

## LOCALIZED NODAL SOLUTIONS FOR SEMICLASSICAL NONLINEAR KIRCHHOFF EQUATIONS

LIXIA WANG

ABSTRACT. In this article, we consider the existence of localized sign-changing solutions for the semiclassical Kirchhoff equation

$$-(\varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla u|^2 dx) \Delta u + V(x)u = |u|^{p-2}u, \quad x \in \mathbb{R}^3, \quad u \in H^1(\mathbb{R}^3)$$

where  $4 < p < 2^* = 6$ ,  $\varepsilon > 0$  is a small parameter,  $V(x)$  is a positive function that has a local minimum point  $P$ . When  $\varepsilon \rightarrow 0$ , by using a minimax characterization of higher dimensional symmetric linking structure via the symmetric mountain pass theorem, we obtain an infinite sequence of localized sign-changing solutions clustered at the point  $P$ .

### 1. INTRODUCTION AND MAIN RESULTS

In this article, we study the semiclassical states of nonlinear Kirchhoff equation

$$\begin{aligned} -\left(\varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla u|^2 dx\right) \Delta u + V(x)u &= |u|^{p-2}u, \quad x \in \mathbb{R}^3, \\ u &\in H^1(\mathbb{R}^3), \end{aligned} \tag{1.1}$$

where  $p \in (4, 2^*)$ ,  $2^* = 6$ ,  $\varepsilon > 0$  is a small parameter and  $V : \mathbb{R}^3 \rightarrow \mathbb{R}$  is a continuous function satisfying the following conditions:

- (A1)  $V \in C^1(\mathbb{R}^3, \mathbb{R})$  and there exist  $n_0 > m_0 > 0$  such that  $m_0 \leq V(x) \leq n_0$  for any  $x \in \mathbb{R}^3$ .
- (A2) There is a bounded domain  $\Lambda \subset \mathbb{R}^3$  with smooth boundary  $\partial\Lambda$  such that

$$\vec{n}(x) \cdot \nabla V(x) > 0 \quad \forall x \in \partial\Lambda, \tag{1.2}$$

where  $\vec{n}(x)$  denotes the outward normal to  $\partial\Lambda$  at  $x$  and  $\cdot$  denotes the inner product in  $\mathbb{R}^3$ .

Note that if  $V$  has an isolated local minimum set, the condition (A2) is satisfied. That is,  $V$  has a local trapping potential well. Under (A2), the set of critical points of  $V$  is

$$A = \{x \in \Lambda \mid \nabla V(x) = 0\} \neq \emptyset, \tag{1.3}$$

and  $A$  is a compact subset of  $\Lambda$ . In the following, we will assume  $0 \in A$ .

---

2020 *Mathematics Subject Classification*. 35J20, 35J60.

*Key words and phrases*. Kirchhoff equations; nodal solutions; penalization method.

©2022. This work is licensed under a CC BY 4.0 license.

Submitted April 18, 2022. Published August 2, 2022.

Equation (1.1) or a more general version of

$$-\left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx\right) \Delta u + V(x)u = f(x, u), \quad x \in \mathbb{R}^N. \quad (1.4)$$

This equation has been studied recently under different conditions on  $f(x, u)$  and  $V(x)$ , where  $N = 1, 2, 3$  and  $a, b$  are two positive constants. It is well known that problem (1.4) is a nonlocal problem since the presence of the term  $b \int_{\mathbb{R}^3} |\nabla u|^2 dx$ . This fact indicates that (1.4) is not a pointwise identity. It causes some mathematical difficulties, and in the mean time, makes the study of such a problem particularly interesting. For a pure power  $f(x, u) := |u|^{p-2}u$  ( $3 < p \leq 6$ ), Li and Ye [13] studied the existence of a positive ground state solution by using a monotonicity trick and a new version of global compactness lemma. The authors used the constrained minimization on a new manifold which is related to the Pohozaev's identity to get a positive ground state solution to (1.4).

We note that if  $V(x) = 0$  and  $\mathbb{R}^N$  is replaced by a bounded domain  $\Omega \subset \mathbb{R}^N$  in (1.4), then we have the Kirchhoff Dirichlet problem

$$\begin{aligned} -\left(a + b \int_{\Omega} |\nabla u|^2 dx\right) \Delta u &= f(x, u), \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega, \end{aligned}$$

which is arises when studying wave solutions of the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right) \frac{\partial^2 u}{\partial x^2} = 0.$$

It is related to the stationary analogue of the Kirchhoff equation

$$u_{tt} - \left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx\right) \Delta u = g(x, t), \quad (1.5)$$

which is proposed by Kirchhoff [12] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. In [5], the authors pointed out that Problem (1.5) models several physical and biological systems, where  $u$  describes a process which depends on the average of itself (for example, population density).

Motivated by the works above, in this paper we study the existence of localized sign-changing solutions to the semiclassical nonlinear Kirchhoff equation (1.1). Before giving our main results, we give some notations. Let  $H^1(\mathbb{R}^3)$  be the usual Sobolev space endowed with the standard scalar and norm

$$(u, v) = \int_{\mathbb{R}^3} (\nabla u \nabla v + uv) dx; \quad \|u\| = \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + |u|^2) dx\right)^{1/2}.$$

$D^{1,2}(\mathbb{R}^3)$  is the completion of  $C_0^\infty(\mathbb{R}^3)$  with respect to the norm

$$\|u\|_D := \|u\|_{D^{1,2}(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^{1/2}.$$

The norm on  $L^s = L^s(\mathbb{R}^3)$  with  $1 < s < \infty$  is given by  $|u|_s = \left(\int_{\mathbb{R}^3} |u|^s dx\right)^{1/s}$ .

Assume the functional space is

$$H_\varepsilon = \left\{u \in H^1(\mathbb{R}^3) : \|u\|_\varepsilon = \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + V(\varepsilon x)u^2) dx\right)^{1/2} < \infty\right\}.$$

Since  $0 < m_0 \leq V(x) \leq n_0$ , we have

$$\begin{aligned} \min(1, m_0) \|u\|^2 &\leq \int_{\mathbb{R}^3} (|\nabla u|^2 + V(\varepsilon x)u^2) dx \leq \max\{1, n_0\} \|u\|^2, \\ H_\varepsilon(\mathbb{R}^3) &\hookrightarrow L^p(\mathbb{R}^3) \quad (2 \leq p \leq 6), \\ |u|_p &\leq C_p \|u\| \leq C'_p \|u\|_\varepsilon. \end{aligned}$$

Moreover we make the following assumptions. For any set  $\Omega \subset \mathbb{R}^3$ ,  $\varepsilon > 0$  and  $\delta > 0$ , we set

$$\Omega_\varepsilon = \{x \in \mathbb{R}^3 : \varepsilon x \in \Omega\}, \quad \Omega^\delta = \{x \in \mathbb{R}^3 : \text{dist}(x, \Omega) := \inf_{z \in \Omega} |x - z| < \delta\}. \quad (1.6)$$

A function  $u \in H^1(\mathbb{R}^3)$  is called sign-changing if  $u^+ \neq 0$  and  $u^- \neq 0$ , where  $u^\pm = \max\{\pm u, 0\}$ .

Our main result reads as follows.

**Theorem 1.1.** *Suppose that  $4 < p < 6$ , (A1) and (A2) hold. Then for any positive integer  $N$ , there exists  $\varepsilon_N > 0$  such that if  $0 < \varepsilon < \varepsilon_N$ , (1.1) has at least  $N$  pairs of sign-changing solutions  $\pm v_{j,\varepsilon}$ ,  $j = 1, 2, \dots, N$ , satisfying that, for any  $\delta > 0$ , there exist  $c = c(\delta, N) > 0$  and  $C = C(\delta, N) > 0$  such that*

$$|v_{j,\varepsilon}(x)| \leq C \exp\left(-\frac{c \text{dist}(x, A^\delta)}{\varepsilon}\right), \quad 1 \leq j \leq N.$$

In recent years, the existence and multiplicity solutions for  $(SK_\varepsilon)$  have been studied by many researchers under different assumptions on the potential and nonlinearity. Figueiredo [8] constructed a family of positive solutions which concentrates around the local minima of  $V$  as  $\varepsilon \rightarrow 0$ , the nonlinearities is subcritical. Motivated by [8], He [11] extended the result of Figueiredo to the case where the nonlinearity is of critical growth, i.e. due to [20]. In [21] the authors consider the stability of ground states to a nonlinear focusing Schrödinger equation in presence of a Kirchhoff term.

Especially, if  $a = 1$ ,  $b = 0$  and  $\mathbb{R}^3$  replaced by  $\mathbb{R}^N$ , (1.1) is reduced to a singular perturbed Schrödinger equation i.e.,

$$-\varepsilon^2 \Delta u + V(x)u = |u|^{p-2}u, \quad x \in \mathbb{R}^N, \quad 2 < p < 2^*, N \geq 1. \quad (1.7)$$

Floer and Weinstein [9] constructed a single peak solution which concentrates around any given non-degenerate critical point of the potential  $V$ . Oh [15] showed the existence of multi-peak solutions which concentrate around any finite subsets of the non-degenerate critical points of  $V$ . The methods in [9, 15] are mainly used a Lyapunov-Schmidt reduction.

The concentration behavior of the positive solutions also has been considered by variational methods. When  $\varepsilon > 0$  small enough, by using the Mountain-Pass Theorem, Rabinowitz [16] proved that (1.7) possesses a positive ground state solution under the condition

$$(A3) \quad V_\infty = \liminf_{|x| \rightarrow \infty} V(x) > V_0 = \inf_{x \in \mathbb{R}^N} V(x) > 0.$$

Other results on the concentration behavior for the family of positive ground solution see [6]. By using the same arguments as in [6, 16], He and Zou [10] considered the existence, concentration and multiplicity of solutions for (1.1) with general nonlinearity  $f(u)$ , and the potential  $V(x)$  satisfy the condition

$$(A4) \quad 0 < V_0 := \inf V(x) < \liminf_{|x| \rightarrow \infty} V(x) = V_\infty, \quad \text{where } V_\infty \leq +\infty$$

and  $f(u) \in C^1(\mathbb{R}^+, \mathbb{R}^+)$  is a subcritical function satisfying the Ambrosetti-Rabinowitz condition, which is concentrate on the minima of  $V(x)$  as  $\varepsilon \rightarrow 0$ .

Now we give an outline of the proof, we set  $v(x) = u(\varepsilon x)$ . Then (1.1) is changed to

$$\begin{aligned} -(a + b \int_{\mathbb{R}^3} |\nabla v|^2 dx) \Delta v + V(\varepsilon x)v &= |v|^{p-2}v, \quad x \in \mathbb{R}^3, \\ v &\in H^1(\mathbb{R}^3), \end{aligned} \quad (1.8)$$

and the corresponding energy functional is

$$I_\varepsilon(v) = \frac{1}{2} \int_{\mathbb{R}^3} (a|\nabla v|^2 + V(\varepsilon x)v^2) dx + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla v|^2 dx \right)^2 - \frac{1}{p} \int_{\mathbb{R}^3} v^p dx. \quad (1.9)$$

It is well known that, by using Rabinowitz [16], we can prove that  $I_\varepsilon$  satisfies the  $(PS)_c$  condition if  $c$  is smaller than the mountain pass value of the limiting functional

$$I(v) = \frac{1}{2} \int_{\mathbb{R}^3} (a|\nabla v|^2 + V_0 v^2) dx + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla v|^2 dx \right)^2 - \frac{1}{p} \int_{\mathbb{R}^3} v^p dx.$$

where  $V_0 = \liminf_{|x| \rightarrow \infty} V(x)$ . However, we will construct the solutions in Theorem 1.1 have larger critical values. The variational problem does not satisfy the compact condition anymore. Instead we use some ideas for nonlinear Schrödinger equations (see [3]) in particular in the recent work of [4] in which for nonlinear Schrödinger equations have an infinite sequence of localized nodal solutions were constructed near a local minimum of the potential function  $V(x)$ . This involves using the Byeon-Wang's penalization method [3], we define  $\Gamma_\varepsilon : H_\varepsilon \rightarrow \mathbb{R}$  by

$$\Gamma_\varepsilon(v) = I_\varepsilon(v) + Q_\varepsilon(v),$$

where

$$\begin{aligned} Q_\varepsilon(v) &= \left( \int_{\mathbb{R}^3} \chi_\varepsilon v^2 dx - 1 \right)_+^\beta, \\ \chi_\varepsilon(x) &= \begin{cases} 0, & \text{if } x \in \Lambda_\varepsilon, \\ \varepsilon^{-6} \xi(\text{dist}(x, \Lambda_\varepsilon)), & \text{if } x \notin \Lambda_\varepsilon. \end{cases} \end{aligned}$$

The function  $Q_\varepsilon$  will act as a penalization to force the concentration phenomena to occur inside the set of  $A$ . The function  $\Gamma_\varepsilon$  has an advantage that it has a higher threshold for  $(PS)_c$  condition to hold. Indeed, for any positive integer  $L$ , there exists  $\varepsilon_L > 0$  such that  $\Gamma_\varepsilon$  satisfies the  $(PS)_c$  condition for every  $c < L$  if  $0 < \varepsilon < \varepsilon_L$ .

By using a minimax theorem for sign-changing solutions (see [14]) and the genus (see [17]), we obtain that, for any positive integer  $N$ , there exists  $\varepsilon_N > 0$  such that  $\Gamma_\varepsilon$  has at least  $N$  pairs of sign-changing critical points  $v_{j,\varepsilon}$  ( $1 \leq j \leq N$ ) if  $0 < \varepsilon < \varepsilon_N$ .

To verify the critical point  $v_{j,\varepsilon}$  of  $\Gamma_\varepsilon$  is a solution of the original problem (1.8), we need a finer asymptotic analysis and the local Pohozaev identity. Moreover, we show that the concentration points of these solutions lie in  $A$  as  $\varepsilon \rightarrow 0$ .

**Remark 1.2.** As it is pointed in [4, 18] considered the critical frequency case, that is,  $V$  satisfies

$$(A5) \quad \liminf_{|x| \rightarrow \infty} V(x) > \inf_{x \in \mathbb{R}^N} V(x) = 0;$$

(A6) There exists a closed subset  $Z$  with a nonempty interior such that  $V(x) = 0$  for  $x \in Z$ .

By using minimax theorem, they obtained that for any integer  $N$ , there exists  $\varepsilon_N > 0$  such that for  $0 < \varepsilon < \varepsilon_N$ , (1.7) has at least  $N$  solutions. Under the assumption of critical frequency, one can use higher dimensional symmetric structures to construct minimax values below the mountain pass value of the limiting functional  $I$ . However, in our case with positive potentials, the energies of the sequence of localized nodal solutions tend to infinity.

To the best of our knowledge, there is no result on the existence and concentration of sign-changing solutions for Kirchhoff type equation under (A1) and (A2). In the present paper, we will adopt the ideas of Chen and Wang [4] to study the existence of sign-changing solutions for (1.1). But their method cannot be used directly because of the nonlocal term and more careful analysis is needed.

Throughout this paper, the letters  $C, C'$  will be used to denote various positive constants which may vary from line to line and are not essential to the problem.  $E'$  is a dual space for a Banach space  $E$ . The closure and the boundary of set  $G$  are denoted by  $\bar{G}$  and  $\partial G$  respectively. For  $F \in C^1(E, \mathbb{R})$ , we denote the Fréchet derivative of  $F$  at  $u$  by  $F'(u)$ , and the Gateaux derivative of  $F$  by  $\langle F'(u), v \rangle$  for all  $u, v \in E$ . We denote  $\rightharpoonup$  for weak convergence, and  $\rightarrow$  for strong convergence. Also if we take a subsequence of a sequence  $\{u_n\}$ , we shall denote it again  $\{u_n\}$ .

This article is organized as follows. In Section 2, we introduce the penalized function  $\Gamma_\varepsilon$ , show that  $\Gamma_\varepsilon$  satisfy  $(PS)_c$  condition for  $c < L$  and  $\varepsilon$  small enough. In Section 3, when  $\varepsilon$  is small, we show the existence of multiple sign-changing solutions of the problem through an abstract critical point theorem. In Section 4, we give the proof of Theorem 1.1. We prove the solutions obtained in Section 3 are in fact solutions of the original problem for  $\varepsilon$  small.

## 2. VARIATIONAL SETTING AND COMPACTNESS CONDITION

Set  $\xi \in C^\infty(\mathbb{R})$  be a cut-off function such that  $0 \leq \xi(t) \leq 1$  and  $\xi'(t) \geq 0$  for any  $t \in \mathbb{R}$ .  $\xi(t) > 0$  if  $t > 0$ ,  $\xi(t) = 1$  if  $t \geq 1$  and  $\xi(t) = 0$  if  $t \leq 0$ . Define

$$\chi_\varepsilon(x) = \begin{cases} 0, & \text{if } x \in \Lambda_\varepsilon, \\ \varepsilon^{-6}\xi(\text{dist}(x, \Lambda_\varepsilon)), & \text{if } x \notin \Lambda_\varepsilon. \end{cases}$$

Obviously, for  $\varepsilon$  small,  $\chi_\varepsilon$  is a  $C^1$  function and

$$\chi_\varepsilon(x) = \begin{cases} 0, & \text{if } x \in \Lambda_\varepsilon, \\ \varepsilon^{-6}, & \text{if } x \notin (\Lambda_\varepsilon)^1. \end{cases}$$

For  $u \in H^1(\mathbb{R}^3)$ , we define the penalization function

$$Q_\varepsilon(v) = \left( \int_{\mathbb{R}^3} \chi_\varepsilon v^2 dx - 1 \right)_+^\beta \quad (2.1)$$

which  $\beta$  satisfies  $2 < 2\beta < p$  and  $(t)_+ = \max\{t, 0\}$ . For  $v \in H^1(\mathbb{R}^3)$ , we define

$$\Gamma_\varepsilon(v) = I_\varepsilon(v) + Q_\varepsilon(v), \quad (2.2)$$

where  $I_\varepsilon$  is defined by (1.9). For  $u, v \in H(\mathbb{R}^3)$ ,

$$\begin{aligned} \langle \Gamma'_\varepsilon(v), u \rangle &= \int_{\mathbb{R}^3} (a \nabla v \nabla u + V(\varepsilon x)vu) dx + b \int_{\mathbb{R}^3} |\nabla v|^2 dx \int_{\mathbb{R}^3} \nabla v \nabla u dx \\ &\quad + 2\beta \left( \int_{\mathbb{R}^3} \chi_\varepsilon v^2 dx - 1 \right)_+^{\beta-1} \int_{\mathbb{R}^3} \chi_\varepsilon v u dx - \int_{\mathbb{R}^3} |v|^{p-2} v u dx. \end{aligned} \quad (2.3)$$

The critical point  $v$  of  $\Gamma_\varepsilon$  is a solution of

$$-\left(a + b \int_{\mathbb{R}^3} |\nabla v|^2 dx\right) \Delta v + V(\varepsilon x)v + 2\beta \left( \int_{\mathbb{R}^3} \chi_\varepsilon v^2 dx - 1 \right)_+^{\beta-1} \chi_\varepsilon v = |v|^{p-2} v, \quad (2.4)$$

for any  $v \in H^1(\mathbb{R}^3)$ . If  $v$  is a critical point of  $\Gamma_\varepsilon$  with  $Q_\varepsilon(v) = 0$ , then  $v$  is a solution of (1.8).

**Lemma 2.1.** *For any  $L > 0$ , there exists  $\varepsilon_L > 0$  such that, for any  $\varepsilon \in (0, \varepsilon_L)$  and  $c < L$ , then  $\Gamma_\varepsilon$  satisfies  $(PS)_c$  condition.*

*Proof.* Let  $\{u_n\} \subset H^1(\mathbb{R}^3)$  satisfy the conditions

$$\Gamma_\varepsilon(u_n) \rightarrow c, \quad \Gamma'_\varepsilon(u_n) \rightarrow 0 \quad \text{in } (H^1(\mathbb{R}^3))'.$$

Now we can show that  $\{u_n\}$  contains a convergent subsequence in  $H^1(\mathbb{R}^3)$ . Note that

$$\begin{aligned} & o(\|u_n\|) + L \\ & \geq o(\|u_n\|) + c \\ & = \Gamma_\varepsilon(u_n) - \frac{1}{p} \langle \Gamma'_\varepsilon(u_n), u_n \rangle \\ & = \frac{1}{2} \int_{\mathbb{R}^3} (a |\nabla u_n|^2 + V(\varepsilon x)u_n^2) dx + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 dx \right)^2 - \frac{1}{p} \int_{\mathbb{R}^3} |u_n|^p dx \\ & \quad + \left( \int_{\mathbb{R}^3} \chi_\varepsilon u_n^2 dx - 1 \right)_+^\beta \\ & \quad - \frac{1}{p} \int_{\mathbb{R}^3} (a |\nabla u_n|^2 + V(\varepsilon x)u_n^2) dx - \frac{b}{p} \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 dx \right)^2 + \frac{1}{p} \int_{\mathbb{R}^3} |u_n|^p dx \\ & \quad - \frac{2\beta}{p} \left( \int_{\mathbb{R}^3} \chi_\varepsilon u_n^2 dx - 1 \right)_+^{\beta-1} \int_{\mathbb{R}^3} \chi_\varepsilon u_n^2 dx \\ & = \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}^3} (a |\nabla u_n|^2 + V(\varepsilon x)u_n^2) dx + b \left( \frac{1}{4} - \frac{1}{p} \right) \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 dx \right)^2 \\ & \quad + \left( \int_{\mathbb{R}^3} \chi_\varepsilon u_n^2 dx - 1 \right)_+^\beta - \frac{2\beta}{p} \left( \int_{\mathbb{R}^3} \chi_\varepsilon u_n^2 dx - 1 \right)_+^{\beta-1} \int_{\mathbb{R}^3} \chi_\varepsilon u_n^2 dx \end{aligned}$$

From this inequality and  $2 < 2\beta < p$ , there exists  $\eta_L > 0$  independent of  $\varepsilon$  such that  $\|u_n\| \leq \eta_L$  and  $Q_\varepsilon(u_n) \leq \eta_L$ . Suppose that  $u_n \rightharpoonup u$  in  $H^1(\mathbb{R}^3)$  as  $n \rightarrow \infty$  and

$$\lambda_n := 2\beta \left( \int_{\mathbb{R}^3} \chi_\varepsilon u_n^2 dx - 1 \right)_+^{\beta-1} \rightarrow \lambda, \quad n \rightarrow \infty.$$

It is easy to prove that  $u$  solves

$$-\left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx\right) \Delta u + V(\varepsilon x)u + \lambda \chi_\varepsilon u = |u|^{p-2} u. \quad (2.5)$$

Hence, for  $v \in H^1(\mathbb{R}^3)$ ,

$$a \int_{\mathbb{R}^3} \nabla(u_n - u) \nabla v dx + b \int_{\mathbb{R}^3} |\nabla u|^2 dx \int_{\mathbb{R}^3} \nabla(u_n - u) \nabla v dx$$

$$\begin{aligned}
& + b \int_{\mathbb{R}^3} (|\nabla u_n|^2 - |\nabla u|^2) dx \int_{\mathbb{R}^3} \nabla u_n \nabla v dx + \int_{\mathbb{R}^3} V(\varepsilon x)(u_n - u)v dx \\
& + \lambda \int_{\mathbb{R}^3} \chi_\varepsilon(u_n - u)v dx + (\lambda_n - \lambda) \int_{\mathbb{R}^3} \chi_\varepsilon u_n v dx - \int_{\mathbb{R}^3} (|u_n|^{p-2}u_n - |u|^{p-2}u)v dx \\
& = \langle \Gamma'_\varepsilon(u_n), v \rangle = o(\|v\|), \quad \text{as } n \rightarrow \infty.
\end{aligned}$$

Since  $\Lambda$  is a bounded set, there exists  $r_0 > 0$  satisfy  $\Lambda \subset B(0, r_0)$ . Let  $\phi_\varepsilon$  be a  $C^\infty$  cut-off function such that  $0 \leq \phi_\varepsilon \leq 1$  and  $|\nabla \phi_\varepsilon| \leq 4$  in  $\mathbb{R}^3$ ,  $\phi_\varepsilon(x) = 1$  if  $|x| \geq \varepsilon^{-1}r_0 + 2$  and  $\phi_\varepsilon(x) = 0$  if  $|x| \leq \varepsilon^{-1}r_0 + 1$ . We choose  $v = \phi_\varepsilon^2(u_n - u)$  in (2.5) we obtain that

$$\begin{aligned}
& \left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx \right) \int_{\mathbb{R}^3} |\nabla(\phi_\varepsilon(u_n - u))|^2 dx + \int_{\mathbb{R}^3} V(\varepsilon x)\phi_\varepsilon^2(u_n - u)^2 dx \\
& + b \int_{\mathbb{R}^3} (|\nabla u_n|^2 - |\nabla u|^2) dx \int_{\mathbb{R}^3} \nabla u_n (2\phi_\varepsilon \nabla \phi_\varepsilon(u_n - u) + \phi_\varepsilon^2 \nabla(u_n - u)) dx \\
& + \lambda \int_{\mathbb{R}^3} \chi_\varepsilon \phi_\varepsilon^2(u_n - u)^2 dx + (\lambda_n - \lambda) \int_{\mathbb{R}^3} \chi_\varepsilon \phi_\varepsilon^2(u_n - u)^2 dx \\
& - (p-1) \int_{\mathbb{R}^3} (\theta u_n + (1-\theta)u)^{p-2} \phi_\varepsilon^2(u_n - u)^2 dx \\
& - \left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx \right) \int_{\mathbb{R}^3} (u_n - u)^2 |\nabla \phi_\varepsilon|^2 dx \\
& = o(1) \quad \text{as } n \rightarrow \infty.
\end{aligned} \tag{2.6}$$

where  $0 < \theta < 1$  comes from the mean value theorem. By  $\lambda_n \rightarrow \lambda$  as  $n \rightarrow \infty$ ,  $|\nabla \phi_\varepsilon|^2$  has compact support, and  $u_n \rightharpoonup u$  in  $H^1(\mathbb{R}^3)$  as  $n \rightarrow \infty$ , then we have

$$(\lambda_n - \lambda) \int_{\mathbb{R}^3} \chi_\varepsilon \phi_\varepsilon^2(u_n - u)u_n dx = o(1), \quad \int_{\mathbb{R}^3} (u_n - u)^2 |\nabla \phi_\varepsilon|^2 = o(1).$$

as  $n \rightarrow \infty$ . Then by  $V \geq m_0$  in  $\mathbb{R}^3$  and (2.6), we obtain

$$\begin{aligned}
& \min \left\{ a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx, m_0 \right\} \|\phi_\varepsilon(u_n - u)\|^2 \\
& \leq \left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx \right) \int_{\mathbb{R}^3} |\nabla \phi_\varepsilon(u_n - u)|^2 dx + \int_{\mathbb{R}^3} V(\varepsilon x)\phi_\varepsilon^2(u_n - u)^2 dx \\
& \quad + b \int_{\mathbb{R}^3} (|\nabla u_n|^2 - |\nabla u|^2) dx \int_{\mathbb{R}^3} \nabla u_n \{2\phi_\varepsilon \nabla \phi_\varepsilon(u_n - u) + \phi_\varepsilon^2 \nabla(u_n - u)\} dx \\
& \leq (p-1) \int_{\mathbb{R}^3} |\theta u_n + (1-\theta)u|^{p-2} \phi_\varepsilon^2(u_n - u)^2 dx + o(1) \\
& \leq (p-1) \left( \int_{\mathbb{R}^3} |\theta u_n + (1-\theta)u|^p dx \right)^{(p-2)/p} \left( \int_{\mathbb{R}^3} \phi_\varepsilon^p(u_n - u)^p dx \right)^{2/p} + o(1) \\
& \leq C(p-1) \left\{ \left( \int_{|x| \geq \varepsilon^{-1}r_0+1} |u_n|^p dx \right)^{(p-2)/p} \right. \\
& \quad \left. + \left( \int_{|x| \geq \varepsilon^{-1}r_0+1} |u|^p dx \right)^{(p-2)/p} \right\} \|\phi_\varepsilon(u_n - u)\|^2 + o(1) \quad \text{as } n \rightarrow \infty,
\end{aligned} \tag{2.7}$$

where  $C > 0$  is a constant independent of  $n$  and  $\varepsilon$ . By Fatou's Lemma, we have

$$\int_{\mathbb{R}^3} (\nabla u_n \phi_\varepsilon^2 \nabla u_n - \nabla u_n \phi_\varepsilon^2 \nabla u) dx = \int_{\mathbb{R}^3} (|\nabla u_n|^2 \phi_\varepsilon^2 - |\nabla u_n| |\nabla u| \phi_\varepsilon^2) dx \geq 0,$$

then

$$\int_{\mathbb{R}^3} 2\nabla u_n \phi_\varepsilon \nabla \phi_\varepsilon (u_n - u) u_n dx = o(1).$$

By  $Q_\varepsilon(u_n) \leq \eta_L$  and  $(A_\varepsilon)^1 \subset B(0, \varepsilon^{-1}r_0 + 1)$ , we have

$$\int_{|x| \geq \varepsilon^{-1}r_0 + 1} u_n^2 dx \leq (1 + \hat{\eta}_L)^{1/\beta} \varepsilon^6.$$

It follows that

$$\int_{|x| \geq \varepsilon^{-1}r_0 + 1} u^2 dx \leq (1 + \hat{\eta}_L)^{1/\beta} \varepsilon^6.$$

Assume  $p < q < 6$ . By the inequality  $|u|_p \leq |u|_2^t |u|_q^{1-t} \leq C' |u|_2^t \|u\|^{1-t}$ , where the positive  $C'$  is independent of  $n$  and  $\varepsilon$ , and  $\frac{1}{p} = \frac{t}{2} + \frac{1-t}{q}$ , by above two inequalities and  $\|u_n\| \leq \hat{\eta}_L$ , we infer that there is a constant  $C_L > 0$  independent of  $\varepsilon$  and  $n$  such that

$$\int_{|x| \geq \varepsilon^{-1}r_0 + 1} u_n^p dx \leq C_L \varepsilon^{3pt}, \quad \int_{|x| \geq \varepsilon^{-1}r_0 + 1} u^p dx \leq C_L \varepsilon^{3pt}.$$

Let  $\varepsilon_L > 0$  satisfying that, for  $0 < \varepsilon < \varepsilon_L$ ,

$$C(p-1)(2C_L^{\frac{p-2}{p}} \varepsilon^{3(p-2)t}) < \frac{1}{2} \min\{a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx, m_0\}.$$

Then by (2.6) we obtain

$$\lim_{n \rightarrow \infty} \|\phi_\varepsilon(u_n - u)\| = 0. \quad (2.8)$$

Choosing  $v = (1 - \phi_\varepsilon)^2(u_n - u)$  in (2.5), we obtain that (2.6) still holds if we replace  $\phi_\varepsilon$  with  $1 - \phi_\varepsilon$ . Indeed,  $1 - \phi_\varepsilon$  has a compact support and  $u_n \rightarrow u$  in  $L_{loc}^q(\mathbb{R}^3)$  for any  $2 \leq q < 2^*$ ,

$$\begin{aligned} & \left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx \right) \int_{\mathbb{R}^3} |\nabla((1 - \phi_\varepsilon)^2(u_n - u))|^2 dx \\ & - \left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx \right) \int_{\mathbb{R}^3} |\nabla(1 - \phi_\varepsilon)|^2 (u_n - u)^2 dx \\ & + b \int_{\mathbb{R}^3} (|\nabla u_n|^2 - |\nabla u|^2) dx \\ & + \int_{\mathbb{R}^3} \{ \nabla u_n \nabla(1 - \phi_\varepsilon)^2(u_n - u) + \nabla u_n (1 - \phi_\varepsilon)^2 \nabla(u_n - u) \} dx \\ & + \int_{\mathbb{R}^3} V(\varepsilon x) (1 - \phi_\varepsilon)^2 (u_n - u)^2 dx + \lambda \int_{\mathbb{R}^3} \chi_\varepsilon (1 - \phi_\varepsilon)^2 (u_n - u)^2 dx \\ & + (\lambda_n - \lambda) \int_{\mathbb{R}^3} \chi_\varepsilon (1 - \phi_\varepsilon)^2 (u_n - u)^2 dx \\ & - (p-1) \int_{\mathbb{R}^3} (\theta u_n + (1 - \theta)u)^{p-2} (1 - \phi_\varepsilon)^2 (u_n - u)^2 dx = o(1), \quad \text{as } n \rightarrow \infty. \end{aligned}$$

This implies

$$\lim_{n \rightarrow \infty} \|(1 - \phi_\varepsilon)(u_n - u)\| = 0 \quad (2.9)$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} \|u_n - u\| &= \lim_{n \rightarrow \infty} \|(1 - \phi_\varepsilon)(u_n - u) + \phi_\varepsilon(u_n - u)\| \\ &\leq \lim_{n \rightarrow \infty} \|(1 - \phi_\varepsilon)(u_n - u)\| + \lim_{n \rightarrow \infty} \|\phi_\varepsilon(u_n - u)\| = 0. \end{aligned}$$

The proof is complete. □

### 3. EXISTENCE OF MULTIPLE SIGN-CHANGING CRITICAL POINTS OF $\Gamma_\varepsilon$

We will use an abstract critical point theorem in [14] to obtain multiple sign-changing critical points for  $\Gamma_\varepsilon$ . First we give some definitions and notation.

Let  $X$  be a Banach space. For  $P \subset X$ , define  $-P = \{-u : u \in P\}$ . The genus (see [17]) of a closed symmetric subset  $B$  (i.e.  $-B = B$ ) of  $X$  is denoted by  $\gamma(B)$ . For  $J \in C^1(X, \mathbb{R})$  and  $c \in \mathbb{R}$ , denote

$$J^c = \{u \in X : J(u) \leq c\},$$

$$K_c = \{u \in X : J(u) = c, J'(u) = 0\}.$$

**Definition 3.1** ([14]). Let  $J \in C^1(X, \mathbb{R})$  be an even functional. Let  $P \subset X$  be a non-empty open set and  $W = P \cup (-P)$ .  $P$  is called an admissible invariant set with respect to  $J$  at level  $c$ , if the following deformation property holds, there is  $\tau_0 > 0$  and a symmetric open neighborhood  $\mathbb{M}$  of  $K_c \setminus W$  with  $\gamma(\mathbb{M}) < \infty$ , such that for  $\tau \in (0, \tau_0)$ , there exists  $\eta \in C(X, X)$  satisfying

- (1)  $\eta(\partial P) \subset P, \eta(\partial(-P)) \subset -P, \eta(P) \subset P, \eta(-P) \subset -P$ ;
- (2)  $\eta(-u) = -\eta(u)$ , for all  $u \in X$ ;
- (3)  $\eta|_{J^{c-2\tau}} = id$ .
- (4)  $\eta(J^{c+\tau} \setminus (\mathbb{M} \cup W)) \subset J^{c-\tau}$ .

**Proposition 3.2.** Assume  $J \in C^1(X, \mathbb{R})$  is an even functional,  $P \subset X$  is a non-empty open set,  $M = P \cap (-P), W = P \cup (-P)$  and  $\Sigma = \partial P \cap \partial(-P)$ . Let  $P$  be an admissible invariant set with respect to  $J$  for  $c \in [c^*, L]$  for some  $L > c^*$ , where  $c^* = \inf_{u \in \Sigma} J(u)$  and for any  $n \in \mathbb{N}$ , there is a continuous map  $\phi_n : B_n := \{x \in \mathbb{R}^n : |x| \leq 1\} \rightarrow X$  satisfying

- (1)  $\phi_n(0) \in M, \phi_n(-t) = -\phi_n(t)$  for all  $t \in B_n$ ;
- (2)  $\phi_n(\partial B_n) \cap M = \emptyset$ ;
- (3)  $\max\{J(0), \sup_{u \in \phi_n(\partial B_n)} J(u)\} < c^*$ .

For  $j \in \mathbb{N}$ , we define

$$c_j = \inf_{B \in \Lambda_j} \sup_{u \in B \setminus W} J(u),$$

where

$$\Lambda_j = \{B : B = \phi(B_n \setminus Y) \text{ for some } \phi \in G_n, n \geq j, \\ \text{and open } Y \subset B_n \text{ such that } -Y = Y \text{ and } \gamma(\overline{Y}) \leq n - j\}$$

and

$$G_n = \{\phi : \phi \in C(B_n, X), \phi(-t) = -\phi(t) \text{ for any } t \in B_n, \phi|_{\partial B_n} = \phi_n|_{\partial B_n}\}.$$

Then for  $j \geq 2$ , if  $L > c_j$ , we have

$$K_{c_j} \setminus W \neq \emptyset. \tag{3.1}$$

Furthermore, if  $j \geq 2$  and  $L > c := c_j = \dots = c_{j+m} \geq c_*$ , we have

$$\gamma(K_c \setminus W) \geq m + 1. \tag{3.2}$$

The above proposition was proved in [14, Theorem 2.5]. If we choose  $k = 1$  and  $G = -id$  in [14], we can obtain (3.1). The result (3.2) is proved in [4] by a variant of the argument in the proof of [14, Theorem 2.5]. So we omit it here.

Let  $P_{\pm} := \{u \in H^1(\mathbb{R}^3) : u \geq (\leq) 0\}$ . For  $\sigma > 0$ , let

$$\begin{aligned} P_+^{\sigma} &:= \{u \in H^1(\mathbb{R}^3) : \text{dist}_{H^1}(u, P_+) < \sigma\}, \\ P_-^{\sigma} &:= \{u \in H^1(\mathbb{R}^3) : \text{dist}_{H^1}(u, P_-) < \sigma\}, \end{aligned}$$

where  $\text{dist}_{H^1}(u, B) := \inf_{v \in B} \|u - v\|$  for  $u \in H^1(\mathbb{R}^3)$  and  $B \subset H^1(\mathbb{R}^3)$ . It is easy to see that  $P_-^{\sigma} = -P_+^{\sigma}$ .

To apply Proposition 3.2 to obtain multiple sign-changing critical points of  $\Gamma_{\varepsilon}$ , we let

$$X = H^1(\mathbb{R}^3), \quad P = P_+^{\sigma}, \quad W = P_-^{\sigma} \cup P_+^{\sigma}, \quad J = \Gamma_{\varepsilon} \quad (3.3)$$

in Definition 3.1 and Proposition 3.2. It is easy to know that  $W$  is a symmetric and open subset of  $H^1(\mathbb{R}^3)$  and sign-changing functions are contained in  $H^1(\mathbb{R}^3) \setminus W$ . Furthermore, since 0 is a strict local minimum point of  $\Gamma_{\varepsilon}$ , the constant  $c^*$  in Proposition 3.2 satisfies

$$c^* = \inf_{\partial(P_-^{\sigma}) \cap \partial(P_+^{\sigma})} \Gamma_{\varepsilon} > 0,$$

when  $\sigma > 0$  is small enough.

Without loss of generality, we assume that

$$0 \in A. \quad (3.4)$$

For  $z \in \mathbb{R}^3$  and  $r > 0$ , we define  $B(z, r) = \{x \in \mathbb{R}^3 : |x - z| < r\}$ . From (3.3), we obtain

$$B(0, 1) \subset \Lambda_{\varepsilon} \quad (3.5)$$

if  $\varepsilon > 0$  small enough.

Now we define a function

$$J_0(u) = \frac{1}{2} \int_{B(0,1)} (a|\nabla u|^2 + n_0 u^2) dx - \frac{1}{p} \int_{B(0,1)} |u|^p dx + \frac{b}{4} \left( \int_{B(0,1)} |\nabla u|^2 dx \right)^2,$$

$u \in H_0^1(B(0, 1))$ .

Assume  $E_n := \text{span}\{e_1, \dots, e_n\}$ , where  $\{e_n\} \subset H_0^1(B(0, 1))$  is an orthonormal basis. From  $p > 2$ , we can infer that there is an increasing sequence of positive numbers  $\{R_n\}$  satisfying

$$J_0(u) < 0, \quad \text{for all } u \in E_n \text{ and } \|u\| \geq R_n.$$

We also we define  $\phi_n \in C(B_n, H_0^1(B(0, 1)))$  as

$$\phi_n(t) = R_n \sum_{i=1}^n t_i e_i, \quad t = (t_1, \dots, t_n) \in B_n. \quad (3.6)$$

One can easily prove that under (3.4),  $\phi_n$  satisfied (1)–(3) in Proposition 3.2. For  $j \in \mathbb{N}$ , we define four sets

$$\begin{aligned} \Lambda_j &= \{B : B = \phi(B_n \setminus Y) \text{ for some } \phi \in G_n, n \geq j, \\ &\quad \text{and open } Y \subset B_n \text{ such that } -Y = Y \text{ and } \gamma(\bar{Y}) \leq n - j\}, \end{aligned}$$

$$\begin{aligned} \tilde{\Lambda}_j &= \{B : B = \phi(B_n \setminus Y) \text{ for some } \phi \in \tilde{G}_n, n \geq j, \\ &\quad \text{and open } Y \subset B_n \text{ such that } -Y = Y \text{ and } \gamma(\bar{Y}) \leq n - j\}, \end{aligned}$$

$$G_n = \{\phi : \phi \in C(B_n, H^1(\mathbb{R}^3)), \phi(-t) = -\phi(t) \text{ for all } t \in B_n, \phi|_{\partial B_n} = \phi_n|_{\partial B_n}\},$$

$$\begin{aligned} \tilde{G}_n &= \{\phi : \phi \in C(B_n, H_0^1(B(0, 1))), \phi(-t) = -\phi(t) \\ &\quad \forall t \in B_n, \phi|_{\partial B_n} = \phi_n|_{\partial B_n}\}. \end{aligned}$$

Then we can give the the minimax values

$$c_j^\varepsilon = \inf_{B \in \Lambda_j} \sup_{u \in B \setminus W} \Gamma_\varepsilon(u), \quad \tilde{c}_j = \inf_{B \in \tilde{\Lambda}_j} \sup_{u \in B \setminus W} J_0(u)$$

We obtain

$$0 < c_2^\varepsilon \leq c_3^\varepsilon \leq \dots, \quad \tilde{c}_2 \leq \tilde{c}_3 \leq \dots \tag{3.7}$$

Because  $\chi_\varepsilon = 0$  in  $\Lambda_\varepsilon$ , from  $V \leq n_0$  and (3.5), for all  $u \in H_0^1(B(0, 1))$ , we can obtain that  $\Gamma_\varepsilon(u) \leq J_0(u)$ . And then by  $\tilde{\Lambda}_j \subset \Lambda_j$ , for any  $j \geq 2$  and sufficiently small  $\varepsilon > 0$ , we have

$$0 < c_j^\varepsilon \leq \tilde{c}_j. \tag{3.8}$$

**Proposition 3.3.** *Assume that  $\sigma_0 > 0$  and  $L > 0$ . Then for any  $\sigma \in (0, \sigma_0)$  and  $\varepsilon \in (0, \varepsilon_L)$ ,  $P_+^\sigma$  is an admissible invariant set with respect to  $\Gamma_\varepsilon$  for  $c < L$ , where  $\varepsilon_L$  is from Lemma 2.1.*

We prove the above proposition in the appendix.

**Proposition 3.4.** *For any  $N \in \mathbb{N}$ , there exists  $\varepsilon'_N > 0$  such that, for any  $\varepsilon \in (0, \varepsilon'_N)$ ,  $\Gamma_\varepsilon$  has at least  $N$  pairs of sign-changing critical points  $\{\pm v_{j,\varepsilon} : 1 \leq j \leq N\}$  satisfying*

$$\Gamma_\varepsilon(v_{j,\varepsilon}) = c_{j+1}^\varepsilon \leq \tilde{c}_{N+1}, \quad 1 \leq j \leq N.$$

The above proposition follows from 3.2, Proposition 3.3, using (3.3), (3.7), and (3.8).

#### 4. PROOF OF THEOREM 1.1

In this part, we first verify that the sign-changing critical points  $\{v_{j,\varepsilon}\}$  obtained in Proposition 3.4 are solutions of (1.8), then we can prove the main theorem.

**Lemma 4.1.** *For any  $N \in \mathbb{N}$  and  $0 < \varepsilon < \varepsilon'_N$ , there exist  $\rho = \rho(a, m_0, p) > 0$  and  $\eta_N > 0$  such that*

$$\rho \leq \|v_{j,\varepsilon}\| \leq \eta_N, \quad Q_\varepsilon(v_{j,\varepsilon}) \leq \eta_N, \quad 1 \leq j \leq N,$$

where  $\eta_N$  is independent of  $\varepsilon$ .

*Proof.* Since

$$\begin{aligned} \tilde{c}_{N+1} &\geq c_{j+1}^\varepsilon = \Gamma_\varepsilon(v_{j,\varepsilon}) - \frac{1}{p} \langle \Gamma'_\varepsilon(v_{j,\varepsilon}), v_{j,\varepsilon} \rangle \\ &= I_\varepsilon(v_{j,\varepsilon}) + Q_\varepsilon(v_{j,\varepsilon}) - \frac{1}{p} \langle \Gamma'_\varepsilon(v_{j,\varepsilon}), v_{j,\varepsilon} \rangle \\ &= \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathbb{R}^3} (a|\nabla v_{j,\varepsilon}|^2 + V(\varepsilon x)v_{j,\varepsilon}^2) dx + b\left(\frac{1}{4} - \frac{1}{p}\right) \left(\int_{\mathbb{R}^3} |\nabla v_{j,\varepsilon}|^2 dx\right)^2 \\ &\quad + \left(\int_{\mathbb{R}^3} \chi_\varepsilon v_{j,\varepsilon}^2 dx - 1\right)_+^\beta - \frac{2\beta}{p} \left(\int_{\mathbb{R}^3} \chi_\varepsilon v_{j,\varepsilon}^2 dx - 1\right)_+^{\beta-1} \int_{\mathbb{R}^3} \chi_\varepsilon v_{j,\varepsilon}^2 dx \end{aligned}$$

and  $2 < 2\beta < p$ , we can have that there exists  $\eta_N > 0$  independent of  $\varepsilon$  such that  $\|v_{j,\varepsilon}\| \leq \eta_N$  and  $Q_\varepsilon(v_{j,\varepsilon}) \leq \eta_N$ .

From  $\langle \Gamma'_\varepsilon(v_{j,\varepsilon}), v_{j,\varepsilon} \rangle = 0$ , we obtain

$$\begin{aligned} \min\{a, m_0\} \|v_{j,\varepsilon}\|^2 &\leq \int_{\mathbb{R}^3} (a|\nabla v_{j,\varepsilon}|^2 + V(\varepsilon x)v_{j,\varepsilon}^2) dx + b\left(\int_{\mathbb{R}^3} |\nabla v_{j,\varepsilon}|^2 dx\right)^2 \\ &\quad + 2\beta \left(\int_{\mathbb{R}^3} \chi_\varepsilon v_{j,\varepsilon}^2 dx - 1\right)_+^{\beta-1} \int_{\mathbb{R}^3} \chi_\varepsilon v_{j,\varepsilon}^2 dx \end{aligned}$$

$$= \int_{\mathbb{R}^3} |v_{j,\varepsilon}|^p dx \leq C \|v_{j,\varepsilon}\|^p,$$

where  $C = C(p)$  is the constant in Sobolev inequality. Since  $v_{j,\varepsilon}$  are sign-changing functions,  $v_{j,\varepsilon} \neq 0$  and  $p > 2$ , it follows that there exists  $\rho = \rho(a, m_0, p) > 0$  such that  $\|v_{j,\varepsilon}\| \geq \rho$  for  $1 \leq j \leq N$ . The proof is complete.  $\square$

**Lemma 4.2.** *If  $\delta > 0$ , then  $\lim_{\varepsilon \rightarrow 0} \|v_{j,\varepsilon}\|_{L^\infty(\mathbb{R}^3 \setminus (\Lambda_\varepsilon)^\delta)} = 0$  for  $1 \leq j \leq N$ .*

*Proof.* From  $Q_\varepsilon(v_{j,\varepsilon}) \leq \eta_N$  and the definition of cut-off function  $\chi_\varepsilon$ , we have that, for any  $\delta > 0$ , there exists a positive constant  $C = C(\delta, N)$  such that

$$\int_{\mathbb{R}^3 \setminus (\Lambda_\varepsilon)^\delta} v_{j,\varepsilon}^2 dx \leq C\varepsilon^6, \quad 1 \leq j \leq N. \tag{4.1}$$

Because  $v_{j,\varepsilon}$  solves (2.4),  $\|v_{j,\varepsilon}\| \leq \eta_N$ . Then by (4.1) and using the bootstrap argument, we have

$$\|v_{j,\varepsilon}\|_{L^\infty(\mathbb{R}^3 \setminus (\Lambda_\varepsilon)^\delta)} \leq C\varepsilon^3, \quad 1 \leq j \leq N.$$

The proof is complete.  $\square$

**Lemma 4.3.** *Assume  $\varsigma > 0$ ,  $\{y_\varepsilon\} \subset \mathbb{R}^3$ , and  $\{v_\varepsilon\} \subset H^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$  satisfy*

$$\sup_{\varepsilon > 0} \|v_\varepsilon\| < +\infty, \tag{4.2}$$

$$\int_{B(y_\varepsilon, 1)} v_\varepsilon^2 dx \geq \varsigma, \tag{4.3}$$

$$\sup\{\langle \Gamma'_\varepsilon(v_\varepsilon), u \rangle : u \in H_0^1(\Lambda_\varepsilon), \|u\|_{H_0^1(\Lambda_\varepsilon)} \leq 1\} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0, \tag{4.4}$$

and for  $\delta > 0$ ,

$$\lim_{\varepsilon \rightarrow 0} \|v_\varepsilon\|_{L^\infty(\mathbb{R}^3 \setminus (\Lambda_\varepsilon)^\delta)} = 0. \tag{4.5}$$

Then  $y_\varepsilon \in \Lambda_\varepsilon$  and  $\lim_{\varepsilon \rightarrow 0} \text{dist}(y_\varepsilon, \partial\Lambda_\varepsilon) = +\infty$ .

*Proof.* It follows from (4.3) and (4.5) that  $y_\varepsilon \in (\Lambda_\varepsilon)^1$ . Assume  $w_\varepsilon = v_\varepsilon(\cdot + y_\varepsilon)$ . Then by (4.3), we obtain

$$\int_{B(0,1)} w_\varepsilon^2 dx \geq \varsigma. \tag{4.6}$$

If not, we suppose

$$\lim_{\varepsilon \rightarrow 0} \text{dist}(y_\varepsilon, \partial\Lambda_\varepsilon) = l < +\infty. \tag{4.7}$$

By changing variables, without loss of generality, we may assume that

$$y_\varepsilon = 0 \tag{4.8}$$

and there exists  $z_\varepsilon = (a_\varepsilon, 0, \dots, 0) \in \partial\Lambda_\varepsilon$  such that

$$|\alpha_\varepsilon| = \text{dist}(y_\varepsilon, \partial\Lambda_\varepsilon) \rightarrow l \text{ as } \varepsilon \rightarrow 0. \tag{4.9}$$

Up to a subsequence, we assume  $\lim_{\varepsilon \rightarrow 0} \alpha_\varepsilon = \alpha$ .

By  $y_\varepsilon \in (\Lambda_\varepsilon)^1$  and (4.7)–(4.9), we can infer that  $-1 \leq \alpha < +\infty$ . Because  $\|w_\varepsilon\| = \|v_\varepsilon\|$  and (4.2), we can set that  $w_\varepsilon \rightharpoonup w$  in  $H^1(\mathbb{R}^3)$  as  $\varepsilon \rightarrow 0$ . From (4.6), we can obtain  $w \neq 0$ . And from (4.5) and (4.9), if  $x_1 \geq \alpha$ , we obtain

$$w(x) = 0, \tag{4.10}$$

where  $x = (x_1, x_2, x_3)$ . By  $\chi_\varepsilon = 0$  in  $\Lambda_\varepsilon$  and (4.8), we obtain

$$\begin{aligned} & a \int_{\mathbb{R}^3} \nabla w_\varepsilon \nabla u \, dx + \int_{\mathbb{R}^3} V(\varepsilon x) w_\varepsilon u \, dx + b \int_{\mathbb{R}^3} |\nabla w_\varepsilon|^2 dx \int_{\mathbb{R}^3} \nabla w_\varepsilon \nabla u \, dx \\ &= \int_{\mathbb{R}^3} |w_\varepsilon|^{p-2} w_\varepsilon u \, dx + \langle \Gamma'_\varepsilon(w_\varepsilon), u \rangle, \quad \forall u \in H_0^1(\Lambda_\varepsilon). \end{aligned} \tag{4.11}$$

By (4.4), (4.10), and (4.11), we infer that  $w$  is a weak solution of

$$\begin{aligned} & -\left(a + b \int_{\mathbb{R}^3} |\nabla w_\varepsilon|^2 dx\right) \Delta w + V(0)w = |w|^{p-2}w \quad \text{in } \{x \in \mathbb{R}^3 : x_1 < \alpha\}, \\ & w|_{x_1=\alpha} = 0. \end{aligned}$$

By [7, Theorem I.1], the only solution of this equation in  $H^1(\mathbb{R}^3)$  is  $w = 0$ . This contradicts with  $w \neq 0$ . The proof is complete.  $\square$

**Lemma 4.4.** *Let  $v_{j,\varepsilon} \rightharpoonup \tilde{v}_0$  in  $H^1(\mathbb{R}^3)$  as  $\varepsilon \rightarrow 0$ . If  $\liminf_{\varepsilon \rightarrow 0} \|v_{j,\varepsilon} - \tilde{v}_0\|_{L^p(\mathbb{R}^3)} > 0$ , then there exists  $m_j \in \mathbb{N}$ ,  $m_j$  nonzero functions  $\tilde{v}_i$  in  $H^1(\mathbb{R}^3)$ ,  $1 \leq i \leq m_j$  and  $m_j$  sequences  $\{y_{j,\varepsilon}^i\} \subset \Lambda_\varepsilon$ ,  $1 \leq i \leq m_j$  satisfy*

$$\begin{aligned} \text{(i)} \quad & \lim_{\varepsilon \rightarrow 0} |y_{j,\varepsilon}^i| = +\infty, \lim_{\varepsilon \rightarrow 0} \text{dist}(y_{j,\varepsilon}^i, \partial\Lambda_\varepsilon) = +\infty, \quad 1 \leq i \leq m_j, \text{ and} \\ & \lim_{\varepsilon \rightarrow 0} |y_{j,\varepsilon}^i - y_{j,\varepsilon}^{i'}| = +\infty, \quad \text{if } i \neq i'; \end{aligned}$$

(ii)  $\tilde{v}_0$  is a solution of

$$-(a + bA_j)\Delta v + V(0)v = |v|^{p-2}v, \quad v \in H^1(\mathbb{R}^3), \tag{4.12}$$

where

$$A_j := \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^3} |\nabla v_{j,\varepsilon}|^2 dx, \quad \int_{\mathbb{R}^3} |\nabla \tilde{v}_0|^2 dx \leq A_j.$$

For every  $1 \leq i \leq m_j$ ,  $\tilde{v}_i$  is a nontrivial solution of

$$-(a + bA_j)\Delta v + V(y_j^i)v = |v|^{p-2}v, \quad v \in H^1(\mathbb{R}^3), \tag{4.13}$$

where  $y_j^i = \lim_{\varepsilon \rightarrow 0} \varepsilon y_{j,\varepsilon}^i \in \bar{\Lambda}$ ;

(iii) For any  $2 < q < 6$ ,

$$\lim_{\varepsilon \rightarrow 0} \|v_{j,\varepsilon} - \tilde{v}_0 - \sum_{i=1}^{m_j} \tilde{v}_i(\cdot - y_{j,\varepsilon}^i)\|_{L^q(\mathbb{R}^3)} = 0. \tag{4.14}$$

*Proof.* Since  $\|v_{j,\varepsilon}\|$  and  $Q(v_{j,\varepsilon})$  are bounded and  $v_{j,\varepsilon}$  solves (2.4), we can prove that  $\tilde{v}_0$  is a solution of (4.12). Indeed, since  $v_{j,\varepsilon} \rightharpoonup \tilde{v}_0$  in  $H^1(\mathbb{R}^3)$  as  $\varepsilon \rightarrow 0$ , we assume that for some constant  $A_j \in \mathbb{R}$ ,

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^3} |\nabla v_{j,\varepsilon}|^2 dx = A_j.$$

For any  $\phi \in C_0^\infty(\mathbb{R}^3)$ , we have that  $\langle \Gamma'_\varepsilon(v_{j,\varepsilon}), \phi \rangle \rightarrow 0$ , i.e.,

$$\begin{aligned} & \left(a + b \int_{\mathbb{R}^3} |\nabla v_{j,\varepsilon}|^2 dx\right) \int_{\mathbb{R}^3} \nabla v_{j,\varepsilon} \nabla \phi \, dx + \int_{\mathbb{R}^3} V(\varepsilon x) v_{j,\varepsilon} \phi \, dx - \int_{\mathbb{R}^3} |v_{j,\varepsilon}|^{p-2} v_{j,\varepsilon} \phi \, dx \\ &+ 2\beta \left(\int_{\mathbb{R}^3} \chi_\varepsilon v_{j,\varepsilon}^2 \, dx - 1\right)_+ \int_{\mathbb{R}^3} \chi_\varepsilon v_{j,\varepsilon} \phi \, dx = o(1) \end{aligned}$$

which implies that as  $\varepsilon \rightarrow 0$ ,

$$(a + bA_j) \int_{\mathbb{R}^3} \nabla \tilde{v}_0 \nabla \phi \, dx + V(0) \int_{\mathbb{R}^3} \tilde{v}_0 \phi \, dx - \int_{\mathbb{R}^3} |v_0|^{p-2} v_0 \phi \, dx = 0.$$

Since  $C_0^\infty(\mathbb{R}^3)$  is dense in  $H^1(\mathbb{R}^3)$ , we have that  $\tilde{v}_0$  solves

$$-(a + bA_j)\Delta v + V(0)v = |v|^{p-2}v, \quad v \in H^1(\mathbb{R}^3).$$

Let  $v_{j,\varepsilon}^1 = v_{j,\varepsilon} - \tilde{v}_0$  and  $\{y_{j,\varepsilon}^1\} \subset \mathbb{R}^3$  be such that

$$\int_{B(y_{j,\varepsilon}^1)} (v_{j,\varepsilon}^1)^2 dx = \sup_{y \in \mathbb{R}^3} \int_{B(y,1)} (v_{j,\varepsilon}^1)^2 dx := \varsigma_\varepsilon^1.$$

Since  $v_{j,\varepsilon}^1 \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , we have  $|y_{j,\varepsilon}^1| \rightarrow \infty$  as  $\varepsilon \rightarrow 0$  if  $\liminf_{\varepsilon \rightarrow 0} \varsigma_\varepsilon > 0$ . Since  $v_{j,\varepsilon}$  solves (2.4) and  $\tilde{v}_0$  solves (4.12), we have

$$\begin{aligned} & -a\Delta v_{j,\varepsilon}^1 - b \int_{\mathbb{R}^3} |\nabla v_{j,\varepsilon}|^2 dx \Delta v_{j,\varepsilon}^1 - b \left( \int_{\mathbb{R}^3} |\nabla v_{j,\varepsilon}|^2 dx - A_j \right) \Delta \tilde{v}_0 \\ & + V(\varepsilon x) v_{j,\varepsilon}^1 + (V(\varepsilon x) - V(0)) \tilde{v}_0 + \xi_\varepsilon \chi_\varepsilon v_\varepsilon^1 + \xi_\varepsilon \chi_\varepsilon \tilde{v}_0 \\ & = |v_{j,\varepsilon}|^{p-2} v_{j,\varepsilon} - |\tilde{v}_0|^{p-2} \tilde{v}_0, \end{aligned} \tag{4.15}$$

where

$$\xi_\varepsilon = 2\beta \left( \int_{\mathbb{R}^3} \chi_\varepsilon v_{j,\varepsilon}^2 dx - 1 \right)_+^{\beta-1}. \tag{4.16}$$

From (2.3) and (4.15), for  $u \in H^1(\Lambda_\varepsilon)$ , we have

$$\begin{aligned} \langle \Gamma'_\varepsilon(v_{j,\varepsilon}^1), u \rangle & = \int_{\mathbb{R}^3} (|v_{j,\varepsilon}|^{p-2} v_{j,\varepsilon} - |\tilde{v}_0|^{p-2} \tilde{v}_0 - |v_{j,\varepsilon}^1|^{p-2} v_{j,\varepsilon}^1) u dx \\ & - \int_{\mathbb{R}^3} (V(\varepsilon x) - V(0)) \tilde{v}_0 u dx \\ & + b \left( \int_{\mathbb{R}^3} (|\nabla v_{j,\varepsilon}^1|^2 - |\nabla v_{j,\varepsilon}|^2) \int_{\mathbb{R}^3} \nabla v_{j,\varepsilon}^1 \nabla u dx \right. \\ & \left. - b \left( \int_{\mathbb{R}^3} |\nabla v_{j,\varepsilon}|^2 - A_j \right) \int_{\mathbb{R}^3} \nabla \tilde{v}_0 \nabla u dx. \right. \end{aligned} \tag{4.17}$$

By [19, Lemma 8.1] and

$$\sup \left\{ \int_{\mathbb{R}^3} (V(\varepsilon x) - V(0)) \tilde{v}_0 u dx : u \in H^1(\mathbb{R}^3), \|u\| \leq 1 \right\} \rightarrow 0 \tag{4.18}$$

as  $\varepsilon \rightarrow 0$ . and (4.17), we obtain

$$\sup \left\{ \langle \Gamma'_\varepsilon(v_{j,\varepsilon}^1), u \rangle : u \in H_0^1(\Lambda_\varepsilon), \|u\|_{H_0^1(\Lambda_\varepsilon)} \leq 1 \right\} \rightarrow 0 \tag{4.19}$$

as  $\varepsilon \rightarrow 0$ . Since  $\tilde{v}_0 \in H_0^1(\mathbb{R}^3)$  and  $\tilde{v}_0$  solves (4.12), we have that  $\lim_{|x| \rightarrow \infty} \tilde{v}_0(x) = 0$ . By Lemma 4.2, for any  $\delta > 0$ , we have

$$\lim_{\varepsilon \rightarrow 0} \|v_{j,\varepsilon}^1\|_{L^\infty(\mathbb{R}^3 \setminus (\Lambda_\varepsilon)^\delta)} = 0. \tag{4.20}$$

By Lions Lemma [19] and  $\liminf_{\varepsilon \rightarrow 0} \|v_{j,\varepsilon} - \tilde{v}_0\|_{L^p(\mathbb{R}^3)} > 0$ , we have  $\liminf_{\varepsilon \rightarrow 0} \varsigma_\varepsilon^1 > 0$ . Then by Lemma 4.3, (4.19) and (4.20), we obtain that

$$y_{j,\varepsilon}^1 \in \Lambda_\varepsilon, \quad \lim_{\varepsilon \rightarrow 0} \text{dist}(y_{j,\varepsilon}^1, \partial \Lambda_\varepsilon) = +\infty. \tag{4.21}$$

Let  $w_{j,\varepsilon}^1 = v_{j,\varepsilon}^1(\cdot + y_{j,\varepsilon}^1)$ . Then

$$\liminf_{\varepsilon \rightarrow 0} \int_{B(0,1)} (w_{j,\varepsilon}^1)^2 dx = \liminf_{\varepsilon \rightarrow 0} \varsigma_\varepsilon^1 > 0. \tag{4.22}$$

Let  $w_{j,\varepsilon}^1 \rightharpoonup \tilde{v}_1$  as  $\varepsilon \rightarrow 0$ . By (2.3) and (4.19), we deduce that for any  $u \in H_0^1(y_{j,\varepsilon}^1 + \Lambda_\varepsilon)$ , as  $\varepsilon \rightarrow 0$ ,

$$\begin{aligned} & \left( a + b \int_{\mathbb{R}^3} |\nabla w_{j,\varepsilon}^1|^2 dx \right) \int_{\mathbb{R}^3} \nabla w_{j,\varepsilon}^1 \nabla u \, dx + \int_{\mathbb{R}^3} V(\varepsilon(x + y_{j,\varepsilon}^1)) w_{j,\varepsilon}^1 u \, dx \\ & - \int_{\mathbb{R}^3} |w_{j,\varepsilon}^1|^{p-2} w_{j,\varepsilon}^1 u \, dx \rightarrow 0, \end{aligned} \tag{4.23}$$

where  $y_{j,\varepsilon}^1 + \Lambda_\varepsilon = \{x + y_{j,\varepsilon}^1 : x \in \Lambda_\varepsilon\}$ . By (4.21) and (4.23), we know that  $\tilde{v}_1$  is a solution of (4.13) with  $i = 1$ . From Lemma 4.1, we obtain that there exists a positive constant  $\rho$  depending only on  $a, m_0$  and  $p$  such that

$$\|\tilde{v}_1\| \geq \rho. \tag{4.24}$$

Let  $v_{j,\varepsilon}^2 = v_{j,\varepsilon}^1 - \tilde{v}_1(\cdot - y_{j,\varepsilon}^1)$ . Since  $\tilde{v}_1 \in H^1(\mathbb{R}^3)$  is a solution of (4.13), we can deduce that  $\lim_{|x| \rightarrow \infty} \tilde{v}_1(x) = 0$ . Then by (4.20) and (4.21), we obtain that, for all  $\delta > 0$ ,

$$\lim_{\varepsilon \rightarrow 0} \|v_{j,\varepsilon}^2\|_{L^\infty(\mathbb{R}^3 \setminus (\Lambda_\varepsilon)^\delta)} = 0. \tag{4.25}$$

Since  $v_{j,\varepsilon}, \tilde{v}_0$  and  $\tilde{v}_1$  solves (2.4), (4.12), and (4.13) with  $i = 1$  respectively, we have

$$\begin{aligned} & -a \Delta v_{j,\varepsilon}^2 - b \int_{\mathbb{R}^3} |\nabla v_{j,\varepsilon}|^2 dx \Delta v_{j,\varepsilon}^2 - b \left( \int_{\mathbb{R}^3} |\nabla v_{j,\varepsilon}|^2 - A_j \right) \Delta \tilde{v}_0 \\ & - b \left( \int_{\mathbb{R}^3} |\nabla v_{j,\varepsilon}|^2 - A_j \right) \Delta \tilde{v}_1 + \xi_\varepsilon \chi_\varepsilon v_{j,\varepsilon}^2 + \xi_\varepsilon \chi_\varepsilon \tilde{v}_0 + \xi_\varepsilon \chi_\varepsilon \tilde{v}_1 \\ & + V(\varepsilon x) v_{j,\varepsilon}^2 + (V(\varepsilon x) - V(0)) \tilde{v}_0 + (V(\varepsilon x) - V(y_j^1)) \tilde{v}_1(\cdot - y_{j,\varepsilon}^1) \\ & = |v_{j,\varepsilon}|^{p-2} v_{j,\varepsilon} - |\tilde{v}_0|^{p-2} \tilde{v}_0 - |\tilde{v}_1|^{p-2} \tilde{v}_1(\cdot - y_{j,\varepsilon}^1). \end{aligned} \tag{4.26}$$

By (2.3), (4.26) and [19, Lemma 8.1], for any  $u \in H_0^1(\Lambda_\varepsilon)$  with  $\|u\|_{H_0^1(\Lambda_\varepsilon)} \leq 1$ , we have

$$\begin{aligned} & \langle \Gamma'_\varepsilon(v_{j,\varepsilon}^2), u \rangle \\ & = \int_{\mathbb{R}^3} (|v_{j,\varepsilon}|^{p-2} v_{j,\varepsilon} - |\tilde{v}_0|^{p-2} \tilde{v}_0 - |\tilde{v}_1(\cdot - y_{j,\varepsilon}^1)|^{p-2} \tilde{v}_1(\cdot - y_{j,\varepsilon}^1) - |v_{j,\varepsilon}^2|^{p-2} v_{j,\varepsilon}^2) u \, dx \\ & - \int_{\mathbb{R}^3} (V(\varepsilon x) - V(0)) \tilde{v}_0 u \, dx - \int_{\mathbb{R}^3} (V(\varepsilon x) - V(y_j^1)) \tilde{v}_1(\cdot - y_{j,\varepsilon}^1) u \, dx \\ & + b \left( \int_{\mathbb{R}^3} (|\nabla v_{j,\varepsilon}^2|^2 - |\nabla v_{j,\varepsilon}|^2) \int_{\mathbb{R}^3} \nabla v_{j,\varepsilon}^2 \nabla u \, dx \right. \\ & - b \left( \int_{\mathbb{R}^3} |\nabla v_{j,\varepsilon}|^2 - A_j \right) \int_{\mathbb{R}^3} \nabla \tilde{v}_0 \nabla u \, dx \\ & \left. - b \left( \int_{\mathbb{R}^3} |\nabla v_{j,\varepsilon}|^2 - A_j \right) \int_{\mathbb{R}^3} \nabla \tilde{v}_1(\cdot - y_{j,\varepsilon}^1) \nabla u \, dx + o(1) \right) \end{aligned} \tag{4.27}$$

as  $\varepsilon \rightarrow 0$ . Since  $\lim_{\varepsilon \rightarrow 0} \varepsilon y_{j,\varepsilon}^1 = y_j^1$ , we obtain

$$\sup \left\{ \int_{\mathbb{R}^3} (V(\varepsilon(x + y_{j,\varepsilon}^1)) - V(y_j^1)) \tilde{v}_1 u \, dx : u \in H^1(\mathbb{R}^3), \|u\| \leq 1 \right\} \rightarrow 0$$

as  $\varepsilon \rightarrow 0$ . It follows that

$$\sup \left\{ \int_{\mathbb{R}^3} (V(\varepsilon x) - V(y_j^1)) \tilde{v}_1(\cdot - y_{j,\varepsilon}^1) u \, dx : u \in H^1(\mathbb{R}^3), \|u\| \leq 1 \right\} \rightarrow 0 \tag{4.28}$$

as  $\varepsilon \rightarrow 0$ . By (4.18), (4.27), (4.28), and  $\int_{\mathbb{R}^3} |\nabla v_{j,\varepsilon}|^2 dx \rightarrow A_j$ , we have

$$\sup \left\{ \langle \Gamma'_\varepsilon(v_{j,\varepsilon}^2), u \rangle : u \in H_0^1(\Lambda_\varepsilon), \|u\|_{H_0^1(\Lambda_\varepsilon)} \leq 1 \right\} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \tag{4.29}$$

Let  $\{y_{j,\varepsilon}^2\} \subset \mathbb{R}^3$  be such that

$$\int_{B(y_{j,\varepsilon}^2, 1)} (v_{j,\varepsilon}^2)^2 dx = \sup_{y \in \mathbb{R}^3} \int_{B(y, 1)} (v_{j,\varepsilon}^2)^2 dx := \zeta_\varepsilon^2.$$

By (4.25), (4.29) and Lemma 4.3, we have that  $y_{j,\varepsilon}^2 \in \Lambda_\varepsilon$ ,  $\text{dist}(y_{j,\varepsilon}^2, \partial\Lambda_\varepsilon) \rightarrow +\infty$  as  $\varepsilon \rightarrow 0$ ,  $\lim_{\varepsilon \rightarrow 0} |y_{j,\varepsilon}^2| = +\infty$ ,  $\lim_{\varepsilon \rightarrow 0} |y_{j,\varepsilon}^2 - y_{j,\varepsilon}^1| = +\infty$  if  $\liminf_{\varepsilon \rightarrow 0} \zeta_\varepsilon^2 > 0$ .

Iterating the above argument we can know that the iteration procedure has to stop in finite number of steps, since  $\|v_{j,\varepsilon}\| \leq \eta_N$ ,  $\|\tilde{v}_i\| \geq \rho$  for all  $1 \leq i \leq m_j$ , and

$$\begin{aligned} \|v_{j,\varepsilon}^i\|^2 &= \|v_{j,\varepsilon}^{i-1}\|^2 - \|\tilde{v}_{i-1}\|^2 + o(1) \\ &= \|v_{j,\varepsilon}\|^2 - \sum_{n=1}^{i-1} \|\tilde{v}_n\|^2 + o(1), \quad \text{as } \varepsilon \rightarrow 0. \end{aligned} \tag{4.30}$$

Hence, we obtain  $m_j \in \mathbb{N}$  such that  $v_{j,\varepsilon}^{m_j+1} = v_{j,\varepsilon}^{m_j} - \tilde{v}_{m_j}(\cdot - y_{j,\varepsilon}^{m_j})$  satisfies

$$\sup_{y \in \mathbb{R}^3} \int_{B(y, 1)} (v_{j,\varepsilon}^{m_j+1})^2 dx = 0 \quad \text{as } \varepsilon \rightarrow 0. \tag{4.31}$$

It follows from the Lions lemma and (4.31) that for any  $2 < q < 2^* = 6$ ,

$$\int_{\mathbb{R}^3} |v_{j,\varepsilon}^{m_j+1}|^q dx = 0 \quad \text{as } \varepsilon \rightarrow 0.$$

Hence, we obtain  $m_j$  nonzero functions  $\tilde{v}_i$  in  $H^1(\mathbb{R}^3)$ ,  $1 \leq i \leq m_j$  and  $m_j$  sequences  $\{y_{j,\varepsilon}^i\} \subset \Lambda_\varepsilon$ ,  $1 \leq i \leq m_j$  such that the results (i), (ii), and (iii) hold. The proof is complete.  $\square$

Next, for each  $\varepsilon > 0$  and  $1 \leq j \leq N$ , we assume that  $y_{j,\varepsilon}^0 = 0$ . Let  $\varepsilon_n > 0$  be such that

$$\lim_{n \rightarrow \infty} \varepsilon_n = 0.$$

Up to a subsequence, we assume that  $\lim_{n \rightarrow \infty} \varepsilon_n y_{j,\varepsilon_n}^i$  exists for every  $i$ . We may write the set of these limiting points by

$$\{x_1^*, \dots, x_{s_j}^*\} = \left\{ \lim_{n \rightarrow \infty} \varepsilon_n y_{j,\varepsilon_n}^i : 0 \leq i \leq m_j \right\} \subset \bar{\Lambda}, \tag{4.32}$$

for some  $1 \leq s_j \leq m_j$ . Set

$$\theta_* = \begin{cases} \frac{1}{100} \min\{|x_s^* - x_{s'}^*| : 1 \leq s < s' \leq s_j\}, & \text{if } s_j \geq 2 \\ +\infty, & \text{if } s_j = 1. \end{cases} \tag{4.33}$$

**Lemma 4.5.** *If  $0 < \delta < \theta_*$ , then there exist two positive constants  $C$  and  $c$  independent of  $n$  such that, for every  $0 \leq i \leq m_j$ , when  $n$  is large enough,*

$$|\nabla v_{j,\varepsilon_n}(x)| + |v_{j,\varepsilon_n}(x)| \leq C \exp(-c\varepsilon_n^{-1}), \quad \text{for } x \in \partial B(y_{j,\varepsilon_n}^i, \delta\varepsilon_n^{-1}).$$

*Proof.* Define  $A_n^i = \overline{B(y_{j,\varepsilon_n}^i, \frac{3}{2}\delta\varepsilon_n^{-1})} \setminus B(y_{j,\varepsilon_n}^i, \frac{1}{2}\delta\varepsilon_n^{-1})$ . From  $0 < \delta < \theta_*$ , we can deduce that for every  $0 \leq i, i' \leq m_j$ ,

$$\text{dist}(y_{j,\varepsilon_n}^{i'}, A_n^i) \rightarrow \infty \quad \text{as } n \rightarrow \infty. \tag{4.34}$$

From Lemma 4.4, (4.34), and

$$\lim_{R \rightarrow \infty} \int_{\mathbb{R}^3 \setminus B(y_{j,\varepsilon_n}^i, R)} |\tilde{v}_i(\cdot - y_{j,\varepsilon_n}^i)|^p dx = 0, \quad 0 \leq i \leq m_j, \tag{4.35}$$

we obtain that

$$\lim_{R \rightarrow \infty} \int_{A_n^i} |v_{j,\varepsilon_n}|^p dx = 0 \quad \text{for } 0 \leq i \leq m_j. \tag{4.36}$$

Then there exists  $n_1 \in \mathbb{N}$  such that for  $n \geq n_1$ ,

$$\|v_{j,\varepsilon_n}\|_{L^\infty(A_n^i)}^{p-2} < a/2. \tag{4.37}$$

For  $m \in \mathbb{N}$ , let

$$R_m = \overline{B(y_{j,\varepsilon_n}^i, \frac{3}{2}\delta\varepsilon_n^{-1} - m)} \setminus B(y_{j,\varepsilon_n}^i, \frac{1}{2}\delta\varepsilon_n^{-1} + m).$$

Let  $\zeta_m$  be a cut-off function satisfying that  $0 \leq \zeta_m(t) \leq 1$  for all  $t \in \mathbb{R}$ ,

$$\zeta_m(t) = \begin{cases} 0, & \text{if } t \leq \frac{1}{2}\delta\varepsilon_n^{-1} + m - 1 \text{ or } t \geq \frac{3}{2}\delta\varepsilon_n^{-1} - m + 1, \\ 1, & \text{if } \frac{1}{2}\delta\varepsilon_n^{-1} + m \leq t \leq \frac{3}{2}\delta\varepsilon_n^{-1} - m, \end{cases}$$

and  $|\zeta'_m(t)| \leq 4$  for all  $t$ . For  $x \in \mathbb{R}^3$ , let  $\psi_m(x) = \zeta_m(|x - y_{j,\varepsilon_n}^i|)$ . Multiplying both sides of (2.4) by  $\psi_m^2 v_{j,\varepsilon_n}$  and integrating on  $\mathbb{R}^3$ , by (4.37) we have that

$$\begin{aligned} & (a + b \int_{\mathbb{R}^3} |\nabla v_{j,\varepsilon_n}|^2 dx) \int_{R_{m-1}} |\nabla v_{j,\varepsilon_n}|^2 \psi_m^2 dx + \int_{R_{m-1}} V(\varepsilon x) v_{j,\varepsilon_n}^2 \psi_m^2 dx \\ & + \xi_n \int_{R_{m-1}} \chi_{\varepsilon_n} v_{j,\varepsilon_n}^2 \psi_m^2 dx - \int_{R_{m-1}} |v_{j,\varepsilon_n}|^p \psi_m^2 dx \\ & \geq \min\{a + b\frac{A_j}{2}, \frac{m_0}{2}\} \int_{R_m} (|\nabla v_{j,\varepsilon_n}|^2 + v_{j,\varepsilon_n}^2) dx, \end{aligned} \tag{4.38}$$

and

$$\begin{aligned} & (a + b \int_{\mathbb{R}^3} |\nabla v_{j,\varepsilon_n}|^2 dx) \int_{R_{m-1}} |\nabla v_{j,\varepsilon_n}|^2 \psi_m^2 dx + \int_{R_{m-1}} V(\varepsilon x) v_{j,\varepsilon_n}^2 \psi_m^2 dx \\ & + \xi_n \int_{R_{m-1}} \chi_{\varepsilon_n} v_{j,\varepsilon_n}^2 \psi_m^2 dx - \int_{R_{m-1}} |v_{j,\varepsilon_n}|^p \psi_m^2 dx \\ & \leq 8(a + bA_j) \int_{R_{m-1} \setminus R_m} (|\nabla v_{j,\varepsilon_n}|^2 + v_{j,\varepsilon_n}^2) dx, \end{aligned} \tag{4.39}$$

where

$$\xi_n := 2\beta \left( \int_{\mathbb{R}^3} \chi_{\varepsilon_n} v_{j,\varepsilon_n}^2 dx - 1 \right)_+^{\beta-1},$$

here we have used the fact,  $\lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} |\nabla v_{j,\varepsilon_n}|^2 dx = A_j$ , then there exists  $n_2 \in \mathbb{N}$ , such that  $\int_{\mathbb{R}^3} |\nabla v_{j,\varepsilon_n}|^2 dx > \frac{A_j}{2}$  when  $n > n_2$ . By above inequalities, letting

$$C = \frac{8(a + bA_j)}{\min\{a + bA_j/2, m_0/2\}},$$

we have

$$\int_{R_m} (|\nabla v_{j,\varepsilon_n}|^2 + v_{j,\varepsilon_n}^2) dx \leq C \int_{R_m \setminus R_{m-1}} (|\nabla v_{j,\varepsilon_n}|^2 + v_{j,\varepsilon_n}^2) dx. \tag{4.40}$$

Let  $a_m = \int_{R_m} (|\nabla v_{j,\varepsilon_n}|^2 + v_{j,\varepsilon_n}^2) dx$ , we obtain that  $a_m \leq C(a_{m-1} - a_m)$  which gives  $a_m \leq \theta a_{m-1}$  with  $\theta = C/(1 + C) < 1$ . Therefore  $a_m \leq a_0 \theta^m$ . By Lemma 4.1, we

obtain  $a_0 \leq \eta_N^2$ . Hence, for sufficiently large  $n$ ,  $a_m \leq \eta_N^2 e^{mln\theta}$ . Denote  $[x]$  be the integer part of  $x$ . Choosing  $m = [\delta\varepsilon^{-1}/2] - 1$  and noting that  $[\delta\varepsilon^{-1}/2] - 1 \leq \delta\varepsilon^{-1}/4$  when  $n$  is large enough, we obtain

$$\begin{aligned} \int_{D_n^i} (|\nabla v_{j,\varepsilon_n}|^2 + v_{j,\varepsilon_n}^2) dx &\leq a_m \leq \eta_N^2 \exp(([\delta\varepsilon^{-1}/2] - 1)ln\theta) \\ &\leq \eta_N^2 \exp\left(\frac{1}{4}\delta\varepsilon_n^{-1}ln\theta\right), \end{aligned} \tag{4.41}$$

where

$$D_n^i = \overline{B(y_{j,\varepsilon_n}^i, \delta\varepsilon_n^{-1} + 1)} \setminus B(y_{j,\varepsilon_n}^i, \delta\varepsilon_n^{-1} - 1).$$

By the standard regularity of elliptic equations, we can obtain the result of this lemma. The proof is complete.  $\square$

**Lemma 4.6.** For each  $0 \leq i \leq m_j$ ,  $\lim_{\varepsilon \rightarrow 0} \text{dist}(\varepsilon y_{j,\varepsilon}^i, A) = 0$ .

*Proof.* If not, we assume that there exist  $1 \leq i_0 \leq m_j$  and  $\varepsilon_n > 0$  such that  $\lim_{n \rightarrow \infty} \varepsilon_n = 0$  and

$$\lim_{n \rightarrow \infty} \text{dist}(\varepsilon_n y_{j,\varepsilon_n}^{i_0}, A) > 0.$$

Without loss of generality, we assume that for every  $i$ ,  $\lim_{\varepsilon \rightarrow \infty} \varepsilon_n y_{j,\varepsilon_n}^i$  exists. By the condition of (A2), we deduce that there exists  $\delta' > 0$  such that, for every  $y \in \Lambda^{\delta'}$ ,

$$\inf_{x \in B(y, \delta') \setminus \Lambda} \nabla V(y) \cdot \nabla \text{dist}(x, \partial\Lambda) > 0. \tag{4.42}$$

Since  $y_j^{i_0} = \lim_{n \rightarrow \infty} \varepsilon_n y_{j,\varepsilon_n}^{i_0} \notin A$ , we infer that there exists  $\delta'' > 0$  such that, for sufficiently large  $n$ ,

$$\inf_{x \in B(y_j^{i_0}, \delta''\varepsilon_n^{-1})} \nabla V(\varepsilon_n x) \cdot \nabla V(\varepsilon_n y_{j,\varepsilon_n}^{i_0}) \geq \frac{1}{2} |\nabla V(y_j^{i_0})|^2 > 0. \tag{4.43}$$

Let  $0 < \delta_0 < \min\{\delta', \delta'', \vartheta_*\}$ . To abbreviate notation, let  $w_n = v_{j,\varepsilon_n}$  and  $\tilde{B} = B(y_{j,\varepsilon_n}^{i_0}, \delta_0\varepsilon_n^{-1})$ . Because  $0 < \delta_0 < \vartheta_*$  and Lemma 4.5, there exist constants  $c, C > 0$  independent of  $n$  such that

$$|\nabla w_n(x)| + |w_n(x)| \leq C \exp(-c\varepsilon_n^{-1}), \quad x \in \partial\tilde{B}, \tag{4.44}$$

for sufficiently large  $n$ . From Lemma 4.1, we infer that there exists  $C > 0$  independent of  $n$  such that

$$0 \leq \xi_n \leq C \quad \forall n. \tag{4.45}$$

We denote  $\vec{t}_n = \nabla V(\varepsilon_n y_{j,\varepsilon_n}^{i_0})$ . Since  $w_n$  solves (2.4) and the coefficients of (2.4) are all  $C^1$  functions, we infer that  $w_n$  is a  $C^2$  function. Multiplying both sides of (2.4) by  $\vec{t}_n \cdot \nabla w_n$  and integrating in  $\tilde{B}$ , we obtain the local Pohozaev type identity

$$\begin{aligned} &\frac{1}{2} \int_{\tilde{B}} (\varepsilon_n \vec{t}_n \cdot (\nabla V)(\varepsilon_n x) + \xi_n \nabla \chi_{\varepsilon_n} \vec{t}_n) w_n^2 dx \\ &= \left(a + b \int_{\mathbb{R}^3} |\nabla w_n|^2 dx\right) \int_{\partial\tilde{B}} \frac{1}{2} |\nabla w_n|^2 \vec{t}_n \cdot \nu \\ &\quad - \left(a + b \int_{\mathbb{R}^3} |\nabla w_n|^2 dx\right) \int_{\partial\tilde{B}} (\nabla w_n \cdot \nu)(\nabla w_n \cdot \vec{t}_n) ds \\ &\quad - \frac{1}{p} \int_{\partial\tilde{B}} |w_n|^p (\vec{t}_n \cdot \nu) ds, \end{aligned} \tag{4.46}$$

where  $\nu$  denotes the unit outward normal to the boundary of  $\tilde{B}$ .

From (4.43) and  $w(\cdot + y_{j,\varepsilon_n}^{i_0}) \rightharpoonup \tilde{v}_{i_0} \neq 0$  in  $H^1(\mathbb{R}^3)$ , we obtain that

$$\begin{aligned} & \varepsilon_n \int_{\tilde{B}} (\vec{t}_n \cdot (\nabla V)(\varepsilon_n x)) w_n^2 dx \\ & \geq \frac{\varepsilon_n}{2} |\nabla V(y_j^{i_0})|^2 \int_{B(0, \delta_0 \varepsilon_n^{-1})} w_n^2(\cdot + y_{j,\varepsilon_n}^{i_0}) dx \geq C\varepsilon_n, \end{aligned} \tag{4.47}$$

where

$$C = \frac{1}{4} |\nabla V(y_j^{i_0})|^2 \int_{\mathbb{R}^3} \tilde{v}_{i_0}^2 > 0.$$

By (4.42), we obtain that, for any  $x \in \tilde{B} \setminus \Lambda_{\varepsilon_n}$ ,

$$\vec{t}_n \cdot \nabla \chi_{\varepsilon_n}(x) \geq 0. \tag{4.48}$$

Furthermore, by (4.44) and (4.45), there exist two positive constants  $C, c$  independent of  $n$  such that, for sufficiently large  $n$ ,

$$\begin{aligned} & \left( a + b \int_{\mathbb{R}^3} |\nabla w_n|^2 dx \right) \int_{\partial \tilde{B}} \frac{1}{2} |\nabla w_n|^2 \vec{t}_n \cdot \nu \\ & - \left( a + b \int_{\mathbb{R}^3} |\nabla w_n|^2 dx \right) \int_{\partial \tilde{B}} (\nabla w_n \cdot \nu) (\nabla w_n \cdot \vec{t}_n) ds \\ & - \frac{1}{p} \int_{\partial \tilde{B}} |w_n|^p (\vec{t}_n \cdot \nu) ds \\ & \leq C \exp(-c\varepsilon_n^{-1}). \end{aligned} \tag{4.49}$$

This contradicts (4.46). The proof is complete.  $\square$

**Lemma 4.7.** *For any  $\delta > 0$ , there exist two positive constants  $C = C(\delta, N)$  and  $c = c(\delta, N) > 0$  independent of  $\varepsilon$  such that for every  $1 \leq j \leq N$ ,*

$$|v_{j,\varepsilon}(x)| \leq C \exp(-c \text{dist}(x, (A_\varepsilon)^\delta)), \quad x \in \mathbb{R}^3.$$

*Proof.* By (4.14), (4.35), and Lemma 4.6, we infer that there is  $R_0 > 0$  independent of  $\varepsilon$  such that, for sufficiently small  $\varepsilon > 0$ ,

$$|v_{j,\varepsilon}(x)|^{p-2} < m_0/2 \text{ if } \text{dist}(x, \overline{(A_\varepsilon)^\delta}) \geq R_0. \tag{4.50}$$

To prove the result, we only need to show that

$$|v_{j,\varepsilon}(x)| \leq C \exp(-c \text{dist}(x, (A_\varepsilon)^\delta)), \quad \text{if } \text{dist}(x, \overline{(A_\varepsilon)^\delta}) \geq R_0. \tag{4.51}$$

For  $m \in \mathbb{N}$ , let  $B_m = \{x \in \mathbb{R}^3 : \text{dist}(x, \overline{(A_\varepsilon)^\delta}) \geq R_0 - m + 1\}$ . Let  $\rho_m$  be a cut-off function satisfying  $0 \leq \rho_m(t) \leq 1$ ,  $|\rho'_m(t)| \leq 4$  for all  $t \in \mathbb{R}$  and

$$\rho_m(t) = \begin{cases} 0, & \text{if } t \leq R_0 + m - 1, \\ 1, & \text{if } t \leq R_0 + m. \end{cases}$$

For  $x \in \mathbb{R}^3$ , set  $\phi_m(x) = \rho_m(\text{dist}(x, \overline{(A_\varepsilon)^\delta}))$ . Multiplying both sides of (2.4) by  $\phi_m^2 v_{j,\varepsilon}$  and integrating on  $\mathbb{R}^3$ , we have

$$\begin{aligned} & \left( a + b \int_{\mathbb{R}^3} |\nabla v_{j,\varepsilon}|^2 dx \right) \int_{B_m} |\nabla v_{j,\varepsilon}|^2 \phi_m^2 dx + \int_{B_m} V(\varepsilon x) v_{j,\varepsilon}^2 \phi_m^2 dx \\ & + \xi_\varepsilon \int_{B_m} \chi_\varepsilon v_{j,\varepsilon}^2 \phi_m^2 dx - \int_{B_m} |v_{j,\varepsilon}|^p \phi_m^2 dx \\ & \leq 8(a + bA_j) \int_{B_m \setminus B_{m+1}} (|\nabla v_{j,\varepsilon}|^2 + v_{j,\varepsilon}^2) dx, \end{aligned} \tag{4.52}$$

and by (4.50), we obtain

$$\begin{aligned}
 & \left( a + b \int_{\mathbb{R}^3} |\nabla v_{j,\varepsilon}|^2 dx \right) \int_{B_{m+1}} |\nabla v_{j,\varepsilon}|^2 \phi_m^2 dx + \int_{B_{m+1}} V(\varepsilon x) v_{j,\varepsilon}^2 \phi_m^2 dx \\
 & + \xi_\varepsilon \int_{B_{m+1}} \chi_\varepsilon v_{j,\varepsilon}^2 \phi_m^2 dx - \int_{B_{m+1}} |v_{j,\varepsilon}|^p \psi_m^2 dx \\
 & \geq \min \left\{ a + b \frac{A_j}{2}, \frac{m_0}{2} \right\} \int_{B_{m+1}} (|\nabla v_{j,\varepsilon}|^2 + v_{j,\varepsilon}^2) dx,
 \end{aligned} \tag{4.53}$$

where  $\xi_\varepsilon$  is defined by (4.15). From the above two inequalities, we have

$$\left( a + b \int_{\mathbb{R}^3} |\nabla v_{j,\varepsilon}|^2 dx \right) \int_{B_m} |\nabla v_{j,\varepsilon}|^2 \phi_m^2 dx \leq C \int_{B_m \setminus B_{m+1}} (|\nabla v_{j,\varepsilon}|^2 + v_{j,\varepsilon}^2) dx, \tag{4.54}$$

where  $C = 8 / \min \{ a + bA_j/2, m_0/2 \}$ . Then similar to the proof of Lemma 4.4, we can obtain (4.51). The proof is complete.  $\square$

**Lemma 4.8.** *There exist  $\varepsilon_N > 0$  such that if  $0 < \varepsilon < \varepsilon_N$ , then for every  $1 \leq j \leq N$ ,  $v_{j,\varepsilon}$  is a solution of (2.4).*

*Proof.* Since  $A$  is a compact subset of  $\Lambda$ ,  $\text{dist}(A, \partial\Lambda) > 0$ . By choosing  $0 < \delta < \text{dist}(A, \partial\Lambda)$ , from Lemma 4.7, we obtain that, for every  $1 \leq j \leq N$ ,

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^3} \chi_\varepsilon v_{j,\varepsilon}^2 dx = 0. \tag{4.55}$$

It follows from that  $Q_\varepsilon(v_{j,\varepsilon}) = 0$  if  $\varepsilon > 0$  is small enough. Hence there exists  $\varepsilon_N > 0$  such that if  $0 < \varepsilon < \varepsilon_N$ , then for every  $1 \leq j \leq N$ ,  $v_{j,\varepsilon}$  is a solution of (2.4). The proof is complete.  $\square$

*Proof of Theorem 1.1.* By Proposition 3.4 and Lemmas 4.7 and 4.8, we can obtain the results for Theorem 1.1.  $\square$

### 5. APPENDIX

In this section, we give the proof of Proposition 3.3. Let  $G$  is an operator on  $H^1(\mathbb{R}^3)$ . For  $u \in H^1(\mathbb{R}^3)$ , we define  $w = G(u)$  by

$$- \left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx \right) \Delta w + V(\varepsilon x)w + 2\beta \left( \int_{\mathbb{R}^3} \chi_\varepsilon u^2 dx - 1 \right)_+^{\beta-1} \chi_\varepsilon w = |u|^{p-2}u, \tag{5.1}$$

where  $w \in H^1(\mathbb{R}^3)$ . We can check that  $G$  is odd on  $H^1(\mathbb{R}^3)$ .

**Lemma 5.1.**  *$G$  is well defined and continuous on  $H^1(\mathbb{R}^3)$ .*

*Proof.* Since

$$\xi(u) := 2\beta \left( \int_{\mathbb{R}^3} \chi_\varepsilon u^2 dx - 1 \right)_+^{\beta-1} \tag{5.2}$$

is non-negative,  $G$  is well defined and continuous on  $H^1(\mathbb{R}^3)$ . If  $u_n \rightarrow u$  in  $H^1(\mathbb{R}^3)$ , we can obtain that

$$\begin{aligned}
 & \min \left\{ a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx, a_0 \right\} \|A(u_n) - A(u)\|^2 \\
 & \leq \int_{\mathbb{R}^3} \left| |u_n|^{p-2}u_n - |u|^{p-2}u \right| |A(u_n) - A(u)| dx \\
 & \quad + |\xi(u_n) - \xi(u)| \int_{\mathbb{R}^3} \chi_\varepsilon |A(u_n) - A(u)| |A(u)| dx.
 \end{aligned}$$

Since  $|\xi(u_n) - \xi(u)| \rightarrow 0$  is obvious, by Sobolev embedding, we can get the conclusion. The proof is complete.  $\square$

**Lemma 5.2.** *For any  $u \in H^1(\mathbb{R}^3)$ ,*

$$\begin{aligned} \langle \Gamma'_\varepsilon(u), u - A(u) \rangle &= \left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx \right) \int_{\mathbb{R}^3} |\nabla(u - A(u))|^2 dx \\ &\quad + \int_{\mathbb{R}^3} V(\varepsilon x)(u - A(u))^2 dx + \xi(u) \int_{\mathbb{R}^3} \chi_\varepsilon(u - A(u))^2 dx. \end{aligned} \tag{5.3}$$

and for any  $u \in H^1(\mathbb{R}^3)$ , there exists a positive constant  $C$  such that

$$\|\Gamma'_\varepsilon(u)\| \leq \|u - A(u)\| \left( \max \left\{ a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx, 1 \right\} + C\|u\|^{2\beta-2} \right). \tag{5.4}$$

*Proof.* By a direct computation, we can get (5.3). In the following, we only need to show (5.4). For any  $\psi \in H^1(\mathbb{R}^3)$ ,

$$\begin{aligned} \langle \Gamma'_\varepsilon(u), \psi \rangle &= \left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx \right) \int_{\mathbb{R}^3} \nabla u \nabla \psi dx + \int_{\mathbb{R}^3} V(\varepsilon x) u \psi dx \\ &\quad + \xi(u) \int_{\mathbb{R}^3} \chi_\varepsilon u \psi dx - \int_{\mathbb{R}^3} |u|^{p-2} u \psi dx \end{aligned} \tag{5.5}$$

Multiplying (5.1) by  $\psi$ , and then then integrating on both sides, we obtain

$$\begin{aligned} &\left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx \right) \int_{\mathbb{R}^3} \nabla u \nabla \psi dx + \int_{\mathbb{R}^3} V(\varepsilon x) u \psi dx + \xi(u) \int_{\mathbb{R}^3} \chi_\varepsilon u \psi dx \\ &= \int_{\mathbb{R}^3} |u|^{p-2} u \psi dx. \end{aligned} \tag{5.6}$$

By (5.5) and (5.6), we have

$$\begin{aligned} \langle \Gamma'_\varepsilon(u), \psi \rangle &= \left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx \right) \int_{\mathbb{R}^3} \nabla(u - w) \nabla \psi dx + \int_{\mathbb{R}^3} V(\varepsilon x)(u - w) \psi dx \\ &\quad + \xi(u) \int_{\mathbb{R}^3} \chi_\varepsilon(u - w) \psi dx. \end{aligned}$$

Then

$$\begin{aligned} |\langle \Gamma'_\varepsilon(u), \psi \rangle| &\leq \max \left\{ a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx, 1 \right\} \|u - A(u)\| \|\psi\| \\ &\quad + C\|u\|^{2\beta-2} \|u - A(u)\| \|\psi\| \end{aligned}$$

that is for any  $u \in H^1(\mathbb{R}^3)$ , we obtain that

$$\|\Gamma'_\varepsilon(u)\| \leq \|u - A(u)\| \left( \max \left\{ a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx, 1 \right\} + C\|u\|^{2\beta-2} \right).$$

The proof is complete.  $\square$

**Lemma 5.3.** *There exists  $\sigma_0 > 0$  such that for  $\sigma \in (0, \sigma_0)$ ,*

$$G(\partial(P_-^\sigma)) \subset P_-^\sigma, \quad G(\partial(P_+^\sigma)) \subset P_+^\sigma.$$

*Proof.* We only proof  $G(\partial(P_-^\sigma)) \subset P_-^\sigma$ . For  $u \in H^1(\mathbb{R}^3)$ , let  $w = G(u), C_1 := (\min\{a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx, m_0\})^{-1}$ . We obtain

$$\text{dist}_{H^1}(w, P_-) \|w^+\| \leq C_1 \|w^+\|^2$$

$$\begin{aligned}
&\leq \left\{ \left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx \right) \int_{\mathbb{R}^3} |\nabla w|^2 dx + \int_{\mathbb{R}^3} V(\varepsilon x) w^2 dx \right\} \\
&= C_1 \left\{ \left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx \right) \int_{\mathbb{R}^3} w w^+ dx + \int_{\mathbb{R}^3} V(\varepsilon x) w w^+ dx \right\} \\
&= C_1 \int_{\mathbb{R}^3} |u|^{p-2} u w^+ dx - C_1 \xi(u) \int_{\mathbb{R}^3} \chi_\varepsilon w w^+ \psi dx \\
&\leq C_1 \int_{\mathbb{R}^3} |u|^{p-2} u w^+ dx \\
&\leq C_1 \int_{\mathbb{R}^3} |u|^{p-2} u^+ w^+ dx \\
&\leq C_1 \|u^+\|_p^{p-1} \|w^+\|_p \\
&= C_1 (\text{dist}_{L^p}(u, P_-))^{p-1} \|w^+\|_p \\
&\leq C_1 C (\text{dist}_{H^1}(u, P_-))^{p-1} \|w^+\|.
\end{aligned}$$

Then we can infer that  $\text{dist}_{H^1}(w, P_-) \leq C\sigma^{p-1}$ . For  $\sigma > 0$  small enough, we can get the conclusion. The proof is complete.  $\square$

We need to have a locally Lipschitz perturbation of  $G$ , here  $G$  may be only continuous. We  $E_0 = H^1(\mathbb{R}^3) \setminus K$ , where  $K$  is the set of fixed points of  $G$ , that is, the set of critical points of  $\Gamma_\varepsilon$ .

**Lemma 5.4.** *There exists a locally Lipschitz continuous operator  $B : E_0 \rightarrow H^1(\mathbb{R}^3)$  such that*

- (1)  $B(\partial(P_+^\sigma)) \subset P_+^\sigma$  and  $B(\partial(P_-^\sigma)) \subset P_-^\sigma$  for  $\sigma \in (0, \sigma_0)$ ;
- (2)  $\frac{1}{2}\|u - B(u)\| \leq \|u - G(u)\| \leq 2\|u - B(u)\|$  for  $u \in E_0$ ;
- (3)  $\langle \Gamma_\varepsilon(u), u - B(u) \rangle \geq \frac{1}{2}\|u - G(u)\|^2$  for  $u \in E_0$ ;
- (4)  $B$  is odd.

Since the proof of the above lemma is similar to the proofs of [1, Lemma 4.1] and [2, Lemma 7], we omit it here.

Note that  $\Gamma_\varepsilon$  satisfies  $(PS)_c$  condition for  $c < L$  if  $0 < \varepsilon < \varepsilon_L$ , by using the map  $B$  and similar argument of [14, Lemma 3.5], we can obtain the following lemma.

**Lemma 5.5.** *Assume that  $0 < \varepsilon < \varepsilon_L$ ,  $c < L$ , and  $N$  is a symmetric closed neighborhood of  $K_c$ . Then there exist a positive constant  $\iota_0$  such that for  $0 < \iota < \iota_0$ , there exists a continuous map  $\zeta : [0, 1] \times H^1(\mathbb{R}^3) \rightarrow H^1(\mathbb{R}^3)$  satisfying*

- (1)  $\zeta(0, u) = u$  for all  $u \in H^1(\mathbb{R}^3)$ ;
- (2)  $\zeta(t, u) = u$  for  $t \in [0, 1]$ ,  $\Gamma_\varepsilon(u) \notin [c - \iota', c + \iota']$ ;
- (3)  $\zeta(t, -u) = -\zeta(t, u)$  for all  $t \in [0, 1]$  and  $u \in H^1(\mathbb{R}^3)$ ;
- (4)  $\zeta(1, (\Gamma_\varepsilon)^{c+\iota}) \subset (\Gamma_\varepsilon)^{c-\iota}$ ;
- (5)  $\zeta(t, \partial(P_+^\sigma)) \subset P_+^\sigma$ ,  $\zeta(t, \partial(P_-^\sigma)) \subset P_-^\sigma$ ,  $\zeta(t, P_+^\sigma) \subset P_+^\sigma$ ,  $\zeta(t, P_-^\sigma) \subset P_-^\sigma$ ,  $t \in [0, 1]$ .

*Proof of Proposition 3.3.* Set  $D$  is a closed symmetric neighborhood of  $K_c \setminus W$ . Notice that  $N = D \cup \bar{P}_+^\sigma \cup \bar{P}_-^\sigma$  is a closed symmetric neighborhood of  $K_c$ . By Lemma 5.5, we can choose  $\eta = \zeta(1, \cdot)$  in Definition 3.1.  $\square$

**Acknowledgments.** This research was supported by the National Natural Science Foundation of China (11801400) and (11571187), and by the Scientific Research Program of Tianjin Education Commission (2020KJ045).

## REFERENCES

- [1] T. Bartsch, Z. L. Liu; *On a superlinear elliptic  $p$ -Laplacian equation*, Differential Equations 198 (2004), 149–175.
- [2] T. Bartsch, Z. L. Liu, T. Weth; *Nodal solutions of a  $p$ -Laplacian equation*, Proc. Lond. Soc. 91 (3) (2005), 129–152.
- [3] J. Byeon, Z. Q. Wang; *Standing waves with a critical frequency for nonlinear Schrödinger equations*, Arch. Ration. Mech. Anal. 165 (4) (2002), 295–316.
- [4] S. W. Chen, Z. Q. Wang; *Localized nodal solutions of higher topological type for semiclassical nonlinear Schrödinger equations*, Calc. Var. 56 (1) (2017), 1–26.
- [5] M. Chipot, B. Lovat; *Some remarks on nonlocal elliptic and parabolic problems*, Nonlinear Anal. 30 (7) (1997), 4619–4627.
- [6] S. Cingolani, N. Lazzo; *Multiple semiclassical standing waves for a class of nonlinear Schrödinger equations*, Topo. Methods Nonlinear Anal. 10 (1997), 1–13.
- [7] M. J. Estaban, P. L. Lions; *Existence and nonexistence results for semilinear elliptic problem in unbounded domains*, Proc. R. Soc. Edinb. Sect. A 93 (1982/83), 1–14.
- [8] G. M. Figueiredo, N. Ikoma, J. R. Santos Junior; *Existence and concentration result for the Kirchhoff type equations with general nonlinearities*, Arch Ration Mech. Anal. 213 (2014), 931–979.
- [9] A. Floer, A. Weinstein; *Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential*, J. Funct. Anal. 69 (1986), 397–408.
- [10] X. M. He, W. M. Zou; *Existence and concentration behavior of positive solutions for Kirchhoff equation in  $\mathbb{R}^3$* , J. Differential Equations 2 (2012), 1813–1834.
- [11] Y. He; *Concentrating bounded states for a class of singularly perturbed Kirchhoff type equations with a general nonlinearity*, J. Differential Equations 261 (2016), 6178–6220.
- [12] G. Kirchhoff; *Mechanik*. Teubner, Leipzig, 1883.
- [13] G. B. Li, H. Y. Ye; *Existence of positive ground state solutions for the nonlinear Kirchhoff type equations in  $\mathbb{R}^3$* , J. Differential equations 257 (2014), 566–600.
- [14] J. Q. Liu, X. Q. Liu, Z. Q. Wang; *Multiple mixed states of nodal solutions for nonlinear Schrödinger systems*, Calc. Var. Partial Differ. Equations, 52 (2015), 565–586.
- [15] Y. G. Oh; *Corrections to existence of semi-classical bounded states of nonlinear Schrödinger equations with potential on the class  $(V)_a$* , Comm. Partial Differential Equations 14 (1989), 833–834.
- [16] P. H. Rabinowitz; *On a class of nonlinear Schrödinger equations*, Z. Angew. Math. Phys. 43(2) (1992), 270–292.
- [17] P. H. Rabinowitz; *Minimax methods in critical point theory with applications to differential equations*. In: CBMS regional conference series in mathematics, Vol. 65. American Mathematical Society, Providence, 1986.
- [18] Y. Sato, K. Tanaka; *Sign-changing multi-bump solutions for nonlinear Schrödinger equations with steep potential wells*, Trans. Am. Math. Soc. 361 (2009), 6205–6253.
- [19] M. Willem; *Minimax Theorems*, Progress in Nonlinear Differential Equations and Their Applications, 24, Birkhäuser, Boston, 1996.
- [20] J. J. Zhang, Z. J. Chen, W. M. Zou; *Standing waves for nonlinear Schrödinger equations involving critical growth*, J. Lond. Math. Soc. 90 (2014), 827–844.
- [21] J. J. Zhang, Z. S. Liu, M. Squassina; *Modulational stability of ground states to nonlinear Kirchhoff equations*, J. Math. Anal. Appl. 477 (2019), 844–859.

LIXIA WANG

SCHOOL OF SCIENCES, TIANJIN CHENGJIAN UNIVERSITY, TIANJIN 300384, CHINA

Email address: wanglixia0311@126.com