Electronic Journal of Differential Equations, Vol. 2007(2007), No. 25, pp. 1–7.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

POSITIVE SOLUTIONS OF A NONLINEAR PROBLEM INVOLVING THE p-LAPLACIAN WITH NONHOMOGENEOUS BOUNDARY CONDITIONS

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Abstract. In this work we consider a boundary-value problem involving the p-Laplacian with nonhomogeneous boundary conditions. We prove the existence of multiple solutions using the quadrature method.

1. Introduction

The p-Laplacian operator arises in the modelling of physical and natural phenomena [11, 15, 16, 21, 22, 23, 24], and has has been considered in many papers; see for example [1, 3, 5, 6, 10, 11, 12, 13, 14, 15, 16, 17, 19, 22, 25]. In this work we consider the boundary-value problem

$$-(|u'(x)|^{p-2}u'(x))' = \lambda f(u(x)), \quad \text{a.e. } 0 < x < 1, \tag{1.1}$$

$$u(0) = u(1) + k(u(1))u'(1) = 0 \tag{1.2}$$

where $\lambda \geq 0$, $p \in (1, 2]$, $k : \mathbb{R}_+ \to \mathbb{R}_+^*$, and $f : \mathbb{R}_+ \to \mathbb{R}_+^*$ smooth enough.

Problem (1.1)–(1.2) was considered by Lakmeche and Hammoudi [19] for k constant, in Anuradha et al. [2][2] and Lakmeche [20] for $p = 2$ and k constant. In this work we generalize [19] by considering the nonhomogeneous boundary conditions. Our aim in this work is to prove existence of solutions of (1.1), (1.2) and their multiplicity, using the quadrature method [8, 9, 12, 18]. In section 2, we give some preliminaries and definitions, in section 3 we give our main results, and we conclude by some remarks in the last section.

2. Preliminaries

In this section we give some definitions and preliminaries.

Definition 2.1. A pair $(u, \lambda) \in \mathrm{C}^1([0, 1]; \mathbb{R}_+) \times [0, +\infty[$ is called a solution of (1.1)-(1.2), if

- $(|u'|^{p-2}u')$ is absolutely continuous, and
- $-(|u'|^{p-2}u')' = \lambda f(u)$ a.e. in $(0, 1)$, and $u(0) = u(1) + k(u(1))u'(1) = 0$.

2000 *Mathematics Subject Classification.* 35J65.
Key words and phrases. p-Laplacian; positive solutions; positone problem; quadrature method, nonhomogeneous boundary conditions.

Submitted July 13, 2006. Published February 12, 2007.

Note that the pair $(0,0)$ is a solution of (1.1), (1.2).

Let $F:\mathbb{R}_+\to\mathbb{R}_+$ be defined by $F(u)=\int_0^\rho f(s)ds$, and $g:\mathbb{R}_+\to\mathbb{R}_+$ be defined by

$$g(\rho)=\begin{cases}2\big(\frac{p-1}{p}\big)^{1/p}\int_0^\rho\frac{ds}{[F(\rho)-F(s)]^{1/p}}, & \text{for } \rho>0,\\ 0 & \text{for } \rho=0.\end{cases}$$

Let $n\geq 0$. Define $h_n:[n,+\infty)\to\mathbb{R}_+^*$, by

$$h_n(\rho)=\big(\frac{p-1}{p}\big)^{1/p}\Big[\int_0^\rho\frac{ds}{[F(\rho)-F(s)]^{1/p}}+\int_n^\rho\frac{ds}{[F(\rho)-F(s)]^{1/p}}\Big].$$

Note that $g\equiv h_0$.

Lemma 2.2. *The functions g and h_n are continuous, and $g(\rho)\leq 2h_n(\rho)\leq 2g(\rho)$, for all $\rho\geq n\geq 0$.*

The proof of the lemma above can be found in [7, Theorem 7].

For $u\in C^1([0,1];\mathbb{R}_+)$, we define $\|u\|:=\sup\{u(s);s\in(0,1)\}$.

Lemma 2.3. *If (u,λ) is a solution of* (1.1), (1.2) *with $\lambda>0$, then*

1. $u'(1)<0$, $u(1)>0$, *and*
2. $\lambda^{1/p}=h_n(\|u\|)$, *where $n=u(1)$.*

Proof. Let (u,λ) be a positive solution of (1.1), (1.2) with $\lambda>0$, then $u\neq 0$. Using the maximum principle [26], we obtain $u>0$ in $(0,1)$, then $u(1)\geq 0$, which implies

$$u'(1)=-\frac{u(1)}{k(u(1))}\leq 0.$$

Since $f(0)>0$, then $u'(1)<0$ and $u(1)>0$. Also there exists a unique $x_0\in(0,1)$ such that $u'(x_0)=0$, $u(x_0)=\|u\|$, $u'(x)>0$ for $x\in(0,x_0)$, and $u'(x)<0$ for $x\in(x_0,1)$.

Let (u,λ) be a solution of (1.1), (1.2), and $u(1)=n$ with $0<n<\rho$, then we have $u(x_0)=\max_{x\in[0,1]}|u(x)|=\rho$. Multiplying (1.1) by $u'(x)$, and integrate it for $x\in[0,x_0]$ and x_0, we obtain

$$-\int_x^{x_0}(|u'(t)|^{p-2}u'(t))'u'(t)dt=\int_x^{x_0}\lambda f(u(t))u'(t)dt. \tag{2.1}$$

We have in one hand

$$\int_x^{x_0}\lambda f(u(t))u'(t)dt=\lambda\int_{u(x)}^{u(x_0)}f(y)dy=\lambda(F(\rho)-F(u(x))), \tag{2.2}$$

and in the other hand

$$-\int_x^{x_0}(|u'(t)|^{p-2}u'(t))'u'(t)dt=\frac{(p-1)}{p}(u'(x))^p. \tag{2.3}$$

From (2.1), (2.2) and (2.3), we have

$$\frac{(p-1)}{p}(u'(x))^p=\lambda(F(\rho)-F(u(x))). \tag{2.4}$$

Then for all $x\in(0,x_0)$, we have

$$(u'(x))^p=\big(\frac{p}{p-1}\big)\lambda(F(\rho)-F(u(x))), \tag{2.5}$$

which implies

$$u'(x) = \big(\frac{p}{p-1}\big)^{1/p}[\lambda(F(\rho) - F(u(x)))]^{1/p} \quad \text{for } x \in [0, x_0], \tag{2.6}$$

and by symmetry

$$u'(x) = -\big(\frac{p}{p-1}\big)^{1/p}[\lambda(F(\rho) - F(u(x)))]^{1/p} \quad \text{for } x \in [x_0, 1]. \tag{2.7}$$

Integrate (2.6) between 0 and x_0, we obtain

$$\lambda^{1/p} x_0 = \big(\frac{p-1}{p}\big)^{1/p} \int_0^\rho \frac{ds}{[F(\rho) - F(s)]^{1/p}}. \tag{2.8}$$

Similarly, by integration of (2.7) between 1 and x_0, we obtain

$$\lambda^{1/p}(1 - x_0) = \big(\frac{p-1}{p}\big)^{1/p} \int_n^\rho \frac{ds}{[F(\rho) - F(s)]^{1/p}}. \tag{2.9}$$

From (2.8) and (2.9), we deduce that

$$\lambda^{1/p} = \big(\frac{p-1}{p}\big)^{1/p}\Big[\int_0^\rho \frac{ds}{[F(\rho) - F(s)]^{1/p}} + \int_n^\rho \frac{ds}{[F(\rho) - F(s)]^{1/p}}\Big]. \tag{2.10}$$

From equation this equation, we deduce the results of lemma 2.3. □

Consider the boundary-value problem consisting of (1.1) and the Dirichlet boundary conditions

$$u(0) = u(1) = 0\,. \tag{2.11}$$

Lemma 2.4. [19] *We have*

1. *If* $\lim_{s\to+\infty} \frac{f(s)}{s^{p-1}} = 0$, *then* $\lim_{s\to+\infty} g(s) = +\infty$
2. *If* $\lim_{s\to+\infty} \frac{f(s)}{s^{p-1}} = +\infty$, *then* $\lim_{s\to+\infty} g(s) = 0$

For the proof of the lemma above, see [19].

Theorem 2.5 ([19])**.** *If* $\lim_{s\to+\infty} f(s)/s^{p-1} = 0$, *then problem* (1.1), (2.11) *has at least one positive solution for all* $\lambda > 0$.

Proof. From lemma 2.4 we have $\lim_{s\to+\infty} g(s) = +\infty$ and $g(0) = 0$. □

Theorem 2.6 ([19])**.** *If* $\lim_{s\to+\infty} f(s)/s^{p-1} = +\infty$, *then there exist* $\lambda^* > 0$ *such that the problem* (1.1), (2.11) *has at least two positive solutions for* $\lambda \in (0, \lambda^*)$, *and no positive solution for* $\lambda > \lambda^*$.

Proof. From lemma 2.4, we have $\lim_{s\to+\infty} g(s) = g(0) = 0$. Then g is bounded and reaches its maximum at some point $\rho_0 > 0$. Further $\lambda^* = (g(\rho_0))^p$. □

3. Main results

Let (u, λ) be a solution of (1.1), (1.2), and $u(1) = n$ with $0 < n < \rho$, then we have $u(x_0) = \max_{x\in[0,1]} |u(x)| = \rho$. Substituting x by 1 in (2.7), we obtain

$$\frac{n}{k(n)} = \big(\frac{p}{p-1}\big)^{1/p}\,[\lambda(F(\rho) - F(s))]^{1/p}\,. \tag{3.1}$$

Hence

$$\lambda^{1/p} = \big(\frac{p-1}{p}\big)^{1/p} \frac{n}{k(n)\,[F(\rho) - F(n)]^{1/p}} \tag{3.2}$$

Then from (2.10) and (3.2), we have

$$\int_0^\rho \frac{ds}{[F(\rho)-F(s)]^{1/p}} + \int_n^\rho \frac{ds}{[F(\rho)-F(s)]^{1/p}} = \frac{n}{k(n)\,[F(\rho)-F(n)]^{1/p}}. \tag{3.3}$$

Theorem 3.1. *Assume that $k \in \mathrm{C}(\mathbb{R}_+;\mathbb{R}_+^*)$. Let $\rho > 0$, then*

1. *there exist at least $n^* \in (0,\rho)$, such that (3.3) is satisfied for $n = n^*$;*
2. *for each n^* satisfying (3.3), there is a unique $\lambda = \lambda(\rho, n^*)$ given by (2.10) or (3.2) such that (1.1), (1.2) has exactly one solution (u,λ), with $\|u\| = \rho$, $u(1) = n^*$, $u'(1) = -\frac{n^*}{h(n^*)}$ and*
$$x_0 = \big(\frac{p-1}{p}\big)^{1/p}\lambda^{-\frac{1}{p}} \int_0^\rho \frac{ds}{[F(\rho)-F(s)]^{1/p}};$$
3. *if k is decreasing, n^* is unique.*

Proof. Equation (3.3) is equivalent to

$$k(n) = \Big(\int_0^\rho \frac{ds}{[F(\rho)-F(s)]^{1/p}} + \int_n^\rho \frac{ds}{[F(\rho)-F(s)]^{1/p}}\Big)^{-1} \frac{n}{[F(\rho)-F(n)]^{1/p}}. \tag{3.4}$$

Let $\gamma : [0,\rho) \to \mathbb{R}_+$ be defined by

$$\gamma(n) := \Big(\int_0^\rho \frac{ds}{[F(\rho)-F(s)]^{1/p}} + \int_n^\rho \frac{ds}{[F(\rho)-F(s)]^{1/p}}\Big)^{-1} \frac{n}{[F(\rho)-F(n)]^{1/p}}. \tag{3.5}$$

We have $\gamma(0) = 0$, $\lim_{n\to\rho^-} \gamma(n) = +\infty$ and γ is differentiable on $(0,\rho)$, with

$$\begin{aligned}\gamma'(n) := {}& \frac{p[F(\rho)-F(n)] + nf(n)}{p[F(\rho)-F(n)]^{1+\frac{1}{p}}\big(\int_0^\rho \frac{ds}{[F(\rho)-F(s)]^{1/p}} + \int_n^\rho \frac{ds}{[F(\rho)-F(s)]^{1/p}}\big)} \\ &+ \frac{n}{[F(\rho)-F(n)]^{1/p}\big(\int_0^\rho \frac{ds}{[F(\rho)-F(s)]^{1/p}} + \int_n^\rho \frac{ds}{[F(\rho)-F(s)]^{1/p}}\big)^2} > 0.\end{aligned}$$

Then γ increases from 0 to $+\infty$ on $(0,\rho)$. Since $k : \mathbb{R}_+ \to \mathbb{R}_+^*$ is continuous and $k(0) > 0$, $k(\rho) < \infty$, then there exist at least $n^* \in (0,\rho)$ such that $k(n^*) = \gamma(n^*)$.

If k decreases, we have $0 < k(n) \le k(0)$ for all $n \in [0,\rho]$, $\gamma(0) = 0$ and $\lim_{s\to\rho^-} \gamma(s) = +\infty$.

Let $n_0 \in (0,\rho)$ such that $k(n_0) = \gamma(n_0)$. Suppose that there exists $n_1 \in (0,\rho)$ such that $k(n_1) = \gamma(n_1)$. Then we have $k(n_0) = \gamma(n_0) < \gamma(n_1) = k(n_1)$ for $n_0 < n_1$ and $k(n_0) \ge k(n_1)$, which is a contradiction. Similarly we find a contradiction if $n_0 > n_1$. Finally, we deduce that $n_0 = n_1$. □

Corollary 3.2. *Assume that $k \in \mathrm{C}(\mathbb{R}_+;\mathbb{R}_+^*)$. Let $\rho > 0$, then the bifurcation diagram (λ,ρ) of the positive solutions of (1.1), (1.2) is given by*

$$\lambda(\rho)^{1/p} = \big(\frac{p-1}{p}\big)^{1/p}\Big[\int_0^\rho \frac{ds}{[F(\rho)-F(s)]^{1/p}} + \int_{n^*}^\rho \frac{ds}{[F(\rho)-F(s)]^{1/p}}\Big],$$

where n^* is the solution of (3.3).

Remark 3.3. If k is not decreasing, then the solution n^* of (3.3) is not necessarily unique, in some cases it could be infinite. This is one of different results with respect to precedent works [2, 18, 19].

Theorem 3.4. *Assume that $k \in \mathrm{C}^1(\mathbb{R}_+;\mathbb{R}_+^*)$. Let $\rho > 0$, and k decreasing, then there exists a unique $n^*(\rho) \in (0,\rho)$ such that $k(n^*) = \gamma(n^*)$. Further n^* is continuously differentiable.*

Proof. Because k is decreasing, $k' \leq 0$; further $\gamma' > 0$, hence $k' - \gamma' < 0$. From the implicit function theorem, there exists a unique $n^*(\rho) \in (0, \rho)$ such that $k(n^*) = \gamma(n^*)$, and $n^*(\rho)$ is continuously differentiable. □

Theorem 3.5. *Assume that $k \in \mathrm{C}(\mathbb{R}_+; \mathbb{R}_+^*)$. Let $\rho > 0$ and k decreasing. Then there exists a unique $n^*(\rho) \in (0, \rho)$, such that (3.3) is satisfied for $n = n^*$. Also there exists a unique $\lambda = \lambda(\rho)$ given by (2.10) or (3.2) for which (1.1), (1.2) has a unique solution (u, λ), with $\|u\| = \rho$, $u(1) = n^*(\rho)$,*

$$u'(1) = -\frac{n^*(\rho)}{h(n^*(\rho))} \quad \text{and} \quad x_0 = \big(\frac{p-1}{p}\big)^{1/p} \lambda^{-\frac{1}{p}} \int_0^\rho \frac{ds}{[F(\rho) - F(s)]^{1/p}}.$$

The result is easily deduced from Theorems 3.1 and 3.4.

Corollary 3.6. *Assume that $k \in \mathrm{C}(\mathbb{R}_+; \mathbb{R}_+^*)$. If k is decreasing, then the bifurcation diagram (λ, ρ) of positive solutions of (1.1), (1.2) is given by*

$$\lambda(\rho)^{1/p} = \big(\frac{p-1}{p}\big)^{1/p} \Big[\int_0^\rho \frac{ds}{[F(\rho) - F(s)]^{1/p}} + \int_{n^*(\rho)}^\rho \frac{ds}{[F(\rho) - F(s)]^{1/p}}\Big],$$

where $n^*(\rho)$ is the unique solution of (3.3).

Theorem 3.7. *Assume that $k \in \mathrm{C}(\mathbb{R}_+; \mathbb{R}_+^*)$. If k is decreasing, then*

(1) *when $\lim_{s \to +\infty} f(s)/s^{p-1} = 0$, (1.1), (1.2) has at least one positive solution for all $\lambda > 0$; and*
(2) *when $\lim_{s \to +\infty} f(s)/s^{p-1} = +\infty$, there exist*
$$\lambda_0^* = \big(\sup\{h_{n^*(s)}(s); s \in (0, +\infty)\}\big)^p$$
such that (1.1), (1.2) has at least two positive solutions for $\lambda \in (0, \lambda_0^*)$, and zero positive solution for $\lambda > \lambda_0^*$.

Proof. We have $g(\rho) \leq 2h_{n^*(\rho)}(\rho) \leq 2g(\rho)$, for all $\rho > 0$. From theorems 2.5 and 2.6, we deduce the results. □

Concluding remarks

In this work we have studied a boundary value problem of the one-dimensional p-Laplacian with nonhomogeneous boundary conditions. We have proved existence of positive solutions using quadrature method, also we have proved the multiplicity of the solutions for $\lim_{s \to +\infty} \frac{f(s)}{s^{p-1}} = +\infty$. In the case where the nonhomogeneous term k is a decreasing function, we proved the uniqueness of the solution (u, λ) for each $\|u\| = \rho > 0$. Our results generalize the works [2, 19]. When k is not a decreasing function we can find, some examples in which the solution n^* of (3.4) is not unique, for example for k given as it follows

$$k(n) = \begin{cases} \gamma(\rho_1), & \text{for } 0 \leq n < \rho_1, \\ \gamma(n), & \text{for } \rho_1 \leq n < \rho_2, \\ \gamma(\rho_2), & \text{for } \rho_2 \leq n, \end{cases}$$

where $0 < \rho_1 < \rho_2 < \rho$, we have an infinite number of solutions of equation (3.4) $(k(n) = \gamma(n))$ which constitutes exactly the interval $[\rho_1, \rho_2]$.

It will be interesting to analyze the ramification of solutions for concrete and simple examples with boundary conditions cited above.

In this work we have considered un autonomous problem, it will be interesting to consider the non-autonomous problem using the fixed point method as the Avery and Peterson fixed point theorem [4].

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