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CONTROLLABILITY AND PERIODIC SOLUTIONS OF NONLINEAR WAVE EQUATIONS

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Communicated by Jesus Ildefonso Diaz
Dedicated to the memory of Felix E. Browder whose guidance is gratefully acknowledged

Abstract. The controllability of time-periodic solutions of a n-dimensional nonlinear wave equation is established with $n = 2, 3$. The result is used to establish the existence of time-periodic solutions of a nonlinear wave equation.

1. Introduction

The purpose of the article is to establish the existence of time-periodic solutions of a nonlinear wave equation in bounded domains of $\mathbb{R}^n$ with $n = 2, 3$, using controllability. Following the pioneering work of Rabinowitz [8, 9] on time-periodic solutions of the one-dimensional nonlinear wave equation, extensive studies of the problem were done by Berti-Bolle [1, 2], Brezis-Nirenberg [3] and others. Controllability and fictitious domains were used by Glowinski and his collaborators [5], Glowinski-Rossi [6] to treat numerically the existence of time-periodic solutions of the linear wave equation in cylindrical domains. For higher spatial dimensions, Berti and Polle [3] used The Nash-Moser iteration to study T-periodic solutions of the problem

$$\begin{gathered} u'' - \Delta u + mu = \varepsilon F(\omega t, x, u) \\ u(t, x) = u(t, x + 2k\pi) \quad \forall k \in Z^n \end{gathered}$$

where F is $2\pi/\omega$ periodic in time and 2π-periodic in x_j, $j = 1, \dots, n$.

In [10, 11] the author established the existence of time-periodic solutions of a nonlinear wave equation in non-cylindrical domains of $\mathbb{R}^n$, $n = 2, 3$ with the forcing term in a non-empty subset of $K^\perp$ with

$$K = \{v : v \in L^2(0, T; L^2(G)), \int_0^T v(\cdot, t)dt = 0\}$$

In this paper we shall show that for any f in $K^\perp$ there exists a time-periodic solution of a nonlinear wave equation in cylindrical domains. The proof is carried out in Section 5.Notations and the basic assumption of the paper are given in Section 2.

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Given f in $K^\perp$ and u_0 in $H_0^1(G)\cap L^p(G)$ we shall establish the existence of a control $g_f(u_0)$ in $(H_0^1(G)\cap L^p(G))^*$ and a time-periodic solution of the nonlinear wave equation

$$\begin{gathered}
u''-\Delta u+|u|^{p-2}u=f-g_f(u_0)\quad\text{in } G\times(0,T),\\
u=0 \text{ on } \partial G\times(0,T),\quad \{u,u'\}\big|_{t=0}=\{u,u'\}\big|_{t=T}=\{u_0,0\}
\end{gathered}$$

The solution and its derivative take prescribed values at $t=0$ and at $t=T$.

In Section 4 we consider a semi-exact controllability problem. Given f in $K^\perp$ and u_0 in $H_0^1(G)\cap L^p(G)$, we shall prove the existence of (i) a control $g_f(u_0)$ and (ii) a time-periodic solution of the problem

$$\begin{gathered}
u''-\Delta u+|u|^{p-2}u=f-g_f(u_0)\quad\text{in } G\times(0,T),\\
u=0 \text{ on } \partial G\times(0,T),\quad u(0)=u_0=u(T),\quad u'(0)=u'(T).
\end{gathered}$$

As the solution u takes a prescribed common value at $t=0$ and at $t=T$, its derivative u' is not required to take a specific value at the two end points, we shall call it a semi-exact controllability problem.

Notation. Let G be a bounded open subset of $\mathbb{R}^n$ with $n=2,3$, and let

$$K=\{v: v\in L^2(0,T;L^2(G)),\ \int_0^T v(\,.,s)ds=0\}.$$

The set K is a closed convex subset of $L^2(0,T;L^2(G))$and let J, be the duality mapping of $L^2(0,T;L^2(G))$ into $L^2(0,T;L^2(G))$ with gauge function $\Phi(r)=r$. The penalty function

$$\beta(v)=J(v-P_Kv)$$

where P_K is the projection of K onto $L^2(0,T;L^2(G))$, is well-defined. For a given u in $L^2(0,T;L^2(G))$ there exists a unique P_Ku in K such that

$$\|u-P_Ku\|_{L^2(0,T;L^2(G))}\le\|u-k\|_{L^2(0,T;L^2(G))}\quad\forall k\in K.$$

In this article, we denote by $(\cdot,\cdot)$ the various pairings between $L^2(G), L^p(G)$ and their duals.

Assumption. We assume that $2\le p<\infty$ if $G\subset R^2$ and $2\le p\le 4$ if $G\subset R^3$.

2. Exact controllability time periodic problem

The main result of the section is the following theorem

Theorem 2.1. *Let $\{f,u_0\}$ be in $K^\perp\times\{H_0^1(G)\cap L^p(G)\}$ then there exist:*

(i) *$g_f(u_0)$ in $[H_0^1(G)\cap L^p(G)]^*$*
(ii) *$\{u,u'\}$ in $L^\infty(0,T;H_0^1(G)\cap L^p(G))\times L^\infty(0,T;L^2(G))$, solution of the problem*

$$\begin{gathered}
u''-\Delta u+|u|^{p-2}u=f-g_f(u_0)\quad\text{in } G\times(0,T)\\
u=0 \text{ on } \partial G\times(0,T),\quad \{u,u'\}\big|_{t=0}=\{u,u'\}\big|_{t=T}=\{u_0,0\}
\end{gathered}\tag{2.1}$$

We consider the initial boundary-value problem

$$\begin{gathered}
u_\varepsilon''-\varepsilon\Delta u_\varepsilon'-\Delta u_\varepsilon+|u_\varepsilon|^{p-2}u_\varepsilon+\varepsilon^{-1}\beta(u_\varepsilon')=f\quad\text{in } G\times(0,T),\\
u_\varepsilon=u_\varepsilon'=0 \text{ on } \partial G\times(0,T),\quad \{u_\varepsilon,u_\varepsilon'\}\big|_{t=0}=\{u_0,u_1\}\,.
\end{gathered}\tag{2.2}$$

Lemma 2.2. *Let $\{f, u_0, u_1\}$ be in $K^\perp \times [H_0^1(G) \cap L^p(G)] \times L^2(G)$ then there exists a unique solution u_ε of (2.2). Moreover*

$$
\begin{aligned}
&\|u'_\varepsilon(t)\|^2_{L^2(G)} + 2\varepsilon\|\nabla u'_\varepsilon\|^2_{L^2(0,t;L^2(G))} + \|\nabla u_\varepsilon(t)\|^2_{L^2(G)} \\
&+ 2p^{-1}\|u_\varepsilon(t)\|^p_{L^p(G)} + 2\varepsilon^{-1}\int_0^t (\beta(u'_\varepsilon), u'_\varepsilon)ds \\
&\le \|u_1\|^2_{L^2(G)} + \|\nabla u_0\|^2_{L^2(G)} + 2p^{-1}\|u_0\|^p_{L^p(G)} + 2\int_0^t (f, u'_\varepsilon)ds
\end{aligned}
$$

The standard Galerkin approximation method gives the existence of a unique solution of (2.2) with the stated estimate. We shall not reproduce the proof.

Lemma 2.3. *Let u_ε be as in Lemma 2.2 then there exists a subsequence such that*

$$\{u_\varepsilon, u'_\varepsilon, \beta(u'_\varepsilon)\} \to \{u, u', 0\}$$

in the space

$$
\begin{aligned}
&\Big\{C(0,T;L^2(G)) \cap [L^\infty(0,T;H_0^1(G)\cap L^p(G))]_{weak^*}\Big\} \\
&\times [L^\infty(0,T;L^2(G))]_{weak^*} \times [L^2(0,T;L^2(G))]_{\rm weak}.
\end{aligned}
$$

Furthermore $\beta(u') = 0$, i.e. u' in K and thus, $u(\cdot,0) = u(\cdot,T) = u_0$.

Proof. (1) From the estimate of Lemma 2.2 and the Gronwalls lemma, there exists a subsequence such that $\{u_\varepsilon, u'_\varepsilon\} \to \{u, u'\}$ in

$$C(0,T;L^2(G)) \cap [L^\infty(0,T;H_0^1(G)\cap L^p(G))]_{weak^*} \times [L^\infty(0,T;L^2(G))]_{weak^*}$$

We have

$$
\begin{aligned}
\|\beta(u'_\varepsilon)\|_{L^2(0,T;L^2(G))} &= \|J(u'_\varepsilon - P_K u'_\varepsilon)\|_{L^2(0,T;L^2(G))} \\
&= \Phi(\|u'_\varepsilon - P_K u'_\varepsilon\|_{L^2(0,T;L^2(G))}) \\
&= \|u'_\varepsilon - P_K u'_\varepsilon\|_{L^2(0,T;L^2(G))} \\
&\le \|u'_\varepsilon\|_{L^2(0,T;L^2(G))} + \|P_K u'_\varepsilon - P_K 0\|_{L^2(0,T;L^2(G))} \\
&\le 2\|u'_\varepsilon\|_{L^2(0,T;L^2(G))} \le M
\end{aligned}
$$

Thus,

$$\beta(u'_\varepsilon) \to \chi \quad \text{in } (L^2(0,T;L^2(G)))_{\rm weak}.$$

(2) We now show that $\chi = 0$. From (2.2) we have

$$
\begin{aligned}
&-\varepsilon\int_0^T (u'_\varepsilon, \varphi')dt + \varepsilon^2\int_0^T (\nabla u'_\varepsilon, \nabla\varphi)dt + \varepsilon\int_0^T (\nabla u_\varepsilon, \nabla\varphi)dt \\
&+ \varepsilon\int_0^T (|u_\varepsilon|^{p-2}u_\varepsilon, \varphi)dt + \int_0^T (\beta(u'_\varepsilon), \varphi)dt \\
&= \varepsilon\int_0^T (f,\varphi)dt \quad \forall \varphi \in C_0^\infty(0,T;H_0^1(G)\cap L^p(G))
\end{aligned}
$$

Thus,

$$\int_0^T (\beta(u'_\varepsilon), \varphi)dt \to 0 \quad \forall \varphi \in C_0^\infty(0,T;H_0^1(G)\cap L^p(G))$$

Since $\beta(u'_\varepsilon) \to \chi$ in $[L^2(0,T;L^2(G)]_{\rm weak}$, we deduce that $\chi = 0$.

(3) We now show that $\beta(u') = 0$. Since β is monotone in $L^2(0,T;L^2(G))$ we have

$$\int_0^T (\beta(u'_\varepsilon) - \beta(v'), u'_\varepsilon - v')dt \geq 0 \;\forall v' \in L^2(0,T;L^2(G)),$$

in particular for all v with

$$v = \int_0^t \varphi(.,s)ds, \;\; \varphi \in L^2(0,T;L^2(G)).$$

Thus,

$$\int_0^T (\beta(u'_\varepsilon) - \beta(\varphi), u'_\varepsilon - \varphi)dt \geq 0 \quad \forall \varphi \in L^2(0,T;L^2(G)).$$

From the estimate of Lemma 2.2 and from the above we have

$$\lim_{\varepsilon\to 0} \int_0^T (\beta(u'_\varepsilon), u'_\varepsilon)dt = 0 = \lim_{\varepsilon} \int_0^T (\beta(u'_\varepsilon), \varphi)dt.$$

Hence

$$-\int_0^T (\beta(\varphi), u' - \varphi)dt \geq 0 \quad \forall \varphi \in L^2(0,T;L^2(G)).$$

Take $\varphi = u' + \lambda w$, $\lambda > 0$ and w in $L^2(0,T;L^2(G))$. We have

$$\int_0^T (\beta(u' + \lambda w), \, w)dt \geq 0 \quad \forall w \in L^2(0,T;L^2(G)).$$

Letting $\lambda \to 0$ we obtain

$$\int_0^T (\beta(u'), w)dt \geq 0 \quad \forall w \in L^2(0,T;L^2(G)).$$

Changing w to $-w$ and we deduce that $\beta(u') = 0$ i.e. $u' \in K$ and $u(\cdot,0) = u(\cdot,T) = u_0$. $\square$

Lemma 2.4. *Let $\{u_\varepsilon, u\}$, be as in Lemmas 2.2 and 2.3. There exists $g_f(u_0,u_1)$ in $[H_0^1(G) \cap L^p(G)]^*$ and associated with $g_f(u_0,u_1)$, a unique solution u, of the problem*

$$\begin{gathered} u'' - \Delta u + |u|^{p-2}u = f - g_f(u_0,u_1) \quad \text{in } G \times (0,T), \\ u = 0 \text{ on } \partial G \times (0,T), \quad \{u, u'\}\big|_{t=0} = \{u_0, u_1\} = \{u(\cdot,T), u_1\} \end{gathered} \tag{2.3}$$

with

$$\int_0^T (g_f(u_0,u_1), \varphi)dt = \lim_{\varepsilon\to 0} \varepsilon^{-1} \int_0^T (\beta(u'_\varepsilon), \varphi)dt$$

for all $\varphi \in C_0^\infty(0,T;H_0^1(G) \cap L^p(G))$. Furthermore,

$$\begin{aligned} &\liminf \|u'_\varepsilon(t)\|^2_{L^2(G)} + \|\nabla u(t)\|^2_{L^2(G)} + 2p^{-1}\|u(t)\|^p_{L^p(G)} \\ &\leq \|u_1\|^2_{L^2(G)} + \|\nabla u_0\|^2_{L^2(G)} + 2p^{-1}\|u_0\|^p_{L^p(G)} + 2\int_0^t (f,u')ds. \end{aligned}$$

Proof. (1) Since $u_\varepsilon \to u$ in $C(0,T;L^2(G)) \cap (L^\infty(0,T;L^p(G)))_{weak^*}$, a standard argument gives

$$|u_\varepsilon|^{p-2}u_\varepsilon \to |u|^{p-2}u \quad \text{in } [L^\infty(0,T;L^q(G))]_{weak^*}.$$

(2) Let φ be in $C_0^\infty(0,T;H_0^1(G)\cap L^p(G))$ then φ' is in K and we have

$$\int_0^T (\beta(u_\varepsilon') - \beta(\varphi'), u_\varepsilon' - \varphi')dt = \int_0^T (\beta(u_\varepsilon'), u_\varepsilon' - \varphi')dt \geq 0.$$

It follows from (2.2) that

$$\begin{aligned}
&\int_0^T (u_\varepsilon'', u_\varepsilon' - \varphi')dt + \int_0^T (\nabla(\varepsilon u_\varepsilon' + u_\varepsilon), \nabla(u_\varepsilon' - \varphi'))dt \\
&+ \int_0^T (|u_\varepsilon|^{p-2}u_\varepsilon, u_\varepsilon' - \varphi')dt + \varepsilon^{-1}\int_0^T (\beta(u_\varepsilon'), u_\varepsilon' - \varphi')dt \\
&= \int_0^T (f, u_\varepsilon' - \varphi')dt
\end{aligned} \tag{2.4}$$

Hence

$$\begin{aligned}
&\|u_\varepsilon'(T)\|^2_{L^2(G)} + 2\varepsilon\|\nabla u_\varepsilon'\|^2_{L^2(0,T:L^2(G))} + \|\nabla u_\varepsilon(T)\|^2_{L^2(G)} + 2p^{-1}\|u_\varepsilon(T)\|^p_{L^p(G)} \\
&- 2\int_0^T (f, u_\varepsilon')dt - \Big\{\|u_1\|^2_{L^2(G)} + \|\nabla u_0\|^2_{L^2(G)} + 2p^{-1}\|u_0\|^p_{L^p(G)}\Big\} \\
&\leq 2\int_0^T (u_\varepsilon'', \varphi')dt + 2\int_0^T (\nabla(\varepsilon u_\varepsilon' + u_\varepsilon), \nabla\varphi')dt + 2\int_0^T (|u_\varepsilon|^{p-2}u_\varepsilon - f, \varphi')dt
\end{aligned}$$

Letting $\varepsilon \to 0$, we obtain

$$\begin{aligned}
&\liminf \|u_\varepsilon'(T)\|^2_{L^2(G)} + \|\nabla u(T)\|^2_{L^2(G)} + 2p^{-1}\|u(T)\|^p_{L^p(G)} \\
&- \{\|u_1\|^2_{L^2(G)} + \|\nabla u_0\|^2_{L^2(G)} + 2p^{-1}\|u_0\|^p_{L^p(G)}\} \\
&\leq 2\int_0^T < u'' - \Delta u + |u|^{p-2}u - f, \varphi' > dt
\end{aligned}$$

for all $\varphi \in C_0^\infty(0,T;H_0^1(G)\cap L^p(G))$. We have used the fact that $f \in K^\perp$ and that u' is in K. Set

$$\Phi(u,\varphi') = 2\int_0^T < u'' - \Delta u + |u|^{p-2}u - f, \varphi' > dt$$

and

$$\begin{aligned}
E(u) = &\liminf \|u_\varepsilon'(T)\|^2_{L^2(G)} + \|\nabla u(T)\|^2_{L^2(G)} + 2p^{-1}\|u(T)\|^p_{L^p(G)} - \|u_1\|^2_{L^2(G)} \\
&- \|\nabla u_0\|^2_{L^2(G)} - 2p^{-1}\|u_0\|^p_{L^p(G)}
\end{aligned}$$

Then

$$E(u) \leq \Phi(u,\varphi') \quad \forall \varphi \in C_0^\infty(0,T;H_0^1(G)\cap L^p(G)).$$

In particular

$$E(u) \leq \Phi(u,-\varphi') \quad \forall \varphi \in C_0^\infty(0,T;H_0^1(G)\cap L^p(G))$$

Hence

$$E(u) \leq \Phi(u,\varphi') \leq -E(u) \quad \forall \varphi \in C_0^\infty(0,T;H_0^1(G)\cap L^p(G))$$

Let $\lambda > 0$ then $\lambda^{-1}\varphi$ is in $C_0^\infty(0,T;H_0^1(G)\cap L^p(G))$ and we have

$$\lambda E(u) \leq \Phi(u,\varphi') \leq -\lambda E(u)$$

Letting $\lambda \to 0$ we obtain

$$\Phi(u,\,\varphi') = \int_0^T \langle u'' - \Delta u + |u|^{p-2}u - f, \varphi'\rangle dt = 0$$

for all $\varphi \in C_0^\infty(0,T; H_0^1(G)\cap L^p(G))$. Therefore

$$\{u'' - \Delta u + |u|^{p-2}u - f\}' = 0 \quad \text{in } \mathcal{D}'(0,T; [H_0^1(G)\cap L^p(G)]^*).$$

It follows that

$$u'' - \Delta u + |u|^{p-2} - f = g_f(u_0,u_1) \quad \text{in } \mathcal{D}'(0,T; [H_0^1(G)\cap L^p(G)]^*) \tag{2.5}$$

for any $g_f(u_0,u_1)$ in $[H_0^1(G)\cap L^p(G)]^*$.

(3) We now show that $g_f(u_0,u_1)$ is uniquely defined. From (2.3) we have

$$\begin{aligned}
&-\int_0^T (u_\varepsilon', \varphi')dt + \int_0^T (\nabla(\varepsilon u_\varepsilon' + u_\varepsilon), \nabla\varphi)dt + \int_0^T (|u_\varepsilon|^{p-2}u_\varepsilon, \varphi)dt \\
&+ \varepsilon^{-1}\int_0^T (\beta(u_\varepsilon'), \varphi)dt - \int_0^T (f,\varphi)dt = 0
\end{aligned}$$

for all $\varphi \in C_0^\infty(0,T; H_0^1(G)\cap L^p(G))$.

Letting $\varepsilon \to 0$ we obtain

$$\begin{aligned}
&-\int_0^T (u', \varphi')dt + \int_0^T (\nabla u, \nabla\varphi)dt \\
&+ \int_0^T (|u|^{p-2}, \varphi)dt + \lim_{\varepsilon\to 0}\varepsilon^{-1}\int_0^T (\beta(u_\varepsilon'), \varphi)dt \\
&= \int_0^T (f, \varphi)dt
\end{aligned}$$

for all $\varphi \in C_0^\infty(0,T; H_0^1(G)\cap L^p(G))$. Thus,

$$u'' - \Delta u + |u|^{p-2}u + \lim_{\varepsilon\to 0}\varepsilon^{-1}\beta(u_\varepsilon') = f \quad \text{in } \mathcal{D}'(0,T; [H_0^1(G)\cap L^p(G)]^*)$$

Comparing with (2.4) and we have

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}\beta(u_\varepsilon') = g_f(u_0,u_1) \quad \text{in } \mathcal{D}'(0,T; [H_0^1(G)\cap L^p(G)]^*)$$

It is clear that if h is any other element of $(H_0^1(G)\cap L^p(G))^*$ in (2.5) then

$$h = g_f(u_0,u_1) = \lim_{\varepsilon\to 0}\varepsilon^{-1}\beta(u_\varepsilon') \quad \text{in } \mathcal{D}'(0,T; [H_0^1\cap L^p(G)]^*)$$

(4) Suppose that v is a solution of the problem

$$\begin{gathered}
v'' - \Delta v + |v|^{p-2}v + g_f(u_0,u_1) = f \quad \text{in} G\times(0,T), \\
v = 0 \text{ on } \partial G\times(0,T), \quad v(\cdot,0) = u_0, \quad v'(\cdot,0) = u_1
\end{gathered}$$

Then an argument as in Lions [11, p.14-15], shows that $u = v$ and completes the proof. $\square$

Lemma 2.5. *Let $g_f(u_0,u_1)$ be as in Lemma 2.4 then*

$$\begin{aligned}
&\|g_f(u_0,u_1)\|_{[H_0^1(G)\cap L^p(G)]^*} \\
&\le C\{1 + \|u_0\|_{H_0^1(G)}^{p-1} + \|u_1\|_{L^2(G)}^{p-1} + \|u_0\|_{L^p(G)}^{p-1} + \|f\|_{L^2(0,T;L^2(G))}\}
\end{aligned}$$

Proof. Let h be in $H_0^1(G)\cap L^p(G)$ and let ζ be in $C_0^\infty(0,T)$ with $\zeta \ge 0$. From Lemma 2.4 we have

$$\int_0^T \zeta(g_f(u_0,u_1),h) = \int_0^T (f,\zeta h)dt + \int_0^T (u',\zeta' h) - \int_0^T (\nabla u, \zeta\nabla h)dt$$

$$
- \int_0^T (|u|^{p-2}u, \zeta h)dt
$$

Hence

$$
\begin{aligned}
\alpha|(g_f(u_0,u_1),h)| \le C\Big\{ &\|f\|_{L^2(0,T;L^2(G))} + \|u'\|_{L^2(0,T;L^2(G))} + \|\nabla u\|_{L^2(0,T;L^2(G))} \\
&+ \|u\|^{p-1}_{L^\infty(0,T;L^p(G))} \Big\} \|h\|_{H_0^1(G)}
\end{aligned}
$$

for all h in $H_0^1(G) \cap L^p(G)$ and where

$$
\alpha = \int_0^T \zeta dt > 0.
$$

Since $2 \le p$, it follows from the estimate of Lemma 2.4 that

$$
\begin{aligned}
&\|g_f(u_0,u_1)\|_{[H_0^1(G)\cap L^p(G)]^*} \\
&\le C\Big\{ 1 + \|u_0\|_{H_0^1(G)} + \|u_1\|_{L^2(G)} + \|u_0\|^{p-1}_{L^p(G)} + \|f\|_{L^2(0,T;L^2(G))} \Big\}
\end{aligned}
$$

The proof is complete. □

Lemma 2.6. *Let u_ε'' be as in Lemma 2.2. Then*

$$
\|u_\varepsilon''\|_{L^2(0,T;[H_0^1(G)\cap L^p(G)]^*)} \le C
$$

where C is independent of ε. Moreover

$$
\begin{gathered}
u_\varepsilon' \to u' \quad \text{in } C(0,T;[H_0^1(G)\cap L^p(G)]^*) \cap [L^\infty(0,T;L^2(G))]_{\text{weak}*}, \\
\|u'(T)\|_{L^2(G)} \le \liminf \|u_\varepsilon'(T)\|_{L^2(G)}
\end{gathered}
$$

Proof. Let φ be in $C_0^\infty(0,T;H_0^1(G)\cap L^p(G))$ and set

$$
\gamma_\varepsilon(\varphi) = \int_0^T (u_\varepsilon'',\varphi)dt.
$$

- Case 1: $\gamma_\varepsilon(\varphi) \ge 0$. We have

$$
\begin{aligned}
&\lim \Big|\int_0^T (u_\varepsilon'',\varphi)dt\Big| \\
&= \lim \int_0^T (u_\varepsilon'',\varphi)dt \\
&= -\int_0^T (\nabla u,\nabla\varphi)dt - \int_0^T (|u|^{p-2}u,\varphi)dt - \lim \varepsilon^{-1}\int_0^T (\beta(u_\varepsilon'),\varphi)dt + \int_0^T (f,\varphi)dt \\
&= -\int_0^T (\nabla u,\nabla\varphi)dt - \int_0^T (|u|^{p-2}u,\varphi)dt - \int_0^T (g_f(u_0,u_1),\varphi)dt + \int_0^T (f,\varphi)dt \\
&\le C\{\|u\|_{L^2(0,T;H_0^1(G))} + \|u\|^{p-1}_{L^\infty(0,T;L^p(G))} + \|f\|_{L^2(0,T;L^2(G))}\} \\
&\quad \times \|\varphi\|_{L^2(0,T;H_0^1(G)\cap L^p(G))}
\end{aligned}
$$

- Case 2: $\gamma_\varepsilon(\varphi) \le 0$. Then we have

$$
\begin{aligned}
&\lim \Big|\int_0^T (u_\varepsilon'',\varphi)dt\Big| \\
&= \lim -\int_0^T (u_\varepsilon'',\varphi)dt
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^T (\nabla u, \nabla\varphi)dt + \int_0^T (|u|^{p-2}u, \varphi)dt + \int_0^T (g_f(u_0,u_1),\varphi)dt - \int_0^T (f,\varphi)dt \\
&\le C\{\|u\|_{L^2(0,T;H_0^1(G))} + \|u\|^{p-1}_{L^\infty(0,T;L^p(G))} + \|f\|_{L^2(0,T;L^2(G))}\} \\
&\quad \times \|\varphi\|_{L^2(0,T;H_0^1(G)\cap L^p(G))}
\end{aligned}
$$

Hence

$$
\lim\Big|\int_0^T (u_\varepsilon'',\varphi)dt\Big| \le M\|\varphi\|_{L^2(0,T;H_0^1(G)\cap L^p(G))} \quad \forall\varphi\in C_0^\infty(0,T;H_0^1(G)\cap L^p(G)).
$$

Since $C_0^\infty(0,T;H_0^1(G)\cap L^p(G))$ is dense in $L^2(0,T;H_0^1(G)\cap L^p(G))$, we have

$$
\|u_\varepsilon''\|_{L^2(0,T;[H_0^1(G)\cap L^p(G)]^*)} \le M
$$

The other assertions of the lemma are trivial to verify. □

Proof of Theorem 2.1. Taking $u_1 = 0$, from Lemma 2.4 there exists $g_f(u_0)$ in $[H_0^1(G)\cap L^p(G)]^*$ and

$$
\{u,u'\}\in L^\infty(0,T;H_0^1(G)\cap L^p(G))\times L^\infty(0,T;L^2(G)),
$$

solution of the problem

$$
\begin{gathered}
u'' - \Delta u + |u|^{p-2}u = f - g_f(u_0) \quad \text{in } G\times(0,T), \\
u = 0 \text{ on } \partial G\times(0,T), \quad u(\cdot,0) = u(\cdot,T) = u_0, \quad u'(\cdot,0) = 0.
\end{gathered}
$$

From the estimate in Lemma 2.4 we obtain

$$
\|u'(T)\|^2_{L^2(G)} \le 0
$$

as f is in $K^\perp$ and u' is in K. Therefore

$$
u'(\cdot,0) = 0 = u'(\cdot,T).
$$

The proof is complete. □

3. Semi exact controllability

In this section we shall establish the existence of time-periodic solutions of a nonlinear wave equation with the solution taking a prescribed value at $t = 0$.

Theorem 3.1. *Let $\{f,u_0\}$ be in $K^\perp\times\{H_0^1(G)\cap L^p(G)\}$. There exists*

(i) *$g_f(u_0)$ in $[H_0^1(G)\cap L^p(G)]^*$*
(ii) *a solution u of the problem*

$$
\begin{gathered}
u'' - \Delta u + |u|^{p-2}u = f - g_f(u_0) \quad \text{in } G\times(0,T), \\
u = 0 \text{ on } \partial G\times(0,T), \quad \{u,u'\}\big|_{t=0} = \{u,u'\}\big|_{t=T} = \{u_0,u'(0)\}
\end{gathered} \tag{3.1}
$$

with $\{u,u'\}$ in $L^\infty(0,T;H_0^1(G)\cap L^p(G))\times L^\infty(0,T;L^2(G))$.

As $u'(\cdot,0)$ and $u'(\cdot,T)$ are not required to take a prescribed value and are allowed to take the same value derived from the equation, we have only half of the exact controllability condition.

A simple corollary of the theorem yields the existence of time-periodic solutions of linear wave equations.

Corollary 3.2. *Let f be in $K^{\perp}$ then there exists $\{\tilde{u}, \tilde{u}'\}$ in $L^{\infty}(0,T;H_0^1(G)) \times L^{\infty}(0,T;L^2(G))$, solution of the problem*

$$
\begin{gathered}
\tilde{u}'' - \Delta\tilde{u} + \tilde{u} = f \quad \text{in } G \times (0,T), \\
\tilde{u} = 0 \ \text{on } \partial G \times (0,T), \quad \{\tilde{u}, \tilde{u}'\}\big|_{t=0} = \{\tilde{u}, \tilde{u}'\}\big|_{t=T}
\end{gathered}
\tag{3.2}
$$

Proof. Given f in $K^{\perp}$ and a u_0 in $H_0^1(G)$ it follows from the theorem that there exists $g_f(u_0)$ in $H^{-1}(G)$ and associated with it a solution u of the problem

$$
\begin{gathered}
u'' - \Delta u + u + g_f(u_0) = f \quad \text{in } G \times (0,T), \\
u = 0 \text{ on } \partial G \times (0,T), \quad \{u, u'\}\big|_{t=0} = \{u, u'\}\big|_{t=T} = \{u_0, u'(0)\}
\end{gathered}
$$

Consider the elliptic boundary problem

$$
-\Delta\hat{u} + \hat{u} = g_f(u_0) \text{ in } G, \quad \hat{u} = 0 \text{ on } \partial G.
$$

There exists a unique solution $\hat{u}$ in $H_0^1(G)$ of the problem. Set $\tilde{u} = u + \hat{u}$ and the corollary is proved ☐

Proof of Theorem 3.1. (1) Let

$$
\{f, u_0, u_1\} \in K^{\perp} \times \{H_0^1(G) \cap L^p(G)\} \times L^2(G)
$$

then there exists $g_f(u_0, u_1)$ in $[H_0^1(G) \cap L^p(G)]^*$ and associated with it, a unique solution u of the problem

$$
\begin{gathered}
u'' - \Delta u + |u|^{p-2}u + g_f(u_0, u_1) = f \quad \text{in } G \times (0,T), \\
u = 0 \text{ on } \partial G \times (0,T), \quad u(\cdot,0) = u_0 = u(\cdot,T), \quad u'(\cdot,0) = u_1
\end{gathered}
\tag{3.3}
$$

Moreover Lemmas 2.5 and 2.6 show that

$$
\|u'(T)\|^2_{L^2(G)} \le \|u_1\|^2_{L^2(G)}
$$

(2) Let $\mathcal{B} = \{v : \|v\|_{L^2(G)} \le 1\}$. Then it is clear that $\mathcal{B}$ is a compact convex subset of $[H_0^1(G) \cap L^p(G)]^*$. Denote by $\mathcal{A}$ the mapping of $\mathcal{B}$ into $\mathcal{B}$ given by

$$
\mathcal{A}(u_1) = u'(T) \tag{3.4}
$$

as $f \in K^{\perp}$ and u' is in K. The mapping is well-defined and takes $\mathcal{B}$ into $\mathcal{B}$.

We now show that $\mathcal{A}$ is a $[H_0^1(G) \cap L^p(G)]^*$-continuous mapping. Let $u_{1,n}$ in $\mathcal{B}$, then corresponding to $\{f, u_0, u_{1,n}\}$, there exists $g_f(u_0, u_{1,n})$ in $[H_0^1(G) \cap L^p(G)]^*$ and u_n, solution of the problem

$$
\begin{gathered}
u_n'' - \Delta u_n + |u_n|^{p-2} + g_f(u_0, u_{1,n}) = f \quad \text{in } G \times (0,T), \\
u_n = 0 \text{ on } \partial G \times (0,T), \quad u_n(0) = u_0 = u_n(T), \quad u_n'(0) = u_{1,n}
\end{gathered}
$$

From Lemmas 2.4–2.6 we get

$$
\|g_f(u_0, u_{1,n})\|_{[H_0^1(G)\cap L^p(G)]^*} + \|u_n\|_{L^{\infty}(0,T;H_0^1(G)\cap L^p(G))} + \|u_n'\|_{L^{\infty}(0,T;L^2(G))} \le C
$$

We have a subsequence such that

$$
\{u_n, u_n', g_f(u_0, u_{1,n})\} \to \{u, u', g_f(u_0, u_1)\}
$$

in

$$
[L^{\infty}(0,T;H_0^1(G) \cap L^p(G))]_{weak^*} \times [L^{\infty}(0,T;L^2(G)]_{weak^*} \times [H_0^1(G) \cap L^p(G)]_{weak^*}
$$

It is clear that $\{u_n,\, u_n'\} \to \{u, u'\}$ in $C(0,T;L^2(G)) \times C(0,T;[H_0^1(G) \cap L^p(G)]^*)$, and therefore

$$
\{u_n(0), u_n'(0), u_n'(T)\} \to \{u(0), u'(0), u'(T)\}
$$

in $L^2(G) \times [H_0^1(G) \cap L^p(G)]^* \times [H_0^1(G) \cap L^p(G)]^*$. Hence $u(0) = u_0 = u(T)$ and $u'(0) = u_1$. A standard argument shows that

$$|u_n|^{p-2}u_n \to |u|^{p-2}u \quad \text{in } [L^q(0,T; L^q(G)]_{\text{weak}}$$

and thus,

$$\begin{gathered} u'' - \Delta u + |u|^{p-2}u + g_f(u_0, u_1) = f \quad \text{in } G \times (0,T), \\ u = 0 \text{ on } \partial G \times (0,T), \quad u(0) = u_0 = u(T), quadu'(0) = u_1 \end{gathered}$$

It follows that $\mathcal{A}(u_1) = u'(T)$.

An application of the Schauder fixed point theorem yields the existence of $\hat{u}_1$ in $\mathcal{B}$ such that $\mathcal{A}(\hat{u}_1) = \hat{u}_1$. With u_0 given and with the fixed point $\hat{u}_1$, there exists as in Lemma 2.4 a control $g_f(u_0, \hat{u}_1) = \hat{g}_f(u_0)$ in $[H_0^1(G) \cap L^p(G)]^*$ and associated with the control, a solution of

$$\begin{gathered} \hat{u}'' - \Delta\hat{u} + |\hat{u}|^{p-2}\hat{u} = f - \hat{g}_f(u_0) \quad \text{in } G \times (0,T), \\ \hat{u} = 0 \text{ on } \partial G \times (0,T), \quad \{\hat{u}, \hat{u}'\}\big|_{t=0} = \{\hat{u}, \hat{u}'\}\big|_{t=T} \end{gathered}$$

with $\hat{u}(0) = \hat{u}(T) = u_0$. The theorem is proved. □

4. Periodic solutions

In this section we shall use u_0 of Theorem 3.1 as a control to show that for any given $f \in K^\perp$, there exists

$$\{\tilde{f}, \tilde{u}_0, g_{\tilde{f}}(\tilde{u}_0)\} \in K^\perp \times H_0^1(G) \cap L^p(G) \times [H_0^1(G) \cap L^p(G)]^*$$

such that $f = \tilde{f} - g_{\tilde{f}}(\tilde{u}_0)$. The main result of the section and of this article is the following theorem.

Theorem 4.1. *Let f be in $K^\perp$. Then there exists a solution $\{u, u'\}$ in the space $L^\infty(0,T; H_0^1(G) \cap L^p(G)) \times L^\infty(0,T; L^2(G))$ for the problem*

$$\begin{gathered} u'' - \Delta u + |u|^{p-2}u = f \quad \text{in } G \times (0,T), \\ u = 0 \text{ on } \partial G \times (0,T), \quad \{u, u'\}\big|_{t=0} = \{u, u'\}\big|_{t=T}. \end{gathered} \tag{4.1}$$

Proof. First we consider the initial boundary-value problem

$$\begin{gathered} w'' - \Delta w + |w|^{p-2}w = f \quad \text{in } G \times (0,T), \\ w = 0 \text{ on } \partial G \times (0,T), \quad \{w, w'\}\big|_{t=0} = \{u_0, u_1\} \end{gathered} \tag{4.2}$$

It is known that for a given

$$\{f, u_0, u_1\} \in L^2(0,T; L^2(G)) \times \{H_0^1(G) \cap L^p(G) \times L^2(G)\},$$

there exists a unique solution of (4.2) with

$$\begin{aligned} &\|w'(t)\|^2_{L^2(G)} + \|\nabla w(t)\|^2_{L^2(G)} + 2/p\|w(t)\|^p_{L^p(G)} \\ &\le e^t\{\|u_1\|^2_{L^2(G)} + \|\nabla u_0\|^2_{L^2(G)} + 2/p\|u_0\|^p_{L^p(G)} + \|f\|^2_{L^2(0,T;L^2(G))}\} \end{aligned}$$

Consider the optimization problem

$$\begin{aligned} \alpha(f) = \inf\Big\{ &\|u(0) - u(T)\|_{L^2(G)} + \|u'(0) - u'(T)\|_{L^2(G)} : u \text{ is the solution of (4.2)} \\ &\forall\{u_0, u_1\} \text{ with } \|u_0\|_{H_0^1(G)\cap L^p(G)} + \|u_1\|_{L^2(G)} \le R\Big\} \end{aligned} \tag{4.3}$$

From Theorem 3.1 we know that for each u_0 in $H_0^1(G)\cap L^p(G)$, for a given f in $K^\perp$ there exists $g_f(u_0)$ in $[H_0^1(G)\cap L^p(G)]^*$ and a solution u of

$$\begin{gathered} u'' - \Delta u + |u|^{p-2}u = f - g_f(u_0) \quad \text{in } G\times(0,T), \\ u = 0 \text{ on } \partial G\times(0,T), \quad u(0)=u_0=u(T), \quad u'(0)=u'(T). \end{gathered}$$

Let

$$S = \cup_{f\in K^\perp}\big\{f\oplus\{-g_f(u_0): u_0\in H_0^1(G)\cap L^p(G)\}\big\},$$

where $g_f(u_0)$ is as in Theorem 3.1 and thus, $\alpha(f-g_f(u_0))=0$.

The set S is non-empty and $L^2(G)=L^2(G)\oplus 0\subset S$. Indeed $L^2(G)\subset K^\perp$ as the stationary solution of the elliptic boundary problem

$$-\Delta w + |w|^{p-2}w = f(x) \text{ in } G, \quad w=0 \text{ on } \partial G$$

is time-periodic. Thus $\alpha(f)=0=\alpha(f-g_f)$ and $g_f=0$, and hence f is in S.

We have

$$S\subset K^\perp\oplus\cup_{h\in K^\perp}\{-g_h(u_0): u_0\in H_0^1(G)\cap L^p(G)\}$$

Thus,

$$\begin{aligned} L^2(G) &= \{L^2(G)\oplus 0\}\cap\{K^\perp\oplus 0\} \\ &\subset S\cap\{K^\perp\oplus 0\} \\ &\subset \big\{K^\perp\oplus\cup_{h\in K^\perp}\{-g_h(u_0): u_0\in H_0^1(G)\cap L^p(G)\}\big\}\cap\{K^\perp\oplus 0\} \\ &\subset K^\perp\oplus 0. \end{aligned}$$

Indeed

$$0\in\cup_{h\in K^\perp}\{-g_h(u_0): u_0\in H_0^1(G)\cap L^p(G)\}$$

as $\alpha(\hat f)=0=g_{\hat f}$ for $\hat f\in L^2(G)$. Hence $\{K^\perp\oplus 0\}\subset S$.

Let f in $\{K^\perp\oplus 0\}$ then there exists h in $K^\perp$ and $g_h(u_0)$ for some u_0 in $H_0^1(G)\cap L^p(G)$ such that

$$f = h - g_h(u_0), \quad \alpha(h-g_h(u_0))=0$$

and therefore $\alpha(f)=0$. Thus for $f\in K^\perp$ there exists $\tilde u$, solution of the problem

$$\begin{gathered} \tilde u'' - \Delta\tilde u + |\tilde u|^{p-2}\tilde u = f \quad \text{in } G\times(0,T), \\ \tilde u = 0 \text{ on } \partial G\times(0,T), \quad \{\tilde u,\tilde u'\}\big|_{t=0} = \{\tilde u,\tilde u'\}\big|_{t=T} \end{gathered}$$

The proof is complete. $\square$

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