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L^p-L^q ESTIMATES FOR DAMPED WAVE EQUATIONS WITH ODD INITIAL DATA

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Dedicated to the memory of Professor Tsutomu Arai

Abstract. We study the Cauchy problem for the damped wave equation. In a previous paper [16] the author has shown the L^p-L^q estimates between the solutions of the damped wave equation and the solutions of the corresponding heat equation. In this paper, we show new L^p-L^q estimates for the damped wave equation with odd initial data.

1. Introduction

Consider the Cauchy problem for the damped wave equation

$$\partial_t^2 u - \Delta u + 2a\partial_t u = 0, \quad (t,x) \in (0,\infty) \times R^n \tag{1.1}$$

with initial data

$$u(0,x) = \varphi_0(x), \quad \partial_t u(0,x) = \varphi_1(x), \quad x \in R^n, \tag{1.2}$$

where a is a positive constant, $\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$ for $j = 1, 2, \dots, n$ and $\Delta = \partial_1^2 + \dots + \partial_n^2$ is the Laplace operator in R^n. Here and after we denote $\partial_x^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$ and $|\alpha| = \alpha_1 + \dots + \alpha_n$ for a multi-index of non-negative integers $\alpha = (\alpha_1, \dots, \alpha_n)$, and $\nabla h = (\partial_1 h, \dots, \partial_n h)$.

Several authors have indicated the diffusive structure of problem (1.1)–(1.2) as $t \to \infty$; see for example [1, 7, 10, 12, 16, 17]. Recently the author has shown the L^p-L^q estimates of the difference between the solution of problem (1.1)–(1.2) and the solution of the corresponding heat equation

$$2a\partial_t \phi - \Delta \phi = 0, \quad (t,x) \in (0,\infty) \times R^n \tag{1.3}$$

with initial data

$$\phi(0,x) = \varphi_0(x) + \varphi_1(x)/2a, \quad x \in R^n. \tag{1.4}$$

We use the standard function spaces $L^p = L^p(R^n)$, $L^p = H_p^0$ and $H_p^s = H_p^s(R^n) = (1-\Delta)^{-s/2} L^p$ equipped with the norms

$$\|f\|_{H_p^s} \equiv \|f\|_{s,p} \equiv \|\mathcal{F}^{-1}((1+|\xi|^2)^{s/2}\hat{f})\|_p,$$

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where $\|f\|_p$ denotes the usual L^p-norm. $\mathcal{F}$ denotes the Fourier transformation:

$$(\mathcal{F}f)(\xi) \equiv \widehat{f}(\xi) \equiv \big(\frac{1}{2\pi}\big)^{n/2} \int_{R^n} e^{-ix\cdot\xi} f(x)\, dx,$$

$\mathcal{F}^{-1}$ denotes an inverse of $\mathcal{F}$, and $*$ denotes the convolution with respect to x;

$$(f * g)(x) = \int_{R^n} f(x-y)g(y)\, dy.$$

Let $X_1 \cap \dots \cap X_m$ be the normed space equipped with norm $\|\cdot\|_{X_1\cap\dots\cap X_m} \equiv \|\cdot\|_{X_1} + \dots + \|\cdot\|_{X_m}$ for normed spaces $X_1, \dots, X_m$, and let $[\mu]$ denote the greatest integer that does not exceed μ.

To illustrate the decay profiles of problem (1.1)–(1.2) we set $\varphi_0(x) = \varphi_1(x) = x_1 \dots x_d \exp(-a|x|^2/2)$, where $d \in [0, n]$ is an integer. Let u and ϕ be the solutions of problem (1.1)–(1.2) and problem (1.3)–(1.4), respectively. Since

$$\phi(t,x) = (1 + 1/2a)(t+1)^{-n/2-d} x_1 \dots x_d \exp\big(-\frac{a|x|^2}{2(t+1)}\big),$$

it follows that

$$\|\phi(t,\cdot)\|_p = C(1+t)^{-n/2(1-1/p)-d/2}, \quad 1 \le p \le \infty, t > 0.$$

Hence, Theorems 1.1–1.2 below show that

$$\widetilde{C}_1(1+t)^{-n(1-1/p)-d/2} \le \|u(t,\cdot)\|_p \le \widetilde{C}_2(1+t)^{-n/2(1-1/p)-d/2} \tag{1.5}$$

for any $p \in [1, \infty]$ and sufficiently large $t > 0$, where $\widetilde{C}_1$ and $\widetilde{C}_2$ are positive constants that depend only on n, d, p and a. When $d = 0$, (1.5) indicates that the optimal decay rate of L^p norm of the solution to (1.1) is $(1+t)^{-n/2(1-1/p)}$ as $t \to \infty$. When $d \ge 1$, (1.5) also shows that the solution decays faster than solutions with general initial data. This faster decay seems to be caused by the fact $(\partial/\partial\xi)^\alpha \widehat{u}(t,0) = 0$ for $|\alpha| < d$. When the initial data are odd in the sense of (1.6) below, the solution u of (1.1) satisfies $(\partial/\partial\xi)^\alpha \widehat{u}(t,0) = 0$ for $|\alpha| < d$. Hence, we may expect a new L^p-L^q estimates of the solutions of problem (1.1)–(1.2), when initial data are odd.

The aim in this paper is to show the new L^p-L^q estimates to the solutions of problem (1.1)–(1.2), when the initial data (φ_0, φ_1) are odd in the sense of (1.6). These new L^p-L^q estimates imply the new decay estimates to the solution of problem (1.1)–(1.2).

Let $d \in [1, n]$ be an integer, and $x \equiv (x', x'') \equiv (x_1, \dots, x_d, x_{d+1}, \dots, x_n)$. A function $f(x)$ defined on R^n is said to be odd with respect to x' when it satisfies

$$f(x_1, \dots, -x_j, \dots, x_n) = -f(x_1, \dots, x_j, \dots, x_n), \quad (j = 1, \dots, d). \tag{1.6}$$

Define the weight function $P(\cdot)$ by

$$P(x) = (1 + x_1^2)^{1/2} \cdot \dots \cdot (1 + x_d^2)^{1/2}. \tag{1.7}$$

Our first result is as follows.

Theorem 1.1 (Estimate of the low frequency part). *Let $1 \le q \le p \le \infty$, $\epsilon > 0$, and let $b > 0$ be constants. Let v be the solution of* (1.1) *with initial data*

$$v(0,x) = v_0(x), \quad \partial_t v(0,x) = v_1(x), \quad x \in R^n.$$

Let V be the solution of (1.3) with initial data

$$V(0,x) = v_0(x) + v_1(x)/2a, \quad x \in R^n.$$

Assume that the function v_i is odd with respect to x' and it satisfies

$$P(\cdot)v_i \in L^q, \quad \operatorname{supp}\widehat{v}_i \subset \{\xi; |\xi| \le b\} \quad (i = 0, 1).$$

Then, for any $\theta \in [0,1]$, for a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ and for a non-negative integer k, the following estimates holds:

$$\begin{aligned} &\|P(\cdot)^\theta \partial_t^k \partial_x^\alpha (v(t,\cdot) - V(t,\cdot))\|_p \\ &\le C(1+t)^{-n\delta(p,q)-|\alpha|/2-k-(1-\theta)d/2-1+\epsilon}(\|P(\cdot)v_0\|_q + \|P(\cdot)v_1\|_q) \end{aligned}$$

for some constant $C = C(p,q,\epsilon) > 0$, where $\delta(p,q) = 1/2q - 1/2p$. When $1 < q < p < \infty$, $p = \infty$ and $q = 1$ or $p = q = 2$, we may take $\epsilon = 0$ in the above estimates.

The decay property of the solution to (1.3) with odd initial data (Proposition 3.1 below, see also [13]) shows the following estimates.

Corollary 1.1. *Under the assumptions of Theorem 1.1,*

$$\|P(\cdot)^\theta \partial_t^k \partial_x^\alpha v(t,\cdot)\|_p \le C(1+t)^{-n\delta(p,q)-|\alpha|/2-k-(1-\theta)d/2}(\|P(\cdot)v_0\|_q + \|P(\cdot)v_1\|_q).$$

Similar arguments to ones in [16] give the following estimates.

Theorem 1.2 (Estimate of high frequency part)**.** *Let $1 < q \le p < \infty$ and $\theta = 0, 1$. Assume that $P(\cdot)^\theta w_i \in L^q$, $\operatorname{supp}\widehat{w}_i \subset \{\xi; |\xi| \ge 2a\}$ for $i = 0, 1$. Then the solution w of (1.1) with initial data*

$$w(0,x) = w_0(x), \quad \partial_t w(0,x) = w_1(x), \quad x \in R^n$$

satisfies

$$\begin{aligned} &\|P(\cdot)^\theta (w(t,\cdot) - e^{-at}\mathcal{F}^{-1}(M_0(t,\cdot)\widehat{w}_0 + M_1(t,\cdot)\widehat{w}_1))\|_p \\ &\le C(p,q)e^{-at}(1+t)^N(\|P(\cdot)^\theta w_0\|_q + \|P(\cdot)^\theta w_1\|_q) \end{aligned}$$

for some constant $N = N(n) > 0$ and $C(p,q) > 0$, where

$$\begin{aligned} M_1(t,\xi) = \frac{1}{\sqrt{|\xi|^2 - a^2}}\Big(&\sin t|\xi| \sum_{0 \le k < (n-1)/4} \frac{(-1)^k}{(2k)!} t^{2k}\Theta(\xi)^{2k} \\ &- \cos t|\xi| \sum_{0 \le k < (n-3)/4} \frac{(-1)^k}{(2k+1)!} t^{2k+1}\Theta(\xi)^{2k+1}\Big), \\ M_0(t,\xi) = \cos t|\xi| &\sum_{0 \le k < (n+1)/4} \frac{(-1)^k}{(2k)!} t^{2k}\Theta(\xi)^{2k} \\ &+ \sin t|\xi| \sum_{0 \le k < (n-1)/4} \frac{(-1)^k}{(2k+1)!} t^{2k+1}\Theta(\xi)^{2k+1} + aM_1(t,\xi), \end{aligned}$$

and $\Theta(\xi) \equiv \Theta(|\xi|) \equiv |\xi| - \sqrt{|\xi|^2 - a^2}$.

Corollary 1.2. *Let $m = [n/2]$ and $\max(0, 1/2 - 1/2m) < 1/p < \min(1, 1/2 + 1/2m)$. Under the assumptions in Theorem 1.2, the following estimate holds;*

$$\|P(\cdot)^\theta w(t,\cdot)\|_p \le Ce^{-at/2}(\|P(\cdot)^\theta w_0\|_{1,p} + \|P(\cdot)^\theta w_1\|_p).$$

2. Preliminaries

In this section we state the preliminary results necessary for the proofs. $J_\mu(s)$ is the Bessel function of order μ. We shall denote $\widetilde{J}_\mu(s) = J_\mu(s)/s^\mu$ according to Levandosky [9]. Here and after we denote $g(s) = O(|s|^\sigma)$ when $|g(s)| \le C|s|^\sigma$ for a constant σ.

Lemma 2.1 ([9, 16]). *Assume that μ is not a negative integer. Then it follows that:*

(1) $s\widetilde{J}'_\mu(s) = \widetilde{J}_{\mu-1}(s) - 2\mu\widetilde{J}_\mu(s)$.
(2) $\widetilde{J}'_\mu(s) = -s\widetilde{J}_{\mu+1}(s)$.
(3) $\widetilde{J}_{-1/2}(s) = \sqrt{\frac{\pi}{2}}\cos s$.
(4) *If* $\operatorname{Re}\mu$ *is fixed, then*

$$|\widetilde{J}_\mu(s)| \le Ce^{\pi|\mathrm{Im}\ \mu|}, \quad (|s| \le 1),$$
$$J_\mu(s) = Cs^{-1/2}\cos(s - \frac{\mu}{2}\pi - \frac{\pi}{4}) + O(e^{2\pi|\mathrm{Im}\ \mu|}|s|^{-3/2}), \quad (|s| \ge 1).$$

(5) $r^2\rho\widetilde{J}_{\mu+1}(r\rho) = -\frac{\partial}{\partial\rho}\widetilde{J}_\mu(r\rho)$.

The following lemmas are well-known.

Lemma 2.2 ([18]). *Assume that $\widehat{f} \in L^p$ $(1 \le p \le 2)$ is a radial function. Then*

$$f(x) = c\int_0^\infty g(\rho)\rho^{n-1}\widetilde{J}_{n/2-1}(|x|\rho)\,d\rho, \quad g(|\xi|) \equiv \widehat{f}(\xi).$$

Lemma 2.3 (Young). *Let $1 \le q \le p \le \infty$ satisfy $1 - 1/r = 1/q - 1/p$, then the following estimate holds for any $f \in L^q$ and $g \in L^r$:*

$$\|f * g\|_p \le C\|f\|_q\|g\|_r.$$

Lemma 2.4 (Hardy-Littlewood-Sobolev). *Let $1 < q < p < \infty$ satisfy $1 - 1/r = 1/q - 1/p$. Assume that $|g(x)| \le A|x|^{-n/r}$, where A is a constant. Then the following estimate holds for any $f \in L^p$:*

$$\|f * g\|_p \le C(p,q)A\|f\|_q.$$

3. Proof of Theorem 1.1

Let V be the solution of the heat equation

$$2a\partial_t V(t,x) - \Delta V(t,x) = 0, \quad t > 0, x \in R^n \tag{3.1}$$

with initial data

$$V(0,x) = V_0(x), \quad x \in R^n. \tag{3.2}$$

Assume that the function V_0 is odd with respect to x' and $V_0 \in L^q$ for some $1 \le q \le \infty$. Then, $V(t,\cdot)$ is also odd with respect to x'. Arguments similar to those in [13] and [16] give the following result.

Proposition 3.1 (Meier [13]). *Let $1 \le q \le p \le \infty$, $0 \le \theta_1, \dots, \theta_d \le 1$ and $b > 0$ be constants. Assume that V_0 is odd with respect to x', $P(\cdot)V_0 \in L^q$ and $\hat{V}_0(\xi) = 0$ for $|\xi| \ge b$. Let V be the solution of the Cauchy problem (3.1)–(3.2). Then, for $t > 0$, $V(t,\cdot)$ is odd with respect to x' and $\hat{V}(t,\xi) = 0$ for $|\xi| \ge b$. Moreover, for any* *multi-index of non-negative integers* $\alpha = (\alpha_1, \dots, \alpha_n)$ *and for any integer* $k \geq 0$, *the following estimates hold;*

$$\begin{aligned}&\|(1+x_1^2)^{\theta_1/2}\dots(1+x_d^2)^{\theta_d/2}\partial_t^k\partial_x^\alpha V(t,\cdot)\|_p\\&\leq C(1+t)^{-n\delta(p,q)-k-|\alpha|/2-(1-\theta_1)/2-\dots-(1-\theta_d)/2}\|P(\cdot)V_0\|_q,\end{aligned}$$

where $\delta(p,q) = 1/2q - 1/2p$.

Choose a function χ_1 of class C^∞ satisfying $\chi_1(\rho) = 1$ for $\rho \leq a/2$ and $\chi_1(\rho) = 0$ for $\rho \geq 2a/3$. Define the functions Θ_1 and g by

$$\Theta_1(\rho) = \frac{\rho^4}{2a(a+\sqrt{a^2-\rho^2})^2}, \tag{3.3}$$

$$g(t,\rho) = (\exp(-t\Theta_1(\rho)) - 1)\exp\big(-\frac{t\rho^2}{4a}\big). \tag{3.4}$$

Here and after we denote $\chi_1(\xi) = \chi_1(|\xi|)$ and $g(t,\xi) = g(t,|\xi|)$. For the proof of Theorem 1.1, we need the following lemmas. Let $\mathcal{I}$ be the set of all multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$ satisfying $\alpha_j = 0, 1$ for $j = 1, \dots, d$ and $\alpha_j = 0$ for $j = d+1, \dots, n$.

Lemma 3.1. *Let* $1 \leq q \leq p \leq \infty$ *and* $b > 0$ *be constants, and let* χ_{11} *be a function of class* C^∞ *satisfying* $\chi_{11}(\xi) = 0$ *for* $|\xi| \geq b$. *Then, the estimates*

$$\|P(\cdot)\mathcal{F}^{-1}(\chi_{11}\widehat{h})\|_p \leq C_b \sup_\xi \sum_{|\alpha|\leq n+d+1} |\partial_\xi^\alpha \chi_{11}(\xi)| \|P(\cdot)h\|_q$$

hold for any h *satisfying* $P(\cdot)h \in L^q$.

Proof. Since $P(x) \leq C_1 \sum_{\alpha\in\mathcal{I}} |x^\alpha| \leq C_2 P(x)$ and $\mathcal{F}(x^\alpha f)(\xi) = c_\alpha \partial_\xi^\alpha \hat{f}(\xi)$, it follows that

$$\|P(\cdot)\mathcal{F}^{-1}(\chi_{11}\widehat{h})\|_p \leq C\sum_{\alpha\in\mathcal{I}} \|x^\alpha \mathcal{F}^{-1}(\chi_{11}\hat{h})\|_p \leq C\sum_{\alpha\in\mathcal{I}}\sum_{\beta+\gamma=\alpha} \|\mathcal{F}^{-1}(\partial_\xi^\beta\chi_{11}\partial_\xi^\gamma\widehat{h})\|_p. \tag{3.5}$$

Since $(1+|x|)^{-(n+1)} \in L^1$ and $\operatorname{supp}\chi_{11} \subset \{\xi : |\xi| \leq b\}$, it follows that

$$\begin{aligned}\|\mathcal{F}^{-1}(\partial_\xi^\beta\chi_{11})\|_{L^1\cap L^\infty} &\leq C\|(1+|x|)^{n+1}\mathcal{F}^{-1}(\partial_\xi^\beta\chi_{11})\|_\infty\\&\leq C\sum_{|\alpha|\leq n+1}\|\partial_\xi^\alpha\partial_\xi^\beta\chi_{11}\|_{L^1}\\&\leq C\sup_\xi\sum_{|\alpha|\leq n+1}\big|\partial_\xi^\alpha\partial_\xi^\beta\chi_{11}(\xi)\big|.\end{aligned}$$

Hence, for any β satisfying $|\beta| \leq d$,

$$\|\mathcal{F}^{-1}(\partial_\xi^\beta\chi_{11})\|_{L^1\cap L^\infty} \leq C\sup_\xi\sum_{|\alpha|\leq n+d+1}|\partial_\xi^\alpha\chi_{11}(\xi)|. \tag{3.6}$$

Since

$$\begin{gathered}\mathcal{F}^{-1}(\partial_\xi^\beta\chi_{11}\partial_\xi^\gamma\widehat{h}) = c\mathcal{F}^{-1}(\partial_\xi^\beta\chi_{11}) * \mathcal{F}^{-1}(\partial_\xi^\gamma\hat{h}),\\ \|\mathcal{F}^{-1}(\partial_\xi^\gamma\hat{h})\|_q \leq C\|P(\cdot)h\|_q, \quad \gamma\in\mathcal{I},\end{gathered}$$

Lemma 2.3 and estimates (3.5)–(3.6) give the desired estimate. □

Note that the function

$$I(t,x)=\mathcal{F}^{-1}(\chi_1 g(t,\cdot))(x)=(\frac{1}{2\pi})^{n/2}\int_{R^n} e^{ix\cdot\xi}\chi_1(\xi)g(t,\xi)\,d\xi$$

is a radial function and belongs to $\mathcal{S}(R^n)$ for any $t\geq 0$.

Lemma 3.2. *For any $t>0$, the following two estimates hold*

$$\sup_x |I(t,x)|\leq C(1+t)^{-n/2-1}, \tag{3.7}$$

$$\sup_x |(1+|x|)^{n+1/2}I(t,x)|\leq C(1+t)^{-3/4}. \tag{3.8}$$

Proof. We prove only the case where $n=1$. For the proof when $n\geq 2$, see [16, Proposition 3.1]. Since

$$I(t,x)=\sqrt{\frac{2}{\pi}}\int_0^\infty \chi_1(\rho)g(t,\rho)\cos\rho|x|\,d\rho \tag{3.9}$$

and

$$|g(t,\rho)|\leq Ct\rho^4\exp\big(-\frac{t\rho^2}{4a}\big),\quad (0\leq\rho\leq 2a/3),$$

easy calculations show that

$$|I(t,x)|\leq C\int_0^{2a/3} t\rho^4\exp\big(-\frac{t\rho^2}{4a}\big)\,d\rho\leq C(1+t)^{-3/2}.$$

Thus we have proved estimate (3.7). Since

$$\cos\rho|x|=-\frac{1}{|x|^2}(\frac{\partial}{\partial\rho})^2\cos\rho|x|,$$

Using integration by parts in (3.9),

$$|I(t,x)|\leq\frac{C}{x^2}\int_0^\infty \big|(\frac{\partial}{\partial\rho})^2(\chi_1(\rho)g(t,\rho))\big|\,d\rho\leq\frac{C}{x^2}(1+t)^{-1/2}, \tag{3.10}$$

where we have used

$$|\frac{\partial g}{\partial\rho}(t,\rho)|+|\frac{\partial^2 g}{\partial\rho^2}(t,\rho)|\leq C\exp\big(-\frac{t\rho^2}{8a}\big),\quad (0\leq\rho\leq a).$$

Estimates (3.7) and (3.10) show that

$$|I(t,x)|\leq\frac{C}{1+x^2}(1+t)^{-1/2}.$$

Therefore,

$$|I(t,x)|\leq C((1+t)^{-3/2})^{1/4}(\frac{1}{1+x^2}(1+t)^{-1/2})^{3/4}\leq\frac{C}{(1+|x|)^{3/2}}(1+t)^{-3/4}.$$

Thus we have proved estimate (3.8). □

Lemma 3.3. *Let $1\leq q\leq p\leq\infty$, and let k be a non-negative integer. Then*

$$\|\partial_t^k I(t,\cdot)*f\|_p\leq C(1+t)^{-n\delta(p,q)-k-1+\epsilon}\|f\|_q,\quad t\geq 0$$

for any $\epsilon>0$, where $C=C(p,q,\epsilon,k)>0$ and $\delta(p,q)=1/2q-1/2p$. We may take $\epsilon=0$ when $1<q<p<\infty$, $p=\infty$ and $q=1$ or $p=q=2$.

Proof. Consider the case where $k = 0$. Lemma 3.2 shows

$$\|I(t,\cdot) * f\|_\infty \le C(1+t)^{-n/2-1}\|f\|_1. \tag{3.11}$$

Since

$$|\chi_1(\xi)g(t,\xi)| \le C|\chi_1(\xi)||\xi|^4 t \exp\big(-\frac{|\xi|^2 t}{4a}\big) \le C\min(\frac{1}{t},\ t) \le \frac{C}{1+t},$$

it follows that

$$\|I(t,\cdot) * f\|_2 \le C\|\chi_1 g_1(t,\cdot)\widehat{f}\|_2 \le C(1+t)^{-1}\|f\|_2. \tag{3.12}$$

Set $r \in [1,\infty]$ by $1 - 1/r = 2\delta(p,q)$, and set $\theta = 2n/((2n+1)r) \in [0,1)$, then Lemma 3.2 shows

$$|I(t,x)| = |I(t,x)|^\theta |I(t,x)|^{1-\theta} \le C(1+t)^{-n\delta(p,q)-1}|x|^{-n/r}.$$

Hence Lemma 2.4 show

$$\|I(t,\cdot) * f\|_p \le C(1+t)^{-n\delta(p,q)-1}\|f\|_q, \quad (1 < q < p < \infty). \tag{3.13}$$

Since Lemma 3.2 also shows $|I(t,x)| \le C(1+t)^{-1+\epsilon}(1+|x|)^{-n-2\epsilon}$ for $0 < \epsilon \le 1/4$, it follows that

$$\|I(t,\cdot)\|_1 \le C(\epsilon)(1+t)^{-1+\epsilon}, \quad (0 < \epsilon \le 1/4).$$

Therefore, Lemma 2.3 gives

$$\|I(t,\cdot) * f\|_p \le C(\epsilon)(1+t)^{-1+\epsilon}\|f\|_p, \quad (1 \le p \le \infty). \tag{3.14}$$

Estimates (3.11)–(3.14) give the desired estimate when $k = 0$.

Now consider the case where $k \ge 1$. Easy calculations show that

$$\partial_t^k \widehat{I}(t,\xi) = |\xi|^{2k}\Big(B_{k,1}(\xi)\widehat{I}(t,\xi) + B_{k,2}(\xi)|\xi|^2\chi_1(\xi)\exp\big(-\frac{|\xi|^2 t}{4a}\big)\Big), \tag{3.15}$$

where $B_{k,1}, B_{k,2} \in C^\infty$ satisfying $B_{k,1}(\xi) = B_{k,2}(\xi) = 0$ when $|\xi| \ge 2a/3$. Since $\mathcal{F}^{-1}B_{k,i} \in \mathcal{S}(R^n)$ for $i = 1,2$, the well-known estimate

$$\|\mathcal{F}^{-1}\Big(|\xi|^{2k+2}\chi_1 \exp\big(-\frac{|\xi|^2 t}{4a}\big)\widehat{f}\Big)\|_p \le C(1+t)^{-n\delta(p,q)-k-1}\|f\|_q$$

hold for $1 \le q \le p \le \infty$. Hence, the estimates when $k = 0$ and (3.15) give the desired estimate in the case where $k \ge 1$. $\square$

From Proposition 3.1 and Lemma 3.3, we obtain the next lemma.

Lemma 3.4. *Let $1 \le q \le p \le \infty$, $0 \le \theta \le 1$ and $\epsilon > 0$. Assume that f is odd with respect to x', $P(\cdot)f \in L^q$ and $\widehat{f}(\xi) = 0$ for $|\xi| \ge a/2$. We set*

$$\widehat{h}(t,\xi) = \exp\big(-\frac{|\xi|^2 t}{4a}\big)\widehat{f}(\xi), \quad t \ge 0.$$

Then, for any integer $k \ge 0$ and a multi-index α, estimates

$$\|P(\cdot)^\theta \partial_t^k \partial_x^\alpha (I(t,\cdot) * h(t,\cdot))\|_p \le C(1+t)^{-n\delta(p,q)-k-|\alpha|/2-(1-\theta)d/2-1+\epsilon}\|P(\cdot)f\|_q$$

hold, where $C = C(p,q,\epsilon,k,\alpha)$ and $\delta(p,q) = 1/2q - 1/2p$. In the above estimates we may take $\epsilon = 0$ when $1 < q < p < \infty$, $p = \infty$ and $q = 1$ or $p = q = 2$.

Proof. Consider the case where $\theta = 0$. Since

$$\partial_t^k \partial_x^\alpha (I(t,x) * h(t,x)) = \sum_{k_1+k_2=k} c(k_1,k_2) \partial_t^{k_1} I(t,x) * \partial_t^{k_2} \partial_x^\alpha h(t,x),$$

Proposition 3.1 and Lemma 3.3 show that

$$\begin{aligned} \|\partial_t^k \partial_x^\alpha (I(t,\cdot) * h(t,\cdot))\|_p &\le C \sum_{k_1+k_2=k} (1+t)^{-n\delta(p,q)-k_1-1+\epsilon} \|\partial_t^{k_2} \partial_x^\alpha h(t,\cdot)\|_q \\ &\le C(1+t)^{-n\delta(p,q)-k-|\alpha|/2-d/2-1+\epsilon} \|P(\cdot) f\|_q. \end{aligned} \tag{3.16}$$

Thus we have obtained the desired estimate when $\theta = 0$.

Now we show the estimate of $\|P(\cdot) I(t,\cdot) * h(t,\cdot)\|_p$. Easy calculations show

$$\chi_1(\xi) \partial_\xi^\beta \big(\exp\big(-t\Theta_1\big) - 1\big) = \xi^\beta \sum_{j=1}^{|\beta|} c t^j |\xi|^{2\sigma(j,\beta)} \Psi_j(|\xi|^2) \exp(-t\Theta_1), \tag{3.17}$$

for $\beta \in \mathcal{I}$ with $|\beta| \ge 1$, where $\sigma(j,\beta) = \max(2j - |\beta|, 0)$ and Ψ_j is a function of class C^∞ satisfying $\Psi_j(|\xi|^2) = 0$ for $|\xi|^2 \ge 2a/3$ for $j = 1, \dots, |\beta|$,

$$\partial_\xi^\gamma \exp\big(-\frac{t|\xi|^2}{4a}\big) = c t^{|\gamma|} \xi^\gamma \exp\big(-\frac{t|\xi|^2}{4a}\big), \tag{3.18}$$

for $\gamma \in \mathcal{I}$, and

$$\exp(-t\Theta_1) \exp\big(-\frac{t|\xi|^2}{4a}\big) = g(t,\xi) + \exp\big(-\frac{t|\xi|^2}{4a}\big). \tag{3.19}$$

Let $\alpha \in \mathcal{I}$ be fixed. Since $\chi_1(\xi) = 1$ on $\operatorname{supp} \widehat{h}(t,\cdot)$ for any $t \ge 0$, (3.17)–(3.19) imply

$$\begin{aligned} &\mathcal{F}(x^\alpha I(t,\cdot) * h(t,\cdot)) \\ &= \sum_{\beta+\gamma+\mu=\alpha} c_{\beta,\gamma,\mu} \chi_1(|\xi|) \partial_\xi^\beta (\exp(-t\Theta_1) - 1) \partial_\xi^\gamma \exp\big(-\frac{t|\xi|^2}{4a}\big) \partial_\xi^\mu \widehat{h}(t,\xi) \\ &= \sum_{\gamma+\mu=\alpha} c_{\gamma,\mu} \xi^\gamma t^{|\gamma|} g(t,\xi) \partial_\xi^\mu \widehat{h}(t,\xi) + \sum_{\beta+\gamma+\mu=\alpha, |\beta|\ge 1} \sum_{j=1}^{|\beta|} c_{\beta,\gamma,\mu,j} \xi^{\beta+\gamma} \\ &\quad \times |\xi|^{2\sigma(j,\beta)} t^{j+|\gamma|} \Big(g(t,\xi) + \exp\big(-\frac{t|\xi|^2}{4a}\big)\Big) \partial_\xi^\mu \widehat{h}(t,\xi). \end{aligned} \tag{3.20}$$

Hence, Proposition 3.1, Lemma 3.1–3.2 and (3.20) imply

$$\begin{aligned} &\|x^\alpha I(t,\cdot) * h(t,\cdot)\|_p \le C \sum_{\beta+\gamma+\mu=\alpha} (1+t)^{|\beta|/2+|\gamma|/2-n\delta(p,q)-1+\epsilon} \|x^\mu h(t,\cdot)\|_q \\ &\le C \sum_{\beta+\gamma+\mu=\alpha} (1+t)^{|\beta|/2+|\gamma|/2-n\delta(p,q)-1+\epsilon-(d-|\mu|)/2} \|P(\cdot) f\|_q \\ &\le C(1+t)^{-n\delta(p,q)-1+\epsilon} \|P(\cdot) f\|_q. \end{aligned}$$

Therefore, we obtain the following estimate for $1 \le q \le p \le \infty$:

$$\|P(\cdot) I(t,\cdot) * h(t,\cdot)\|_p \le C(1+t)^{-n\delta(p,q)-1+\epsilon} \|P(\cdot) f\|_q. \tag{3.21}$$

Since

$$\begin{aligned}&\mathcal{F}\big\{\partial_t^j\partial_x^\alpha(I(t,\cdot)*h(t,\cdot))\big\}(\xi)\\&=\{g_1(t,\xi)m_1(\xi)+m_2(\xi)|\xi|^2\}|\xi|^{2j}\xi^\alpha\exp\big(-\frac{t|\xi|^2}{4a}\big)\widehat{h}(t,\xi)\end{aligned}$$

for some functions m_1 and m_2 defined on R^n of class C^∞ with compact support, estimate (3.21) shows that

$$\|P(\cdot)\partial_t^j\partial_x^\alpha(I(t,\cdot)*h(t,\cdot))\|_p\le C(1+t)^{-n\delta(p,q)-j-|\alpha|/2-1+\epsilon}\|P(\cdot)f\|_q. \tag{3.22}$$

Since

$$\begin{aligned}&\|P(\cdot)^\theta\partial_t^j\partial_x^\alpha(I(t,\cdot)*h(t,\cdot))\|_p\\&\le\|P(\cdot)\partial_t^j\partial_x^\alpha(I(t,\cdot)*h(t,\cdot))\|_p^\theta\|\partial_t^j\partial_x^\alpha(I(t,\cdot)*h(t,\cdot))\|_p^{1-\theta}\end{aligned}$$

for $0\le\theta\le1$, estimates (3.16) and (3.22) give the desired estimate. □

Proof of Theorem 1.1. The Fourier transformation of (1.1) yields

$$\widehat{v}(t,\xi)=e^{-at}\Big(\cos t\sqrt{|\xi|^2-a^2}\widehat{v}_0(\xi)+\frac{\sin t\sqrt{|\xi|^2-a^2}}{\sqrt{|\xi|^2-a^2}}(a\widehat{v}_0(\xi)+\widehat{v}_1(\xi))\Big) \tag{3.23}$$

for $t\ge0$. Since

$$\cos\sqrt{z}=\sum_{k=0}^\infty\frac{(-z)^k}{(2k)!},\quad \frac{\sin\sqrt{z}}{\sqrt{z}}=\sum_{k=0}^\infty\frac{(-z)^k}{(2k+1)!},\quad z\in\mathbb{C},$$

$\cos t\sqrt{|\xi|^2-a^2}$ and $\sin t\sqrt{|\xi|^2-a^2}/\sqrt{|\xi|^2-a^2}$ are smooth functions of (t,ξ) in $R\times R^n$, and they satisfy

$$\begin{aligned}\cos t\sqrt{|\xi|^2-a^2}&=\sum_{k=0}^\infty\frac{(-1)^kt^{2k}}{(2k)!}(|\xi|^2-a^2)^k,\\ \frac{\sin t\sqrt{|\xi|^2-a^2}}{\sqrt{|\xi|^2-a^2}}&=\sum_{k=0}^\infty\frac{(-1)^kt^{2k+1}}{(2k+1)!}(|\xi|^2-a^2)^k.\end{aligned}$$

First we consider the case where $b\ge a/2$ and $\operatorname{supp}\widehat{v}_0\cup\operatorname{supp}\widehat{v}_1\subset\{\xi;a/3\le|\xi|\le b\}$. Since

$$|\partial_\xi^\alpha(|\xi|^2-a^2)^k|\le C_\alpha\frac{k!}{(k-|\alpha|)!}(1+|\xi|^{|\alpha|})||\xi|^2-a^2|^{k-|\alpha|},\quad |\alpha|\le k,$$

it follows that

$$\begin{aligned}&\big|\partial_\xi^\alpha\big(\frac{\sin t\sqrt{|\xi|^2-a^2}}{\sqrt{|\xi|^2-a^2}}\big)\big|+|\partial_\xi^\alpha\cos t\sqrt{|\xi|^2-a^2}|\\&=\Big|\partial_\xi^\alpha\sum_{k=0}^\infty\frac{(-1)^k}{(2k+1)!}t^{2k+1}(|\xi|^2-a^2)^k\Big|+\Big|\partial_\xi^\alpha\sum_{k=0}^\infty\frac{(-1)^k}{(2k)!}t^{2k}(|\xi|^2-a^2)^k\Big|\\&\le C_\alpha(1+t^{2|\alpha|+1})(1+|\xi|^\alpha)\exp(t||\xi|^2-a^2|^{1/2})\\&\le C_\alpha(1+t^{2|\alpha|+1})\exp(at\sqrt{15}/4),\quad a/4\le|\xi|\le5a/4,\quad t\ge0.\end{aligned} \tag{3.24}$$

From the estimates

$$|\partial_\xi^\alpha\sqrt{|\xi|^2-a^2}|+\big|\partial_\xi^\alpha\big(\frac{1}{\sqrt{|\xi|^2-a^2}}\big)\big|\le C_\alpha,\quad |\xi|\ge5a/4,\quad |\alpha|\ge1,$$

we obtain

$$\big|\partial_\xi^\alpha\big(\frac{\sin t\sqrt{|\xi|^2-a^2}}{\sqrt{|\xi|^2-a^2}}\big)\big| + |\partial_\xi^\alpha \cos t\sqrt{|\xi|^2-a^2}| \le C_\alpha(1+t^{|\alpha|}) \tag{3.25}$$

for $t \ge 0$ and $|\xi| \ge 5a/4$.

Choose a smooth function χ_{12} satisfying $\chi_{12}(\xi)\hat{v}_i(\xi) = \hat{v}_i(\xi)$ $(i = 0, 1)$ and $\operatorname{supp}\chi_{12} \subset \{\xi : a/4 < |\xi| < b + 1\}$. (3.24)–(3.25) and Lemma 3.1 with

$$\chi_{11}(\xi) = \chi_{12}(\xi)\cos t\sqrt{|\xi|^2-a^2}, \quad \chi_{11}(\xi) = \chi_{12}(\xi)\frac{\sin t\sqrt{|\xi|^2-a^2}}{\sqrt{|\xi|^2-a^2}}$$

give the following estimate

$$\begin{aligned} &\big\|P(\cdot)\partial_x^\beta\mathcal{F}^{-1}\chi_{12}\Big(\cos t\sqrt{|\xi|^2-a^2}\hat{v}_0 + \frac{\sin t\sqrt{|\xi|^2-a^2}}{\sqrt{|\xi|^2-a^2}}(a\hat{v}_0+\hat{v}_1)\Big)\|_p \\ &\le C_\beta(1+t)^{2n+2d+3}(\|P(\cdot)v_0\|_q + \|P(\cdot)v_1\|_q) \end{aligned} \tag{3.26}$$

for $t \ge 0$ and β. Since $\hat{v}(t,\xi) = \chi_{12}\hat{v}(t,\xi)$, (3.23) and (3.26) shows

$$\|P(\cdot)\partial_x^\beta v(t,\cdot)\|_p \le C_\beta e^{-\lambda t}(\|P(\cdot)v_0\|_q + \|P(\cdot)v_1\|_q), \quad t \ge 0 \tag{3.27}$$

for $1 \le q \le p \le \infty$, where $\lambda \equiv (4-\sqrt{15})a/5$.

Set $v_k(t,x) = \partial_t^k v(t,x)$ for $k = 0, 1, 2, \dots$. Then $v_k(t,x)$ satisfies

$$\partial_t v_k(t,x) + 2av_k(t,x) = \Delta v_{k-1}(t,x), \quad v_k(0,x) = \partial_t^k v(0,x), \quad t > 0, x \in R^n,$$

hence

$$v_k(t,x) = e^{-2at}v_k(0,x) + \int_0^t e^{-2a(t-\tau)}\Delta v_{k-1}(\tau,x)\,d\tau, \quad t \ge 0, x \in R^n \tag{3.28}$$

for $k = 1, 2, \dots$. Moreover, it satisfies $\hat{v}_k(0,\xi) = 0$ for $|\xi| \ge b$, and

$$\|P(\cdot)\partial_x^\beta v_k(0,\cdot)||_q \le C_{k,\beta}\big(\|P(\cdot)v_0\|_q + \|P(\cdot)v_1\|_q\big) \quad 1 \le q \le \infty, \tag{3.29}$$

for $k = 1, 2, \dots$. Therefore, (3.27)– (3.29) show

$$\|P(\cdot)\partial_t^k\partial_x^\beta v(t,\cdot)\|_p \le C_{k,\beta}e^{-\lambda t}\big(\|P(\cdot)v_0\|_q + \|P(\cdot)v_1\|_q\big), \quad t \ge 0 \tag{3.30}$$

for $1 \le q \le p \le \infty$, $k = 0, 1, 2, \dots$ and β. Since $\hat{v}_i(\xi) = \chi_{12}(\xi)\hat{v}_i(\xi)$ $(i = 0, 1)$, the solution formula

$$\hat{V}(t,\xi) = \exp\big(-\frac{t|\xi|^2}{2a}\big)\big(\hat{v}_0(\xi) + \frac{1}{2a}\hat{v}_1(\xi)\big), \quad t \ge 0,$$

shows that $\hat{V}(t,\xi) = \chi_{12}\hat{V}(t,\xi)$. Hence, the similar arguments to the above estimates and Lemma 3.1 shows

$$\|P(\cdot)\partial_t^k\partial_x^\beta V(t,\cdot)\|_p \le C_{k,\beta}e^{-\lambda t}(\|P(\cdot)v_0\|_q + \|P(\cdot)v_1\|_q), \quad t \ge 0 \tag{3.31}$$

for $1 \le q \le p \le \infty$, $k = 0, 1, 2, \dots$ and β. Since $P(x) \ge 1$, (3.30)–(3.31) give the desired result in Theorem 1.1, in the case where $b \ge a/2$ and $\operatorname{supp}\widehat{v_0} \cup \operatorname{supp}\widehat{v_1} \subset \{\xi : a/3 \le |\xi| \le b\}$.

Now we consider the case where $\operatorname{supp}\widehat{v_0} \cup \widehat{v_1} \subset \{\xi : |\xi| \le a/2\}$. The solution formula (3.23) shows that

$$\widehat{v}(t,\xi) = \widehat{V}(t,\xi) + \frac{1}{2}\widehat{\phi}_1(t,\xi) + \widehat{\phi}_2(t,\xi) + \widehat{\phi}_3(t,\xi), \tag{3.32}$$

where

$$\widehat{\phi}_1(t,\xi) = g(t,\xi)\chi_1(\xi)\exp\big(-\frac{t|\xi|^2}{4a}\big)\big(\widehat{v}_0(\xi) + \frac{a\widehat{v}_0(\xi)+\widehat{v}_1(\xi)}{\sqrt{a^2-|\xi|^2}}\big),$$

$$\widehat{\phi}_2(t,\xi) = \exp\big(-\frac{t|\xi|^2}{2a}\big)\frac{|\xi|^2\chi_1(\xi)}{\sqrt{a^2-|\xi|^2}(a+\sqrt{a^2-|\xi|^2})}\cdot\frac{a\widehat{v}_0(\xi)+\widehat{v}_1(\xi)}{2a},$$

$$\widehat{\phi}_3(t,\xi) = \frac{1}{2}\exp\big(-at - t\sqrt{a^2-|\xi|^2}\big)\chi_1(\xi)\big(\widehat{v}_0(\xi) - \frac{a\widehat{v}_0(\xi)+\widehat{v}_1(\xi)}{\sqrt{a^2-|\xi|^2}}\big).$$

It follows that $\widehat{v}_i(\xi) = \chi_1(\xi)\widehat{v}_i(\xi)$ for $i=0,1$, the function $\xi \mapsto \chi_1(\xi)/\sqrt{a^2-|\xi|^2}$ is a radial function that belongs to $S(R^n)$, and the function v_i $(i=0,1)$ is odd with respect to x'. Hence the function

$$\mathcal{F}^{-1}\big(\widehat{v}_0 + \frac{a\widehat{v}_0+\widehat{v}_1}{\sqrt{a^2-|\xi|^2}}\big) = v_0 + c\mathcal{F}^{-1}\big(\frac{\chi_1(\xi)}{\sqrt{a^2-|\xi|^2}}\big) * (av_0+v_1)$$

is also odd with respect to x', and moreover, Lemma 2.3 shows

$$\|P(\cdot)\mathcal{F}^{-1}(\widehat{v}_0 + \frac{a\widehat{v}_0+\widehat{v}_1}{\sqrt{a^2-|\xi|^2}})\|_q \le C(\|P(\cdot)v_0\|_q + \|P(\cdot)v_1\|_q) \tag{3.33}$$

for $t \ge 0$. Set

$$\widehat{h}(t,\xi) = \exp\big(-\frac{t|\xi|^2}{4a}\big)\big(\widehat{v}_0(\xi) + \frac{a\widehat{v}_0(\xi)+\widehat{v}_1(\xi)}{\sqrt{a^2-|\xi|^2}}\big), \quad t \ge 0.$$

Since $\phi_1(t,\cdot) = cI(t,\cdot) * h(t,\cdot)$, Lemma 3.4 and estimate (3.33) show that

$$\begin{aligned} &\|P(\cdot)^\theta\partial_t^k\partial_x^\alpha\phi_1(t,\cdot)\|_p \\ &\le C(1+t)^{-n\delta(p,q)-k-|\alpha|/2-(1-\theta)d/2-1+\epsilon}(\|P(\cdot)v_0\|_q + \|P(\cdot)v_1\|_q) \end{aligned} \tag{3.34}$$

for $t \ge 0$, $1 \le q \le p \le \infty$ and $\theta = 0,1$. Proposition 3.1 and Lemma 3.1 show

$$\begin{aligned} &\|P(\cdot)^\theta\partial_t^k\partial_x^\alpha\phi_2(t,\cdot)\|_p + \|P(\cdot)^\theta\partial_t^k\partial_x^\alpha\phi_3(t,\cdot)\|_p \\ &\le C(1+t)^{-n\delta(p,q)-k-|\alpha|/2-(1-\theta)d/2-1+\epsilon}(\|P(\cdot)v_0\|_q + \|P(\cdot)v_1\|_q) \end{aligned} \tag{3.35}$$

for $t \ge 0$, $1 \le q \le p \le \infty$ and $\theta = 0,1$. Hence, (3.32) and estimates (3.34)–(3.35) give the desired estimate. $\square$

4. Proof of Theorem 1.2

Let N be a positive integer. Then the function

$$h_N(y) = e^{iy} - \sum_{k=0}^{N}\frac{(iy)^k}{k!} \tag{4.1}$$

satisfies $|\partial_y^k h_N(y)| \le C|y|^{N-k}$, for $k \in [0,N]$. Let χ_2 be a radial function of class C^∞ that satisfies $\chi_2(\xi) = 0$ for $|\xi| \le 3a/2$, and $\chi_2(\xi) = 1$ for $|\xi| \ge 2a$. Here and after we denote $\chi_2(\rho) = \chi_2(\rho\omega)$ for $\rho \ge 0$ and $\omega \in R^n$, $|\omega| = 1$.

Define the function

$$II_N(t,x) = \mathcal{F}^{-1}(\chi_2(\cdot)h_N(t\Theta(\cdot))e^{it|\xi|})(x)\,.$$

Then Lemma 2.2 shows that

$$II_N(t,x) = c\int_0^\infty \chi_2(\rho)h_N(t\Theta(\rho))\rho^{n-1}\widetilde{J}_{-1+n/2}(\rho|x|)\,d\rho. \tag{4.2}$$

Lemma 4.1 (cf. in [16, Lemma 4.1])**.** *Let $N \ge n+1$ and $m = [n/2]$, then*

(1) $\|II_N(t,\cdot)\|_\infty \le C|t|^N$,
(2) $\|II_N(t,\cdot)\|_1 \le C(|t|^N + |t|^{N+m+2})$.

Proof. (1) Since $|h_N(t\Theta(\rho))| \le C|t|^N\Theta(\rho)^N \le C|t|^N/\rho^N$, for $\rho \ge 3a/2$, Lemma 2.1 (4) and (4.2) show the desired estimate

$$|II_N(t,x)| \le C\int_{3a/2}^\infty \frac{|t|^N}{\rho^{N-n+1}}\,d\rho \le C|t|^N.$$

(2) Since

$$|(\frac{d}{dy})^k h_N(y)| \le C|y|^{N-k}, \quad |(\frac{d}{d\rho})^k\Theta(\rho)| \le C\rho^{-k-1}$$

for $\rho \ge 3a/2$ and $0 \le k \le N$, easy calculations show

$$|(\frac{\partial}{\partial\rho})^k h_N(t\Theta(\rho))| \le C|t|^N\rho^{-k-N}, \quad (0 \le k \le N, \rho \ge 3a/2). \tag{4.3}$$

The differential operator X defined by

$$Xv(t,\rho) = \frac{\partial}{\partial\rho}(\frac{1}{\rho}v(t,\rho))$$

satisfies

$$X^k(v(t,\rho)\rho^l) = \sum_{j=0}^k c_{jkl}\partial_\rho^j v(t,\rho)\rho^{l-2k+j}. \tag{4.4}$$

Then (4.3)–(4.4) read

$$(\frac{\partial}{\partial\rho})^l(\rho^i X^k(\chi_2(\rho)h_N(t\Theta(\rho))e^{it\rho}\rho^{n-1}))\Big|_0^\infty = 0 \tag{4.5}$$

for $i = 0, 1$, $0 \le k \le m$ and $0 \le l \le 2$. Hence, Lemma 2.1 (5), (4.2), (4.5) and integration by parts give

$$\begin{aligned} II_N(t,x) &= \frac{c}{|x|^2}\int_0^\infty \chi_2(\rho)h_N(t\Theta(\rho))e^{it\rho}\rho^{n-1}\frac{1}{\rho}(\frac{\partial}{\partial\rho})\widetilde{J}_{n/2-2}(\rho|x|)\,d\rho \\ &= \frac{c}{|x|^2}\int_0^\infty X(\chi_2(\rho)h_N(t\Theta(\rho))e^{it\rho}\rho^{n-1})\widetilde{J}_{n/2-2}(\rho|x|)\,d\rho, \end{aligned}$$

when $(n/2-2)$ is not a negative integer. Repeating the above integration by parts, we obtain

$$II_N(t,x) = \frac{c}{|x|^{2\mu}}\int_0^\infty X^\mu(\chi_2(\rho)h_N(t\Theta(\rho))e^{it\rho}\rho^{n-1})\widetilde{J}_{n/2-1-\mu}(\rho|x|)\,d\rho, \tag{4.6}$$

where $\mu = [(n-1)/2]$. In the case where $n = 2m$, equation (4.6) reads

$$II_N(t,x) = \frac{c}{|x|^{n-2}}\int_0^\infty X^{m-1}(\chi_2(\rho)h_N(t\Theta(\rho))e^{it\rho}\rho^{n-1})J_0(\rho|x|)\,d\rho. \tag{4.7}$$

Lemma 2.1 (1) shows that

$$J_0(\rho|x|) = 2\widetilde{J}_1(\rho|x|) + \rho(\frac{\partial}{\partial\rho})\widetilde{J}_1(\rho|x|),$$

hence (4.5), (4.7) and integration by parts give

$$
\begin{aligned}
&|II_N(t,x)| \\
&\le \frac{c}{|x|^{n-2}} \sum_{k=0}^{1} \Big| \int_0^\infty \rho^k (\frac{\partial}{\partial\rho})^k X^{m-1}(\chi_2(\rho) h_N(t\Theta(\rho)) e^{it\rho}\rho^{n-1}) \widetilde{J}_1(\rho|x|)\, d\rho \Big|.
\end{aligned} \tag{4.8}
$$

Since Lemma 2.1 (4) shows

$$
|\widetilde{J}_1(\rho|x|) - c\rho^{-3/2}|x|^{-3/2}\cos(\rho|x| - \frac{3\pi}{4})| \le C\rho^{-5/2}|x|^{-5/2},
$$

estimate (4.8) and integration by parts show

$$
\begin{aligned}
&|II_N(t,x)| \\
&\le \frac{C}{|x|^{n+1/2}} \sum_{k=0}^{2} \int_0^\infty \rho^{k-5/2} \big| (\frac{\partial}{\partial\rho})^k X^{m-1}(\chi_2(\rho) h_N(t\Theta(\rho)) e^{it\rho}\rho^{n-1}) \big|\, d\rho,
\end{aligned} \tag{4.9}
$$

where we have used

$$
\cos(\rho|x| - \frac{3\pi}{4}) = \frac{1}{|x|} \frac{\partial}{\partial\rho} \sin(\rho|x| - \frac{3\pi}{4}).
$$

Hence, estimate (4.4), (4.9) and (4.3) show

$$
|II_N(t,x)| \le \frac{C}{|x|^{n+1/2}} |t|^N (1+|t|^{m+2}). \tag{4.10}
$$

Since

$$
\|II_N(t,\cdot)\|_1 = \int_{|x|\le 1} |II_N(t,x)|\, dx + \int_{|x|\ge 1} |II_N(t,x)|\, dx,
$$

estimate (4.10) and Lemma 4.1 (1) give the desired estimate in Lemma 4.1 (2) when $n = 2m$.

Now let us consider the case where $n = 2m+1$. Since Lemma 2.1 (3) shows

$$
\widetilde{J}_{-1/2}(\rho|x|) = \sqrt{\frac{\pi}{2}} \cos\rho|x| = -\sqrt{\frac{\pi}{2}} \frac{1}{|x|^2} (\frac{\partial}{\partial\rho})^2 \cos\rho|x|,
$$

(4.4)–(4.6) and integration by parts give

$$
\begin{aligned}
|II_N(t,x)| &= \frac{c}{|x|^{n+1}} \Big| \int_0^\infty X^{m-1}(\chi_2(\rho) h_N(t\Theta(\rho)) e^{it\rho}\rho^{n-1}) (\frac{\partial}{\partial\rho})^2 \cos\rho|x|\, d\rho \Big| \\
&= \frac{c}{|x|^{n+1}} \int_0^\infty \big| (\frac{\partial}{\partial\rho})^2 X^m(\chi_2(\rho) h_N(t\Theta(\rho)) e^{it\rho}\rho^{n-1}) \cos\rho|x| \big|\, d\rho \\
&\le \frac{C}{|x|^{n+1}} |t|^N (1+|t|^{m+2}).
\end{aligned} \tag{4.11}
$$

Estimates (4.11) and Lemma 4.1 (1) give the desired estimate when $n = 2m+1$. $\square$

Corollary 4.1. *Let $1 \le q \le p \le \infty$. Under the assumptions in Lemma 4.1, the following estimates hold;*

$$
\|II_N(t,\cdot) * g\|_p \le C|t|^N(1+|t|^{m+2})\|g\|_q, \quad g \in L^q.
$$

Proof. Set $r \in [0,\infty]$ by $1 - 1/r = 1/q - 1/p$. Lemma 4.1 shows

$$
\|II_N(t,\cdot)\|_r \le \|II_N(t,\cdot)\|_1^{1/r} \|II_N(t,\cdot)\|_\infty^{1-1/r} \le C|t|^N(1+|t|^{m+2}),
$$

hence Lemma 2.3 gives the desired estimate. $\square$

Note that Corollary 4.1 shows the following estimates:

Lemma 4.2. *Let $N \geq n+d+1$, $1 < q \leq p < \infty$ and $m = [n/2]$. Assume that $f \in L^q$ and $\operatorname{supp} \widehat{f} \subset \{\xi; |\xi| \geq 2a\}$, then*

$$\|P(\cdot)\mathcal{F}^{-1}(\chi_2 h_N(t\Theta)\widehat{f})\|_p \leq C|t|^N(1+|t|^{m+d+2})\|P(\cdot)f\|_q.$$

Proof. Let $\alpha \in \mathcal{I}$ be fixed. Since $\chi_2(\xi) = 1$ on $\operatorname{supp} \widehat{f}$,

$$\begin{aligned} &\|x^\alpha \mathcal{F}^{-1}\Big(\chi_2 h_N(t\Theta)e^{it|\xi|}\widehat{f}\Big)\|_p = c\|\mathcal{F}^{-1}\Big(\chi_2 \partial_\xi^\alpha\big(h_N(t\Theta)e^{it|\xi|}\widehat{f}\big)\Big)\|_p \\ &\leq C \sum_{\beta+\gamma+\mu=\alpha} \|\mathcal{F}^{-1}\Big(\chi_2 \partial_\xi^\beta h_N(t\Theta)\chi_2 \partial_\xi^\gamma e^{it|\xi|}\partial_\xi^\mu \widehat{f}\Big)\|_p. \end{aligned} \tag{4.12}$$

Easy calculations show

$$\chi_2(\xi)\partial_\xi^\beta h_N(t\Theta(\xi)) = \sum_{0\leq k\leq|\beta|} ct^k H_{\beta,k,1}(\xi)h_{N-k}(t\Theta(\xi)) \tag{4.13}$$

when $|\beta| \geq 1$, and

$$\chi_2(\xi)\partial_\xi^\gamma e^{it|\xi|} = \sum_{0\leq k\leq|\gamma|} ct^k H_{\gamma,k,2}(\xi)e^{it|\xi|} \tag{4.14}$$

when $|\gamma| \geq 1$, where

$$H_{\beta,k,1}(\xi) = \chi_2(\xi) \sum_{\widetilde{\beta}_1+\dots+\widetilde{\beta}_k=\beta, |\widetilde{\beta}_1|\geq1,\dots,|\widetilde{\beta}_k|\geq1} c\partial_\xi^{\widetilde{\beta}_1}\Theta(\xi)\dots\partial_\xi^{\widetilde{\beta}_k}\Theta(\xi),$$

and

$$H_{\gamma,k,2}(\xi) = \chi_2(\xi) \sum_{\widetilde{\gamma}_1+\dots+\widetilde{\gamma}_k=\gamma, |\widetilde{\gamma}_1|\geq1,\dots,|\widetilde{\gamma}_k|\geq1} c\partial_\xi^{\widetilde{\gamma}_1}|\xi|\dots\partial_\xi^{\widetilde{\gamma}_k}|\xi|.$$

Since $H_{\beta,k,1}, H_{\gamma,k,2} \in C^\infty(R^n)$ satisfying

$$|\partial_\xi^\nu H_{\beta,k,1}(\xi)| + |\partial_\xi^\nu H_{\gamma,k,2}(\xi)| \leq C_{\nu,\beta,\gamma,k}$$

for any k, $\beta, \gamma \in \mathcal{I}$ with $|\beta| \geq 1$ and $|\gamma| \geq 1$ and any multi-index ν, Hörmander's multiplier theorem(see [2] for example) shows that

$$\|\mathcal{F}^{-1}(H_{\beta,k,1}\widehat{g})\|_p + \|\mathcal{F}^{-1}(H_{\gamma,k,2}\widehat{g})\|_p \leq C_{\beta,\gamma,p,k}\|g\|_p \tag{4.15}$$

for $1 < p < \infty$ and $k \geq 0$ when $\beta, \gamma \in \mathcal{I}$ satisfy $|\beta| \geq 1$ and $|\gamma| \geq 1$.

Since $N-k \geq n+1$ when $0 \leq k \leq d$, (4.12)–(4.15) and Corollary 4.1 show that

$$\begin{aligned} \|x^\alpha \mathcal{F}^{-1}(\chi_2 h_N(t\Theta)e^{it|\xi|}\widehat{f})\|_p &\leq C \sum_{0\leq k+l\leq d, \mu\in\mathcal{I}} |t|^{k+l}\|\mathcal{F}^{-1}(\chi_2 h_{N-k}(t\Theta)e^{it|\xi|}\partial_\xi^\mu \widehat{f})\|_p \\ &\leq C \sum_{k+l\leq d, \mu\in\mathcal{I}} |t|^{k+l}\|II_{N-k}(t,\cdot) * (x^\mu f)\|_p \\ &\leq C \sum_{l=0}^{d} |t|^{N+l}(1+|t|^{m+2}) \sum_{\mu\in\mathcal{I}} \|x^\mu f\|_q \\ &\leq C|t|^N(1+|t|^{m+d+2})\|P(\cdot)f\|_q \end{aligned}$$

for $1 < q \leq p < \infty$. Since

$$\|P(\cdot)\psi\|_p \leq C \sum_{\alpha\in\mathcal{I}} \|x^\alpha \psi\|_p,$$

estimate (4.16) gives the desired estimate. □

For $t \in R$ and a constant k, define the operator $T_k(t)$ by

$$T_k(t)h = \mathcal{F}^{-1}\big(\chi_2|\xi|^{-k}e^{it|\xi|}\widehat{h}\big). \tag{4.16}$$

Set $m = [n/2]$ and $[2/q-1]_+ = \max(2/q-1, 0)$.

Lemma 4.3 ([16, Proposition 4.4])**.** *Let k be an integer, $k \ge (n+1)/2$. For $1 < q \le p < \infty$ and $t \ne 0$, the operator $T_k(t)$ is bounded from L^q to L^p. Moreover, the following estimates hold:*

$$\|T_k(t)h\|_p \le C(1+|t|)^m\|h\|_q$$

when $k \ge n+1$, and

$$\|T_k(t)h\|_p \le C(1+|t|)^m|t|^{[2/q-1]_+(k-n)}\|h\|_q$$

when $(n+1)/2 \le k \le n$.

Lemma 4.3 and arguments similar to those in the proof of Lemma 4.2 give the following result.

Corollary 4.2. *Let k be an integer, $k \ge (n+1)/2$. For $1 < q \le p < \infty$, $t \ne 0$, the following estimates hold:*

$$\|P(\cdot)T_k(t)h\|_p \le C(1+|t|)^{m+d}\|P(\cdot)h\|_q$$

when $k \ge n+1$, and

$$\|P(\cdot)T_k(t)h\|_p \le C(1+|t|)^{m+d}|t|^{[2/q-1]_+(k-n)}\|h\|_q$$

when $(n+1)/2 \le k \le n$.

Proof of Theorem 1.2. When k is a non-negative integer, Hörmander's multiplier theorem [2] shows that the function $\xi \mapsto \chi_2(\xi)\Theta^k|\xi|^k$ is a Fourier multiplier on L^p for $1 < p < \infty$. Hence

$$\|P(\cdot)\mathcal{F}^{-1}(\chi_2\Theta^k|\xi|^k\widehat{h})\|_p \le C\|P(\cdot)h\|_p, \quad (1 < p < \infty). \tag{4.17}$$

Since $\chi_2 = 1$ on $\operatorname{supp}\widehat{w}_1 \cup \widehat{w}_2$, Corollary 4.2 and estimate (4.17) give the following estimates for $j = 0, 1$ and $1 < q \le p < \infty$,

$$\begin{aligned}
\|P(\cdot)\mathcal{F}^{-1}(\chi_2 t^k\Theta^k e^{it|\xi|}\widehat{w}_j)\|_p &= |t|^k\|P(\cdot)\mathcal{F}^{-1}(\chi_2\Theta^k|\xi|^k\chi_2|\xi|^{-k}e^{it|\xi|}\widehat{w}_j)\|_p \\
&\le C|t|^k\|P(\cdot)T_k(t)w_j\|_p \\
&\le C(1+|t|)^k(1+|t|)^{m+d}\|P(\cdot)w_j\|_q \\
&\le C(1+|t|)^{n+m+2d+1}\|P(\cdot)w_j\|_q
\end{aligned} \tag{4.18}$$

for any t and any integer k in $[n+1, n+d+1]$,

$$\begin{aligned}
\|P(\cdot)\mathcal{F}^{-1}(\chi_2 t^k\Theta^k e^{it|\xi|}\widehat{w}_j)\|_p &\le C|t|^k\|P(\cdot)T_k(t)w_j\|_p \\
&\le C|t|^k|t|^{[2/q-1]_+(k-n)}(1+|t|)^{m+d}\|P(\cdot)w_j\|_q \\
&\le C(1+|t|)^{n+m+2d+1}\|P(\cdot)w_j\|_q.
\end{aligned} \tag{4.19}$$

for any $t \ne 0$ and any integer k in $[(n+1)/2, n+d+1]$. Since

$$e^{it\Theta} = h_{n+d+1}(t\Theta) + \sum_{k=0}^{n+d+1}\frac{(it\Theta)^k}{k!},$$

Lemma 4.2 with $N = n + d + 1$ and estimates (4.18)–(4.19) show

$$\begin{aligned} &\|P(\cdot)\mathcal{F}^{-1}\Big\{\Big(e^{it\Theta} - \sum_{0\le k<(n+1)/2} (it\Theta)^k/k!\Big)e^{it|\xi|}\widehat{w}_j\Big\}\|_p \\ &\le C(1+|t|)^{n+m+2d+3}\|P(\cdot)w_j\|_q \end{aligned} \tag{4.20}$$

for any t, where $j = 0, 1$ and $1 < q \le p < \infty$. Since $\cos\rho = (e^{i\rho} + e^{-i\rho})/2$ and $\sin\rho = (e^{i\rho} - e^{-i\rho})/2i$, estimate (4.20) gives the following estimates:

$$\begin{aligned} &\|P(\cdot)\mathcal{F}^{-1}\Big\{\Big(\cos t\Theta - \sum_{0\le 2k<(n+1)/2} (-1)^k(t\Theta)^{2k}/(2k)!\Big)Y(t,\cdot)\widehat{w}_j\Big\}\|_p \\ &\le C(1+|t|)^{n+m+2d+3}\|P(\cdot)w_j\|_q, \end{aligned} \tag{4.21}$$

and

$$\begin{aligned} &\|P(\cdot)\mathcal{F}^{-1}\Big\{\Big(\sin t\Theta - \sum_{0\le 2k+1<(n+1)/2} (-1)^k(t\Theta)^{2k+1}/(2k+1)!\Big)Y(t,\cdot)\widehat{w}_j\Big\}\|_p \\ &\le C(1+|t|)^{n+m+2d+3}\|P(\cdot)w_j\|_q, \end{aligned} \tag{4.22}$$

where $j = 0, 1$, $Y(t,\xi) = \cos t|\xi|$ or $Y(t,\xi) = \sin t|\xi|$. Since

$$\widehat{w}(t,\xi) = e^{-at}\big(\cos t(|\xi| - \Theta)\widehat{w}_0 + \frac{\sin t(|\xi| - \Theta)}{\sqrt{|\xi|^2 - a^2}}(a\widehat{w}_0 + \widehat{w}_1)\big),$$

estimates (4.21)–(4.22) give the desired estimate in Theorem 1.2. □

Proof of Corollary 1.2. Let $T_k(t)$ be the operator defined in Lemma 4.3. $w(t,\cdot) \equiv T_m(t)f$ satisfies the Cauchy problem to the wave equation

$$\partial_t^2 w - \Delta w = 0, \quad w(0,\cdot) = \mathcal{F}^{-1}(\chi_2|\xi|^{-m}\widehat{f}), \quad \partial_t w(0,\cdot) = i\mathcal{F}^{-1}(\chi_2|\xi|^{-m+1}\widehat{f})$$

for $t > 0$, $x \in R^n$. Hence, the solution formula for the Cauchy problem to the wave equation

$$\partial_t^2 W - \Delta W = 0$$

shows that $T_m(t)$ is a bounded operator on L^p for any $1 < p < \infty$, and it satisfies

$$\|T_m(t)f\|_p \le C_p(1+|t|)^m\|f\|_p\,. \tag{4.23}$$

For any t, the operator $T_0(t)$ is bounded on L^2, and satisfies

$$\|T_0(t)f\|_2 \le C\|f\|_2\,. \tag{4.24}$$

Hence, the Stein interpolation theorem between estimates (4.23)–(4.23) shows that

$$\|T_1(t)f\|_p \le C_p(1+|t|)^m\|f\|_p \tag{4.25}$$

holds for $\max(0, 1/2 - 1/2m) < 1/p < \min(1, 1/2 + 1/2m)$. For any $p \in (1,\infty)$ and α, the functions

$$|\xi|\partial_\xi^\alpha(\chi_2(\xi)\Theta), \quad |\xi|\partial_\xi^\alpha\big(\frac{\chi_2(\xi)}{\sqrt{|\xi|^2 - a^2}}\big)$$

are Fourier-multipliers on L^p (see [2]). Therefore estimate (4.25) and similar calculations to ones in the proof of Lemma 4.2 show that

$$\|P(\cdot)\mathcal{F}^{-1}\big(e^{it|\xi|}\Theta^k\frac{|\xi|}{\sqrt{|\xi|^2 - a^2}}\widehat{f}(\xi)\big)\|_p \le C(1+|t|)^{m+d}\|P(\cdot)f\|_p \tag{4.26}$$

and

$$\|P(\cdot)\mathcal{F}^{-1}\big(e^{it|\xi|}\Theta^l|\xi|\widehat{f}(\xi)\big)\|_p \le C(1+|t|)^{m+d}\|P(\cdot)f\|_p \tag{4.27}$$

for any p satisfying $\max(0, 1/2 - 1/2m) < 1/p < \min(1, 1/2 + 1/2m)$, and for positive integers $k \ge 0$, $l \ge 1$, provided that $\operatorname{supp}\widehat{f} \subset \{\xi : |\xi| \ge 2a\}$. Estimate (4.26)–(4.27) and Theorem 1.2 give the desired estimate. □

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