburg-Landau equation

$$\Delta u = -\frac{1}{\epsilon^2}\big(1-|u|^2\big)u\,.$$

The vortices of solutions of this equation were studied by Bethuel, Brezis, and Hélein in [5]. (We recall that x_0 is a vortex if it is an isolated zero of u, and if the degree of u at x_0 is nonzero.) They prescribed nonhomogeneous boundary conditions $u|_{\partial\Omega} = g$ with $g\colon \partial\Omega \to S^1$ such that $\deg g = d > 0$. If Ω is convex, and if ϵ is sufficiently small, they proved that a minimizing solution u has precisely d distinct vortices of degree 1. This result has been extended to include all bounded smooth domains by Struwe [17]. It was further shown in [5] that there exist non-minimizing solutions of the Ginzburg-Landau equation whose vortex at the origin is an arbitrarily prescribed nonzero integer.

In this paper, we find a sharp upper bound in terms of $1/\epsilon$ for the degree of vortices for solutions which are not necessarily minimizing. Chanillo and Kiessling proved in [7, Lemma 6] that if x_0 is a vortex of degree $d \in \mathbb{N}$, then the vanishing order of u is at least d. Therefore, we may use unique continuation

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methods to address this problem. Using the result of Chanillo and Kiessling, the paper [12] implies that in the homogeneous Dirichlet and periodic cases the degree of u at any vortex x_0 is less than $C\epsilon^{-2}$ where C depends only on Ω. In Theorem 3.1 below we improve this bound to $C\epsilon^{-1}$ and then show that this bound is best possible. The main tool in the proof is a new logarithmically convex quantity for the Laplacian operator. More precisely, for any $\alpha > -1$ and any harmonic function u, the quantity

$$H(r) = \int_{B_r(0)} u(x)^2 \big(r^2 - |x|^2\big)^\alpha \, dx$$

is logarithmically convex, i.e., $\log H(r)$ is a convex function of $\log r$. Due to cancellations of terms involving α in (2.10)–(2.12) below, and due to a gradient structure of the Ginzburg-Landau equation, we can choose an appropriate optimal α which gives our result. Inspired by an example in [5], we construct in Remark 3.3 a solution which shows that our bound $C\epsilon^{-1}$ can not be improved upon. Theorem 3.2 contains an estimate concerning the boundary condition $u|_{\partial\Omega} = g$ where $|g| = 1$, while Theorem 3.5 covers the Neumann case.

For properties of stationary Ginzburg-Landau equation, cf. [5, 8, 15, 16, 17] and to [1, 2, 3, 4, 9, 10, 12, 13] for various results on logarithmic convexity and unique continuation.

2 Quantitative uniqueness for systems

In this section, we consider nontrivial solutions u of the system

$$\begin{aligned} \Delta u &= F'\big(|u|^2\big)u \\ u|_{\partial\Omega} &= 0 \end{aligned} \tag{2.1}$$

where $u \in C^2(\Omega, \mathbb{R}^D) \cap C(\overline{\Omega}, \mathbb{R}^D)$ with $D \in \mathbb{N}$. We assume that $\Omega \subseteq \mathbb{R}^d$, where $d \geq 2$, and one of the following:

(a) Ω is a convex bounded domain;

(b) Ω is a Dini domain; Dini domains are bounded domains with the following property: Around any point there is a neighborhood N, such that after a rotation of coordinates $\Omega \cap N$ lies below a graph of a function whose normal is Dini continuous (see [14] for details);

(c) Ω is a periodic cube $[0, L]^d$; in this case, $\partial\Omega = \emptyset$.

As in [12], we are mainly interested in periodic boundary conditions; the papers [1] and [14] enable us to consider homogeneous Dirichlet conditions without much change. As far as the Ginzburg-Landau equation is concerned, the nonhomogeneous boundary conditions $u|_{\partial\Omega} = g$ (with $|g| = 1$) and homogeneous Neumann conditions $(du/d\nu)|_{\partial\Omega} = 0$ are more physically relevant and more widely studied. Theorem 3.1 addresses the homogeneous Dirichlet boundary conditions; the non-homogeneous boundary conditions are considered in Theorem 3.2, while Theorem 3.5 covers the Neumann case. Let $M = \max_{\overline{\Omega}} |u|^2$. On $F\colon [0,M] \to \mathbb{R}$, we make the following assumptions:

(i) $F \in C^1\big([0,M]\big)$ and
$$\big|F'(x)\big| \le \lambda\,, \quad x \in [0,M] \tag{2.2}$$
for some $\lambda > 0$;

(ii) $F(0) = 0$;

(iii) F is convex on $[0,M]$.

Conditions (ii) and (iii) imply
$$F(x) \le xF'(x)\,, \quad x \in [0,M] \tag{2.3}$$
and
$$\big|F(x)\big| \le \lambda x\,, \quad x \in [0,M]\,. \tag{2.4}$$

The following is the main result of this section.

We recall that the order of vanishing at $x_0 \in \overline{\Omega}$ is defined as the largest integer $n \in \mathbb{N}_0 = \{0, 1, \dots\}$ such that
$$\frac{1}{|B_r(x_0) \cap \Omega|} \int_{B_r(x_0)\cap\Omega} |u|^2 = \mathcal{O}(r^{2n})\,, \quad \text{as} r \to 0\,.$$
(Here and in each subsequent occurrence, one needs to replace $B_r(x_0) \cap \Omega$ with $B_r(x_0)$ in the case of periodic boundary conditions (c).) In particular, if u does not vanish at x_0, then the order of vanishing is 0. We also add that u may not have any zeros in Ω.

Theorem 2.1 *Let $x_0 \in \overline{\Omega}$. The order of vanishing of u at x_0 is less than $C(\sqrt{\lambda}+1)$ where C is a constant depending only on Ω.*

If λ is sufficiently small, and if Ω satisfies (a) or (b), then there are no nontrivial solutions of (2.1). In these cases, the bound $C(\sqrt{\lambda}+1)$ may be replaced by $C\sqrt{\lambda}$.

Let $x_0 \in \overline{\Omega}$ and $R > 0$ be such that $B_R(x_0) \cap \Omega$ is starshaped with respect to x_0. For an arbitrary $\alpha > -1$ and $r > 0$, denote
$$H_{x_0}(r) = \int_{B_r(x_0)\cap\Omega} \big|u(x)\big|^2 \big(r^2 - |x - x_0|^2\big)^\alpha \, dx$$
where $|u|^2 = u_j u_j$. We will omit the dependency on x_0 when it is clear from the context.

Lemma 2.2 *Let $q \ge 1$, and let $0 < r_1 < r_2$ be such that $qr_2 \le R$. Then*
$$\log \frac{H_{x_0}(qr_1)}{H_{x_0}(r_1)} \le \log \frac{H_{x_0}(qr_2)}{H_{x_0}(r_2)} + \frac{(q^2-1)r_2^2 d\lambda}{\alpha+1}\,.$$

Proof of Lemma 2.2 Without loss of generality, $x_0 = 0$. Let $r \in (0, R)$ be arbitrary. Denoting $B_r = B_r(0)$, we get

$$\begin{aligned} H'(r) &= 2\alpha r \int_{B_r\cap\Omega} |u|^2 \big(r^2 - |x|^2\big)^{\alpha-1}\, dx \\ &= \frac{2\alpha}{r} H(r) - \frac{1}{r}\int_{B_r\cap\Omega} x_j |u|^2\, \partial_j\Big(\big(r^2-|x|^2\big)^{\alpha}\Big)\, dx \end{aligned} \tag{2.5}$$

whence, by the divergence theorem,

$$H'(r) = \frac{2\alpha + d}{r} H(r) + \frac{1}{(\alpha+1)r} I(r) \tag{2.6}$$

where

$$I(r) = 2(\alpha+1)\int_{B_r\cap\Omega} x_j u_k\, \partial_j u_k \big(r^2 - |x|^2\big)^{\alpha}\, dx\,. \tag{2.7}$$

By the divergence theorem,

$$\begin{aligned} I(r) &= -\int_{B_r\cap\Omega} u_k\, \partial_j u_k\, \partial_j\Big(\big(r^2-|x|^2\big)^{\alpha+1}\Big)\, dx \\ &= \int_{B_r\cap\Omega} \partial_j u_k\, \partial_j u_k \big(r^2-|x|^2\big)^{\alpha+1}\, dx \\ &\quad + \int_{B_r\cap\Omega} |u|^2 F'(|u|^2)\big(r^2-|x|^2\big)^{\alpha+1}\, dx\,. \end{aligned}$$

As in (2.5) above, we get

$$\begin{aligned} I'(r) &= \frac{2(\alpha+1)}{r}\int_{B_r\cap\Omega} \partial_j u_k\, \partial_j u_k \big(r^2-|x|^2\big)^{\alpha+1}\, dx \\ &\quad -\frac{1}{r}\int_{B_r\cap\Omega} x_m\, \partial_j u_k\, \partial_j u_k\, \partial_m\Big(\big(r^2-|x|^2\big)^{\alpha+1}\Big)\, dx \\ &\quad +2(\alpha+1)r\int_{B_r\cap\Omega} |u|^2 F'(|u|^2)\big(r^2-|x|^2\big)^{\alpha}\, dx\,. \end{aligned}$$

Using the divergence theorem on the second integral leads to

$$\begin{aligned} I'(r) &= \frac{2(\alpha+1)+d}{r}\int_{B_r\cap\Omega} \partial_j u_k\, \partial_j u_k \big(r^2-|x|^2\big)^{\alpha+1}\, dx \\ &\quad +\frac{2}{r}\int_{B_r\cap\Omega} x_m\, \partial_j u_k\, \partial_{jm} u_k \big(r^2-|x|^2\big)^{\alpha+1}\, dx \\ &\quad -\frac{1}{r}\int_{B_r\cap\partial\Omega} x_m\, \partial_j u_k\, \partial_j u_k \big(r^2-|x|^2\big)^{\alpha+1}\nu_m\, d\sigma(x) \\ &\quad +2(\alpha+1)r\int_{B_r\cap\Omega} |u|^2 F'(|u|^2)\big(r^2-|x|^2\big)^{\alpha}\, dx \end{aligned}$$

where $\nu=(\nu_1,\dots,\nu_d)$ denotes the outward unit normal and where $d\sigma(x)$ stands for the surface measure on $\partial\Omega$. Applying the divergence theorem in the second integral, this time with respect to the variable x_j, we get

$$\begin{aligned} I'(r) \;=\;& \frac{2\alpha+d}{r}\int_{B_r\cap\Omega} \partial_j u_k\,\partial_j u_k\big(r^2-|x|^2\big)^{\alpha+1}\,dx \\ &-\frac{2}{r}\int_{B_r\cap\Omega} x_m\partial_m u_k\,F'\big(|u|^2\big)u_k\big(r^2-|x|^2\big)^{\alpha+1}\,dx \\ &+\frac{4(\alpha+1)}{r}\int_{B_r\cap\Omega} x_m\partial_m u_k\,x_j\partial_j u_k\big(r^2-|x|^2\big)^{\alpha}\,dx \\ &+\frac{2}{r}\int_{B_r\cap\partial\Omega} x_m\partial_m u_k\,\partial_j u_k\big(r^2-|x|^2\big)^{\alpha+1}\nu_j\,d\sigma(x) \\ &-\frac{1}{r}\int_{B_r\cap\partial\Omega} x_m\partial_j u_k\,\partial_j u_k\big(r^2-|x|^2\big)^{\alpha+1}\nu_m\,d\sigma(x) \\ &+2(\alpha+1)r\int_{B_r\cap\Omega} |u|^2F'\big(|u|^2\big)\big(r^2-|x|^2\big)^{\alpha}\,dx\,. \end{aligned} \tag{2.8}$$

The second term on the right hand side of (2.8) equals

$$\begin{aligned} &-\frac{1}{r}\int_{B_r\cap\Omega} x_m\,\partial_m\big(F\big(|u|^2\big)\big)\big(r^2-|x|^2\big)^{\alpha+1}\,dx \\ &= \frac{d}{r}\int_{B_r\cap\Omega} F\big(|u|^2\big)\big(r^2-|x|^2\big)^{\alpha+1}\,dx \\ &\quad -\frac{2(\alpha+1)}{r}\int_{B_r\cap\Omega} F\big(|u|^2\big)|x|^2\big(r^2-|x|^2\big)^{\alpha}\,dx\,; \end{aligned}$$

since the boundary integral vanishes by $F(0)=0$. On the other hand, the sum of the fourth and the fifth term is

$$\frac{1}{r}\int_{B_r\cap\partial\Omega} x_k\nu_k\frac{\partial u_m}{\partial\nu}\frac{\partial u_m}{\partial\nu}\big(r^2-|x|^2\big)^{\alpha+1}\,d\sigma(x)$$

which is due to the fact $\partial_k u=(\partial u/\partial\nu)\nu_k$ resulting from $u|_{\partial\Omega}=0$. This integral is nonnegative since $B_r\cap\Omega$ is starshaped with respect to 0. Then

$$\begin{aligned} I'(r) \;\ge\;& \frac{2\alpha+d}{r}I(r)-\frac{2\alpha+d}{r}\int_{B_r\cap\Omega}|u|^2F'\big(|u|^2\big)\big(r^2-|x|^2\big)^{\alpha+1}\,dx \\ &+\frac{d}{r}\int_{B_r\cap\Omega} F\big(|u|^2\big)\big(r^2-|x|^2\big)^{\alpha+1}\,dx \\ &-\frac{2(\alpha+1)}{r}\int_{B_r\cap\Omega} F\big(|u|^2\big)|x|^2\big(r^2-|x|^2\big)^{\alpha}\,dx \\ &+\frac{4(\alpha+1)}{r}\int_{B_r\cap\Omega} x_m\,\partial_m u_k x_j\,\partial_j u_k\big(r^2-|x|^2\big)^{\alpha}\,dx \\ &+2(\alpha+1)r\int_{B_r\cap\Omega}|u|^2F'\big(|u|^2\big)\big(r^2-|x|^2\big)^{\alpha}\,dx\,. \end{aligned} \tag{2.9}$$

The sum of the second, the third, the fourth, and the sixth term is

$$\frac{1}{r}\int_{B_r\cap\Omega} E(x)\big(r^2-|x|^2\big)^\alpha\,dx$$

where

$$\begin{aligned} E(x) &= -(2\alpha+d)|u|^2F'\big(|u|^2\big)\big(r^2-|x|^2\big)+dF\big(|u|^2\big)\big(r^2-|x|^2\big)\\ &\quad -(2\alpha+2)F\big(|u|^2\big)|x|^2+(2\alpha+2)F'\big(|u|^2\big)|u|^2r^2\,. \end{aligned}\tag{2.10}$$

We get

$$\begin{aligned} E(x) &= r^2\Big(-2\alpha|u|^2F'\big(|u|^2\big)-d|u|^2F'\big(|u|^2\big)+dF\big(|u|^2\big)\\ &\quad +2\alpha F'\big(|u|^2\big)|u|^2+2F'\big(|u|^2\big)|u|^2\Big)\\ &\quad +|x|^2\Big(2\alpha F'\big(|u|^2\big)|u|^2+dF'\big(|u|^2\big)|u|^2\\ &\quad -dF\big(|u|^2\big)-2\alpha F\big(|u|^2\big)-2F\big(|u|^2\big)\Big) \end{aligned}\tag{2.11}$$

from where, using (iii),

$$E(x) \geq r^2\Big((2-d)|u|^2F'\big(|u|^2\big)+dF\big(|u|^2\big)\Big)-2|x|^2F\big(|u|^2\big)\,. \tag{2.12}$$

By (2.2) and (2.4) and using $|x|^2\leq r^2$, we get

$$E(x)\geq -4d\lambda r^2|u|^2$$

which leads to

$$I'(r)\geq \frac{2\alpha+d}{r}I(r)+\frac{4(\alpha+1)}{r}\int_{B_r\cap\Omega} x_m\partial_m u_k\,x_j\partial_j u_k\big(r^2-|x|^2\big)^\alpha\,dx-4d\lambda rH(r)\,.$$

Now, let $N(r)=I(r)/H(r)$. Then

$$\begin{aligned} N'(r) &\geq -4d\lambda r+\frac{4(\alpha+1)}{rH(r)^2}\bigg(\int_{B_r\cap\Omega} x_m\partial_m u_k\,x_j\partial_j u_k\big(r^2-|x|^2\big)^\alpha\,dx\\ &\quad\times\int_{B_r\cap\Omega}|u|^2\big(r^2-|x|^2\big)^\alpha\,dx-\Big(\int_{B_r\cap\Omega}x_ju_k\,\partial_ju_k\big(r^2-|x|^2\big)^\alpha\,dx\Big)^2\bigg) \end{aligned}$$

whence, by the Cauchy-Schwarz inequality, $N'(r)\geq -4d\lambda r$, i.e.,

$$N(r_2)-N(r_1)\geq -2d\lambda(r_2-r_1)(r_2+r_1)\,,\quad 0<r_1\leq r_2\leq R\,. \tag{2.13}$$

Let $q\geq 1$, and let $0<r_1\leq r_2\leq R$ be such that $qr_2\leq R$. Dividing (2.6) by $H(r)$ and integrating the resulting equality between r_1 and qr_1 leads to

$$\begin{aligned} \log\frac{H(qr_1)}{H(r_1)} &= (2\alpha+d)\log q+\frac{1}{\alpha+1}\int_{r_1}^{qr_1}\frac{N(\rho)}{\rho}\,d\rho\\ &= (2\alpha+d)\log q+\frac{1}{\alpha+1}\int_{r_2}^{qr_2}\frac{N(r_1\rho/r_2)}{\rho}\,d\rho\,. \end{aligned}$$

Using (2.13), we get

$$\begin{aligned}\log\frac{H(qr_1)}{H(r_1)} &\leq (2\alpha+d)\log q+(r_2^2-r_1^2)\frac{(q^2-1)d\lambda}{\alpha+1}+\frac{1}{\alpha+1}\int_{r_2}^{qr_2}\frac{N(\rho)}{\rho}\,d\rho\\ &= \log\frac{H(qr_2)}{H(r_2)}+\frac{(r_2^2-r_1^2)(q^2-1)d\lambda}{\alpha+1}\end{aligned}$$

and the lemma follows. □

Again, let $x_0\in\overline{\Omega}$ be fixed, and denote

$$h_{x_0}(r)=\int_{B_r(x_0)\cap\Omega}|u(x)|^2\,dx\,.$$

Let $R>0$ be such that $B_R(x_0)\cap\Omega$ is starshaped with respect to $x_0\in\overline{\Omega}$.

Lemma 2.3 *Let $\alpha\geq 0$, $0<r_1<4r_2/3$ and $4r_2\leq R$. Then, with the above assumptions,*

$$\log\frac{h_{x_0}(2r_1)}{h_{x_0}(r_1)}\leq\log\frac{h_{x_0}(4r_2)}{h_{x_0}(r_2)}+C\left(\alpha+\frac{dr_2^2\lambda}{\alpha+1}\right)$$

where C is a universal constant.

Proof of Lemma 2.3 Denote $h(r)=h_{x_0}(r)$ and $H(r)=H_{x_0}(r)$. If $0<r<\rho$, then

$$H(r)\leq r^{2\alpha}h(r) \tag{2.14}$$

and

$$h(r)\leq\frac{H(\rho)}{(\rho^2-r^2)^\alpha} \tag{2.15}$$

Therefore,

$$\log\frac{h(2r_1)}{h(r_1)}\leq\log\frac{H(3r_1)}{H(r_1)}-\alpha\log 5\leq\log\frac{H(3r_1)}{H(r_1)}$$

since $\alpha\geq 0$. By Lemma 2.2,

$$\log\frac{h(2r_1)}{h(r_1)}\leq\log\frac{H(4r_2)}{H(4r_2/3)}+\frac{Cr_2^2d\lambda}{\alpha+1}\,.$$

Using (2.14) and (2.15) again, we get our assertion. □

In Theorem 3.2 below, we will need an interior version of the above lemma, which we state for sake of completeness.

Lemma 2.4 *Let u be a solution of $\Delta u=F'\big(|u|^2\big)u$, where F is as above, in $B_R(x_0)$. Let $\alpha\geq 0$, $0<r_1<4r_2/3$ and $4r_2\leq R$ Then*

$$\log\frac{h_{x_0}(2r_1)}{h_{x_0}(r_1)}\leq\log\frac{h_{x_0}(4r_2)}{h_{x_0}(r_2)}+C\left(\alpha+\frac{dr_2^2\lambda}{\alpha+1}\right)$$

where $h_{x_0}(r)=\int_{B_r(x_0)}|u(x)|^2\,dx$ and C is a universal constant.

Proof of Lemma 2.4 The proof is the same as that of Lemma 2.3. □

Next lemma will be used in the overlapping chain of balls argument.

Lemma 2.5 *Let $\alpha \geq 0$. Assume that $x_1, x_2 \in \overline{\Omega}$ and $r > 0$ are such that $B_{20r}(x_1) \cap \Omega$ is starshaped with respect to $x_2 \in \overline{\Omega}$. If $B_r(x_1)$ and $B_r(x_2)$ intersect, and if*

$$\int_\Omega |u(x)|^2 \, dx \leq K H_{x_1}(r)$$

for some $K \geq 0$, then

$$\int_\Omega |u(x)|^2 \, dx \leq K^3 \exp\left(C \left(\alpha + \frac{\lambda}{\alpha + 1} \right) \right) H_{x_2}(r)$$

where C is a constant which depends only on d and $\mathrm{diam}(\Omega)$.

Proof of Lemma 2.5 It is easy to check that $H_{x_1}(r) \leq H_{x_2}(4r)$. Therefore,

$$\int_\Omega |u|^2 \leq K H_{x_1}(r) \leq K H_{x_2}(4r) \tag{2.16}$$

which, by (2.14), implies

$$H_{x_2}(8r) \leq (8r)^{2\alpha} h(8r) \leq (8r)^{2\alpha} \int_\Omega |u|^2 \leq C^\alpha K H_{x_2}(4r)$$

where C denotes a generic constant which depends only on d and $\mathrm{diam}\,\Omega$. Lemma 2.2 then implies

$$\log \frac{H_{x_2}(4r)}{H_{x_2}(2r)} \leq \log K + C \left(\alpha + \frac{\lambda}{\alpha + 1} \right) \tag{2.17}$$

and similarly

$$\log \frac{H_{x_2}(2r)}{H_{x_2}(r)} \leq 2 \log K + C \left(\alpha + \frac{\lambda}{\alpha + 1} \right) . \tag{2.18}$$

The inequalities (2.16), (2.17), and (2.18) then give

$$\log \frac{\int_\Omega |u|^2}{H_{x_2}(4r)} \leq 3 \log K + C \left(\alpha + \frac{\lambda}{\alpha + 1} \right)$$

which gives our assertion. □

Proof of Theorem 2.1 In the cases (a) and (c), we can take R to be arbitrarily large. Note that, in the case (a),

$$\log \frac{h_{x_0}(4r)}{h_{x_0}(r)} = 0 , \quad x_0 \in \overline{\Omega} \tag{2.19}$$

provided $r \geq \operatorname{diam}\Omega$. Therefore, by Lemma 2.3, there is a numerical constant C such that

$$\log \frac{h_{x_0}(2r_1)}{h_{x_0}(r_1)} \leq C\left(\alpha + \frac{d\operatorname{diam}(\Omega)^2\lambda}{\alpha+1}\right), \quad x_0 \in \overline{\Omega}$$

for every $\alpha \geq 0$ and $r_1 \in (0, \operatorname{diam}\Omega)$. Choosing $\alpha = \sqrt{d\lambda}\operatorname{diam}\Omega$, we get

$$\log \frac{h_{x_0}(2r)}{h_{x_0}(r)} \leq C\sqrt{\lambda d}\operatorname{diam}\Omega\,, \quad x_0 \in \overline{\Omega}$$

for $r \in (0, \operatorname{diam}\Omega)$, and Theorem 2.1 follows. In the case (c), the argument is the same. The only difference is that (2.19) is replaced by

$$\log \frac{h_{x_0}(4r)}{h_{x_0}(r)} \leq C\,, \quad x_0 \in \overline{\Omega}$$

provided $r \geq \operatorname{diam}\Omega$, where C is a constant depending only on d. In this case we therefore obtain

$$\log \frac{h_{x_0}(2r)}{h_{x_0}(r)} \leq C(1 + \sqrt{\lambda}\operatorname{diam}\Omega)\,, \quad x_0 \in \overline{\Omega}$$

for $r \in (0, \operatorname{diam}\Omega)$, where C is a constant which depends only on dimension d.

The proof in the case (b) involves a standard argument employing overlapping chain of balls (cf. [11, 13]). Below, the symbol C denotes a generic constant depending only on Ω. First, we choose $r > 0$ and $x_1, \ldots, x_m \in \overline{\Omega}$ such that

(1) $B(x_1, r/2), \ldots, B(x_m, r/2)$ cover $\overline{\Omega}$;

(2) for every $j \in \{1, \ldots, m\}$, the region $\Omega \cap B(x_j, 10r)$ is starshaped with respect to x_j;

(3) if $B(x_j, 10r)$ intersects $\partial\Omega$, it is assumed that the variation of the normal ν is sufficiently small (cf. [14, p. 444]).

We fix $\alpha = \sqrt{\lambda} + 1$. There exists $j_0 \in \{1, \ldots, m_0\}$ such that

$$\int_{B_{r/2}(x_{j_0})} |u|^2 \geq \frac{1}{m}\int_\Omega |u|^2$$

whence

$$\int_\Omega |u|^2 \leq C^\alpha H_{x_{j_0}}(r)\,.$$

For every $j \in \{1, \ldots, m\}$, there exists an overlapping chain of (distinct) balls from (1) connecting $B_r(x_j)$ and $B_r(x_{j_0})$. Repeated use of Lemma 2.4 then gives

$$\int_\Omega |u|^2 \leq C^{\sqrt{\lambda}+1} H_{x_j}(r)\,, \quad j = 1, \ldots, m\,.$$

Therefore,

$$H_{x_j}(2r) \le C^{\sqrt{\lambda}+1} H_{x_j}(r)\,, \quad j = 1, \dots, m\,.$$

An argument parallel to [14, p. 445] then leads to

$$H_x(2\rho) \le C^{\sqrt{\lambda}+1} H_x(\rho)$$

for every $x \in \overline{\Omega}$ and arbitrary $\rho \in (0, r/2)$. Using (2.14) and (2.15), we get the theorem. $\square$

3 The degree of Ginzburg-Landau vortices

Now, we apply Theorem 2.1 to the Ginzburg-Landau equation

$$\begin{aligned} \Delta u &= -\tfrac{1}{\epsilon^2}\big(1 - |u|^2\big)u \\ u|_{\partial\Omega} &= 0\,, \end{aligned} \tag{3.1}$$

where $u\colon \overline{\Omega} \to \mathcal{C}$ is assumed to be nontrivial. The domain $\Omega \subseteq \mathbb{R}^2$ is as in the beginning of Section 2 and $\epsilon > 0$.

Theorem 3.1 *The order of vanishing of u at $x_0 \in \overline{\Omega}$ is less than*

$$C\left(\frac{1}{\epsilon} + 1\right) \tag{3.2}$$

where C is a constant which depends only on Ω.

As it was pointed out in the remark following Theorem 2.1, the above bound (3.2) can be replaced by C/ϵ if Ω satisfies (a) or (b).

By [7], (3.2) then provides an estimate for the the degree of u at any vortex $x_0 \in \Omega$. (Recall that x_0 is a vortex if $u(x_0) = 0$ and the degree of u at x_0 is nonzero.) Namely, by [7, Lemma 6] and our Theorem 3.1, the degree at every vortex is less than (3.2).

Proof By the maximum principle, we conclude $\big|u(x)\big| \le 1$ for $x \in \overline{\Omega}$, i.e., $M = 1$. Taking

$$F(x) = -\frac{1}{\epsilon^2}x + \frac{1}{2\epsilon^2}x^2$$

we easily verify that (i)–(iii) are satisfied with $\lambda = \epsilon^{-2}$. Theorem 3.1 then follows from Theorem 2.1. $\square$

Next, we present a result concerning the nonhomogeneous boundary conditions $u|_{\partial\Omega} = g$ where $g\colon \partial\Omega \to S^1$ is sufficiently regular, e.g. continuous. We assume that Ω is starshaped. In this case, Bethuel, Brézis, and Hélein proved in [5, Lemma X.1] that

$$\int_\Omega \big(1 - |u|^2\big)^2 \le C_0 \epsilon^2 \tag{3.3}$$

where C_0 depends only on g and Ω.

Theorem 3.2 *The order of vanishing of u at $x_0 \in \Omega$ is less than C/ϵ where C depends on Ω, the boundary function g, and the distance from x_0 to $\partial\Omega$.*

Proof It is easy to check that if ϵ is sufficiently large, then u does not vanish. (For instance, we may use the inequality $\left|\nabla u(x)\right| \le C/\epsilon$ from [5] where C depends on g and Ω.) Let $x_0 \in \Omega$, denote $R = \operatorname{dist}(x_0, \partial\Omega)$ and $r_0 = R/4$. We distinguish two cases.

Case 1: $\epsilon \ge R^2/(C \cdot C_0)$ where C is a large enough numerical constant and C_0 is as in (3.3). In this case, we can use analyticity arguments to show that the order of vanishing is bounded by a constant depending only on Ω, g, and R (cf. [12]).

Case 2: $\epsilon \le R^2/(C \cdot C_0)$ where C is large enough. Then (3.3) implies

$$\int_{B_{R/4}(x_0)} |u|^2 \ge \frac{R^2}{C}$$

as can be readily checked. Since also $\max_{\overline{\Omega}} |u| = 1$, we get

$$\int_{B_R(x_0)} |u|^2 \le C \int_{B_{R/4}(x_0)} |u|^2$$

where C depends on Ω, g, and R. Since $R = \operatorname{dist}(x_0, \partial\Omega)$, we have $B_R(x_0) \cap \partial\Omega = \emptyset$. Therefore, by Lemma 2.4, we get

$$\log \frac{\int_{B_{2r}(x_0)} |u|^2}{\int_{B_r(x_0)} |u|^2} \le C + C\left(\alpha + \frac{2R^2\epsilon^{-2}}{\alpha + 1}\right)$$

for all $\alpha \ge 0$ provided $r < R/3$. Choosing $\alpha = R/\epsilon$, we get

$$\log \frac{\int_{B_{2r}(x_0)} |u|^2}{\int_{B_r(x_0)} |u|^2} \le C\left(\frac{R}{\epsilon} + 1\right) .$$

Note that $1 \le CR/\epsilon$ due to the fact $\epsilon \le R^2/CC_0$. Therefore,

$$\log \frac{\int_{B_{2r}(x_0)} |u|^2}{\int_{B_r(x_0)} |u|^2} \le C \frac{\operatorname{dist}(x_0, \partial\Omega)}{\epsilon}$$

and the statement follows. □

Remark 3.3 Here we show by means of an example that Theorem 3.1 is sharp. Let $\Omega = B_1(0)$. We shall show that there exists $\epsilon_0 > 0$ with the following property: For every $\epsilon \in (0, \epsilon_0)$, there exists a solution u of (3.1) such that the degree of u at 0 is at least $1/C\epsilon$.

We seek this solution in the form $u(x) = f(r)e^{id\theta}$, where $x = re^{i\theta}$, with a suitable fixed integer d. We find f as a global minimizer of the functional

$$\Phi(f) = \int_0^1 \left(rf'^2 + \frac{d^2}{r} f^2 + \frac{r}{2\epsilon^2}(f^2 - 1)^2\right) dr$$

in the space

$$V = \left\{ f \in H^1_{\mathrm{loc}}(0,1) : \sqrt{r} f', \frac{f}{\sqrt{r}} \in L^2(0,1), f(1) = 0 \right\} .$$

What remains to be shown is that if d is suitably chosen, then the minimizer f is not identically zero. Choose an arbitrary $g \in V$ such that $0 < g(r) < 1$ for $r \in (0,1)$, say $g(r) = r(1-r)$. Then if $\epsilon \in (0, \epsilon_0)$, where ϵ_0 is sufficiently small, and if $d = [1/C\epsilon]$ where C is large enough, then

$$\Phi(g) < \frac{1}{2\epsilon^2} \int_0^1 r \, dr = \Phi(0)$$

and 0 can not be the global minimizer.

Now, $u(x) = f(r)e^{id\theta}$, where f and d are as above, is not identically 0, it satisfies $u \in C(B_1) \cap C^\infty\big(B_1 \backslash \{0\}\big)$ and solves the Ginzburg-Landau equation for $x \neq 0$. But then it can be readily checked that 0 is a removable singularity, and consequently (3.1) holds. It also follows immediately that 0 is an isolated zero. Indeed, in the opposite case, there would be a sequence $r_1, r_2, \ldots$ which converges to 0 such that $u(x) = 0$ if $|x| = r_j$ for $j \in \mathbb{N}$. Therefore, 0 would be a zero of infinite order, which is not possible since $u \not\equiv 0$.

The rest of the paper is concerned with the Ginzburg-Landau equation

$$\Delta u = -\frac{1}{\epsilon^2}\big(1 - |u|^2\big)u \tag{3.4}$$

with homogeneous Neumann boundary conditions

$$\frac{du}{d\nu}\bigg|_{\partial\Omega} = 0 \tag{3.5}$$

where Ω is a connected bounded C^3 domain. The treatment is similar to (but not completely the same as) the Dirichlet case. We need to consider a conformal straightening of the boundary, which, in turn, leads us to consider the following analog of (2.1). Let

$$\Delta u = vF'\big(|u|^2\big)u$$
$$\tfrac{du}{d\nu}\Big|_{\partial'\Omega} = 0 \, ,$$

where $\Omega = B^+_{R_0}(0) = \big\{(x_1, \ldots, x_d) \in B_{R_0} = B_{R_0}(0) : x_d > 0\big\}$ and $\partial'\Omega = \big\{(x_1, \ldots, x_d) \in B_{R_0} : x_d = 0\big\}$. As in Section 2, we denote $M = \max_{\overline{\Omega}} |u|^2$ and we make same assumptions on $F \colon [0, M] \to \mathbb{R}$ as before. We assume that v is a nonnegative function such that $\max_{x \in \Omega} v(x) \leq M_0$ and $\max_{x\in\Omega} \big|\nabla v(x)\big| \leq M_1$.

Lemma 3.4 *Let $\alpha > 0$, $0 < r_1 < 4r_2/3$, and $4r_2 \leq R_0$. Then*

$$\log \frac{h(2r_1)}{h(r_1)} \leq \log \frac{h(4r_2)}{h(r_2)} + C\left(\alpha + \frac{dr_2^2 \lambda}{\alpha + 1}\right)$$

where C depends only on M_0 and M_1, and where $h(r) = \int_{B_r^+} \big|u(x)\big|^2 \, dx$.

Proof The proof is similar to that of Lemma 2.3; we only indicate the main steps and point out the main differences. As before, we let

$$H(r) = \int_{B_r^+} |u|^2 \big(r^2 - |x|^2\big)^\alpha \, dx .$$

Then (2.6) holds with (2.7) (where $B_r \cap \Omega = B_r^+$). After a short computation, we get

$$\begin{aligned}
I'(r) \quad &= \quad \frac{2\alpha+d}{r} I(r) - \frac{2\alpha+d}{r} \int_{B_r^+} |u|^2 v F'\big(|u|^2\big)\big(r^2 - |x|^2\big)^{\alpha+1} \, dx \\
&\quad + \frac{d}{r} \int_{B_r^+} vF\big(|u|^2\big)\big(r^2 - |x|^2\big)^{\alpha+1} \, dx \\
&\quad - \frac{2(\alpha+1)}{r} \int_{B_r^+} |x|^2 vF\big(|u|^2\big)\big(r^2 - |x|^2\big)^{\alpha} \, dx \\
&\quad + \frac{1}{r} \int_{B_r^+} x_m \partial_m v \, F\big(|u|^2\big)\big(r^2 - |x|^2\big)^{\alpha+1} \, dx \\
&\quad + \frac{4(\alpha+1)}{r} \int_{B_r^+} x_j \partial_j u_k \, x_m \partial_m u_k \, \big(r^2 - |x|^2\big)^{\alpha} \, dx \\
&\quad + 2(\alpha+1) r \int_{B_r^+} |u|^2 v F'\big(|u|^2\big)\big(r^2 - |x|^2\big)^{\alpha} \, dx
\end{aligned}$$

in place of (2.9). From here, we get

$$\begin{aligned}
I'(r) \quad &\geq \quad \frac{2\alpha+d}{r} I(r) - 4d\lambda r \max_\Omega v \, H(r) \\
&\quad + \frac{4(\alpha+1)}{r} \int_{B_r^+} x_j \partial_j u_k \, x_m \partial_m u_k \, \big(r^2 - |x|^2\big)^{\alpha} \, dx \\
&\quad - \lambda(\max_\Omega |\nabla v|) \int_{B_r^+} |u|^2 \big(r^2 - |x|^2\big)^{\alpha+1} \, dx
\end{aligned}$$

where we also used nonnegativity of v. Instead of $N'(r) \geq -4d\lambda r$, which we had before, we now conclude

$$N'(r) \geq -rd\lambda r \max_\Omega v - \lambda r^2 \max_\Omega |\nabla v| .$$

The rest follows as before. □

Now, we return to the Ginzburg-Landau equation (3.4) with the Neumann boundary conditions (3.5), where Ω is a C^3 bounded connected domain in $\mathbb{R}^2$.

Theorem 3.5 *The order of vanishing of u at $x_0 \in \overline{\Omega}$ is less than (3.2) where C is a constant which depends only on Ω.*

Proof (sketch) The proof of the theorem is analogous to the proof os Theorem 3.1. The main difference is that we use a conformal map to straighten the boundary. Namely, let $x_0 \in \partial\Omega$. Then, by the Riemann mapping theorem, there exists $R_0 > 0$ and $r_0 > 0$ and a conformal map

$$f\colon B_{r_0}(x_0) \cap \Omega \to B^+_{R_0}(0)$$

such that $f(x_0) = 0$. The equation (3.4) then transfers to

$$\Delta u = -\frac{1}{\epsilon^2} v\big(1 - |u|^2\big)u$$

with $v = 1/|f'|^2$. The boundary of Ω being C^3 guarantees that v and ∇v are bounded up to the lower boundary $\partial' B^+_{R_0}$ [6]. The rest is then established as in the proof of Theorem 3.1, except that we use Lemmas 3.4 and 2.4 instead of Lemma 2.3. $\square$

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Igor Kukavica
Department of Mathematics
University of Southern California
Los Angeles, CA 90089
e-mail: kukavica@math.usc.edu