Electronic Journal of Differential Equations, Vol. 2022 (2022), No. 46, pp. 1–22.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXISTENCE AND NONEXISTENCE OF GLOBAL SOLUTIONS TO THE CAHN-HILLIARD EQUATION WITH VARIABLE EXPONENT SOURCES

QUACH V. CHUONG, LE C. NHAN, LE X. TRUONG

Abstract. In this article, we study the existence and nonexistence of global solutions to the Cahn-Hilliard equation with variable exponent sources and arbitrary initial energy. We also study the asymptotic behavior of weak solutions. Our results extend some recent results of Han [10] to PDEs with variable exponent sources.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$. In this article, we study the initial-boundary value problem

$$\begin{gathered}
u_t + \Delta^2 u - \Delta_{p(x)} u = |u|^{q(x)-2} u, \quad (x,t) \in Q_T, \\
u(x,t) = \frac{\partial u}{\partial \nu}(x,t) = 0, \quad (x,t) \in \Gamma_T, \\
u(x,0) = u_0(x), \quad x \in \Omega,
\end{gathered} \tag{1.1}$$

where $Q_T := \Omega \times (0,T)$, $\Gamma_T := \partial\Omega \times (0,T)$, ν is the unit outward normal vector on $\partial\Omega$ and initial data $u_0 \in H_0^2(\Omega)$. It will also be assumed throughout this paper that $p(\cdot)$ and $q(\cdot)$ are measurable functions satisfying

$$1 < p^- := \operatorname{ess\,inf}_{x\in\Omega} p(x) \le p^+ := \operatorname{ess\,sup}_{x\in\Omega} p(x) < \begin{cases} \infty, & \text{if } N \le 2, \\ \frac{2N}{N-2}, & \text{if } N > 2, \end{cases} \tag{1.2}$$

and

$$\begin{aligned}
\max\{2, p^+\} < q^- &:= \operatorname{ess\,inf}_{x\in\Omega} q(x) \\
&\le q^+ := \operatorname{ess\,sup}_{x\in\Omega} q(x) < \begin{cases} \infty, & \text{if } N \le 4, \\ \frac{2N}{N-4}, & \text{if } N > 4. \end{cases}
\end{aligned} \tag{1.3}$$

It is well known that the fourth-order parabolic equations

$$u_t + \Delta^2 u - \nabla \cdot f(\nabla u) = h(x,t,u) \tag{1.4}$$

can be used to model a variety of important physical processes. For example, it can be used to describe the evolution of the epitaxial growth of nanoscale thin films, see [19, 25, 30] and references therein. Equation (1.4) is also known as classical

---

2020 *Mathematics Subject Classification.* 35B35, 35B40, 35K57, 35Q92, 92C17.
Key words and phrases. Fourth-order parabolic equation; potential well method; global solution; variable exponents.

Submitted September 4, 2021. Published July 5, 2022.

Cahn-Hilliard equation arising from modeling of phase transitions in binary systems such as alloys, glasses, thin film epitaxy, and polymer-mixtures; see for example [2, 24, 26, 29].

When the nonlinearities f and h satisfy some constant growth conditions, there have been many results on the existence, uniqueness, and some other properties of the solutions of (1.4). We refer the interested readers to the bibliography given in [3, 4, 14, 12, 16, 15]. Concerning to the blow-up property, Han [10] used the potential well method proposed by Sattinger [23] (see [20, 17]) to study the problem

$$\begin{gathered} u_t + \Delta^2 u - \Delta_p u = |u|^{q-2}u, \quad (x,t) \in Q_T, \\ u(x,t) = \frac{\partial u}{\partial \nu}(x,t) = 0, \quad (x,t) \in \Gamma_T, \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{gathered}$$

In that paper, the author showed a threshold result on global solutions, and on the blow-up in finite time when the initial energy is subcritical critical and supercritical, respectively. In addition, the author also studied the decay rate of the L^2-norm of global solutions. This result was then improved by Zhou [31] in studying the exponential decay property of the energy functional.

To the best of our knowledge there are very few results on parabolic equations involving variable exponent sources. In [21] the author used the eigenfunction argument of Kaplan [11] to study the blow-up property of solutions of the homogeneous Dirichlet problem for the semilinear parabolic equation

$$u_t - \Delta u = f(x,u),$$

where the source term is either

$$f(x,u) = a(x)u^{p(x)} \quad \text{or} \quad f(x,u) = a(x)\int_\Omega u^{q(x)}(y,t)dy.$$

Then in [32, 27] the authors established the blow-up results for solutions of the evolution m-Laplace equation involving the variable exponent sources,

$$u_t - \Delta_m u = |u|^{p(x)-1}u.$$

Recently, in [22] by using the concavity method, the authors established a blow up in finite time result, in the case of non-positive initial energy $J(u_0)$, for fourth-order parabolic equation

$$u_t + \Delta^2 u = u^{q(x)}.$$

Motivated by the above papers, we consider a more general problem with variable exponent nonlinearities of the form (1.1). Our results are twofold: Firstly, different from the previous results [21, 32, 27, 22] in which the authors only concerned about the blow-up property (global nonexistence). We establish in this paper some sharp results on the existence and nonexistence of global weak solutions for arbitrary initial energy $J(u_0)$. As far as we know, such results in the case of PDEs with variable exponent sources are new. Secondly, the decay rate of H_0^2-norm of global solutions which start from potential wells is also concerned. It is noticed that even in the case of constant exponent, Han [10] only proved decay rate of L^2-norm of global solutions in case $J(u_0) < d$ where d is the potential well depth (see (3.5)). Then Zhou [31] proved the decay of energy functional when $J(u_0) < d_0 \le d$ with

$$d_0 = \frac{q-2}{2q} S^{-\frac{2q}{q-2}},$$

where $S > 0$ is the optimal constant of the embedding $H_0^2(\Omega) \hookrightarrow L^q(\Omega)$. In this paper, we improve this result by showing the exponential decay of energy functional under the condition $J(u_0) < d$. In addition, our techniques are different from [31] where the author's arguments depend either on $p \le 2$ or $p > 2$. Because (1.3), we shall prove in Theorem 4.5 that the decay property of energy functional depends strongly on $q(x)$ instead of $p(x)$. It is also worth noticing that this is not a trivial generalization of similar problems with the constant exponent sources. The substantial difficulties in treating the above problem are caused by the complicated nonlinearities $-\Delta_{p(x)}u$ and $|u|^{q(x)-2}u$ (it is non-homogeneous) and the lack of a maximum principle and comparison principle for fourth-order equations. The key point is to treat the gap between the norm and the integral in variable exponent spaces. Our method presented here can be used to treat the problem in [21, 22, 27, 32].

This article is organized as follows: In Section 2 we recall some facts about $H_0^2(\Omega)$ space and Ozlicz-Sobolev type spaces. In Section 3 we study the stationary state of (1.1) and construct the stable sets and unstable sets; In Section 4 we present our main results on the evolution problem. The proof of main results are given in the rest of the paper.

2. Preliminaries

Let Ω be as in Section 1. We denote by $\|\cdot\|_r$ the usual norm of the space $L^r(\Omega)$ for $1 \le r \le \infty$ and $\langle\cdot,\cdot\rangle$ the usual inner product of the Hilbert space $L^2(\Omega)$. We also denote by $\|\cdot\|_{H_0^2}$ the norm of $H_0^2(\Omega)$. That is

$$\|u\|_{H_0^2} = \sqrt{\|u\|_2^2 + \|\nabla u\|_2^2 + \|\Delta u\|_2^2}.$$

As in [10], $H_0^2(\Omega)$ is a Hilbert space with inner product

$$\langle u, v\rangle_{H_0^2} = \langle \Delta u, \Delta v\rangle\,.$$

Then $H_0^2(\Omega)$ is uniformly convex and the norm $\|\cdot\|_{H_0^2}$ is equivalent to the norm $\|\Delta(\cdot)\|_2$ due to Poincare's inequality.

We next introduce some preliminary results on Lebesgue and Sobolev spaces with variable exponents (see [5, 6, 7, 8, 13]). Denote by $\mathcal{P}(\Omega)$ the set of all measurable functions $p : \Omega \to [1, \infty]$. Define the Lebesgue space with a variable exponent $p(\cdot)$ which is the so-called Nakano space and a special case of Musielak-Orlicz spaces (see [18]), as follows:

$$L^{p(\cdot)}(\Omega) := \big\{u : \Omega \to \mathbb{R} \text{ measurable}, \ \rho(u) := \int_\Omega |u(x)|^{p(x)}\,\mathrm{d}x < \infty\big\},$$

where $p \in \mathcal{P}(\Omega)$. The space $L^{p(\cdot)}(\Omega)$ is equipped with the Luxemburg-type norm

$$\|u\|_{p(\cdot)} := \inf\big\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}\,\mathrm{d}x \le 1\big\}\,.$$

The following proposition shows the relation between the norm $\|u\|_{p(\cdot)}$ and the modular $\rho(u)$.

Proposition 2.1 ([5]). *Let $p \in \mathcal{P}(\Omega)$. It holds that*

$$\min\big\{\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\big\} \le \rho(u) \le \max\big\{\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\big\}, \quad \textit{for all } u \in L^{p(\cdot)}(\Omega).$$

For $p^+ < \infty$, the dual space of $L^{p(\cdot)}(\Omega)$ is identified with $L^{p'(\cdot)}(\Omega)$ with the dual variable exponent $p' \in \mathcal{P}(\Omega)$ given by

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1 \quad \text{for a.e. } x \in \Omega,$$

where we write $1/\infty = 0$.

The Hölder inequality also holds for variable Lebesgue spaces.

Proposition 2.2 (Hölder inequality,[5])**.** *Let $p, q, s \in \mathcal{P}(\Omega)$, it holds that*

$$\|uv\|_{s(\cdot)} \le 2\|u\|_{p(\cdot)}\|v\|_{q(\cdot)} \quad \textit{for all } u \in L^{p(\cdot)}(\Omega),\ v \in L^{q(\cdot)}(\Omega),$$

provided that

$$\frac{1}{s(x)} = \frac{1}{p(x)} + \frac{1}{q(x)} \quad \textit{for a.e. } x \in \Omega.$$

Proposition 2.3 ([5])**.** *Let $p, q \in \mathcal{P}(\Omega)$. If $p(x) \le q(x)$ for a.e. $x \in \Omega$, then the embedding $L^{q(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is continuous.*

We next define variable exponent Sobolev spaces

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\},$$

endowed with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \Big(\|u\|_{p(\cdot)}^2 + \|\nabla u\|_{p(\cdot)}^2\Big)^{1/2}.$$

Furthermore, let $W_0^{1,p(\cdot)}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. It is noticed that if $1 < p^- \le p^+ < \infty$, then $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ are uniformly convex Banach spaces and therefore they are reflexive.

3. Stationary problem and potential wells

In this section, we consider the stationary solutions of (1.1) which solve the problem

$$\begin{gathered} \Delta^2 u - \Delta_{p(x)} u = |u|^{q(x)-2}u \quad \text{in } \Omega, \\ u(x) = \frac{\partial u}{\partial \nu}(x) = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.1}$$

where $p(x)$ and $q(x)$ satisfy (1.2)-(1.3). Consider the energy functional J and the Nehari functional I given by

$$\begin{gathered} J(u) = \frac{1}{2}\|\Delta u\|_2^2 + \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,\mathrm{d}x - \int_\Omega \frac{1}{q(x)}|u|^{q(x)}\,\mathrm{d}x, \\ I(u) = \|\Delta u\|_2^2 + \int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x - \int_\Omega |u|^{q(x)}\,\mathrm{d}x. \end{gathered}$$

Then J and I are of class C^1 over $H_0^2(\Omega)$ and critical points of J are weak solutions of (3.1). Moreover, we can estimate J and I as follows:

$$\begin{aligned} J(u) &\ge \frac{1}{2}\|\Delta u\|_2^2 + \frac{1}{p^+}\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x - \frac{1}{q^-}\int_\Omega |u|^{q(x)}\,\mathrm{d}x \\ &= \big(\frac{1}{2} - \frac{1}{q^-}\big)\|\Delta u\|_2^2 + \big(\frac{1}{p^+} - \frac{1}{q^-}\big)\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x + \frac{1}{q^-}I(u), \end{aligned} \tag{3.2}$$

$$\begin{aligned} J(u) &\le \frac{1}{2}\|\Delta u\|_2^2 + \frac{1}{p^-}\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x - \frac{1}{q^+}\int_\Omega |u|^{q(x)}\,\mathrm{d}x \\ &= \big(\frac{1}{2} - \frac{1}{q^+}\big)\|\Delta u\|_2^2 + \big(\frac{1}{p^-} - \frac{1}{q^+}\big)\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x + \frac{1}{q^+}I(u), \end{aligned} \tag{3.3}$$

$$\begin{aligned} J(u) &= \big(\frac{1}{2} - \frac{1}{q^-}\big)\|\Delta u\|_2^2 + \int_\Omega \big(\frac{1}{p(x)} - \frac{1}{q^-}\big)|\nabla u|^{p(x)}\,\mathrm{d}x \\ &\quad + \int_\Omega \big(\frac{1}{q^-} - \frac{1}{q(x)}\big)|u|^{q(x)}\,\mathrm{d}x + \frac{1}{q^-}I(u). \end{aligned} \tag{3.4}$$

Let $u \in H_0^2(\Omega)\backslash\{0\}$ and consider the fibering map $\lambda \mapsto j(\lambda) := J(\lambda u)$ for $\lambda > 0$ given by

$$j(\lambda) = \frac{\lambda^2}{2}\|\Delta u\|_2^2 + \int_\Omega \frac{\lambda^{p(x)}}{p(x)}|\nabla u|^{p(x)}\,\mathrm{d}x - \int_\Omega \frac{\lambda^{q(x)}}{q(x)}|u|^{q(x)}\,\mathrm{d}x.$$

Lemma 3.1. *Let* (1.2)-(1.3) *hold and* $u \in H_0^2(\Omega)\backslash\{0\}$. *Then the following results hold:*

(i) $\lim_{\lambda\to 0^+} j(\lambda) = 0$ *and* $\lim_{\lambda\to\infty} j(\lambda) = -\infty$.

(ii) *There exists a* $\lambda_* = \lambda_*(u) > 0$ *such that* $j(\lambda)$ *attains the maximum at* $\lambda = \lambda_*$, *then* $I(\lambda_* u) = 0$. *In addition, we have* $0 < \lambda_* < 1, \lambda_* = 1$ *and* $\lambda_* > 1$ *provided that* $I(u) < 0, I(u) = 0$ *and* $I(u) > 0$, *respectively.*

Proof. It is easily seen that

$$\begin{aligned} j(\lambda) &\ge \frac{1}{2}\lambda^2\|\Delta u\|_2^2 + \min\{\lambda^{p^-}, \lambda^{p^+}\}\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,\mathrm{d}x \\ &\quad - \max\{\lambda^{q^-}, \lambda^{q^+}\}\int_\Omega \frac{1}{q(x)}|u|^{q(x)}\,\mathrm{d}x, \end{aligned}$$

and

$$\begin{aligned} j(\lambda) &\le \frac{1}{2}\lambda^2\|\Delta u\|_2^2 + \max\{\lambda^{p^-}, \lambda^{p^+}\}\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,\mathrm{d}x \\ &\quad - \min\{\lambda^{q^-}, \lambda^{q^+}\}\int_\Omega \frac{1}{q(x)}|u|^{q(x)}\,\mathrm{d}x. \end{aligned}$$

This, together with $q^- > \max\{2, p^+\}$ and $\int_\Omega \frac{1}{q(x)}|u|^{q(x)}\,\mathrm{d}x > 0$, implies (i). Furthermore, we also have $j(\lambda) > 0$ for sufficiently small $\lambda > 0$. Hence, there exists a $\lambda_* > 0$ such that $j(\lambda_*) = \sup_{\lambda>0} j(\lambda)$. Then by Fermat's Theorem, we obtain $j'(\lambda_*) = 0$. This gives $I(\lambda_* u) = 0$ by the relation $I(\lambda u) = \lambda j'(\lambda)$.

Finally, we prove the last statement of (ii). By the definition of I, we obtain

$$\begin{aligned} 0 &= I(\lambda_* u) \\ &= \lambda_*^2\|\Delta u\|_2^2 + \int_\Omega \lambda_*^{p(x)}|\nabla u|^{p(x)}\,\mathrm{d}x - \int_\Omega \lambda_*^{q(x)}|u|^{q(x)}\,\mathrm{d}x \\ &= \big(\lambda_*^2 - \lambda_*^{q^-}\big)\|\Delta u\|_2^2 + \int_\Omega \big(\lambda_*^{p(x)} - \lambda_*^{q^-}\big)|\nabla u|^{p(x)}\,\mathrm{d}x \\ &\quad + \int_\Omega (\lambda_*^{q^-} - \lambda_*^{q(x)})|u|^{q(x)}\,\mathrm{d}x + \lambda_*^{q^-} I(u), \end{aligned}$$

which can be rewritten in the form

$$\lambda_*^{q^-} I(u) = \big(\lambda_*^{q^-} - \lambda_*^2\big)\|\Delta u\|_2^2 + \int_\Omega \big(\lambda_*^{q^-} - \lambda_*^{p(x)}\big)|\nabla u|^{p(x)}\,\mathrm{d}x$$

$$+ \int_\Omega \big(\lambda_*^{q(x)} - \lambda_*^{q^-}\big)|u|^{q(x)}\,\mathrm{d}x.$$

Since $q^- > \max\{2, p^+\}$, the above equality shows that $0 < \lambda_* < 1, \lambda_* = 1$ and $\lambda_* > 1$ provided that $I(u) < 0, I(u) = 0$ and $I(u) > 0$, respectively. This completes the proof. □

We now define the so-called Nehari manifold associated to the energy functional J by

$$\mathcal{N} = \{u \in H_0^2(\Omega)\backslash\{0\} : I(u) = 0\}.$$

It follows from Lemma 3.1 that $\mathcal{N}$ is not empty set. Thus, we can define

$$d = \inf_{u\in\mathcal{N}} J(u). \tag{3.5}$$

The following lemma plays an important role in the proofs of our main results for the low initial energy case.

Lemma 3.2. *Let* (1.2)-(1.3) *hold and* $u \in H_0^2(\Omega)\backslash\{0\}$. *Then*

$$J(u) - \frac{1}{q^-}I(u) \geq \frac{d}{\max\{\lambda_*^2, \lambda_*^{p-}, \lambda_*^{q^+}\}},$$

where λ_* *is as in Lemma 3.1.*

Proof. For any $u \in H_0^2(\Omega)\backslash\{0\}$, by Lemma 3.1, there exists $\lambda_* \in (0,\infty)$ such that $I(\lambda_* u) = 0$. By the definition of d and replacing u by $\lambda_* u$ in (3.4), one has

$$\begin{aligned}
d &\leq J(\lambda_* u)\\
&= \big(\frac{1}{2} - \frac{1}{q^-}\big)\lambda_*^2\|\Delta u\|_2^2 + \int_\Omega \lambda_*^{p(x)}\big(\frac{1}{p(x)} - \frac{1}{q^-}\big)|\nabla u|^{p(x)}\,\mathrm{d}x\\
&\quad + \int_\Omega \lambda_*^{q(x)}\big(\frac{1}{q^-} - \frac{1}{q(x)}\big)|u|^{q(x)}\,\mathrm{d}x\\
&\leq \max\{\lambda_*^2, \lambda_*^{p-}, \lambda_*^{q^+}\}\Big[\big(\frac{1}{2} - \frac{1}{q^-}\big)\|\Delta u\|_2^2 + \int_\Omega \big(\frac{1}{p(x)} - \frac{1}{q^-}\big)|\nabla u|^{p(x)}\,\mathrm{d}x\\
&\quad + \int_\Omega \big(\frac{1}{q^-} - \frac{1}{q(x)}\big)|u|^{q(x)}\,\mathrm{d}x\Big]\\
&= \max\big\{\lambda_*^2, \lambda_*^{p-}, \lambda_*^{q^+}\big\}[J(u) - \frac{1}{q^-}I(u)].
\end{aligned}$$

This implies the required result. The proof is complete. □

Based on the above two lemmas, we can prove the following lemma, which shows that d is positive and is actually attained at some $u \in \mathcal{N}$.

Lemma 3.3. *Let* (1.2)–(1.3) *hold. Then we have*

(i) $d = \inf_{u\in H_0^2(\Omega)\backslash\{0\}} \sup_{\lambda>0} J(\lambda u)$.
(ii) *d is a positive number.*
(iii) *There exists $u^* \in \mathcal{N}$, $u^*(x) \geq 0$ a.e. in Ω such that $J(u^*) = d$.*

Proof. For any $u \in H_0^2(\Omega)\backslash\{0\}$. By Lemma 3.1, we have

$$\sup_{\lambda>0} J(\lambda u) = J(\lambda_* u). \tag{3.6}$$

By the definition of $\mathcal{N}$, it follows from Lemma 3.1 that $\lambda_* u \in \mathcal{N}$. Hence

$$J(\lambda_* u) \geq \inf_{u\in\mathcal{N}} J(u) = d. \tag{3.7}$$

Combining (3.6) and (3.7), one has

$$\inf_{u\in H_0^2(\Omega)\setminus\{0\}} \sup_{\lambda>0} J(\lambda u) \geq d. \tag{3.8}$$

On the other hand, for any $u \in \mathcal{N}$, by Lemma 3.1, one has $\lambda_* = 1$, that is,

$$\sup_{\lambda>0} J(\lambda u) = J(u).$$

Therefore,

$$\inf_{u\in H_0^2(\Omega)\setminus\{0\}} \sup_{\lambda>0} J(\lambda u) \leq \inf_{u\in\mathcal{N}} \sup_{\lambda>0} J(\lambda u) = \inf_{u\in\mathcal{N}} J(u) = d. \tag{3.9}$$

We deduce from (3.8) and (3.9) that (i) holds.

We next prove (ii). Since $q(x)$ satisfies (1.3), $H_0^2(\Omega)$ can be embedded into $L^{q(\cdot)}(\Omega)$ continuously. Denote by $S_{q(\cdot)}$ the optimal embedding constant, i.e.,

$$S_{q(\cdot)} = \sup_{u\in H_0^2(\Omega)\setminus\{0\}} \frac{\|u\|_{q(\cdot)}}{\|\Delta u\|_2}.$$

Let any $u \in H_0^2(\Omega)\setminus\{0\}$ such that $I(u) \leq 0$. Then it follows that

$$\begin{aligned}
\|\Delta u\|_2^2 &\leq \int_\Omega |u|^{q(x)}\,\mathrm{d}x \\
&\leq \max\big\{\|u\|_{q(\cdot)}^{q^-}, \|u\|_{q(\cdot)}^{q^+}\big\} \\
&\leq \max\big\{S_{q(\cdot)}^{q^-}\|\Delta u\|_2^{q^-}, S_{q(\cdot)}^{q^+}\|\Delta u\|_2^{q^+}\big\}.
\end{aligned}$$

Taking this fact into account and notice that $\|\Delta u\|_2 > 0$ and $q^- > 2$, we obtain

$$\|\Delta u\|_2 \geq \delta_1, \tag{3.10}$$

where

$$\delta_1 = \min\big\{S_{q(\cdot)}^{\frac{q^-}{2-q^-}}, S_{q(\cdot)}^{\frac{q^+}{2-q^+}}\big\}.$$

Fixing $u \in \mathcal{N}$, we have $u \in H_0^2(\Omega)\setminus\{0\}$ and $I(u) = 0$. By using (3.2) and (3.10), we obtain

$$\begin{aligned}
J(u) &\geq \big(\frac{1}{2} - \frac{1}{q^-}\big)\|\Delta u\|_2^2 + \big(\frac{1}{p^+} - \frac{1}{q^-}\big)\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x + \frac{1}{q^-}I(u) \\
&\geq \big(\frac{1}{2} - \frac{1}{q^-}\big)\|\Delta u\|_2^2 \\
&\geq \big(\frac{1}{2} - \frac{1}{q^-}\big)\delta_1^2.
\end{aligned} \tag{3.11}$$

Then by the definition of d, we obtain

$$d \geq \big(\frac{1}{2} - \frac{1}{q^-}\big)\delta_1^2 > 0.$$

Finally, we prove (iii). By (3.5), there exists $\{u_n\}_{n=1}^\infty \subset \mathcal{N}$ is a minimizing sequence of J such that $\lim_{n\to\infty} J(u_n) = d$. Clearly, $|u_n| \in \mathcal{N}$ and $J(|u_n|) = J(u_n)$. For this reason, we may assume that $u_n(x) \geq 0$ a.e. in Ω for all $n \in \mathbb{N}^*$.

Since $\lim_{n\to\infty} J(u_n) = d$ and using (3.11), we infer that $\{u_n\}$ is bounded in $H_0^2(\Omega)$. Then, since $H_0^2(\Omega)$ is reflexive, $H_0^2(\Omega) \hookrightarrow W_0^{1,p(\cdot)}(\Omega)$ and $H_0^2(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$

are compact embeddings (by (1.2) and (1.3)), there exists a sub-sequence of $\{u_n\}$, still denoted by $\{u_n\}$ and a $u^* \in H_0^2(\Omega)$ such that

$$
\begin{gathered}
u_n \rightharpoonup u^* \quad \text{weakly in } H_0^2(\Omega), \\
u_n \to u^* \quad \text{strongly in } W_0^{1,p(\cdot)}(\Omega), \\
u_n \to u^* \quad \text{strongly in } L^{q(\cdot)}(\Omega), \\
u_n(x) \to u^*(x) \quad \text{a.e. in } \Omega.
\end{gathered}
$$

Then we have $u^*(x) \geq 0$ a.e. in Ω and

$$
\begin{gathered}
\|\Delta u^*\|_2 \leq \liminf_{n\to\infty} \|\Delta u_n\|_2, \\
\int_\Omega |\nabla u^*|^{p(x)}\,\mathrm{d}x = \lim_{n\to\infty} \int_\Omega |\nabla u_n|^{p(x)}\,\mathrm{d}x, \\
\int_\Omega \big(\frac{1}{p(x)} - \frac{1}{q^-}\big)|\nabla u^*|^{p(x)}\,\mathrm{d}x = \lim_{n\to\infty} \int_\Omega \big(\frac{1}{p(x)} - \frac{1}{q^-}\big)|\nabla u_n|^{p(x)}\,\mathrm{d}x, \\
\int_\Omega |u^*|^{q(x)}\,\mathrm{d}x = \lim_{n\to\infty} \int_\Omega |u_n|^{q(x)}\,\mathrm{d}x, \\
\int_\Omega \big(\frac{1}{q^-} - \frac{1}{q(x)}\big)|u^*|^{q(x)}\,\mathrm{d}x = \lim_{n\to\infty} \int_\Omega \big(\frac{1}{q^-} - \frac{1}{q(x)}\big)|u_n|^{q(x)}\,\mathrm{d}x.
\end{gathered}
$$

Replacing u by u_n in (3.4) and notice that $u_n \in \mathcal{N}$, one has

$$
\begin{aligned}
d &= \liminf_{n\to\infty} J(u_n) \\
&= \liminf_{n\to\infty} \Big[\big(\frac{1}{2} - \frac{1}{q^-}\big)\|\Delta u_n\|_2^2 + \int_\Omega \big(\frac{1}{p(x)} - \frac{1}{q^-}\big)|\nabla u_n|^{p(x)}\,\mathrm{d}x \\
&\quad + \int_\Omega \big(\frac{1}{q^-} - \frac{1}{q(x)}\big)|u_n|^{q(x)}\,\mathrm{d}x\big] \\
&\geq \big(\frac{1}{2} - \frac{1}{q^-}\big)\|\Delta u^*\|_2^2 + \int_\Omega \big(\frac{1}{p(x)} - \frac{1}{q^-}\big)|\nabla u^*|^{p(x)}\,\mathrm{d}x \\
&\quad + \int_\Omega \big(\frac{1}{q^-} - \frac{1}{q(x)}\big)|u^*|^{q(x)}\,\mathrm{d}x \\
&= J(u^*) - \frac{1}{q^-} I(u^*).
\end{aligned} \tag{3.12}
$$

Suppose that $I(u^*) < 0$. Then by Lemmas 3.1 and 3.2, there exists $\lambda_* \in (0,1)$ such that

$$
J(u^*) - \frac{1}{q^-} I(u^*) \geq \frac{d}{\max\{\lambda_*^2, \lambda_*^{p-}, \lambda_*^{q^+}\}} > d.
$$

This contradicts (3.12), and so

$$
I(u^*) \geq 0. \tag{3.13}
$$

Since $u_n \in \mathcal{N}$, we have $I(u_n) = 0$, it follows that

$$
\begin{aligned}
0 &= \liminf_{n\to\infty} I(u_n) \\
&= \liminf_{n\to\infty} \Big(\|\Delta u_n\|_2^2 + \int_\Omega |\nabla u_n|^{p(x)}\,\mathrm{d}x - \int_\Omega |u_n|^{q(x)}\,\mathrm{d}x\Big) \\
&\geq \|\Delta u^*\|_2^2 + \int_\Omega |\nabla u^*|^{p(x)}\,\mathrm{d}x - \int_\Omega |u^*|^{q(x)}\,\mathrm{d}x \\
&= I(u^*),
\end{aligned}
$$

which, together with (3.13), implies that $I(u^*) = 0$. We now prove $u^* \in \mathcal{N}$. It remains to show that $u^* \neq 0$. Indeed, since $u_n \in \mathcal{N}$ and (3.10), we obtain

$$\int_\Omega |u_n|^{q(x)}\,\mathrm{d}x = \|\Delta u_n\|_2^2 + \int_\Omega |\nabla u_n|^{p(x)}\,\mathrm{d}x \geq \delta_1^2.$$

Passing to the limit, we have

$$\int_\Omega |u^*|^{q(x)}\,\mathrm{d}x \geq \delta_1^2 > 0,$$

which gives $u^* \neq 0$. Hence, $u^* \in \mathcal{N}$ and therefore $J(u^*) \geq d$. By (3.12) and $I(u^*) = 0$, we have $J(u^*) \leq d$. So, $J(u^*) = d$. The proof is complete. □

We now define the so-called stable set $\mathcal{W}$ and unstable set $\mathcal{U}$ which is similar to Sattinger [23], Payne and Sattinger [20].

$$\begin{gathered}\mathcal{W} = \{u \in H_0^2(\Omega) : J(u) < d, I(u) > 0\} \cup \{0\},\\ \mathcal{U} = \{u \in H_0^2(\Omega) : J(u) < d, I(u) < 0\},\end{gathered}$$

We also introduce

$$\mathcal{N}_- = \{u \in H_0^2(\Omega) : I(u) < 0\}, \quad \mathcal{N}_+ = \{u \in H_0^2(\Omega) : I(u) > 0\},$$

and the open sub levels of J,

$$J^k = \{u \in H_0^2(\Omega) : J(u) < k\}.$$

The variational characterization of d also shows that

$$\mathcal{N}_k := \mathcal{N} \cap J^k \neq \emptyset \quad \text{for all} \quad k > d.$$

For $k > d$, we now define

$$\lambda_k = \inf\{\|u\|_2 : u \in \mathcal{N}_k\} \quad \text{and} \quad \Lambda_k = \sup\{\|u\|_2 : u \in \mathcal{N}_k\}. \tag{3.14}$$

It is obvious that $k \mapsto \lambda_k$ is non-increasing, and $k \mapsto \Lambda_k$ is non-decreasing. The next lemma shows that λ_k and Λ_k are finite positive numbers, and therefore the result of Theorem 4.9 is nontrivial.

Lemma 3.4. *Let* (1.2)-(1.3) *hold. Then for any $k > d$, λ_k and Λ_k defined in* (3.14) *satisfy* $0 < \lambda_k \leq \Lambda_k < \infty$.

Proof. Firstly, we prove $\Lambda_k < \infty$. For any $k > d$ and $u \in \mathcal{N}_k$, we have $J(u) < k$ and $I(u) = 0$. Then by (3.2) and using the embedding $H_0^2(\Omega) \hookrightarrow L^2(\Omega)$, we obtain

$$\begin{aligned} k > J(u) &\geq \big(\frac{1}{2} - \frac{1}{q^-}\big)\|\Delta u\|_2^2 + \big(\frac{1}{p^+} - \frac{1}{q^-}\big)\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x + \frac{1}{q^-}I(u)\\ &\geq \big(\frac{1}{2} - \frac{1}{q^-}\big)\|\Delta u\|_2^2\\ &\geq \big(\frac{1}{2} - \frac{1}{q^-}\big)S_2^{-2}\|u\|_2^2, \end{aligned} \tag{3.15}$$

which yields

$$\|u\|_2 \leq S_2\sqrt{\frac{2kq^-}{q^- - 2}},$$

where $S_2 > 0$ is the optimal embedding constant, i.e.,

$$S_2 = \sup_{u \in H_0^2(\Omega)\setminus\{0\}} \frac{\|u\|_2}{\|\Delta u\|_2}. \tag{3.16}$$

This shows that

$$\Lambda_k \leq S_2\sqrt{\frac{2kq^-}{q^- - 2}} < \infty.$$

Secondly, we prove that $\lambda_k > 0$. By the Gagliardo-Nirenberg inequality, there exists a positive constant A_0 depending only on Ω, N, q^- and q^+ such that

$$\|u\|_{q^-}^{q^-} \leq A_0\|\Delta u\|_2^{\theta^- q^-}\|u\|_2^{(1-\theta^-)q^-},$$
$$\|u\|_{q^+}^{q^+} \leq A_0\|\Delta u\|_2^{\theta^+ q^+}\|u\|_2^{(1-\theta^+)q^+},$$

where $\theta^\pm = \frac{N(q^\pm - 2)}{4q^\pm} \in (0;1)$ by (1.3). Then, since $u \in \mathcal{N}_k$ and $\mathcal{N}_k \subset \mathcal{N}$, it follows that

$$\begin{aligned}
\|\Delta u\|_2^2 &\leq \int_\Omega |u|^{q(x)}\,\mathrm{d}x \\
&\leq \int_\Omega \left(|u|^{q^-} + |u|^{q^+}\right)\mathrm{d}x \\
&\leq 2\max\left\{\|u\|_{q^-}^{q^-}, \|u\|_{q^+}^{q^+}\right\} \\
&\leq 2A_0\max\left\{\|\Delta u\|_2^{\theta^- q^-}\|u\|_2^{(1-\theta^-)q^-}, \|\Delta u\|_2^{\theta^+ q^+}\|u\|_2^{(1-\theta^+)q^+}\right\}.
\end{aligned}$$

Taking this into account and noticing that $\|\Delta u\|_2 > 0$ and $\theta^\pm < 1$, we obtain

$$\|u\|_2 \geq \min\left\{(2A_0)^{\frac{1}{(\theta^- - 1)q^-}}\|\Delta u\|_2^{\frac{2-\theta^- q^-}{(1-\theta^-)q^-}}, (2A_0)^{\frac{1}{(\theta^+ - 1)q^+}}\|\Delta u\|_2^{\frac{2-\theta^+ q^+}{(1-\theta^+)q^+}}\right\}. \tag{3.17}$$

On the other hand, it follows from (3.10) and (3.15) that

$$\delta_1 \leq \|\Delta u\|_2 \leq \sqrt{\frac{2kq^-}{q^- - 2}} := \delta_2, \quad \text{for all } u \in \mathcal{N}_k.$$

This, together with (3.17), implies

$$\begin{aligned}
\|u\|_2 \geq \min\Big\{&(2A_0)^{\frac{1}{(\theta^- - 1)q^-}}\min\left\{\delta_1^{\frac{2-\theta^- q^-}{(1-\theta^-)q^-}}, \delta_2^{\frac{2-\theta^- q^-}{(1-\theta^-)q^-}}\right\}, \\
&(2A_0)^{\frac{1}{(\theta^+ - 1)q^+}}\min\left\{\delta_1^{\frac{2-\theta^+ q^+}{(1-\theta^+)q^+}}, \delta_2^{\frac{2-\theta^+ q^+}{(1-\theta^+)q^+}}\right\}\Big\} > 0.
\end{aligned}$$

Hence, $\lambda_k > 0$ by the definition of λ_k. This completes the proof. $\square$

Finally, we give the following lemma, which is necessary for our proofs of the main results in case of the high initial energy.

Lemma 3.5. *Let* (1.2)-(1.3) *hold. Then we have*

(i) *0 is away from both $\mathcal{N}$ and $\mathcal{N}_-$, that is,* $\operatorname{dist}(0,\mathcal{N}) > 0$ *and* $\operatorname{dist}(0,\mathcal{N}_-) > 0$.
(ii) *The set $\mathcal{N}_+ \cap J^k$ is bounded in $H_0^2(\Omega)$ for any $k > 0$.*

Proof. By (3.10), it is easy to see that

$$\operatorname{dist}(0,\mathcal{N}) = \inf_{u\in\mathcal{N}}\|\Delta u\|_2 \geq \delta_1 > 0,$$
$$\operatorname{dist}(0,\mathcal{N}_-) = \inf_{u\in\mathcal{N}_-}\|\Delta u\|_2 \geq \delta_1 > 0.$$

We now prove (ii). For any $u \in \mathcal{N}_+ \cap J^k$, we have $J(u) < k$ and $I(u) > 0$. Then by using (3.2), it follows that

$$\begin{aligned} k > J(u) &\geq \big(\frac{1}{2} - \frac{1}{q^-}\big)\|\Delta u\|_2^2 + \big(\frac{1}{p^+} - \frac{1}{q^-}\big) \int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x + \frac{1}{q^-} I(u) \\ &\geq \big(\frac{1}{2} - \frac{1}{q^-}\big)\|\Delta u\|_2^2, \end{aligned}$$

which implies that

$$\|\Delta u\|_2 < \sqrt{\frac{2kq^-}{q^- - 2}}$$

and completes the proof. $\square$

4. Evolution problem

We first give the precise meaning of solution to problem (1.1).

Definition 4.1. Let $T > 0$, a function $u = u(t) \in L^\infty(0,T; H_0^2(\Omega))$ with $u_t \in L^2(0,T; L^2(\Omega))$ is said to be a *weak solution* to (1.1) in $\Omega \times [0,T)$, if $u(0) = u_0 \in H_0^2(\Omega)$ and satisfies

$$\langle u_t, v\rangle + \langle \Delta u, \Delta v\rangle + \langle |\nabla u|^{p(x)-2}\nabla u, \nabla v\rangle = \langle |u|^{q(x)-2}u, v\rangle, \quad \text{a.e. } t \in (0,T), \tag{4.1}$$

for any $v \in H_0^2(\Omega)$. Moreover,

$$\int_0^t \|u'(s)\|_2^2\,\mathrm{d}s + J(u(t)) = J(u_0), \quad 0 \leq t < T. \tag{4.2}$$

Definition 4.2. Let $u(t)$ be a weak solution to the problem (1.1). We define the maximal existence time $T_{\max}$ of $u(t)$ as follows

- (i) If $u(t)$ exists for $0 \leq t < \infty$, then $T_{\max} = \infty$.
- (ii) If there exists $t_0 > 0$ such that $u(t)$ exists for $0 \leq t < t_0$, but does not exist at t_0, then $T_{\max} = t_0$.

Remark 4.3. Under the assumption $u_0 \in H_0^2(\Omega)$, by using the standard Galerkin's method as in [10, 31], we can prove immediately the existence of local weak solutions $u(t)$. For the uniqueness of solution to (1.1), it can be obtained under some suitable assumptions on the initial data u_0, $p(\cdot)$ and $q(\cdot)$. For example, in [10] Han proved the uniqueness of bounded weak solution (L^∞-bounded solution), such kind of solution can be obtained by either in one dimensional space or $p(\cdot) > N$.

The next lemma shows the invariant of stable and unstable sets.

Lemma 4.4. *Let* (1.2)-(1.3) *hold and* $J(u_0) < d$. *Then we possess the following statements:*

- (i) *If* $I(u_0) < 0$, *then* $I(u(t)) < 0$ *for all* $t \in [0, T_{\max})$.
- (ii) *If* $I(u_0) \geq 0$, *then* $I(u(t)) \geq 0$ *for all* $t \in [0, T_{\max})$.

Proof. Note that $u(t) \notin \mathcal{N}$, for all $t \in [0, T_{\max})$ since $J(u(t)) \leq J(u_0)) < d$.

For (i). Assume to the contrary that there exists $t_0 \in (0, T_{\max})$ such that $I(u(t)) < 0$ for all $t \in [0, t_0)$ and $I(u(t_0)) = 0$. Then by (3.10), we obtain $\|\Delta u(t)\|_2 \geq \delta_1$, for all $t \in [0, t_0)$. Letting $t \to t_0$, we have $\|\Delta u(t_0)\|_2 \geq \delta_1$, which gives $u(t_0) \neq 0$. Hence, $u(t_0) \in \mathcal{N}$. We thus arrive at a contradiction.

For (ii). By contradiction, we assume that there exists $t_1 \in (0, T_{\max})$ such that $I(u(t_1)) < 0$. This and $I(u_0) \geq 0$ imply that there exists $t_2 \in [0, t_1)$ such that

$I(u(t_2)) = 0$. It gives $u(t_2) = 0$ due to $u(t_2) \notin \mathcal{N}$. Then we obtain $u(t) = 0$ for $t \in [t_2, T_{\max})$. Thus $u(t_1) = 0$. This contradicts $I(u(t_1)) < 0$. The proof is complete. $\square$

We introduce the set

$$\mathcal{S} = \{\phi \in H_0^2(\Omega) : \phi \text{ is a stationary solution of (1.1)}\},$$

and define the ω-limit set $\omega(u_0)$ of the initial data $u_0 \in W_0^{1,p(\cdot)}(\Omega)$ by

$$\omega(u_0) = \{w \in H_0^2(\Omega) : \exists\{t_n\} \text{ with } t_n \to \infty \text{ such that } u(t_n) \to w\}.$$

Let $u(t)$ be a solution to (1.1) associated with $u_0 \in H_0^2(\Omega)$ on the maximal existence time interval $[0, T_{\max})$. We then introduce the sets

$$\mathcal{G} = \{u_0 \in H_0^2(\Omega) : u(t) \text{ exists globally, i.e. } T_{\max} = \infty\},$$
$$\mathcal{G}_0 = \{u_0 \in \mathcal{G} : u(t) \to 0 \text{ in } H_0^2(\Omega) \text{ as } t \to \infty\},$$
$$\mathcal{B} = \{u_0 \in H_0^2(\Omega) : u(t) \text{ blows up in finite time, i.e. } T_{\max} < \infty\}.$$

Our main results read as follows.

Theorem 4.5. *Let (1.2)-(1.3) hold. If $J(u_0) < d$ and $I(u_0) \geq 0$, then the maximal existence time $T_{\max} = \infty$. Moreover, $u(t)$ holds following decay estimates:*

$$\|u(t)\|_2 \leq \|u_0\|_2 e^{-\alpha t},$$
$$\|\Delta u(t)\|_2 \leq \sqrt{\frac{2q^-}{q^- - 2}(J(u_0) + \|u_0\|_2^2)}e^{-\beta t},$$
$$\sqrt{J(u(t)) + \|u(t)\|_2^2} \leq \sqrt{J(u_0) + \|u_0\|_2^2}e^{-\beta t},$$

where α and β are some positive constants.

Theorem 4.6. *Let (1.2)-(1.3) hold. Then*

(i) *If $u_0 \in H_0^2(\Omega)\backslash\{0\}$ holds $J(u_0) \leq 0$, then $T_{\max} < \infty$. Furthermore, we can get an upper bound for the maximal existence time*

$$T_{\max} \leq C \max\left\{\|u_0\|_2^{2-q^-}, \|u_0\|_2^{2-q^+}\right\},$$

where

$$C = \frac{q^- \max\{S_{q(\cdot),2}^{q^-}, S_{q(\cdot),2}^{q^+}\}}{(q^- - 2)(q^- - \max\{2, p^+\})} > 0, \tag{4.3}$$

and $S_{q(\cdot),2}$ is the optimal embedding constant of $L^{q(\cdot)}(\Omega) \hookrightarrow L^2(\Omega)$ when $q^- > 2$, i.e.,

$$S_{q(\cdot),2} = \sup_{u \in L^{q(\cdot)}(\Omega)\backslash\{0\}} \frac{\|u\|_2}{\|u\|_{q(\cdot)}}. \tag{4.4}$$

(ii) *If $0 < J(u_0) < d$ and $I(u_0) < 0$, then $T_{\max} < \infty$.*

Remark 4.7. As a consequence of Theorem 4.5 and 4.6, we have a sharp result in the case $J(u_0) < d$, that is, the weak solution to (1.1) exists globally and blows up in finite time provided that $I(u_0) \geq 0$ and $I(u_0) < 0$, respectively.

Theorem 4.5 shows that any global weak solution which starts from the potential wells $\mathcal{W}$ tends to zero. And the next theorem shows the asymptotic behavior of any global weak solution of (1.1).

Theorem 4.8. *Let* (1.2)-(1.3) *hold and* $u(t)$ *be a global solution of* (1.1). *Then there exists a sequence* $\{t_n\}$ *with* $t_n \to \infty$ *as* $n \to \infty$ *and* $\phi \in \mathcal{S}$ *such that*

$$\lim_{n\to\infty} \|\Delta u(t_n) - \Delta\phi\|_2 = 0.$$

Our next result gives an abstract criterion for vanishing and global nonexistence of solutions to (1.1) in terms of the variational values λ_k and Λ_k.

Theorem 4.9. *Let* (1.2)-(1.3) *hold and* $J(u_0) > d$. *If* $u_0 \in \mathcal{N}_+$ *and* $\|u_0\|_2 \le \lambda_{J(u_0)}$, *then* $u_0 \in \mathcal{G}_0$. *If* $u_0 \in \mathcal{N}_-$ *and* $\|u_0\|_2 \ge \Lambda_{J(u_0)}$, *then* $u_0 \in \mathcal{B}$.

As a consequence, one has a characterization on the data u_0 with arbitrary high energy $J(u_0)$ that leads to blow-up in finite time phenomena.

Theorem 4.10. *Let* (1.2)-(1.3) *hold and assume that* $u_0 \in H_0^2(\Omega)$ *holds* $J(u_0) > d$ *and*

$$\|u_0\|_2^2 \ge \frac{2q^- S_2^2}{q^- - 2} J(u_0), \tag{4.5}$$

then $u_0 \in \mathcal{N}_- \cap \mathcal{B}$. *Here* S_2 *is the constant given in* (3.16).

5. Proof of Theorem 4.5

Let $u(t) := u(x,t)$ be a solution of (1.1) on the interval $[0, T_{\max})$ associated with to the initial data u_0. We first prove the uniform boundedness in time of $u(t)$ in $H_0^2(\Omega)$, which implies $T_{\max} = \infty$ by the continuation principle. Indeed, since $J(u_0) < d$ and $I(u_0) \ge 0$, by Lemma 4.4 we have that $I(u(t)) \ge 0$. Then by using the non-increasing property of $J(u(t))$ and (3.2), we obtain

$$\begin{aligned} J(u_0) &\ge J(u(t)) \\ &\ge \big(\frac{1}{2} - \frac{1}{q^-}\big)\|\Delta u(t)\|_2^2 + \big(\frac{1}{p^+} - \frac{1}{q^-}\big)\int_\Omega |\nabla u(t)|^{p(x)}\,\mathrm{d}x + \frac{1}{q^-} I(u(t)) \\ &\ge \big(\frac{1}{2} - \frac{1}{q^-}\big)\|\Delta u(t)\|_2^2, \end{aligned} \tag{5.1}$$

which implies

$$\|\Delta u(t)\|_2 \le \sqrt{\frac{2q^- J(u_0)}{q^- - 2}}.$$

We next show the decay estimates of $u(t)$. If $u(t_0) = 0$ for some $t_0 \ge 0$, then we have $u(t) = 0$ for all $t \ge t_0$, and the proof is complete. So we may assume $u(t) \ne 0$ for all $t \ge 0$. Then due to $I(u(t)) \ge 0$, by Lemma 3.1, there exists $\lambda_* \ge 1$ such that $I(\lambda_* u(t)) = 0$. And therefore

$$\begin{aligned} \lambda_*^{q^-} I(u(t)) &= \lambda_*^{q^-} I(u(t)) - I(\lambda_* u(t)) \\ &= \big(\lambda_*^{q^-} - \lambda_*^2\big)\|\Delta u(t)\|_2^2 + \int_\Omega \big(\lambda_*^{q^-} - \lambda_*^{p(x)}\big)|\nabla u(t)|^{p(x)}\,\mathrm{d}x \\ &\quad + \int_\Omega \big(\lambda_*^{q(x)} - \lambda_*^{q^-}\big)|u(t)|^{q(x)}\,\mathrm{d}x \\ &\ge \big(\lambda_*^{q^-} - \lambda_*^2\big)\|\Delta u(t)\|_2^2 + \big(\lambda_*^{q^-} - \lambda_*^{p^+}\big)\int_\Omega |\nabla u(t)|^{p(x)}\,\mathrm{d}x. \end{aligned}$$

Dividing the above inequality by $\lambda_*^{q^-}$, we obtain

$$I(u(t)) \geq (1-\lambda_*^{2-q^-})\|\Delta u(t)\|_2^2 + \big(1-\lambda_*^{p^+-q^-}\big)\int_\Omega |\nabla u(t)|^{p(x)}\,\mathrm{d}x. \tag{5.2}$$

We next estimate for λ_*. By applying Lemma 3.2 and notice that $\lambda_* \geq 1$, one has

$$J(u(t)) - \frac{1}{q^-}I(u(t)) \geq \frac{d}{\max\{\lambda_*^2, \lambda_*^{p-}, \lambda_*^{q^+}\}} = \frac{d}{\lambda_*^{q^+}}. \tag{5.3}$$

On the other hand, by using the non-increasing property of $J(u(t))$ and notice that $I(u(t)) \geq 0$, we have

$$J(u(t)) - \frac{1}{q^-}I(u(t)) \leq J(u_0).$$

This together with (5.3), implies that

$$\lambda_* \geq (\frac{d}{J(u_0)})^{1/q^+} > 1. \tag{5.4}$$

It follows from (5.2) and (5.4) that

$$\begin{aligned} I(u(t)) \geq &\Big(1-(\frac{d}{J(u_0)})^{\frac{2-q^-}{q^+}}\Big)\|\Delta u(t)\|_2^2 \\ &+ \Big(1-(\frac{d}{J(u_0)})^{\frac{p^+-q^-}{q^+}}\Big)\int_\Omega |\nabla u(t)|^{p(x)}\,\mathrm{d}x, \end{aligned}$$

which yields

$$I(u(t)) \geq C_1\|\Delta u(t)\|_2^2 \quad \text{and} \quad I(u(t)) \geq C_2\int_\Omega |\nabla u(t)|^{p(x)}\,\mathrm{d}x, \tag{5.5}$$

where

$$C_1 = 1-\Big(\frac{d}{J(u_0)}\Big)^{\frac{2-q^-}{q^+}} \quad \text{and} \quad C_2 = 1-\Big(\frac{d}{J(u_0)}\Big)^{\frac{p^+-q^-}{q^+}}.$$

We now consider the exponential decay of $\|u(t)\|_2$. Taking $v = u$ in (4.1), we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_2^2 &= -2\Big(\|\Delta u(t)\|_2^2 + \int_\Omega |\nabla u(t)|^{p(x)}\,\mathrm{d}x - \int_\Omega |u(t)|^{q(x)}\,\mathrm{d}x\Big) \\ &= -2I(u(t)). \end{aligned}$$

From this and (5.5), it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_2^2 \leq -2C_1\|\Delta u(t)\|_2^2 \leq -2C_1S_2^{-2}\|u(t)\|_2^2,$$

where S_2 is the constant given in (3.16). This implies that

$$\|u(t)\|_2 \leq \|u_0\|_2 e^{-\alpha t},$$

where $\alpha = C_1S_2^{-2} > 0$.

We next consider the exponential decay of $J(u(t))$ and $\|\Delta u(t)\|_2$. Using (3.3), we obtain

$$J(u(t)) \leq (\frac{1}{2}-\frac{1}{q^+})\|\Delta u(t)\|_2^2 + (\frac{1}{p^-}-\frac{1}{q^+})\int_\Omega |\nabla u(t)|^{p(x)}\,\mathrm{d}x + \frac{1}{q^+}I(u(t)).$$

This together with (5.5) immediately yields

$$J(u(t)) \leq C_3 I(u(t)), \tag{5.6}$$

where

$$C_3 = \frac{1}{C_1}\Big(\frac{1}{2} - \frac{1}{q^+}\Big) + \frac{1}{C_2}\Big(\frac{1}{p^-} - \frac{1}{q^+}\Big) + \frac{1}{q^+} > 0.$$

Let us define an auxiliary function

$$L(t) = J(u(t)) + \|u(t)\|_2^2, \quad \text{for } t \geq 0. \tag{5.7}$$

Then by (5.1) and (5.7), we obtain

$$L(t) \leq J(u(t)) + S_2^2\|\Delta u(t)\|_2^2 \leq C_4 J(u(t)). \tag{5.8}$$

Here $C_4 = 1 + \frac{2q^-}{q^- -2}S_2^2 > 0$ and S_2 is the constant given in (3.16). It follows from (5.6), (5.7) and (5.8) that

$$\frac{\mathrm{d}}{\mathrm{d}t}L(t) = -\|u'(t)\|_2^2 - 2I(u(t)) \leq -\frac{2}{C_3}J(u(t)) \leq -\frac{2}{C_3C_4}L(t),$$

which implies that

$$L(t) \leq L(0)e^{-2\beta t},$$

where $\beta = \frac{1}{C_3C_4} > 0$. The above inequality can be rewritten as

$$J(u(t)) + \|u(t)\|_2^2 \leq (J(u_0) + \|u_0\|_2^2)e^{-2\beta t}. \tag{5.9}$$

By (5.1) and (5.9), we obtain

$$\|\Delta u(t)\|_2^2 \leq \frac{2q^-}{q^- - 2}J(u(t)) \leq \frac{2q^-}{q^- - 2}\big(J(u_0) + \|u_0\|_2^2\big)e^{-2\beta t}.$$

The proof is complete.

6. Proof of Theorem 4.6

We consider following two cases by using different methods:

Case 1: $u_0 \in H_0^2(\Omega)\backslash\{0\}$ with $J(u_0) \leq 0$. We define the function

$$f(t) = \|u(t)\|_2^2, \quad \text{for all } t \in [0, T_{\max}).$$

By the definition of J and I, we have

$$\begin{aligned}
J(u(t)) &\geq \frac{1}{2}\|\Delta u(t)\|_2^2 + \frac{1}{p^+}\int_\Omega |\nabla u(t)|^{p(x)}\,\mathrm{d}x - \frac{1}{q^-}\int_\Omega |u(t)|^{q(x)}\,\mathrm{d}x \\
&\geq \frac{1}{\max\{2,p^+\}}\Big(\|\Delta u(t)\|_2^2 + \int_\Omega |\nabla u(t)|^{p(x)}\,\mathrm{d}x\Big) - \frac{1}{q^-}\int_\Omega |u(t)|^{q(x)}\,\mathrm{d}x \\
&= \Big(\frac{1}{\max\{2,p^+\}} - \frac{1}{q^-}\Big)\int_\Omega |u(t)|^{q(x)}\,\mathrm{d}x + \frac{1}{\max\{2,p^+\}}I(u(t)).
\end{aligned}$$

Taking this fact into account and notice that $J(u(t)) \leq J(u_0) \leq 0$, one has

$$\begin{aligned}
f'(t) &= -2I(u(t)) \\
&\geq -2\max\{2,p^+\}J(u(t)) + 2\Big(1 - \frac{\max\{2,p^+\}}{q^-}\Big)\int_\Omega |u(t)|^{q(x)}\,\mathrm{d}x \\
&\geq 2\Big(1 - \frac{\max\{2,p^+\}}{q^-}\Big)\int_\Omega |u(t)|^{q(x)}\,\mathrm{d}x.
\end{aligned} \tag{6.1}$$

From this and $q^- > \max\{2,p^+\}$, we obtain that $f'(t) \geq 0$ for all $t \in [0, T_{\max})$, which implies that

$$f(t) \geq f(0) = \|u_0\|_2^2 > 0, \quad \text{for all } t \in [0, T_{\max}). \tag{6.2}$$

Then by (6.2), we can estimate $\int_\Omega |u(t)|^{q(x)}\,\mathrm{d}x$ as follows:

$$\begin{aligned}
\int_\Omega |u(t)|^{q(x)}\,\mathrm{d}x &\geq \min\big\{\|u(t)\|_{q(\cdot)}^{q^-}, \|u(t)\|_{q(\cdot)}^{q^+}\big\}\\
&\geq \min\big\{S_{q(\cdot),2}^{-q^-}\|u(t)\|_2^{q^-}, S_{q(\cdot),2}^{-q^+}\|u(t)\|_2^{q^+}\big\}\\
&\geq \min\big\{S_{q(\cdot),2}^{-q^-}, S_{q(\cdot),2}^{-q^+}\big\}\min\big\{\|u(t)\|_2^{q^-}, \|u(t)\|_2^{q^+}\big\}\\
&= \min\big\{S_{q(\cdot),2}^{-q^-}, S_{q(\cdot),2}^{-q^+}\big\}\min\big\{1, f^{\frac{q^+-q^-}{2}}(t)\big\}f^{\frac{q^-}{2}}(t)\\
&\geq \min\big\{S_{q(\cdot),2}^{-q^-}, S_{q(\cdot),2}^{-q^+}\big\}\min\big\{1, \|u_0\|_2^{q^+-q^-}\big\}f^{\frac{q^-}{2}}(t),
\end{aligned} \tag{6.3}$$

where $S_{q(\cdot),2}$ is defined in (4.4). It follows from (6.1) and (6.3) that

$$f'(t) \geq C_0 f^{\frac{q^-}{2}}(t), \tag{6.4}$$

where

$$C_0 = 2\Big(1-\frac{\max\{2,p^+\}}{q^-}\Big)\min\big\{S_{q(\cdot),2}^{-q^-}, S_{q(\cdot),2}^{-q^+}\big\}\min\big\{1, \|u_0\|_2^{q^+-q^-}\big\} > 0.$$

Since $f(t) > 0$, dividing the inequality (6.4) by $f^{\frac{q^-}{2}}(t)$, we obtain

$$f'(t)f^{-q^-/2}(t) \geq C_0.$$

Integrating the above inequality over $[0,t]$, one has

$$f^{1-\frac{q^-}{2}}(t) \leq f^{1-\frac{q^-}{2}}(0) - \big(\frac{q^-}{2}-1\big)C_0 t, \quad \text{for all } t \in [0, T_{\max}).$$

This and $f^{1-\frac{q^-}{2}}(t) > 0$ imply

$$t < \frac{2}{(q^- - 2)C_0}\|u_0\|_2^{2-q^-}, \quad \text{for all } t \in [0, T_{\max}).$$

Thus, we obtain

$$T_{\max} \leq \frac{2}{(q^- - 2)C_0}\|u_0\|_2^{2-q^-} = C\max\{\|u_0\|_2^{2-q^-}, \|u_0\|_2^{2-q^+}\},$$

where C is the constant given in (4.3).

Case 2: $0 < J(u_0) < d$ and $I(u_0) < 0$. By contradiction, we assume that $T_{\max} = \infty$. Thanks to $I(u_0) < 0$, by Lemma 4.4 we have $I(u(t)) < 0$. Then by Lemma 3.1 and 3.2, there exists $\lambda_* \in (0,1)$ such that

$$J(u(t)) - \frac{1}{q^-}I(u(t)) \geq \frac{d}{\max\big\{\lambda_*^2, \lambda_*^{p-}, \lambda_*^{q^+}\big\}} > d,$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_2^2 = -2I(u(t)) > 2q^-(d - J(u(t))) \geq 2q^-(d - J(u_0)).$$

Then we have

$$\|u(t)\|_2^2 = \|u_0\|_2^2 + \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\|u(s)\|_2^2\,\mathrm{d}s \geq \|u_0\|_2^2 + 2q^-(d - J(u_0))t.$$

From this and $J(u_0) < d$, we obtain $\lim_{t\to\infty} \|u(t)\|_2^2 = \infty$. Hence, we can choose sufficiently large $t_0 > 0$ such that

$$\|u(t_0)\|_2^2 > \frac{q^-}{q^- - 2}\|u_0\|_2^2.$$

Let

$$T = \frac{\int_0^{t_0} \|u(s)\|_2^2 \,\mathrm{d}s}{\big(\frac{q^-}{2} - 1\big)\big(\|u(t_0)\|_2^2 - \frac{q^-}{q^- -2}\|u_0\|_2^2\big)} + t_0 \ge t_0 > 0. \tag{6.5}$$

We now define the auxiliary function $F : [0,T] \to (0,\infty)$ by

$$F(t) = \int_0^t \|u(s)\|_2^2 \,\mathrm{d}s + (T-t)\|u_0\|_2^2. \tag{6.6}$$

Then

$$F'(t) = \|u(t)\|_2^2 - \|u_0\|_2^2 = 2\int_0^t \langle u'(s), u(s)\rangle \,\mathrm{d}s,$$

and

$$\begin{aligned} F''(t) = 2\langle u'(t), u(t)\rangle &= -2I(u(t)) \\ &> 2q^-(d - J(u(t))) \\ &= 2q^-(d - J(u_0)) + 2q^- \int_0^t \|u'(s)\|_2^2 \,\mathrm{d}s \\ &\ge 2q^- \int_0^t \|u'(s)\|_2^2 \,\mathrm{d}s. \end{aligned} \tag{6.7}$$

We deduce from (6.6) and (6.7) that

$$F(t)F''(t) \ge 2q^- \int_0^t \|u'(s)\|_2^2 \,\mathrm{d}s \int_0^t \|u(s)\|_2^2 \,\mathrm{d}s. \tag{6.8}$$

On the other hand, by Cauchy-Schwarz inequality, we have

$$\int_0^t \|u'(s)\|_2^2 \,\mathrm{d}s \int_0^t \|u(s)\|_2^2 \,\mathrm{d}s \ge \Big(\int_0^t \langle u'(s), u(s)\rangle \,\mathrm{d}s\Big)^2 = \frac{1}{4}(F'(t))^2. \tag{6.9}$$

Combining (6.8)–(6.9), we obtain

$$F(t)F''(t) \ge \frac{q^-}{2}(F'(t))^2, \quad \text{for all } t \in [0,T]. \tag{6.10}$$

Setting $G(t) = F^{1-\frac{q^-}{2}}(t)$, we obtain

$$G'(t) = \big(1 - \frac{q^-}{2}\big)\frac{F'(t)}{F^{\frac{q^-}{2}}(t)}, \quad G''(t) = \big(1 - \frac{q^-}{2}\big)\frac{F(t)F''(t) - \frac{q^-}{2}(F'(t))^2}{F^{1+\frac{q^-}{2}}(t)}.$$

Then we have $G''(t) \le 0$, for all $t \in [0,T]$ due to (6.10). Thus, $G(t)$ is concave on $[0,T]$. This implies that

$$G(t) \le G(t_0) + G'(t_0)(t - t_0), \quad \text{for all } t \in [0,T].$$

Replacing t by T in the above inequality and notice that (6.5), we obtain

$$\begin{aligned} G(T) &\le G(t_0) + G'(t_0)(T - t_0) \\ &= F^{-\frac{q^-}{2}}(t_0)[F(t_0) - \big(\frac{q^-}{2} - 1\big)(T - t_0)F'(t_0)] = 0. \end{aligned}$$

This contradicts $G(T) > 0$, and the proof is complete.

7. Proof of Theorem 4.8

Assume that $u = u(t)$ is a global weak solution to (1.1). Then by (i) in Theorem 4.6, we obtain $J(u(t)) \geq 0$ for all $t \geq 0$. Therefore,

$$\int_0^t \|u'(s)\|_2^2 \, \mathrm{d}s = J(u_0) - J(u(t)) \leq J(u_0).$$

Letting $t \to \infty$, one has

$$\int_0^\infty \|u'(s)\|_2^2 \, \mathrm{d}s \leq J(u_0) < \infty.$$

Therefore, there exists a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that

$$\lim_{n\to\infty} \|u'(t_n)\|_2 = 0, \tag{7.1}$$

which implies that $\|u'(t_n)\|_2 \leq A$ for all $n \in \mathbb{N}$, for some a constant A. Then

$$\begin{aligned} |I(u(t_n))| &= |\langle u'(t_n), u(t_n)\rangle| &(7.2)\\ &\leq \|u'(t_n)\|_2 \|u(t_n)\|_2 &(7.3)\\ &\leq \|u'(t_n)\|_2 S_2 \|\Delta u(t_n)\|_2 &(7.4)\\ &\leq AS_2 \|\Delta u(t_n)\|_2, &(7.5) \end{aligned}$$

where S_2 is the constant given in (3.16). Using the non-increasing property of $J(u(t))$, (7.5) and replacing u by $u(t_n)$ in (3.2), we obtain

$$\begin{aligned} J(u_0) &\geq J(u(t_n)) \\ &\geq \big(\frac{1}{2} - \frac{1}{q^-}\big)\|\Delta u(t_n)\|_2^2 + \big(\frac{1}{p^+} - \frac{1}{q^-}\big)\int_\Omega |\nabla u(t_n)|^{p(x)} \, \mathrm{d}x + \frac{1}{q^-} I(u(t_n)) \\ &\geq \big(\frac{1}{2} - \frac{1}{q^-}\big)\|\Delta u(t_n)\|_2^2 - \frac{AS_2}{q^-}\|\Delta u(t_n)\|_2, \end{aligned}$$

which implies that

$$\|\Delta u(t_n)\|_2 \leq \frac{AS_2 + \sqrt{A_2^2 S_2^2 + 2q^-(q^- - 2)J(u_0)}}{q^- - 2}. \tag{7.6}$$

The above inequality ensures that $\{u(t_n)\}$ is bounded in $H_0^2(\Omega)$. Then, since $H_0^2(\Omega)$ is reflexive, $H_0^2(\Omega) \hookrightarrow W_0^{1,p(\cdot)}(\Omega)$ and $H_0^2(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ are compact embeddings (by (1.2) and (1.3)), there exists a sub-sequence of $\{u(t_n)\}$, still denoted by $\{u(t_n)\}$ and a $\phi \in H_0^2(\Omega)$ such that

$$u(t_n) \rightharpoonup \phi \quad \text{weakly in } H_0^2(\Omega), \tag{7.7}$$

$$u(t_n) \to \phi \quad \text{strongly in } W_0^{1,p(\cdot)}(\Omega), \tag{7.8}$$

$$u(t_n) \to \phi \quad \text{strongly in } L^{q(\cdot)}(\Omega). \tag{7.9}$$

For any $v \in H_0^2(\Omega)$. Replacing u by $u(t_n)$ in the equation (1.1), by multiplying (1.1) by v and integrating by parts, we have

$$\begin{aligned} &|\langle \Delta u(t_n), \Delta v\rangle + \langle |\nabla u(t_n)|^{p(x)-2}\nabla u(t_n), \nabla v\rangle - \langle |u(t_n)|^{q(x)-2}u(t_n), v\rangle| \\ &= |\langle u'(t_n), v\rangle| \leq \|u'(t_n)\|_2 \|v\|_2. \end{aligned}$$

From this and (7.1), it follows that

$$\lim_{n\to\infty} \Big(\langle \Delta u(t_n), \Delta v\rangle + \langle |\nabla u(t_n)|^{p(x)-2}\nabla u(t_n), \nabla v\rangle - \langle |u(t_n)|^{q(x)-2}u(t_n), v\rangle\Big) = 0,$$

which, together with (7.7), (7.8) and (7.9) yields

$$\phi \in \mathcal{S}. \tag{7.10}$$

By (7.1), (7.4) and (7.6), we obtain

$$\lim_{n\to\infty} I(u(t_n)) = 0,$$

which, together with (7.8), (7.9) and (7.10), implies

$$\begin{aligned}\lim_{n\to\infty} \|\Delta u(t_n)\|_2 &= -\lim_{n\to\infty}\int_\Omega |\nabla u(t_n)|^{p(x)}\,\mathrm{d}x + \lim_{n\to\infty}\int_\Omega |u(t_n)|^{q(x)}\,\mathrm{d}x \\ &= -\int_\Omega |\nabla \phi|^{p(x)}\,\mathrm{d}x + \int_\Omega |\phi|^{q(x)}\,\mathrm{d}x = \|\Delta\phi\|_2.\end{aligned} \tag{7.11}$$

Note that $H_0^2(\Omega)$ is uniformly convex. Then by (7.7) and (7.11), we imply (see [1, Proposition 3.32])

$$u(t_n) \to \phi \quad \text{strongly in } H_0^2(\Omega).$$

The proof is complete.

8. Proof of Theorems 4.9 and 4.10

By borrowing the ideas from [9, 28] we can prove the Theorem 4.9 and 4.10 as follows.

Proof of Theorem 4.9. Assume that $u_0 \in \mathcal{N}_+$ and $\|u_0\|_2 \le \lambda_{J(u_0)}$. We first prove that $u(t) \in \mathcal{N}_+$ for all $t \in [0, T_{\max})$. Indeed, assume on the contrary that there is $t_0 > 0$ such that $u(t) \in \mathcal{N}_+$ for all $t \in [0, t_0)$ and $u(t_0) \in \mathcal{N}$. Then for all $t \in [0, t_0)$, we have

$$0 < |I(u(t))| = |\langle u'(t), u(t)\rangle| \le \|u'(t)\|_2 \|u(t)\|_2,$$

which gives $\|u'(t)\|_2 > 0$. From this and (4.2), we obtain $J(u(t_0)) < J(u_0)$, i.e., $u(t_0) \in J^{J(u_0)}$. So $\|u(t_0)\|_2 \ge \lambda_{J(u_0)}$. On the other hand, for all $t \in [0, t_0)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_2^2 = -2I(u(t)) < 0,$$

which implies that $\|u(t_0)\|_2 < \|u_0\|_2 \le \lambda_{J(u_0)}$. We thus arrive at a contradiction and therefore it proves the claim $u(t) \in \mathcal{N}_+$ for all $t \in [0, T_{\max})$. This gives $u(t) \in \mathcal{N}_+ \cap J^{J(u_0)}$ for all $t \in [0, T_{\max})$ by using the strictly decreasing property of $J(u(t))$. Then by (ii) in Lemma 3.5, $u(t)$ remains bounded in $H_0^2(\Omega)$ for all $t \in [0, T_{\max})$ so that $T_{\max} = \infty$, i.e., $u_0 \in \mathcal{G}$. We next prove $u_0 \in \mathcal{G}_0$. Let any $w \in \omega(u_0)$, we obtain

$$\|w\|_2 < \lambda_{J(u_0)} \quad \text{and} \quad J(w) < J(u_0),$$

which implies that $\omega(u_0) \cap \mathcal{N} = \emptyset$ by definition of $\lambda_{J(u_0)}$. And therefore, $\omega(u_0) = \{0\}$, i.e., $u_0 \in \mathcal{G}_0$.

Now we assume that $u_0 \in \mathcal{N}_-$ and $\|u_0\|_2 \ge \Lambda_{J(u_0)}$. By analogous arguments as above, we also have $u(t) \in \mathcal{N}_-$ for all $t \in [0, T_{\max})$. Assume on the contrary that $T_{\max} = \infty$, then for every $w \in \omega(u_0)$, one has

$$\|w\|_2 > \Lambda_{J(u_0)} \quad \text{and} \quad J(w) < J(u_0),$$

which gives $\omega(u_0) \cap \mathcal{N} = \emptyset$ by the definition of $\Lambda_{J(u_0)}$. However, since $\mathrm{dist}(0, \mathcal{N}_-) > 0$, we also have $0 \notin \omega(u_0)$. And hence $\omega(u_0) = \emptyset$, which contradicts $T_{\max} = \infty$. Thus $u_0 \in \mathcal{B}$. The proof is complete. $\square$

Proof of Theorem 4.10. Let any $u \in H_0^2(\Omega)\backslash\{0\}$. By using (3.2), we obtain

$$\begin{aligned} J(u) &\geq \big(\frac{1}{2}-\frac{1}{q^-}\big)\|\Delta u\|_2^2 + \big(\frac{1}{p^+}-\frac{1}{q^-}\big)\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x + \frac{1}{q^-}I(u) \\ &> \big(\frac{1}{2}-\frac{1}{q^-}\big)\|\Delta u\|_2^2 + \frac{1}{q^-}I(u) \\ &\geq \big(\frac{1}{2}-\frac{1}{q^-}\big)S_2^{-2}\|u\|_2^2 + \frac{1}{q^-}I(u). \end{aligned} \tag{8.1}$$

where S_2 is defined in (3.16). Replacing u by u_0 in (8.1) and using (4.5), we obtain

$$J(u_0) > \big(\frac{1}{2}-\frac{1}{q^-}\big)S_2^{-2}\|u_0\|_2^2 + \frac{1}{q^-}I(u_0) \geq J(u_0) + \frac{1}{q^-}I(u_0),$$

which gives $I(u_0) < 0$, i.e.,

$$u_0 \in \mathcal{N}_-. \tag{8.2}$$

For any $u \in \mathcal{N}_{J(u_0)}$, we have $I(u) = 0$ and $J(u) < J(u_0)$. Then by using (8.1), we obtain

$$\|u\|_2^2 \leq \frac{2q^- S_2^2}{q^- - 2} J(u_0),$$

which, together with (4.5), implies $\|u\|_2 \leq \|u_0\|_2$. Taking the supremum over $u \in \mathcal{N}_{J(u_0)}$, we obtain

$$\Lambda_{J(u_0)} \leq \|u_0\|_2. \tag{8.3}$$

Then by Theorem 4.9, it follows from (8.2) and (8.3) that $u_0 \in \mathcal{N}_- \cap \mathcal{B}$. The proof is complete. □

Acknowledgments. L. C. Nhan was supported by the Ho Chi Minh City University of Technology and Education, Vietnam. L. X. Truong was supported by the University of Economics Ho Chi Minh City, Vietnam.

References

[1] H. Brezis; *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer New York, (2010).

[2] J. W. Cahn, J. E. Hilliard; Free energy of a nonuniform system. I. Interfacial free energy, *J. Chem. Phys.*, **28** (1958), 258–267.

[3] J. A. Carrillo, G. Toscani; Long-time asymptotics for strong solutions of the thin film equation, *Comm. Math. Phys.*, **225** (2002), 551–571.

[4] E. Dibenedetto, R. E. Showalter; Implicit degenerate evolution equations and applications, *SIAM J. Math. Appl.*, **12**(5) (1981), 731–751.

[5] L. Diening, P. Harjulehto, P. Hästö, M. Růžička; *Lebesgue and Sobolev spaces with variable exponents*, Lecture Notes in Mathematics 2017, Springer-Verlag, Heidelberg, (2011).

[6] D. E. Edmunds, J. Rákosník; Sobolev embeddings with variable exponent, *Studia Math.*, **143** (2000), 267–293.

[7] D. E. Edmunds, J. Rákosník; Sobolev embeddings with variable exponent, *Math. Nachr.*, **246**(247) (2002), 53–67.

[8] X. Fan, J. Shen, D. Zhao; Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$, *J. Math. Anal. Appl.*, **262** (2001), 749–760.

[9] F. Gazzola, T. Weth; Finite time blow up and global solutions for semilinear parabolic equations with initial data at high energy level, *Diff. Int. Eq.*, **18** (2005), 961–990.

[10] Y. Han; A class of fourth-order parabolic equation with arbitrary initial energy, *Nonlinear Anal. RWA*, **43** (2018), 451–466.

[11] S. Kaplan; On the growth of the solutions of quasilinear parabolic equations, *Comm. Pure Appl. Math.*, **16** (1963), 305–330.

[12] B. B. King, O. Stein, M. Winkler; A fourth-order parabolic equation modeling epitaxial thin-film growth, *J. Math. Anal. Appl.*, **286** (2003), 459–490.

[13] O. Kováčik, J. Rákosník; On spaces $L^{p(x)}$ and $W^{k,p(x)}$, *Czechoslovak Math. J.*, **41** (1991), 592–618.
[14] B. Liang, S. N. Zheng; Existence and asymptotic behavior of solutions to nonlinear parabolic equations of fourth order, *J. Math. Anal. Appl.*, **348** (2008), 234–243.
[15] C. C. Liu; A fourth order parabolic equation with nonlinear principal part, *Nonlinear Anal.*, **68** (2008), 393–401.
[16] C. C. Liu; Regularity of solutions for a fourth order parabolic equation, *Bull. Belg. Math. Soc. Simon Stevin*, **13**(3) (2006), 527–535.
[17] Y. Liu, J. Zhao; On potential wells and applications to semilinear hyperbolic equations and parabolic equations, *Nonlinear Anal.*, **64**(12) (2006), 2665–2687.
[18] J. Musielak; *Orlicz spaces and modular spaces*, Lecture Notes in Mathematics, vol.1034. Springer-Verlag, Berlin, 1983.
[19] M. Ortiz, E. A. Repetto, H. Si; A continuum model of kinetic roughening and coarsening in thin films, *J. Mech. Phys. Solids*, **47**(4) (1999), 697–730.
[20] L. E. Payne, D. H. Sattinger; Saddle points and instability of nonlinear hyperbolic equations, *Israel J. Math.*, **22** (1975), 273–303.
[21] J. P. Pinasco; Blow-up for parabolic and hyperbolic problems with variable exponents, *Nonlinear Anal. Theory Methods Appl.*, **71**(3–4) (2009), 1094–1099.
[22] C. Qu, W. Zhou, B. Liang; Asymptotic behavior for a fourth-order parabolic equation modeling thin film growth, *Appl. Math. Lett.*, **78** (2018), 141–146.
[23] D. H. Sattinger; Stability of nonlinear hyperbolic equations, *Arch. Rational Mech. Anal.*, **28** (1968), 226–244.
[24] G. Schimperna; Global attractors for Cahn-Hilliard equations with nonconstant mobility, *Nonlinearity.*, **20** (2007), 2365–2387.
[25] T. P. Schulze, R. V. Kohn; A geometric model for coarsening during spiral-mode growth of thin films, *Physica D*, **132**(4) (1999), 520–542.
[26] Y. Su, B. Liu, Z. Feng; Ground state solution of the thin film epitaxy equation, *J. Math. Anal. Appl.*, **503**(2) (2021), 125357.
[27] X. Wu; The blow-up of solutions for m-Laplacian equations with variable sources under positive initial energy, *Comput. Math. Appl.*, **72** (2016), 2516–2524.
[28] R. Z. Xu, J. Su; Global existence and finite time blow-up for a class of semilinear pseudo-parabolic equations, *J. Funct. Anal.*, **264** (2013), 2732–2763.
[29] J. Yin; On the existence of nonnegative continuous solutions of the Cahn-Hilliard equation, *J. Differ. Equ.*, **97** (1992), 310–327.
[30] A. Zangwill; Some causes and a consequence of epitaxial roughening, *J. Cryst. Growth* **163**(1–2) (1996), 8–21.
[31] J. Zhou; Global asymptotical behavior of solutions to a class of fourth order parabolic equation modeling epitaxial growth, *Nonlinear Anal. RWA*, **48** (2019), 54–70.
[32] J. Zhou, D. Yang; Upper bound estimate for the blow-up time of an evolution m-Laplace equation involving variable source and positive initial energy, *Comput. Math. Appl.*, **69** (2015), 1463–1469.

Quach V. Chuong
Faculty of Mathematics and Computer Science, University of Science, Ho Chi Minh City, Vietnam.
Vietnam National University, Ho Chi Minh City, Vietnam.
Department of Mathematics, Dong Nai University, Bien Hoa City, Dong Nai Province, Vietnam
Email address: quachuong1812@dnpu.edu.vn

Le C. Nhan (corresponding author)
Faculty of Applied Sciences, Ho Chi Minh City University of Technology and Education (UTE), Ho Chi Minh City, Vietnam
Email address: nhanlc@hcmute.edu.vn

Le X. Truong
Department of Mathematics, Univerisity of Economics Ho Chi Minh city (UEH), Ho Chi Minh city, Vietnam
Email address: lxuantruong@ueh.edu.vn