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MAXIMAL ESTIMATES FOR FRACTIONAL SCHRÖDINGER EQUATIONS WITH SPATIAL VARIABLE COEFFICIENT

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Abstract. Let $v(r,t) = \mathcal{T}_t v_0(r)$ be the solution to a fractional Schrödinger equation where the coefficient of Laplacian depends on the spatial variable. We prove some weighted L^q estimates for the maximal operator generated by $\mathcal{T}_t$ with initial data in a Sobolev-type space.

1. Introduction

In this article, we study the maximal estimates of solutions for the fractional Schrödinger equation with spatial variable coefficient,

$$\begin{gathered} i\partial_t v(r,t) + [-r^{p_0}(\partial_{rr} + \frac{p_1}{r}\partial_r - \frac{p_2}{r^2})]^{\alpha/2} v(r,t) = 0, \\ (r,t) \in \mathbb{R}^+ \times \mathbb{R},\ \alpha \in \mathbb{R}^+, \\ v(r,0) = v_0(r), \quad r \in \mathbb{R}^+, \end{gathered} \tag{1.1}$$

where v is a complex-valued function, $r = |x|$, $(x \in \mathbb{R}^n)$ is the radius, and the array (p_0, p_1, p_2) satisfies the assumptions

$$p_0 < 2, \quad p_1 > 1, \quad p_2 = (\frac{2-p_0}{2}\mu)^2 - (\frac{p_1 - 1}{2})^2, \quad \mu \ge 0. \tag{1.2}$$

The difficulty in this equation comes from the spatial variable coefficient term r^{p_0} in front of the Laplacian operator. Such a r^{p_0}-factor arises in the problem of the integrability of the inhomogeneous spherically symmetric Heisenberg ferromagnetic spin system (HFSS)

$$\vec{S}_t(r,t) = \rho(r)\vec{S} \times [\vec{S}_{rr} + \frac{n-1}{r}\vec{S}_r] + \rho_r(r)[\vec{S} \times \vec{S}_r], \tag{1.3}$$

where the spin $\vec{S} = (S^x, S^y, S^z)$ is constrained by $\vec{S}^2 = 1$, $\rho(r)$ is a scalar function, $r = |x|$, $0 < r < \infty$.

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By a known geometrical process [8, 13], the spin evolution equation (1.3) is equivalent to the following generalized nonlinear Schrödinger equation

$$\begin{aligned} & iv_t + \rho(v_{rr} + \frac{n-1}{r}v_r - \frac{n-1}{r^2}v + 2|v|^2v) + 2\rho_r v_r \\ & + [\rho_{rr} + \frac{n-1}{r}\rho_r + 2\int_0^r \rho_{r'}|v|^2dr' + 4(n-1)\int_0^r \frac{\rho}{r'}|v|^2dr']v = 0, \end{aligned} \tag{1.4}$$

and the integrability of (1.3) holds for the conditions $\rho(r) = \epsilon_1 r^{-2(n-1)} + \epsilon_2 r^{-(n-2)}$, where ϵ_1, ϵ_2 are arbitrary constants. Obviously, the factor r^{p_0} corresponds to the term $\rho(r)$ in the (1.4).

In the case of the non-fractional (i.e. $\alpha = 2$) Schrödinger equation without the spatial variable coefficient (i.e. $p_0 = 0$), the (1.1) reduces to the classical Schrödinger equation with(out) the inverse-square potential under the assumption of the spherical symmetry:

$$\begin{gathered} i\partial_t u(x,t) - \Delta u(x,t) + \frac{a}{|x|^2}u(x,t) = 0, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}, \\ u(x,0) = f(x), \quad x \in \mathbb{R}^n. \end{gathered} \tag{1.5}$$

As we know, when $a = 0$, there is a large body of literature studying values of s for which the estimates

$$\|S^*f\|_{L^q(wdx)} \le C\|f\|_{H^s(\mathbb{R}^n)}, \quad (S^*f)(x) := \sup_{t\in\mathbb{R}} |e^{it\Delta}f(x)| \tag{1.6}$$

holds for some q and weight $w(x)$. This has implications for the existence almost everywhere of $\lim_{t\to 0} u(x,t)$ for its solution $u(x,t) = e^{it\Delta}f(x)$, which can be formally expressed as

$$e^{it\Delta}f(x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi - t|\xi|^2)}(\mathcal{F}f)(\xi)d\xi, \tag{1.7}$$

where $\mathcal{F}$ is the usual spatial Fourier transform defined by $\mathcal{F}f = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}f(x)dx$.

The maximal estimate (1.6) and related questions were raised by Carleson [4] who proved convergence for $s \ge \frac{1}{4}$ when $n = 1$. Dahlberg and Kenig [7] showed that this result is sharp. In higher dimension, the question of identifying the optimal exponent s has been studied by several authors and our state of knowledge may be summarized as follows. For $n = 2$, the strongest result to date appears in [10] for $s > 3/8$. For $n \ge 2$, the convergence is shown to hold for $s > \frac{2n-1}{4n}$ (see [1, 2]). More generally, it should also be observed that the maximal estimates (1.6) developed for (1.5) with $a = 0$ can be extended to the case of fractional Schrödinger equation without the spatial variable coefficient (i.e. $\alpha > 0$, $p_0 = 0$). Some positive partial results were obtained by Sjölin [14], Heinig-Wang [9], Cho-Lee-Shim [5, 6] and Bourgain [1].

In the case when $p_0 \neq 0$ and $\alpha > 0$, equation (1.1) can be viewed as the general fractional Schrödinger equation with spatial variable coefficient proposed by authors in [19], which is a simplified version of (1.4). Inspired by the results of the papers [18, 19] and equation (1.5), we try to explore the maximal estimate for the more general equation (1.1), which seems that there is no previous literature on it. In this paper, we try to derive some maximal estimates of solution to the general equation (1.1).

Let $v(r,t) = \mathcal{T}_t v_0(r)$ be the solution to (1.1), we define the maximal operator $\mathcal{T}^*$ as

$$(\mathcal{T}^* v_0)(r) = \sup_{t\in\mathbb{R}} |\mathcal{T}_t v_0(r)|. \tag{1.8}$$

Our aim is to investigate the mapping properties of $\mathcal{T}^*$, which are from a Sobolev-type space X to a weighted L^q space. The estimates are of the form

$$\|\mathcal{T}^* v_0\|_{L^q_{\omega,\varrho}(\mathbb{R}^+)} \le C\|v_0\|_X, \quad X = \mathcal{W}^{s,2}(\mathbb{R}^+) \text{ or } \dot{H}^s_{\rm rad}(\mathbb{R}^n), \tag{1.9}$$

where $\mathcal{W}^{s,2}(\mathbb{R}^+)$ is the inhomogeneous Hankel-Sobolev space in Definition 1.2 and $\dot{H}^s_{\rm rad}(\mathbb{R}^n)$ is the usual homogeneous Sobolev space

$$\dot{H}^s_{\rm rad}(\mathbb{R}^n) = \{f \text{ is radial, } \|f\|^2_{\dot{H}^s_{\rm rad}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\xi|^{2s} |(\mathcal{F}f)(\xi)|^2 d\xi < \infty\}. \tag{1.10}$$

We also note that the norm $\|F\|_{L^q_{\omega,\varrho}(\mathbb{R}^+)}$ is abbreviated by

$$\|F\|_{L^q_{\omega,\varrho}(\mathbb{R}^+)} := \Big(\int_{\mathbb{R}^+} |F(r)|^q \varrho(r) d\omega_r\Big)^{1/q}, \tag{1.11}$$

where $d\omega_r = r^{p_1-p_0}dr$ is the Lebesgue measure. For simplicity, $\|F\|_{L^q_\omega(\mathbb{R}^+)} := \|F\|_{L^q_{\omega,1}(\mathbb{R}^+)}$ and $\|F\|_{L^q(\mathbb{R}^+)} := (\int_{\mathbb{R}^+} |F(r)|^q dr)^{1/q}$.

For (1.1), the presence of the factor r^{p_0} makes it difficult to give the expression of the solution by using the usual Fourier transform, which is only a well-suited tool to analyze constant coefficient Schrödinger equation such as (1.5). Inspired by [18, 19], we introduce a suitable Hankel transform.

Definition 1.1. Suppose $f(r)$ is an integrable function in $\mathbb{R}^+$, we define the Hankel transform

$$(\mathcal{H}_\mu f)(\lambda) = \int_{\mathbb{R}^+} (\lambda r)^{\frac{1-p_1}{2}} \mathcal{J}_\mu\big(\frac{2}{2-p_0}(\lambda r)^{\frac{2-p_0}{2}}\big) f(r) d\omega_r, \tag{1.12}$$

where $\mathcal{J}_\mu(z)$ is the first Bessel function of order μ defined as

$$\mathcal{J}_\mu(z) = \frac{(z/2)^\mu}{\Gamma(\mu+\frac12)\pi^{1/2}} \int_{-1}^1 e^{izy}(1-y^2)^{\mu-\frac12} dy.$$

We define the fractional power of the second-order operator $\mathcal{A}_\mu := -r^{p_0}(\partial_{rr} + \frac{p_1}{r}\partial_r - \frac{p_2}{r^2})$ in (1.1) by

$$\mathcal{A}_\mu^{\alpha/2} g(r) = \mathcal{H}_\mu[\lambda^{\frac{2-p_0}{2}\alpha}(\mathcal{H}_\mu g)(\lambda)](r). \tag{1.13}$$

It should be noticed that the definition of $\mathcal{A}_\mu^{\alpha/2}$ can be referred in [12, 18] and makes sense.

For our purpose, we also introduce the Hankel-Sobolev space via the Hankel transform.

Definition 1.2. The homogeneous *Hankel-Sobolev space* $\dot{\mathcal{W}}^{s,2}(\mathbb{R}^+)$ consists of tempered distributions f for which $\mathcal{H}_\mu[\lambda^{\frac{2-p_0}{2}s}(\mathcal{H}_\mu f)(\lambda)](r)$ exists and is in $L^2_\omega(\mathbb{R}^+)$ function. That is,

$$\dot{\mathcal{W}}^{s,2}(\mathbb{R}^+) = \big\{f \in \mathcal{S}'(\mathbb{R}^+), \|f\|^2_{\dot{\mathcal{W}}^{s,2}(\mathbb{R}^+)} = \int_{\mathbb{R}^+} |\lambda^{\frac{2-p_0}{2}s}(\mathcal{H}_\mu f)(\lambda)|^2 d\omega_\lambda < \infty\big\}.$$

We also define the *inhomogeneous Hankel-Sobolev space* $\mathcal{W}^{s,2}(\mathbb{R}^+)$ as

$$\mathcal{W}^{s,2}(\mathbb{R}^+) = \big\{f \in \mathcal{S}'(\mathbb{R}^+), \ \|f\|^2_{\mathcal{W}^{s,2}(\mathbb{R}^+)} = \int_{\mathbb{R}^+} (1+\lambda^{2-p_0})^s |(\mathcal{H}_\mu f)(\lambda)|^2 d\omega_\lambda < \infty\big\}.$$

Note that the space $\dot{H}^s_{\rm rad}(\mathbb{R}^n)$ is the special case of $\dot{\mathcal{W}}^{s,2}(\mathbb{R}^+)$ when $(p_0,p_1,p_2)=(0,n-1,0)$. Our first result is to derive an weighted L^2 estimate for the maximal function $\mathcal{T}^*v_0$, which is stated as follows.

Theorem 1.3. *Suppose $p_2=0$. Let $b\in(\frac{2-p_0}{2},2-p_0)$, $2\le n<\frac{2|p_1-1|}{2-p_0}+2$ and $s\in(1/2,1)$. Then*

$$\|\mathcal{T}^*v_0\|_{L^2_{\omega,\varrho}(\mathbb{R}^+)}\le C(p_0,p_1,\alpha)\|v_0\|_{\dot{H}^s_{\rm rad}(\mathbb{R}^n)}, \tag{1.14}$$

where $\varrho(r)=(1+r)^{-b}$.

As a consequence, we obtain the almost convergence result for $v_0\in\dot{H}^s_{\rm rad}(\mathbb{R}^n)$.

Corollary 1.4. *Let $v_0\in\dot{H}^s_{\rm rad}(\mathbb{R}^n)$ with $s\in(\frac12,1)$ and $2\le n<\frac{2|p_1-1|}{2-p_0}+2$. Then*

$$\lim_{t\to0}v(r,t)=v_0(r),\quad a.e.\ r\in\mathbb{R}^+.$$

If the initial data v_0 lies in the space $\mathcal{W}^{s,2}(\mathbb{R}^+)$, we improve the integrability of the maximal function $\mathcal{T}^*v_0$ for (1.1).

Theorem 1.5. *For $0<\alpha\neq1$. If the initial data $v_0\in\mathcal{W}^{s,2}(\mathbb{R}^+)$ with $s\in[\frac14,\frac12)$, Then the estimates*

$$\|\mathcal{T}^*v_0\|_{L^q_\omega(\mathbb{R}^+)}\le C(p_0,p_1)\|v_0\|_{\mathcal{W}^{s,2}(\mathbb{R}^+)}, \tag{1.15}$$

$$\|\mathcal{T}^*v_0\|_{L^q_{\omega,\varrho}(\mathbb{R}^+)}\le C(p_0,p_1)\|v_0\|_{\mathcal{W}^{s,2}(\mathbb{R}^+)}, \tag{1.16}$$

hold for

$$\frac{8(p_1-p_0+1)}{4p_1-3p_0+2}\le q<\frac{2(p_1-p_0+1)}{p_1-p_0+1-(2-p_0)s}\quad and\quad q=\frac{2(p_1-p_0+1)}{p_1-p_0+1-(2-p_0)s}$$

respectively, where $\varrho(r)=r^b(1+r)^{-b}$, $b>0$.

The plan of this paper is as follows: Section 2 is devoted to the preliminaries, including the properties of Bessel function and the relation between $\dot{\mathcal{W}}^{s,2}(\mathbb{R}^+)$ and $\dot{H}^s_{\rm rad}(\mathbb{R}^n)$. In Section 3, through delicate computation, we give the complete argument about the weighted L^q maximal estimates of the (1.1). If not specified, throughout this paper, the notations $M\ll N$ and $M\sim N$ denote $M\le C^{-1}N$ and $CM\le N\le\tilde{C}M$ respectively for some large constants C and $\tilde{C}$. We also denote $\lesssim_\beta$ as $\le C(\beta)$, where $C(\beta)$ denotes various constant that only depends on β. We abbreviate by writing $A+\epsilon$ as $A+$ or $A-\epsilon$ as $A-$ for $0<\epsilon\ll1$.

2. Preliminaries

In this section, we collect some basic facts which will be used in the later context. We recall some asymptotic properties of the first Bessel function $\mathcal{J}_\mu(z)$ (see [17]). For fixed μ, if $z\ll1$, a simple calculation gives the rough estimate

$$|\mathcal{J}_\mu(z)|\le\frac{Cz^\mu}{2^\mu\Gamma(\mu+\frac12)\Gamma(1/2)}(1+\frac{1}{\mu+1/2}), \tag{2.1}$$

where C is a absolute constant. Another well known asymptotic expansion about the Bessel function is

$$\mathcal{J}_\mu(z)=z^{-1/2}(b_+e^{iz}+b_-e^{-iz})+\Phi_\mu(z),\quad z\gg1, \tag{2.2}$$

where $|\Phi_\mu(z)| \le Cz^{-1}$, $|b_\pm| \le C$ and the constant C is independent of μ. As pointed out in [16], if one seeks a uniform bound for large μ and z, then the best one can do is

$$|\mathcal{J}_\mu(z)| \le Cz^{-1/3}, \quad z \ge 1. \tag{2.3}$$

A simple consequence of the above properties is the following Lemma.

Lemma 2.1. *For $R \gg 1$, there exists a constant $C(p_0)$ independent of μ, R such that*

$$\int_R^{2R} |\mathcal{J}_\mu(r^{\frac{2-p_0}{2}})|^2 dr \le C(p_0) R^{p_0/2}.$$

Next we review some properties of the Hankel transform, which appear in [3, 19].

Lemma 2.2. *The Hankel transform $\mathcal{H}_\mu$ satisfies:*

(i) $\mathcal{H}_\mu = \mathcal{H}_\mu^{-1}$,
(ii) *$\mathcal{H}_\mu$ is an L^2 isometry, i.e.* $\|\mathcal{H}_\mu \phi\|_{L^2_\omega(\mathbb{R}^+)} = \|\phi\|_{L^2_\omega(\mathbb{R}^+)}$,
(iii) $\mathcal{H}_\mu(\mathcal{A}_\mu\phi)(\lambda) = \lambda^{2-p_0}(\mathcal{H}_\mu\phi)(\lambda)$, *where the operator $\mathcal{H}_\mu^{-1}$ is the inverse operator of $\mathcal{H}_\mu$.*

For the Hankel-Sobolev space $\dot{\mathcal{W}}^{\sigma,2}(\mathbb{R}^+)$, there exists the following embedding theorem with $\dot{H}^\sigma_{\rm rad}(\mathbb{R}^n)$, which is proved in the paper [18].

Lemma 2.3. *Let $n \ge 2$ and $\mu > \frac{n-2}{2}$. If $f \in \dot{H}^\sigma_{\rm rad}(\mathbb{R}^n)$, $0 \le \sigma < \frac{n}{2}$, then*

$$\|f\|_{\dot{\mathcal{W}}^{\sigma,2}(\mathbb{R}^+)} \le C(\sigma,\mu,n)\|f\|_{\dot{H}^\sigma_{\rm rad}(\mathbb{R}^n)}. \tag{2.4}$$

Proof. We give only an outline of the proof. From the definition of Hankel transform, (1.13) and using the integral formula of Bessel function [17, p. 385], we obtain

$$\begin{aligned}
&\mathcal{M}[\mathcal{A}_\mu^{\sigma/2} f](z)\\
&= (2-p_0)^\sigma \frac{\Gamma(\frac{2z-p_1+1}{2(2-p_0)} + \frac{\mu}{2})}{\Gamma(1-\frac{2z-p_1+1}{2(2-p_0)} + \frac{\mu}{2})} \frac{\Gamma(1-\frac{2z-p_1+1}{2(2-p_0)} + \frac{\sigma+\mu}{2})}{\Gamma(\frac{2z-p_1+1}{2(2-p_0)} - \frac{\sigma-\mu}{2})} \mathcal{M}[f](z - \frac{2-p_0}{2}\sigma)
\end{aligned} \tag{2.5}$$

where $\mathcal{M}[f(r)](z) = \int_{\mathbb{R}^+} r^{z-1} f(r) dr$ is the Mellin transform.

Denote $B^\sigma_{\mu,w} := \mathcal{A}_\mu^{\sigma/2}\mathcal{A}_w^{-\sigma/2}$. Writing $\tilde{z} = \frac{2z}{2-p_0}$ and $\tilde{\kappa} = \frac{p_1-1}{2-p_0}$, by (2.5), we obtain

$$\begin{aligned}
\mathcal{M}[B^\sigma_{\mu,w} f](z) &= \frac{\Gamma((\tilde{z}-\tilde{\kappa}+\mu)/2)\Gamma(1-(\tilde{z}-\sigma-\tilde{\kappa}-\mu)/2)}{\Gamma(1-(\tilde{z}-\tilde{\kappa}-\mu)/2)\Gamma((\tilde{z}-\sigma-\tilde{\kappa}+\mu)/2)}\\
&\quad\times \frac{\Gamma((\tilde{z}-\sigma-\tilde{\kappa}+w)/2)\Gamma(1-(\tilde{z}-\tilde{\kappa}-w)/2)}{\Gamma(1-(\tilde{z}-\sigma-\tilde{\kappa}-w)/2)\Gamma((\tilde{z}-\tilde{\kappa}+w)/2)} \mathcal{M}[f](z)\\
&:= F(z)\mathcal{M}[f](z).
\end{aligned}$$

For $z = \frac{p_1-p_0+1}{2} + iy$ and $\tilde{z} = \tilde{\kappa} + 1 + \frac{2}{2-p_0} iy$, using the following properties of Gamma function $\Gamma(z)$:

$$\overline{\Gamma(z)} = \Gamma(\bar{z}), \quad \forall z \in \mathbb{C},$$

$$|\Gamma(x+iy)| = \Gamma(x)\prod_{k=0}^\infty (1 + \frac{y^2}{(x+k)^2})^{-1/2}, \quad \forall x > 0,\ \forall y \in \mathbb{R},$$

we obtain

$$
\begin{aligned}
|F(\frac{p_1-p_0+1}{2}+iy)| &= |\frac{\Gamma((\mu+\sigma+1-\frac{2}{2-p_0}iy)/2)}{\Gamma((\mu-\sigma+1+\frac{2}{2-p_0}iy)/2)}\frac{\Gamma((w-\sigma+1+\frac{2}{2-p_0}iy)/2)}{\Gamma((w+\sigma+1-\frac{2}{2-p_0}iy)/2)}| \\
&= |\frac{\Gamma((\mu+\sigma+1)/2)}{\Gamma((\mu-\sigma+1)/2)}\frac{\Gamma((w-\sigma+1)/2)}{\Gamma((w+\sigma+1)/2)}|\prod_0^\infty [R_k(\tilde{y})]^{1/2},
\end{aligned}
$$

where

$$
\begin{aligned}
R_k(\tilde{y}) &= \frac{(1+\tilde{y}^2/(\mu-\sigma+1+2k)^2)(1+\tilde{y}^2/(w+\sigma+1+2k)^2)}{(1+\tilde{y}^2/(\mu+\sigma+1+2k)^2)(1+\tilde{y}^2/(w-\sigma+1+2k)^2)} \\
&= \frac{(1+\tilde{y}^2/(M_k-\sigma)^2)(1+\tilde{y}^2/(N_k+\sigma)^2)}{(1+\tilde{y}^2/(M_k+\sigma)^2)(1+\tilde{y}^2/(N_k-\sigma)^2)} \le 1.
\end{aligned}
$$

and $\tilde{y}=\frac{2y}{2-p_0}$, $M_k=\mu+1+2k, N_k=w+1+2k$. Therefore, for $n>2\sigma$, we have

$$
\sup_y |F(\frac{p_1-p_0+1}{2}+iy)| < \infty.
$$

Hence, using [18, Lemma 2.5], we obtain

$$
\|B^\sigma_{\mu,\kappa} f\|_{L^2_\omega(\mathbb{R}^+)} \le C\|f\|_{L^2_\omega(\mathbb{R}^+)},
$$

which is the desired result. □

At the end of this section, we show the oscillatory integral estimate [6, 15].

Lemma 2.4. *Suppose $\varphi\in C^2(\mathbb{R}^n\backslash\{0\})$ is a radial function such that $|\varphi^{(k)}(\xi)|\sim|\xi|^{a-k}$, $k=0,1,2$ for $0<a\neq 1$. Let A,B,σ be the real numbers such that $A,B\neq 0$, $\sigma\in[1/2,1)$, then there exists a constant $C(a,\sigma)$, independent of A,B, such that*

$$
\Big|\int_{\mathbb{R}} e^{i(A\varphi(\xi)+B\xi)}|\xi|^{-\sigma}d\xi\Big| \le C|B|^{-(1-\sigma)}. \tag{2.6}
$$

3. Proof of main results

Applying the Hankel transform (1.12) to the (1.1), by Definition 1.1, (1.13) and Lemma 2.2 (i), we have

$$
\begin{gathered}
i\partial_t\tilde{v}+\lambda^{\frac{2-p_0}{2}\alpha}\tilde{v}=0 \\
\tilde{v}(\lambda,0)=\tilde{v}_0(\lambda),
\end{gathered}
$$

where

$$
\tilde{v}(\lambda,t)=(\mathcal{H}_\mu v)(\lambda,t), \quad \tilde{v}_0(\lambda)=(\mathcal{H}_\mu v_0)(\lambda).
$$

Solving the ODE and inverting the Hankel transform, we obtain the formal solution

$$
\mathcal{T}_t v_0(r)=\int_{\mathbb{R}^+}(\lambda r)^{\frac{1-p_1}{2}}\mathcal{J}_\mu(\frac{2}{2-p_0}(\lambda r)^{\frac{2-p_0}{2}})e^{it\lambda^{\frac{2-p_0}{2}\alpha}}\tilde{v}_0(\lambda)d\omega_\lambda. \tag{3.1}
$$

Proof of Theorem 1.3. The key ingredients are the asymptotic behavior of the Bessel function, and the properties of Hankel transform.

By the continuity of the embedding $\dot{H}^{\frac{1}{2}-}(\mathbb{R}) \cap \dot{H}^{\frac{1}{2}+}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, it suffices to prove

Proposition 3.1. *Let $p_2 = 0$. For $b \in (\frac{2-p_0}{2}, 2-p_0)$, and $a \in [\frac{1}{2} - \frac{b}{(2-p_0)\alpha}, \frac{1}{2} + \frac{n}{2\alpha} - \frac{b}{(2-p_0)\alpha})$, there exists a constant C independent of v_0 such that*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^+} |\partial_t^a (\mathcal{T}_t v_0(r))|^2 \frac{d\omega_r dt}{(1+r)^b} \le C(p_0, p_1, a, \alpha) \|v_0\|^2_{\dot{\mathcal{W}}^{\sigma,2}(\mathbb{R}^+)},$$

where $\sigma := \frac{(2a-1)\alpha}{2} + \frac{b}{2-p_0}$.

This proposition, NS Lemma 2.3, yield Theorem 1.3. Indeed, under the assumption of a, b above, we have $\sigma \in [0, \frac{n}{2})$ and $\frac{n-2}{2} < \frac{|p_1 - 1|}{2-p_0}$, then

$$\int_{\mathbb{R}} \int_{\mathbb{R}^+} |\partial_t^a (\mathcal{T}_t v_0(r))|^2 \frac{d\omega_r dt}{(1+r)^b} \le C(p_0, p_1, a, \alpha) \|v_0\|^2_{\dot{H}^\sigma_{\rm rad}(\mathbb{R}^n)}. \tag{3.2}$$

Choosing $a = \frac{1}{2}+$ and $a = \frac{1}{2}-$ in (3.2), we obtain

$$\int_{\mathbb{R}^+} |\mathcal{T}^* v_0(r)|^2 \frac{d\omega_r}{(1+r)^b} \le C(p_0, p_1, \alpha) \|v_0\|^2_{\dot{H}_{\rm rad}^{\frac{b}{2-p_0}}(\mathbb{R}^n)}.$$

Hence, Theorem 1.3 is proved.

Proof of Proposition 3.1. Using the Plancherel theorem of the usual Fourier transform with respect to time t, we have

$$\int_{\mathbb{R}} \int_{\mathbb{R}^+} |\partial_t^a (\mathcal{T}_t v_0)|^2 \frac{d\omega_r dt}{(1+r)^b} = \int_{\mathbb{R}^+} \int_{\mathbb{R}} |\tau^a \int_{\mathbb{R}} e^{-it\tau} (\mathcal{T}_t v_0)(r,t) dt|^2 \frac{d\tau d\omega_r}{(1+r)^b}. \tag{3.3}$$

Using (3.1), we obtain

$$\begin{aligned}
&\text{left-hand side of (3.3)}\\
&= \int_{\mathbb{R}^+} \int_{\mathbb{R}} \Big| \tau^a \int_{\mathbb{R}} \int_{\mathbb{R}^+} (\lambda r)^{\frac{1-p_1}{2}} \mathcal{J}_\mu\big(\frac{2}{2-p_0} (\lambda r)^{\frac{2-p_0}{2}}\big) e^{it(\lambda^{\frac{2-p_0}{2}\alpha} - \tau)} \tilde{v}_0(\lambda) d\omega_\lambda dt \Big|^2 \\
&\quad \times \frac{d\tau d\omega_r}{(1+r)^b} \\
&\le \int_{\mathbb{R}^+} \int_{\mathbb{R}} \Big| \tau^a \int_{\mathbb{R}^+} (\lambda r)^{\frac{1-p_1}{2}} \mathcal{J}_\mu\big(\frac{2}{2-p_0} (\lambda r)^{\frac{2-p_0}{2}}\big) \delta(\lambda^{\frac{2-p_0}{2}\alpha} - \tau) \tilde{v}_0(\lambda) d\omega_\lambda \Big|^2 \\
&\quad \frac{d\tau d\omega_r}{(1+r)^b} \\
&\le_{p_0,\alpha} \int_{\mathbb{R}^+} \int_{\mathbb{R}} \Big| \tau^a r^{\frac{1-p_1}{2}} \int_{\mathbb{R}^+} \lambda^{\frac{p_1 - 2p_0 + 3}{(2-p_0)\alpha} - 1} \mathcal{J}_\mu\big(\frac{2r^{\frac{2-p_0}{2}}}{2-p_0} \lambda^{\frac{1}{\alpha}}\big) \tilde{v}_0(\lambda^{\frac{2}{(2-p_0)\alpha}}) \delta(\lambda - \tau) d\lambda \Big|^2 \\
&\quad \frac{d\tau d\omega_r}{(1+r)^b} \\
&\le_{p_0,\alpha} \int_{\mathbb{R}^+} \int_{\mathbb{R}} \Big| r^{\frac{1-p_1}{2}} \lambda^{\frac{p_1 - 2p_0 + 3}{(2-p_0)\alpha} - 1 + a} \mathcal{J}_\mu\big(\frac{2r^{\frac{2-p_0}{2}}}{2-p_0} \lambda^{\frac{1}{\alpha}}\big) \tilde{v}_0(\lambda^{\frac{2}{(2-p_0)\alpha}}) \Big|^2 \frac{d\lambda d\omega_r}{(1+r)^b},
\end{aligned}$$

where the above inequality can be rewritten as

$$C(p_0, \alpha) \iint_{\mathbb{R}^+} \big| r^{\frac{1-p_1}{2}} \lambda^{\frac{p_1 - 2p_0 + 2}{2} + (2a-1)\frac{(2-p_0)\alpha}{4}} \mathcal{J}_\mu\big(\frac{2}{2-p_0} (\lambda r)^{\frac{2-p_0}{2}}\big) \tilde{v}_0(\lambda) \big|^2 \frac{d\lambda d\omega_r}{(1+r)^b}.$$

We make a dyadic decomposition by choosing χ, which is a smoothing function supported in $[\frac{1}{2},2]$, and change the variable $\frac{\lambda}{M}\mapsto\lambda,\ Mr\mapsto r$,

$$
\begin{aligned}
&\text{left-hand side of (3.3)}\\
&\le_{p_0,\alpha}\sum_{M\in 2^{\mathbb{Z}}}\iint_{\mathbb{R}^+}\big|\lambda^{\tilde d+\frac{2p_1-2p_0+1}{2}}(\lambda r)^{\frac{1-p_1}{2}}\mathcal{J}_\mu(\frac{2}{2-p_0}(\lambda r)^{\frac{2-p_0}{2}})\tilde v_0(\lambda)\chi(\frac{\lambda}{M})\big|^2\frac{d\omega_r d\lambda}{(1+r)^b}\\
&\le_{a,\alpha,p_0}\sum_{M\in 2^{\mathbb{Z}}}M^{2\tilde d+p_1-p_0+1}\iint_{\mathbb{R}^+}\big|(\lambda r)^{\frac{1-p_1}{2}}\mathcal{J}_\mu(\frac{2}{2-p_0}(\lambda r)^{\frac{2-p_0}{2}})\tilde v_0(M\lambda)\chi(\lambda)\big|^2\\
&\quad\times\frac{d\omega_r d\omega_\lambda}{(1+\frac{r}{M})^b}\\
&\le_{a,\alpha,p_0}\sum_{M\in 2^{\mathbb{Z}}}\sum_{R\in 2^{\mathbb{Z}}}M^{2\tilde d+p_1-p_0+1}R^{p_1-p_0}\mathcal{K}_R^M,
\end{aligned}
\tag{3.4}
$$

where $\tilde d:=(2a-1)\frac{(2-p_0)\alpha}{4}$. By integrating,

$$
\mathcal{K}_R^M:=\int_{\mathbb{R}^+}\int_R^{2R}|(\lambda r)^{\frac{1-p_1}{2}}\mathcal{J}_\mu(\frac{2}{2-p_0}(\lambda r)^{\frac{2-p_0}{2}})\tilde v_0(M\lambda)\chi(\lambda)|^2\frac{dr d\omega_\lambda}{(1+\frac{r}{M})^b}.
$$

Now we need to prove the bound of $\mathcal{K}_R^M$, which is divided in two cases:

Case 1: $R\ll 1$. Since $\lambda\sim 1$, we have $(r\lambda)^{\frac{2-p_0}{2}}\ll 1$. Using the property of Bessel function (2.1), we have

$$
\begin{aligned}
\mathcal{K}_R^M&\le_{p_0,\mu}\int_{\mathbb{R}^+}\int_R^{2R}|(\lambda r)^{\frac{1-p_1+(2-p_0)\mu}{2}}\tilde v_0(M\lambda)\chi(\lambda)|^2\frac{dr d\omega_\lambda}{(1+\frac{r}{M})^b}\\
&\le_{p_0,\mu}R^{2-p_1+(2-p_0)\mu}\min\{1,(\frac{M}{R})^b\}\int_{\mathbb{R}^+}|\tilde v_0(M\lambda)\chi(\lambda)|^2 d\omega_\lambda.
\end{aligned}
\tag{3.5}
$$

Case 2: $R\gg 1$. Since $\lambda\sim 1$, we obtain $(r\lambda)^{\frac{2-p_0}{2}}\gg 1$. Then $\mathcal{K}_R^M$ can be bounded by

$$
C(p_1)R^{1-p_1}\int_{\mathbb{R}^+}|\tilde v_0(M\lambda)\chi(\lambda)|^2(\int_R^{2R}|\mathcal{J}_\mu(\frac{2}{2-p_0}(\lambda r)^{\frac{2-p_0}{2}})|^2\frac{dr}{(1+\frac{r}{M})^b})d\omega_\lambda.
$$

Noticing that $\lambda\sim 1$, so

$$
\begin{aligned}
&\int_R^{2R}|\mathcal{J}_\mu(\frac{2}{2-p_0}(\lambda r)^{\frac{2-p_0}{2}})|^2\frac{dr}{(1+\frac{r}{M})^b}\\
&\le\min\{1,(\frac{M}{R})^b\}\int_R^{2R}|\mathcal{J}_\mu(\frac{2}{2-p_0}(\lambda r)^{\frac{2-p_0}{2}})|^2 dr\\
&\le_{p_0}\min\{1,(\frac{M}{R})^b\}R^{p_0/2},
\end{aligned}
$$

where the last inequality is obtained by Lemma 2.1. Then

$$
\mathcal{K}_R^M\le_{p_0,p_1}R^{\frac{p_0-2p_1+2}{2}}\min\{1,(\frac{M}{R})^b\}\int_{\mathbb{R}^+}|\tilde v_0(M\lambda)\chi(\lambda)|^2 d\omega_\lambda.
\tag{3.6}
$$

Hence, combining (3.5) and (3.6), for $p_2=0$, we obtain the estimate

$$
\mathcal{K}_R^M\le_{p_0,p_1}\begin{cases}R^{2-p_1+|p_1-1|}M^{p_0-p_1-1}\min\{1,(\frac{M}{R})^b\}\|\tilde v_0(\lambda)\chi(\frac{\lambda}{M})\|^2_{L^2_\omega(\mathbb{R}^+)}, & R\ll 1,\\ R^{\frac{p_0-2p_1+2}{2}}M^{p_0-p_1-1}\min\{1,(\frac{M}{R})^b\}\|\tilde v_0(\lambda)\chi(\frac{\lambda}{M})\|^2_{L^2_\omega(\mathbb{R}^+)}, & R\gg 1.\end{cases}
$$

Now, returning to (3.4), we have

$$\begin{aligned}
&\text{left-hand side of (3.3)}\\
&\le_{a,\alpha,p_0,p_1} \sum_{M\in 2^{\mathbb{Z}}} \sum_{R\in 2^{\mathbb{Z}}:R\ll 1} M^{2\bar{d}} R^{2-p_0+|p_1-1|} \min\{1,(\frac{M}{R})^b\} \|\tilde{v}_0(\lambda)\chi(\frac{\lambda}{M})\|^2_{L^2_\omega(\mathbb{R}^+)}\\
&\quad + \sum_{M\in 2^{\mathbb{Z}}} \sum_{R\in 2^{\mathbb{Z}}:R\gg 1} M^{2\bar{d}} R^{\frac{2-p_0}{2}} \min\{1,(\frac{M}{R})^b\} \|\tilde{v}_0(\lambda)\chi(\frac{\lambda}{M})\|^2_{L^2_\omega(\mathbb{R}^+)}\\
&\le_{a,\alpha,p_0,p_1} \sum_{M\in 2^{\mathbb{Z}}} M^{2\bar{d}+b}[\sum_{R\ll 1} R^{|p_1-1|+2-p_0-b} + \sum_{R\gg 1} R^{\frac{2(1-b)-p_0}{2}}] \|\tilde{v}_0(\lambda)\chi(\frac{\lambda}{M})\|^2_{L^2_\omega(\mathbb{R}^+)}.
\end{aligned}$$

From the assumption $b\in(\frac{2-p_0}{2},2-p_0)$, the above inequality can be further controlled by

$$\sum_{M\in 2^{\mathbb{Z}}} M^{(2a-1)\frac{(2-p_0)\alpha}{2}+b} \|\tilde{v}_0(\lambda)\chi(\frac{\lambda}{M})\|^2_{L^2_\omega(\mathbb{R}^+)}.$$

Finally, by Lemma 2.2 (ii) and letting $M=2^j$, we have

$$\begin{aligned}
\text{left-hand side of (3.3)} &\le_{a,\alpha,p_0,p_1} \sum_{j\in\mathbb{Z}} 2^{j(2-p_0)\sigma} \|\mathcal{H}_\mu[\chi(\frac{\lambda}{2^j})\mathcal{H}_\mu v_0]\|^2_{L^2_\omega(\mathbb{R}^+)}\\
&\le_{a,\alpha,p_0,p_1} \|\Big(\sum_{j\in\mathbb{Z}} |2^{\frac{(2-p_0)\sigma}{2}j}\mathcal{H}_\mu[\chi(\frac{\lambda}{2^j})\mathcal{H}_\mu v_0]|^2\Big)^{1/2}\|^2_{L^2_\omega(\mathbb{R}^+)},
\end{aligned}$$

where $\sigma:=\frac{(2a-1)\alpha}{2}+\frac{b}{2-p_0}$ and Proposition 3.1 follows from Lemma 3.2 below. $\square$

Lemma 3.2 ([18]). *Let $\varsigma\in\mathbb{R}$. Then there exists a constant C that depends only on ς and β such that for all $f\in\dot{\mathcal{W}}^{\frac{2\varsigma}{2-p_0},2}(\mathbb{R}^+)$, we have*

$$\|\Big(\sum_{j\in\mathbb{Z}} |2^{j\varsigma}\mathcal{H}_\mu[\beta_j(\lambda)\mathcal{H}_\mu f]|^2\Big)^{1/2}\|_{L^2_\omega(\mathbb{R}^+)} \le C\|f\|_{\dot{\mathcal{W}}^{\frac{2\varsigma}{2-p_0},2}}, \tag{3.7}$$

where the function $\beta\in C_0^\infty(\mathbb{R}^+)$ is supported in the interval $[1/2,2]$, $\beta_j(\lambda)=\beta(\frac{\lambda}{2^j})$, $\sum_{j\in\mathbb{Z}}\beta_j=1$.

Proof of Theorem 1.5. We consider the solution (3.1):

$$\mathcal{T}_t v_0(r)=\int_{\mathbb{R}^+}(\lambda r)^{\frac{1-p_1}{2}}\mathcal{J}_\mu(\frac{2}{2-p_0}(\lambda r)^{\frac{2-p_0}{2}})e^{it\lambda^{\frac{2-p_0}{2}\alpha}}\tilde{v}_0(\lambda)d\omega_\lambda.$$

By changing the variable $\frac{2}{2-p_0}\lambda^{\frac{2-p_0}{2}}\mapsto\lambda, r^{\frac{2-p_0}{2}}\mapsto r$, the solution becomes

$$\begin{aligned}
&(\frac{2-p_0}{2})^{\frac{p_1-p_0+1}{2-p_0}}\int_{\mathbb{R}^+}\lambda^{\frac{p_1-p_0+1}{2-p_0}}r^{\frac{1-p_1}{2-p_0}}\mathcal{J}_\mu(\lambda r)e^{it(\frac{2-p_0}{2}\lambda)^\alpha}\tilde{v}_0\big((\frac{2-p_0}{2}\lambda)^{\frac{2}{2-p_0}}\big)d\lambda\\
&:=\mathfrak{T}[\tilde{v}_0](r),
\end{aligned}$$

To prove (1.15) in Theorem 1.5, it suffices to prove that for $q=\frac{2(p_1-p_0+1)}{p_1-p_0+1-(2-p_0)s'}$ and $s'\in[\frac14,s)$, the norm $\|\mathfrak{T}[\tilde{v}_0](r)r^{\frac{2p_1-p_0}{(2-p_0)q}}\|_{L^qL^\infty_t(\mathbb{R}^+\times\mathbb{R})}$ is bounded by

$$\Big(\int_{\mathbb{R}^+}|\tilde{v}_0((\frac{2-p_0}{2}\lambda)^{\frac{2}{2-p_0}})[1+(\frac{2-p_0}{2}\lambda)^2]^{s/2}\lambda^{\frac{2p_1-p_0}{2(2-p_0)}}|^2d\lambda\Big)^{1/2},$$

where the norm $\|h(r,t)\|_{L^qL^\infty_t(\mathbb{R}^+\times\mathbb{R})}:=\sup_{t\in\mathbb{R}}\|h(\cdot,t)\|_{L^q(\mathbb{R}^+)}$.

We write $g(\lambda)=\tilde{v}_0((\frac{2-p_0}{2}\lambda)^{\frac{2}{2-p_0}})[1+(\frac{2-p_0}{2}\lambda)^2]^{\frac{s}{2}}\lambda^{\frac{2p_1-p_0}{2(2-p_0)}}$. Then the above estimate is equivalent to

$$\|\mathcal{P}(g)|\|_{L^qL_t^\infty(\mathbb{R}^+\times\mathbb{R})}\le_{q,p_0,p_1}\|g\|_{L^2(\mathbb{R}^+)},$$

where

$$\begin{aligned}\mathcal{P}(g)(r,t)=C(p_0,p_1)r^{\frac{2p_1-p_0}{(2-p_0)q}+\frac{1-p_1}{2-p_0}}\int_{\mathbb{R}^+}\mathcal{J}_\mu(\lambda r)e^{it(\frac{2-p_0}{2}\lambda)^\alpha}g(\lambda)\\ \times[1+(\frac{2-p_0}{2}\lambda)^2]^{-s/2}\lambda^{1/2}d\lambda.\end{aligned}$$

Therefore, utilizing the dual argument, our main task is to prove that

$$\|\mathcal{P}^*(f)\|_{L^2(\mathbb{R}^+)}\le_{q,p_0,p_1}\|f\|_{L^pL_t^1(\mathbb{R}^+\times\mathbb{R})}, \tag{3.8}$$

where $p=\frac{2(p_1-p_0+1)}{p_1-p_0+1+(2-p_0)s'}$ and

$$\begin{aligned}\mathcal{P}^*(f)(\lambda)&=C(p_0,p_1)[1+(\frac{2-p_0}{2}\lambda)^2]^{-s/2}\lambda^{1/2}\\ &\quad\times\int_{\mathbb{R}}\int_{\mathbb{R}^+}\mathcal{J}_\mu(\lambda r)e^{-it(\frac{2-p_0}{2}\lambda)^\alpha}f(r,t)r^{\frac{2p_1-p_0}{(2-p_0)q}+\frac{1-p_1}{2-p_0}}\,dr\,dt.\end{aligned}$$

Now we decompose $\mathcal{P}^*(f)(\lambda)=\sum_{j=0,1,2}(\mathcal{P}_j^*f)(\lambda)$, where

$$\begin{aligned}(\mathcal{P}_j^*f)(\lambda)&=C(p_0,p_1)[1+(\frac{2-p_0}{2}\lambda)^2]^{-s/2}\lambda^{1/2}\\ &\quad\times\int_{\mathbb{R}}\int_{\mathbb{R}^+}\mathcal{J}_\mu(\lambda r)e^{-it(\frac{2-p_0}{2}\lambda)^\alpha}\phi_j(\frac{\lambda r}{\mu})f(r,t)r^{\frac12-\gamma}\,dr\,dt,\end{aligned}$$

and ϕ_j are smooth cut-off functions such that $\phi_0=1$ on $\{|\eta|<\frac14\}$, $\phi_0=0$ on $\{|\eta|<1/4\}$, $\phi_1=1$ on $\{|\eta|\sim1\}$, $\phi_1=0$ otherwise, $\phi_2=0$ on $\{|\eta|<2\}$, $\phi_2=1$ on $\{|\eta|>3\}$, and $\phi_0+\phi_1+\phi_2=1$. The symbol $\gamma=\frac{2p_1-p_0}{2-p_0}(\frac12-\frac1q)$.

Non-endpoint case: $s'\in[\frac14,s)$**.** (i) $j=0$. Using the property of Bessel function (2.1), and $s'\in[\frac{2-p_0}{2},s)$, we obtain

$$\begin{aligned}|(\mathcal{P}_0^*f)(\lambda)|&\le_{p_0,p_1}[1+(\frac{2-p_0}{2}\lambda)^2]^{-s/2}\int_0^{\mu/(2\lambda)}(\lambda r)^{\mu+\frac12}r^{-\gamma}\|f(r,\cdot)\|_{L_t^1}dr\\ &\le_{p_0,p_1,\mu}\lambda^{-s'}\int_0^{\mu/(2\lambda)}r^{-\gamma}\|f(r,\cdot)\|_{L_t^1}dr.\end{aligned}$$

From the basic relation that $\|(\mathcal{P}_0^*f)(\lambda)\|_{L^2(\mathbb{R}^+)}=\|(\mathcal{P}_0^*f)(\frac1\lambda)\frac1\lambda\|_{L^2(\mathbb{R}^+)}$, we have

$$\|(\mathcal{P}_0^*f)(\lambda)\|_{L^2(\mathbb{R}^+)}\le_{p_0,p_1,\mu}\|\mu^{1-s'}\int_0^{(\mu\lambda)/2}\frac{r^{-\gamma}}{(\frac{\mu\lambda}{2})^{1-s'}}\|f(r,\cdot)\|_{L_t^1}dr\|_{L^2(\mathbb{R}^+)}. \tag{3.9}$$

By extending the function $r^{-\gamma}\|f(r,\cdot)\|_{L_t^1}$ to 0 for $r\le0$, the right hand side of (3.9) can be controlled by the Riesz potential operator I_β, $0<\beta<1$, where

$$I_\beta(g)(\lambda)=C_\beta\int_{\mathbb{R}}|\lambda-r|^{\beta-1}g(r)dr,\quad \lambda\in\mathbb{R},$$

and C_β is chosen so that $(I_\beta)^\wedge(\xi)=|\xi|^{-\beta}\widehat{g}(\xi)$. So we further obtain

$$\|\mathcal{P}_0^*f\|_{L^2(\mathbb{R}^+)}\le_{p_0,p_1,\mu,s'}\|I_{s'}(r^{-\gamma}\|f(r,\cdot)\|_{L_t^1})(\frac{\mu\lambda}{2})\|_{L^2(\mathbb{R}^+)} \tag{3.10}$$

$$\begin{aligned} &\leq_{p_0,p_1,\mu,s'} \Big(\int_{\mathbb{R}} |\xi|^{-2s'}|(r^{-\gamma}\|f(r,\cdot)\|_{L^1_t})^\wedge(\xi)|^2 d\xi\Big)^{1/2} &(3.11)\\ &\leq_{p_0,p_1,\mu,s'} \|f\|_{L^pL^1_t(\mathbb{R}^+\times\mathbb{R})} &(3.12)\end{aligned}$$

where the last inequality is proved by using Pitt's inequality for the usual Fourier transform:

Lemma 3.3 ([11]). *If $l \geq p$, $0 \leq a < 1-1/p$, $0 \leq d < 1/l$ and $d = a+1/p+1/l-1$, then*

$$\Big(\int_{\mathbb{R}} |\widehat{f}(\xi)|^l|\xi|^{-dl}d\xi\Big)^{1/l} \leq C(\int_{\mathbb{R}} |f(x)|^p|x|^{ap}dx)^{1/p},$$

where $\widehat{f}(\xi)$ is the usual Fourier transform.

(ii) $j = 1$. The estimate of $\mathcal{P}_1^*f$ is similar to $\mathcal{P}_0^*f$. When $|\lambda r| \sim \mu$, the property (2.3) implies $|\mathcal{J}_\mu(\lambda r)\phi_1(\frac{\lambda r}{\mu})| \leq C(\lambda r)^{-1/3}$, we obtain

$$\begin{aligned}\|(\mathcal{P}_1^*f)(\frac{1}{\lambda})\frac{1}{\lambda}\|_{L^2(\mathbb{R}^+)} &\leq_{p_0,p_1,\mu,s'} \|\int_{(\mu\lambda)/2}^{2\mu\lambda} \frac{r^{\frac{1}{6}-\gamma}}{(2\mu\lambda)^{\frac{7}{6}-s'}}\|f(r,\cdot)\|_{L^1_t}dr\|_{L^2(\mathbb{R}^+)}\\ &\leq_{p_0,p_1,\mu,s'} \|I_{s'}(r^{-\gamma}\|f(r,\cdot)\|_{L^1_t})(2\mu\lambda)\|_{L^2(\mathbb{R}^+)}\\ &\leq_{p_0,p_1,\mu,s'} \|f\|_{L^pL^1_t(\mathbb{R}^+\times\mathbb{R})}.\end{aligned} \tag{3.13}$$

(iii) $j = 2$. Applying the asymptotic expansion of Bessel function (2.2), we write

$$\begin{aligned}(\mathcal{P}_2^*f)(\lambda) &= C(p_0,p_1)b_\pm[1+(\frac{2-p_0}{2}\lambda)^2]^{-s/2}\\ &\quad\times\int_{\mathbb{R}}\int_{\mathbb{R}^+} e^{i[\pm\lambda r - t(\frac{2-p_0}{2}\lambda)^\alpha]}\phi_2(\frac{\lambda r}{\mu})f(r,t)r^{-\gamma}\,dr\,dt\\ &\quad+ C(p_0,p_1)[1+(\frac{2-p_0}{2}\lambda)^2]^{-s/2}\int_{\mathbb{R}}\int_{\mathbb{R}^+} e^{-it(\frac{2-p_0}{2}\lambda)^\alpha}(r\lambda)^{1/2}\Phi_\mu(r\lambda)\\ &\quad\times\phi_2(\frac{\lambda r}{\mu})f(r,t)r^{-\gamma}\,dr\,dt\\ &:= C(p_0,p_1)[(P_\pm f)(\lambda)+(Qf)(\lambda)],\end{aligned} \tag{3.14}$$

where $|\Phi_\mu(r\lambda)| \leq C(r\lambda)^{-1}$, $|b_\pm| \leq C$ and the constant C is independent of μ.

For the estimate of $(P_\pm f)(\lambda)$, it is sufficient to consider $(P_+f)(\lambda)$. We decompose

$$(P_+f)(\lambda) = S_1(\lambda) + S_2(\lambda),$$

where

$$S_1(\lambda) = b_+[1+(\frac{2-p_0}{2}\lambda)^2]^{-s/2}\int_{\mathbb{R}}\int_{\mathbb{R}^+} e^{i[\lambda r - t(\frac{2-p_0}{2}\lambda)^\alpha]}f(r,t)r^{-\gamma}\,dr\,dt,$$

$$S_2(\lambda) = b_+[1+(\frac{2-p_0}{2}\lambda)^2]^{-s/2}\int_{\mathbb{R}}\int_{\mathbb{R}^+} e^{i[\lambda r - t(\frac{2-p_0}{2}\lambda)^\alpha]}(\phi_2(\frac{\lambda r}{\mu})-1)f(r,t)r^{-\gamma}\,dr\,dt.$$

For $S_2(\lambda)$, arguing as $\mathcal{P}_0^*f$, we obtain

$$\|S_2(\lambda)\|_{L^2(\mathbb{R}^+)} \leq \big\|\lambda^{-s'}\int_0^{\frac{3\mu}{\lambda}}\|f(r,\cdot)\|_{L^1_t}r^{-\gamma}\,dr\big\|_{L^2(\mathbb{R}^+)} \leq \|f\|_{L^pL^1_t(\mathbb{R}^+\times\mathbb{R})}. \tag{3.15}$$

For $S_1(\lambda)$, we extend S_1 to $\mathbb{R}$ by setting

$$S_1(\xi) = b_+[1+(\frac{2-p_0}{2}y)^2]^{-s/2}\iint_{\mathbb{R}^2} e^{i[ry - t(\frac{2-p_0}{2}|y|)^\alpha]}f(r,t)r^{-\gamma}drdt, \quad y<0,$$

so

$$\|S_1\|^2_{L^2(\mathbb{R}^+)} \leq_{p_0} \iiiint K(r,r',t,t') r^{-\gamma} f(r,t)(r')^{-\gamma} f(r',t')\,dr\,dt\,dr'\,dt',$$

where

$$K(r,r',t,t') = \int e^{-i[\lambda(r'-r)-(t'-t)(\frac{2-p_0}{2}\lambda)^{\alpha}]}\lambda^{-2s'}d\lambda.$$

Since $s' \in [\frac{1}{4},\frac{1}{2})$, by Lemma 2.4, we obtain $|K(r,r',t,t')| \leq_{s',\alpha} |r-r'|^{2s'-1}$. Using the theory of Riesz potential and the Plancherel theorem of the usual Fourier transform, we have

$$\begin{aligned} \|S_1\|_{L^2(\mathbb{R}^+)} &\leq_{p_0,s',\alpha} \Big(\int_{\mathbb{R}} |\xi|^{-2s'} |(r^{-\gamma}\|f(r,\cdot)\|_{L^1_t})^{\wedge}(\xi)|^2 d\xi\Big)^{1/2} \\ &\leq_{p_0,s',\alpha} \|f\|_{L^pL^1_t(\mathbb{R}^+\times\mathbb{R})}. \end{aligned} \tag{3.16}$$

It remains to estimate $(Qf)(\lambda)$. The uniform decay of the function Φ_μ on μ shows that

$$|(Qf)(\lambda)| \leq_{p_0,p_1} [1+(\frac{2-p_0}{2}\lambda)^2]^{-s/2}\lambda^{-1/2}\int_{2\mu/\lambda}^{\infty} r^{-\frac{1}{2}-\gamma}\|f\|_{L^1_t}dr. \tag{3.17}$$

Applying Hölder's inequality to (3.17), we obtain

$$\begin{aligned} \|Qf\|_{L^2(\mathbb{R}^+)} \leq_{p_0,p_1} \mu^{-s'} \|(&\lambda^{-\frac{1}{2}+s'}\chi_{(0,\mu)}(\lambda) \\ &+ \lambda^{s'-s-\frac{1}{2}}\chi_{(\mu,\infty)}(\lambda))\|_{L^2(\mathbb{R}^+)}\|f\|_{L^pL^1_t(\mathbb{R}^+\times\mathbb{R})} \\ \leq_{p_0,p_1,\mu,s'} &\|f\|_{L^pL^1_t(\mathbb{R}^+\times\mathbb{R})} \end{aligned} \tag{3.18}$$

Combining (3.15)-(3.18), we conclude that

$$\|\mathcal{P}_2^* f\|_{L^2(\mathbb{R}^+)} \leq C(p_0,p_1,\mu,s',\alpha)\|f\|_{L^pL^1_t(\mathbb{R}^+\times\mathbb{R})}, \tag{3.19}$$

Therefore, the claim (3.8) follows from the estimates (3.12), (3.13) and (3.19).

Endpoint case: $s'=s$. For the endpoint case $q=\frac{2(p_1-p_0+1)}{p_1-p_0+1-(2-p_0)s}$, $s\in[\frac{1}{4},\frac{1}{2})$, we follow the almost same line as in the argument of (3.8) by replacing s' with s and $r^{-\gamma}$ with $r^{-\gamma}\varrho(r^{\frac{2}{2-p_0}})^{1/q}$ except for the estimate (3.18), where $\varrho(r^{\frac{2}{2-p_0}}) = r^{\frac{2b}{2-p_0}}(1+r^{\frac{2}{2-p_0}})^{-b}$. We only need to check the part of $\mathcal{P}_2^* f$.

Replacing $r^{-\gamma}$ with $r^{-\gamma}\varrho(r^{\frac{2}{2-p_0}})^{1/q}$ in (3.17) and using Hölder inequality, we obtain

$$\begin{aligned} &\int_{2\mu/\lambda}^{\infty} r^{-\frac{1}{2}-\gamma}\varrho(r^{\frac{2}{2-p_0}})^{1/q}\|f\|_{L^1_t}dr \\ &\leq \Big(\int_{2\mu/\lambda}^{\infty}\min\{1, r^{\frac{2bq}{2-p_0}}\} r^{-(\frac{1}{2}+\gamma)q}dr\Big)^{1/q}\|f\|_{L^pL^1_t(\mathbb{R}^+\times\mathbb{R})}. \end{aligned}$$

Then for $\gamma = \frac{2p_1-p_0}{2-p_0}(\frac{1}{2}-\frac{1}{q})$ and $b>0$,

$$\begin{aligned} |(Qf)(\lambda)| \leq_{p_0,p_1} &[1+(\frac{2-p_0}{2}\lambda)^2]^{-s/2}\lambda^{-1/2} \\ &\times \{((\frac{2\mu}{\lambda})^{-s}\chi_{(0,\mu)}(\lambda) + (\frac{2\mu}{\lambda})^{\frac{2b}{(2-p_0)}-s}\chi_{(\mu,\infty)}(\lambda)\}\|f\|_{L^pL^1_t(\mathbb{R}^+\times\mathbb{R})}, \end{aligned}$$

which yields the estimate (3.18). Hence (1.16) follows. This completes the proof of Theorem 1.5.

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