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UNIFORM STABILIZATION AND EXACT CONTROLLABILITY FOR HYPERBOLIC SYSTEMS WITH DISCONTINUOUS COEFFICIENTS

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Abstract. This paper considers a hyperbolic system with discontinuous coefficients in a bounded, open, connected set with smooth boundary and controlled through the Robin boundary condition. Uniform stabilization of the solutions are established. Exact boundary controllability is obtained through the Russell's "Controllability via Stabilizability" principle.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with a smooth boundary S which consists of the disjoint closed surfaces S_0 and S_1 (the case $S_1 = \emptyset$ is not excluded). In the cylinder $\Omega\times]0, T[$ we consider the mixed problem

$$\partial_t^2 \mathbf{u}(\mathbf{x}, t) - \sum_{i=1}^{n} \partial_{x_i} [P(\mathbf{x})\partial_{x_i}\mathbf{u}(\mathbf{x}, t)] = 0 \quad (\mathbf{x}, t) \in \Omega\times]0, T[\tag{1.1}$$

$$\mathbf{u}(\mathbf{x}, 0) = f_1(\mathbf{x}), \quad \partial_t\mathbf{u}(\mathbf{x}, 0) = f_2(\mathbf{x}) \quad \mathbf{x} \in \Omega \tag{1.2}$$

$$P\partial_\nu\mathbf{u}(\mathbf{x}, t) + a\mathbf{u}(\mathbf{x}, t) + b\partial_t\mathbf{u}(\mathbf{x}, t) = 0 \quad (\mathbf{x}, t) \in \Sigma_0 = S_0\times]0, T[, \tag{1.3}$$

$$\mathbf{u}(\mathbf{x}, t) = 0 \quad (\mathbf{x}, t) \in \Sigma_1 = S_1\times]0, T[\tag{1.4}$$

Here $\mathbf{u} = (u^1(\mathbf{x}, t), \dots, u^m(\mathbf{x}, t))$, $\mathbf{x} = (x_1, \dots, x_n)$, $P(\mathbf{x}) = P^*(\mathbf{x})$ are square matrices of order m, $\nu = (\nu_1, \dots, \nu_n)$ is the unit outward normal to the boundary S, and a, b are positive constants.

Assume that

$$P(\mathbf{x})\xi \cdot \xi \geq c_0|\xi|^2, \quad c_0 > 0$$

where $\xi = (\xi^1, \dots, \xi^m)$ is an arbitrary vector.

Assume that $\Omega_0 \subset \Omega$ is a bounded domain with sufficiently smooth boundary Γ. We set $\Omega_1 = \Omega \setminus \overline{\Omega}_0$ and assume that the entries $a_{pq}(\mathbf{x})$ of the matrix $P(\mathbf{x})$ lose continuity on the surface Γ.

We shall use the notation

$$P(\mathbf{x}) = \begin{cases} A(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega_0, \\ B(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega_1. \end{cases} \quad \mathbf{u}(\mathbf{x}, t) = \begin{cases} w(\mathbf{x}, t) & \text{if } \mathbf{x} \in \Omega_0, \\ v(\mathbf{x}, t) & \text{if } \mathbf{x} \in \Omega_1. \end{cases}$$

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For simplicity we assume that A and B are constants matrices. We add to (1.1) the following interface conditions on Γ:

$$w\big|_{\Sigma} = v\big|_{\Sigma}, \quad A\partial_\nu w\big|_{\Sigma} = B\partial_\nu v\big|_{\Sigma} \quad \text{in } \Sigma = \Gamma\times]0,T[\tag{1.5}$$

where ν is the unit outward (with respect to Ω_0) normal to the surface Γ.

In one space dimension it is well known that stabilization holds for wave operators with piecewise smooth but possibly discontinuous coefficients (BV is the right class) regardless of the sign of the jump. Thus, one dimension is much better than several dimensions. The proof of this is based on simple sidewise energy estimates. See [1] and [19] and the references therein.

Our purpose is to prove the uniform stabilization of solutions to the problem (1.1)-(1.4) and (1.5). Using this result we obtain exact boundary controllability for the corresponding evolution system. Several approaches are known to solve the problem of exact boundary controllability. A systematic method (named HUM) was proposed by Lions [13] and [14].

In [7] we obtained exact controllability for the system (1.1)-(1.4) using HUM. The exact controllability for a system in elasticity theory is established by Lagnese, with method HUM in [11]. We obtain the same result for the class of systems $\partial_t^2 u - \partial_{x_i}(A_{ij}\partial_{x_i} u) = 0$ which includes the system in elasticty theory.

Here we use another approach which is based on D. Russell's "controllability via stabilizability" principle [16], which is different from of Lagnese's in [11]. Both techniques are well known.

There is an extensive number of publications on these topics. Exact controllability and uniform energy decay (boundary damping) are obtained for various equations and systems: the wave equation, the Schrödinger equation, Euler-Bernoulli beam equation, the system of elasticity, Maxwell's equation and others [2], [4]-[15], [18]. Although for equations with discontinuous coefficients very few results are known: Maxwell's equations in multilayered media [6], Euler-Bernoulli beam equation in the one-dimensional case [3].

2. Well-Posedness

Denote by $\mathcal{H}$ the Hilbert space of pairs $\{\mathbf{u}, \mathbf{u}_1\}$ of m-component vector-functions such that

$$\mathbf{u} \in H^1(\Omega_k), \quad \mathbf{u}_1 \in L^2(\Omega_k), \quad k = 0, 1, \quad \mathbf{u}\Big|_{S_1} = 0.$$

The scalar product in $\mathcal{H}$ is defined by the formula

$$\langle\{\mathbf{u}, \mathbf{u}_1\}, \{f, f_1\}\rangle = \int_{S_0} a\,\mathbf{u}\cdot f\,dS + \int_\Omega \Big(P\,\partial_{x_i}\mathbf{u}\cdot\partial_{x_i} f + \mathbf{u}_1\cdot f_1\Big)dx.$$

Define an unbounded operator $\mathcal{A}$ in $\mathcal{H}$ whose domain $D(\mathcal{A})$ consists of the elements $\{\mathbf{u}, \mathbf{u}_1\} \in \mathcal{H}$ such that $\mathbf{u} \in H^2(\Omega_k)$, $\mathbf{u}_1 \in H^1(\Omega_k)$, $k = 0, 1$,

$$P\partial_\nu \mathbf{u}(\mathbf{x}, t) + a\mathbf{u} + b\mathbf{u}_1\Big|_{S_0} = 0, \quad \mathbf{u}_1\Big|_{S_1} = 0, \quad \mathbf{u}\Big|_{S_1} = 0 \tag{2.1}$$

$$\mathbf{u}^0\Big|_{\Gamma} = \mathbf{u}^1\Big|_{\Gamma}, \quad \mathbf{u}_1^0\Big|_{\Gamma} = \mathbf{u}_1^1\Big|_{\Gamma}, \quad A\partial_\nu\mathbf{u}^0\Big|_{\Gamma} = B\partial_\nu\mathbf{u}^1\Big|_{\Gamma}, \tag{2.2}$$

where $\mathbf{u}^k$, $\mathbf{u}_1^k$ are the restrictions of the functions $\mathbf{u}$, $\mathbf{u}_1$ on Ω_k. For $\{\mathbf{u}, \mathbf{u}_1\} \in D(\mathcal{A})$ we set

$$\mathcal{A}\{\mathbf{u}, \mathbf{u}_1\} = \{\mathbf{u}_1,\ \partial_{x_i}(P\partial_{x_i}\mathbf{u})\}.$$

In a standard way we construct the adjoint operator $\mathcal{A}^*$. The domain of the operator $\mathcal{A}^*$ consists of elements $\{\mathbf{v},\mathbf{v}_1\}\in\mathcal{H}$ such that $\mathbf{v}\in H^2(\Omega_k)$, $\mathbf{v}_1\in H^1(\Omega_k)$, $k=0,1$,

$$P\partial_\nu \mathbf{v}(\mathbf{x},t)+a\mathbf{v}-b\mathbf{v}_1\Big|_{S_0}=0,\quad \mathbf{v}_1\Big|_{S_1}=0,\quad \mathbf{v}\Big|_{S_1}=0$$
$$\mathbf{v}^0\Big|_\Gamma=\mathbf{v}^1\Big|_\Gamma,\quad \mathbf{v}_1^0\Big|_\Gamma=\mathbf{v}_1^1\Big|_\Gamma,\quad A\,\partial_\nu\mathbf{v}^0\Big|_\Gamma=B\,\partial_\nu\mathbf{v}^1\Big|_\Gamma,$$

where $\mathbf{v}^k$, $\mathbf{v}_1^k$ are the restrictions of the functions $\mathbf{v}$, $\mathbf{v}_1$ on Ω_k.

For $\{\mathbf{v},\mathbf{v}_1\}\in D(\mathcal{A}^*)$ we set

$$\mathcal{A}^*\{\mathbf{v},\mathbf{v}_1\}=-\left\{\mathbf{v}_1,\ \partial_{x_i}\left(P\partial_{x_i}\mathbf{v}\right)\right\}.$$

It can be shown that $\mathcal{A}$ and $\mathcal{A}^*$ are dissipative operators in $\mathcal{H}$; i.e.,

$$\langle\mathcal{A}\{\mathbf{u},\mathbf{u}_1\},\{\mathbf{u},\mathbf{u}_1\}\rangle\le 0\quad \{\mathbf{u},\mathbf{u}_1\}\in D(\mathcal{A})$$
$$\langle\mathcal{A}^*\{\mathbf{v},\mathbf{v}_1\},\{\mathbf{v},\mathbf{v}_1\}\rangle\le 0\quad \{\mathbf{v},\mathbf{v}_1\}\in D(\mathcal{A}^*).$$

Assume that $\{\mathbf{u},\mathbf{u}_1\}\in D(\mathcal{A})$. Then

$$\frac{d}{dt}\langle\mathcal{A}\{\mathbf{u},\mathbf{u}_1\},\{\mathbf{u},\mathbf{u}_1\}\rangle=-\int_{S_0}b|\mathbf{u}_1|^2d\,S\le 0.$$

Similarly,

$$\frac{d}{dt}\langle\mathcal{A}^*\{\mathbf{v},\mathbf{v}_1\},\{\mathbf{v},\mathbf{v}_1\}\rangle=-\int_{S_0}b|\mathbf{v}_1|^2d\,S\le 0,\quad \{\mathbf{v},\mathbf{v}_1\}\in D(\mathcal{A}^*).$$

Indeed, if $\{\mathbf{u},\mathbf{u}_1\}\in D(\mathcal{A})$, then

$$\begin{aligned}
&\langle\mathcal{A}\{\mathbf{u},\mathbf{u}_1\},\{\mathbf{u},\mathbf{u}_1\}\rangle\\
&=\int_{S_0}a\mathbf{u}_1\cdot\mathbf{u}\,dS+\int_{\Omega_0}\left(A\frac{\partial\mathbf{u}_1^0}{\partial x_i}\cdot\frac{\partial\mathbf{u}^0}{\partial x_i}+\frac{\partial}{\partial x_i}\Big(A\frac{\partial\mathbf{u}^0}{\partial x_i}\Big)\cdot\mathbf{u}_1^0\right)dx\\
&\quad+\int_{\Omega_1}\left(B\frac{\partial\mathbf{u}_1^1}{\partial x_i}\cdot\frac{\partial\mathbf{u}^1}{\partial x_i}+\frac{\partial}{\partial x_i}\Big(B\frac{\partial\mathbf{u}^1}{\partial x_i}\Big)\cdot\mathbf{u}_1^1\right)dx\\
&=\int_{S_0}a\mathbf{u}_1\cdot\mathbf{u}\,dS+\int_{\Omega_0}\left(A\frac{\partial\mathbf{u}_1^0}{\partial x_i}\cdot\frac{\partial\mathbf{u}^0}{\partial x_i}-\frac{\partial\mathbf{u}^0}{\partial x_i}A\frac{\partial\mathbf{u}_1^0}{\partial x_i}\right)dx+\int_\Gamma A\frac{\partial\mathbf{u}^0}{\partial\nu}\mathbf{u}_1^0dS\\
&\quad+\int_{\Omega_1}\left(B\frac{\partial\mathbf{u}_1^1}{\partial x_i}\cdot\frac{\partial\mathbf{u}^1}{\partial x_i}-\frac{\partial\mathbf{u}^1}{\partial x_i}B\frac{\partial\mathbf{u}_1^1}{\partial x_i}\right)dx-\int_\Gamma B\frac{\partial\mathbf{u}^1}{\partial\nu}\mathbf{u}_1^1dS+\int_S B\frac{\partial\mathbf{u}^1}{\partial\nu}\mathbf{u}_1^1dS\\
&=\int_\Gamma\left(A\frac{\partial\mathbf{u}^0}{\partial\nu}\mathbf{u}_1^0-B\frac{\partial\mathbf{u}^1}{\partial\nu}\mathbf{u}_1^1\right)dS+\int_{S_0}a\mathbf{u}_1\cdot\mathbf{u}\,dS+\int_{S_0}B\frac{\partial\mathbf{u}^1}{\partial\nu}\mathbf{u}_1^1dS\\
&=\int_{S_0}a\mathbf{u}_1\cdot\mathbf{u}\,dS+\int_{S_0}P\frac{\partial\mathbf{u}}{\partial\nu}\cdot\mathbf{u}_1\,dS\\
&=\int_{S_0}[a\mathbf{u}_1\cdot\mathbf{u}+(-a\mathbf{u}-b\mathbf{u}_1)\mathbf{u}_1]\,dS=\int_{S_0}b|\mathbf{u}_1|^2\,dS\le 0.
\end{aligned}$$

It can be shown in a similar way that $\mathcal{A}^*$ is dissipative.

Thus, $\mathcal{A}$ generates a C_0-semigroup of contractions $U(t)\colon\mathcal{H}\to\mathcal{H}$, $t>0$ where

$$U(t)\{f_1,f_2\}\in C([0,\infty);D(\mathcal{A}))\cup\ C^1([0,\infty);\mathcal{H})$$

when $\{f_1,f_2\}\in D(\mathcal{A})$ and $U(t)\{f_1,f_2\}$ is strongly differentiable with respect to t for $\{f_1,f_2\}\in D(\mathcal{A})$. Moreover,

$$\frac{d}{dt}U(t)\{f_1,f_2\}=\mathcal{A}U(t)\{f_1,f_2\}$$

and $U(t)$ carries $D(\mathcal{A})$ onto $D(\mathcal{A})$ and commutes with $\mathcal{A}$.

Let $\{f_1, f_2\} \in D(\mathcal{A})$ and $\{\mathbf{u}, \mathbf{u}_1\} = U(t)\{f_1, f_2\}$. Then $\mathbf{u} = \mathbf{u}_1$ and $\mathbf{u}_{1t} = \sum \partial_{x_i}(P\partial_{x_i}\mathbf{u})$; i.e, the first component of $U(t)\{f_1, f_2\}$ is a solution to the problem (1.1), (1.5).

Observe that, for $F = \{f_1, f_2\} \in \mathcal{H}$, $U(t)F$ is a weak solution in $\mathcal{H}$ to the abstract Cauchy problem

$$\frac{d}{dt}\{\mathbf{u}, \mathbf{u}_1\} = \{\mathbf{u}_1,\ \partial_{x_i}(P\partial_{x_i}\mathbf{u})\} = \{\mathbf{u}_1, \mathcal{P}\mathbf{u}\}$$

in the following sense

$$\int_0^T \Big(\langle U(t)F, \frac{d\phi}{dt}\rangle + \langle U(t)F, \mathcal{A}^*\phi\rangle\Big) dt = -\langle F, \phi(0)\rangle$$

for every $\phi \in L^2(0,T; D(\mathcal{A}^*))$, $\phi_t \in L^2(0,T;\mathcal{H})$, $\phi(T) = 0$.

3. Stabilization

We start from geometrical conditions on Ω. We consider the problem:

$$\Delta\Psi = \frac{a_0}{c_0}, \quad x \in \Omega, \quad \partial_\nu\Psi\big|_{S_0} = \frac{a_0\,\mathrm{meas}(\Omega)}{c_0\,\mathrm{meas}(S_0)}, \quad \partial_\nu\Psi\big|_{S_1} = 0, \tag{3.1}$$

where $\Psi(x) \in C^2(\Omega) \cup C^1(\overline{\Omega})$ be a solution to the problem, $a_0 = \max|a_{pq}|$, a_{pq} are the entries of the matrix P, and c_0 is a constant defined as above (observe that for the wave operator $P = I$ and $c_0 = a_0 = 1$).

For an arbitrary bounded domain Ω with smooth boundary S we define the quantity

$$\kappa = \max_{i,j}\sup_{x\in\overline{\Omega}}|\partial^2_{x_ix_j}\Psi(x)|.$$

Suppose that Ω satisfies the conditions: There is a point $x^0 \in \mathbb{R}^n$ such that

(a) S_1 is star-like with respect to x^0: $(x - x^0, \nu) \le 0$ for $x \in S_1$;
(b) for some $0 < \epsilon \le 1$

$$(x - x^0, \nu) > -\frac{1}{\epsilon + n\kappa}\frac{\mathrm{meas}(\Omega)}{\mathrm{meas}(S)}, \quad x \in S_0.$$

Clearly, (b) holds if S_0 is star-like with respect to the point x^0 when it be taken sufficiently close to the domain, see Figure 1.

Theorem 3.1. *Let a domain Ω and surface Γ satisfy the above-listed conditions with a parameter $0 < \epsilon \le 1$ and let the coefficient a in the boundary conditions satisfy $0 < a < \delta c_0 n^2\kappa/(3r)$, where $r = \sup_{x\in\overline{\Omega}}|\nabla\varphi|$. Suppose that $AB = BA$ and matrix $A - B$ is nonnegative. Then there are $T^* > 0$ and $C^* > 0$ such that for $t > T^*$*

$$\|U(t)\{f_1, f_2\}\|^2 \le C^*(T^*)^{\epsilon-1}\frac{1}{t^\epsilon}\|\{f_1, f_2\}\|^2$$

for every $\{f_1, f_2\} \in \mathcal{H}$.

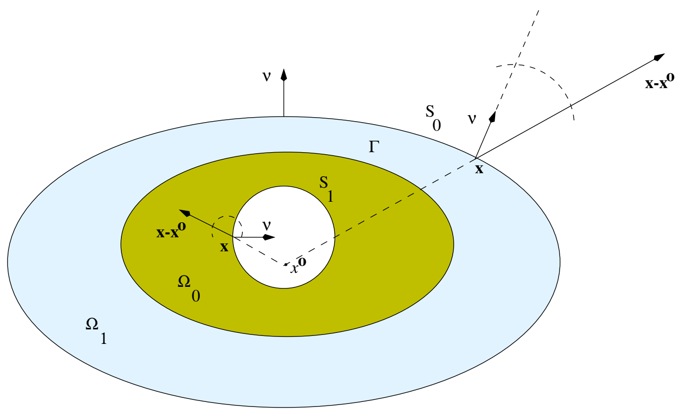

FIGURE 1. Surface $\mathbf{S}_0$ is starlike with respect to the point $\mathbf{x}^o$

Proof. The following identity is proved in the Appendix:

$$
\begin{aligned}
&2\Big[t\partial_t u + (\nabla\varphi, \nabla)u + \frac{n-1}{2}u\Big]\cdot\Big[\partial_t^2 u - \partial_{x_i}\left(P\partial_{x_i}u\right)\Big]\\
&= \partial_t\Big[t\Big(|\partial_t u|^2 + \sum_{i=1}^n P\partial_{x_i}u\cdot\partial_{x_i}u\Big) + 2(\nabla\varphi,\nabla)u\cdot\partial_t u + (n-1)u\cdot\partial_t u\Big]\\
&\quad - \partial_{x_i}\Big[P\partial_{x_i}u\cdot\Big(2t\partial_t u + 2(\nabla\varphi,\nabla)u + (n-1)u\Big)\\
&\quad + \partial_{x_i}\varphi\Big(|\partial_t u|^2 - \sum_{i=1}^n P\partial_{x_i}u\cdot\partial_{x_i}u\Big)\Big]\\
&\quad - \Big[(\Delta\varphi - n + 2)P\partial_{x_i}u\cdot\partial_{x_i}u - (\Delta\varphi - n)|\partial_t u|^2 - 2\partial^2_{x_p x_i}\varphi\,\partial_{x_p}u\cdot P\partial_{x_i}u\Big].
\end{aligned} \tag{3.2}
$$

For $\varphi = 2^{-1}|x - x^o|$, it represents a conservation law, a consequence of invariance of the system relative to the one-parameter group of dilations in all variables with the infinitesimal operator

$$
t\partial_t + (x_i - x_i^o)\partial_{x_i} - \frac{n-1}{2}u^j\partial_{u^j}.
$$

Let $\{f_1, f_2\} \in D(\mathcal{A})$ and $\mathbf{u}(\mathbf{x}, t)$ be a solution of (1.1), (1.5). After integration by parts over $\Omega_0\times]0, T[$ and $\Omega_1\times]0, T[$ using (1.5), we obtain the formula

$$
\begin{aligned}
&-\int_\Omega u\partial_t u\,dx\Big|_{t=T_0}^{t=T} - \int_{S_0}\frac{1}{2}b|u|^2\,dS\Big|_{t=T_0}^{t=T}\\
&= \int_{T_0}^T\int_\Omega\Big(\sum_{i=1}^n P\partial_{x_i}u\cdot\partial_{x_i}u - |\partial_t u|^2\Big)dx\,dt + \int_{T_0}^T\int_{S_0} a|u|^2\,dSdt.
\end{aligned} \tag{3.3}
$$

An application of (3.2), together with (3.3) multiplied by the constant γ, leads to the formula

$$
\begin{aligned}
&\Big\{ \int_\Omega [t\mathcal{I}(u) + 2(\nabla\varphi, \nabla)u \cdot \partial_t u + (n-1-\gamma)u \cdot \partial_t u]\, dx \\
&+ \int_{S_0} \Big[ta|u|^2 + \frac{n-1-\gamma}{2} b|u|^2 \Big] dS \Big\}\Big|_{t=T_0}^{t=T} \\
&= \int_{T_0}^T \int_\Omega \Big[(\Delta\varphi - n + 2 + \gamma)\Phi(u) - (\Delta\varphi - n + \gamma)|\partial_t u|^2 \\
&- 2\partial^2_{x_p x_i}\varphi P \partial_{x_i} u \partial_{x_p} u \Big] dxdt + \int_{T_0}^T \int_{S_1} \partial_\nu \varphi\ \Phi(u)\, dS\, dt \\
&+ \int_{T_0}^T \int_{S_0} \Big\{ \partial_\nu\varphi \left(|\partial_t u|^2 - \Phi(u) \right) - 2tb|\partial_t u|^2 - (n-2-\gamma)a|u|^2 \\
&- 2b(\nabla\varphi, \nabla)u \cdot \partial_t u - 2a(\nabla\varphi, \nabla)u \cdot u \Big\} dS\, dt \\
&+ \int_{T_0}^T \int_\Gamma \Big\{ -\partial_\nu\varphi\,(A-B)\,\partial_{x_i} w \cdot \partial_{x_i} w - \partial_\nu \varphi (AB^{-1}A \\
&+ B - 2A)\partial_\nu w \cdot \partial_\nu w \Big\} d\Gamma\, dt,
\end{aligned}
\tag{3.4}
$$

here we use the notation:

$$
\Phi(u) = \sum_{i=1}^n P\partial_{x_i} u \cdot \partial_{x_i} u, \quad \mathcal{I}(u) = |\partial_t u|^2 + \Phi(u).
$$

Choose the function $\varphi(x)$ in (3.4) as follows

$$
\varphi(x) = \frac{c_0}{a_0}\Psi(x) + \frac{1}{2\theta}|x - x^0|^2, \quad \theta > 0, \quad x^0 \in \mathbb{R}^n
$$

We obtain

$$
\begin{aligned}
\mathcal{K} &\equiv (\Delta\varphi - n + 2 + \gamma)\Phi(u) - (\Delta\varphi - n + \gamma)|\partial_t u|^2 - 2\partial^2_{x_p x_i}\varphi P\partial_{x_i} u \cdot \partial_{x_p} u \\
&\le \Big(n - 1 - \frac{n}{\theta} - \gamma \Big)\, |\partial_t u|^2 + \Big(3 + \frac{n-2}{\theta} + 2\kappa n + \gamma - n \Big) \Phi(u).
\end{aligned}
$$

Set $\theta = (\epsilon + n\kappa)^{-1}$ and $\gamma = n - 2 + \epsilon - n(\epsilon + \kappa n)$. Then $\mathcal{K} \le (1-\epsilon)\mathcal{I}(u)$, and for $x \in S_0$ we have

$$
\partial_\nu\, \varphi = (\epsilon + n\kappa) \left[(x - x^0, \nu) + \frac{1}{(\epsilon + n\kappa)} \frac{\text{meas}(\Omega)}{\text{meas}(S_0)} \right] > 0,
$$

which by compactness of S_0 leads to the inequality

$$
\partial_\nu\, \varphi \ge |\nabla\varphi|\delta, \quad \delta > 0.
$$

We now assume that the surface Γ satisfies the condition

$$
\partial_\nu\ \varphi\big|_\Gamma \ge 0.
$$

Note that if S_0 is strictly star-shaped with respect to $x^0 \in \mathbb{R}^n$; i.e,

$$
(x - x^0,\, \nu) > 0
$$

we can choose $\varphi(\mathbf{x}) = \frac{1}{2}|\mathbf{x} - \mathbf{x}^0|^2$. In this case, Γ is an arbitrary star-shaped surface with respect to $\mathbf{x}^0$.

Moreover, we assume that matrices A and B are constant and

$$AB = BA, \quad (A - B)\xi \cdot \xi \geq 0, \quad \forall \xi \in \mathbb{R}^m,$$

for examples on the last condition, see [11] and [14]. For examples where monotonicity fails, so that the uniform decay does not hold, see [17].

Then we obtain that integral over $\Gamma \times]T_0, T[$ in (3.4) is non positive. Denote by $\mathcal{G}$ the integrand of the integral over $S_0 \times]T_0, T[$ on the right side of the formula (3.4). We have the following estimate for $\mathcal{G}$:

$$\mathcal{G} \leq |u|^2 \Big[n^2 \kappa a - |\nabla \varphi| \frac{3a^2}{\delta c_0} \Big] - |u_t|^2 \Big[2tb - |\nabla \varphi| - |\nabla \varphi| \frac{3b^2}{\delta c_0} \Big]. \tag{3.5}$$

By hypotheses we have

$$0 < a < \frac{\delta c_0 n^2 \kappa}{3r}, \tag{3.6}$$

where $r = \sup_{x \in \overline{\Omega}} |\nabla \varphi|$.

Choose T_1 so large that for $t \geq T_1$ the last term in (3.5) is non positive. Since $(\nabla \varphi, \nu) \leq 0$ for $x \in S_1$, the surface integrals on the right-hand side of (3.4) are nonnegative as $T_0 \geq T_1$. Thus, we obtain that for $T_0 \geq T_1$:

$$\begin{aligned} &\Big\{ \int_\Omega \Big[t\mathcal{I}(u) + 2(\nabla \varphi, \nabla) u \cdot \partial_t u + (n - 1 - \gamma) u \cdot \partial_t u \Big] dx \\ &+ \int_{S_0} \Big[ta|u|^2 + \frac{n-1-\gamma}{2} b|u|^2 \Big] dS \Big\} \Big|_{t=T_0}^{t=T} \\ &\leq (1 - \epsilon) \int_{T_0}^T \int_\Omega \mathcal{I}(u)\, dx dt. \end{aligned} \tag{3.7}$$

here $\gamma = n + \epsilon - 2 - n(\epsilon + \kappa n)$. Denote by τ_0 the smallest constant for which the following inequality holds

$$\int_\Omega \big(|u|^2 + |\nabla u|^2 \big)\, dx \leq \tau_0 \Big(\int_\Omega P \partial_{x_i} u \cdot \partial_{x_i} u \, dx + \int_{S_0} a|u|^2 \, dS \Big), \quad u \in H^1(\Omega).$$

We have

$$\int_\Omega \big[2(\nabla \varphi, \nabla) u \cdot \partial_t u + (n - 1 - \gamma) u \cdot \partial_t u \big] dx \leq C_0 \|U(T_0)\{f_1, f_2\}\|^2, \quad t \geq T_0$$

Combining this estimate with (3.7), we arrive at the inequality

$$T\|U(T)\{f_1, f_2\}\|^2 - (T_0 + C_1)\|U(T_0)\{f_1, f_2\}\|^2 \leq (1 - \epsilon) \int_{T_0}^T \|U(t)\{f_1, f_2\}\|^2 dt \tag{3.8}$$

in which $T_0 \geq T_1$. With the help of Gronwall's inequality, and (3.8) we obtain

$$t\|U(t)\{f_1, f_2\}\|^2 \leq C_2 \big(\frac{t}{T_2} \big)^{1-\epsilon} \|U(T_2)\{f_1, f_2\}\|^2$$

for $t > T_2$.

Given an arbitrary element $\{f_1, f_2\} \in \mathcal{H}$, approximate it by smooth data for which the inequality of the theorem was established above. Taking the limit finishes the proof. ☐

Corollary 3.2. *The operator $U(t)$ takes $\mathcal{H}$ into itself and*

$$\|U(t)\| < 1 \quad \text{for } t > t^* = (C^*(T^*)^{\epsilon - 1})^{1/\epsilon}.$$

By applying semigroup properties, we obtain the following result.

Corollary 3.3. *Assume* $\{f_1, f_2\} \in \mathcal{H}$. *There are* $C,\ \beta > 0$ *such that*
$$\|U(t)\{f_1, f_2\}\|^2 \le C\exp(-\beta t)\|\{f_1, f_2\}\|^2.$$

4. Exact Controllability

In this section, we shall use the estimate of the Theorem 3.1 to prove exact controllability of the evolution system studied in the previous sections. In $\Omega\times]0, T[$ we consider the problem

$$\partial_t^2 \mathbf{u}(\mathbf{x}, t) - \sum_{i=1}^{n} \partial_{x_i}\left[P(\mathbf{x})\partial_{x_i}\mathbf{u}(\mathbf{x}, t)\right] = 0 \quad (\mathbf{x}, t) \in \Omega\times]0, T[\tag{4.1}$$
$$\mathbf{u}(\mathbf{x}, 0) = f_1(\mathbf{x}), \quad \partial_t\mathbf{u}(\mathbf{x}, 0) = f_2(\mathbf{x}) \quad \mathbf{x} \in \Omega \tag{4.2}$$
$$P\partial_\nu\mathbf{u}(\mathbf{x}, t) + a\mathbf{u}(\mathbf{x}, t) = {\scriptstyle ||}(\mathbf{x}, t) \quad (\mathbf{x}, t) \in \Sigma_0 = S_0\times]0, T[, \tag{4.3}$$
$$\mathbf{u}(\mathbf{x}, t) = 0 \quad (\mathbf{x}, t) \in \Sigma_1 = S_1\times]0, T[\tag{4.4}$$
$$\mathbf{w} = \mathbf{v}, \quad A\partial_\nu\mathbf{w} = B\partial_\nu\mathbf{v}, \quad (\mathbf{x}, t) \in \Sigma = \Gamma\times]0, T[\tag{4.5}$$

where $A(B)$ and $\mathbf{w}(\mathbf{v})$ are the restrictions of matrix P and vector-function $\mathbf{u}$ on Ω_0 $(\Omega_1), \mathbf{f} = \{f_1, f_2\}$ is an arbitrary element of the space $\mathcal{H}$.

For every $\mathbf{g} = \{g_1, g_2\} \in \mathcal{H}$, we have to find a vector-function $\mathbf{q}(x, t)$ such that the solution of (4.1) satisfies the conditions
$$\mathbf{u}\Big|_{t=T} = g_1(x), \quad \partial_t\mathbf{u}\Big|_{t=T} = g_2(x), \quad \text{for } T > t^*.$$

Theorem 4.1. *Let the coefficient a in the boundary conditions of problem* (4.1) *satisfies* (3.6). *There is a $t^* > 0$ such that, for $T > t^*$, arbitrary initial data $\mathbf{f} = \{f_1, f_2\} \in \mathcal{H}$, and any element $\mathbf{g} = \{g_1, g_2\} \in \mathcal{H}$, there exists a boundary control $\mathbf{q}(x, t) \in L^2(S_0\times]0, T[)$ transferring a solution of* (4.1) *to the state $\mathbf{g} = \{g_1, g_2\}$ at time T. Moreover,*
$$\|{\scriptstyle ||}\|^2_{L^2(\Gamma_0\times]0,T[)} \le C(\|\mho\|^2 + \|\eth\|^2).$$

Proof. Let $U(t)$ be the semigroup defined above and let $U^*(t)$ be semigroup constructed from the operator $\mathcal{A}^*$. Consider the following equation in $\mathcal{H}$:
$$\{h, h_1\} - U^*(T)U(T)\{h, h_1\} = f - U^*(T)g.$$
The operator $G(T) = U^*(T)U(T)$ takes $\mathcal{H}$ into itself and $\|G(T)\| < 1$ for $T > t^*$. Thus we can solve this equation for any $\mathbf{f},\ \mathbf{g} \in \mathcal{H}$ and
$$\|\mathbf{h}\| = \|\{h, h_1\}\| \le C(\|\mathbf{f}\| + \|\mathbf{g}\|).$$
Consequently, if we choose $\mathbf{h} = (I - G(T))^{-1}(\mathbf{f} - U^*(T)\mathbf{g})$, then
$$\{\alpha, \alpha_1\} = U(t)\mathbf{h} \quad \textit{and} \quad \{\beta, \beta_1\} = U^*(T - t)(U(T)\mathbf{h} - \mathbf{g})$$
are weak solutions to the problems
$$\frac{d}{dt}\{\alpha, \alpha_1\} = \{\alpha_1, \mathcal{P}\alpha\}$$
$$P\partial_\nu\alpha + a\alpha + b\alpha_1\Big|_{S_0} = 0, \quad \alpha\Big|_{S_1} = 0,$$
and
$$\frac{d}{dt}\{\beta, \beta_1\} = \{\beta_1, \mathcal{P}\beta\}$$

$$P\partial_\nu\beta + a\,\beta - b\,\beta_1\Big|_{S_0} = 0, \quad \beta\Big|_{S_1} = 0\,.$$

By the energy identity, the following estimates hold

$$\int_0^T \int_{S_0} b|\alpha_1|^2\,dS\,dt \le C\|\mathbf{h}\|^2, \quad \int_0^T \int_{S_0} b|\beta_1|^2 dS\,dt \le C(\|\mathbf{h}\|^2 + \|\mathbf{g}\|^2).$$

Clearly, $\{u, u_1\} = \{\alpha, \alpha_1\} - \{\beta, \beta_1\}$ is a solution to problem (4.1) with boundary data on S_0:

$$\mathbf{q}(x,t) = -b(\alpha_1 + \beta_1),$$

which belongs to $L^2(S_0\times]0,T[)$ and

$$\|\mathbf{q}\|^2_{L^2(S_0\times(0,T))} \le C(\|\mathbf{f}\|^2 + \|\mathbf{g}\|^2).$$

$\square$

Remark 4.2. *We can study in the same way the more general case. Assume that $B_k \subset \Omega$ is a bounded domain with boundary $\Gamma_k, \overline{B}_k \subset B_{k+1}$ for $k = 1,\dots,n$.*

Assume that $\Gamma_1,\dots,\Gamma_n$ and S_0, S_1 are star-shaped with respect to the point $x^0 \in \mathbb{R}^n$. Suppose that matrix $P(x)$ lose the continuity on $\Gamma_1,\dots,\Gamma_n$. We set

$$\Omega_0 = B_1, \quad \Omega_k = B_{k+1} \setminus \overline{B}_k, \quad k = 1,\dots,n-1, \quad \Omega_n = \Omega \setminus \overline{B}_n$$

The interface conditions are

$$\mathbf{u}^{k-1}\Big|_{\Gamma_k\times]0,T[} = \mathbf{u}^k\Big|_{\Gamma_k\times]0,T[}$$

$$P^{k-1}\partial_\nu\mathbf{u}^{k-1}\Big|_{\Gamma_k\times]0,T[} = P^k\partial_\nu\mathbf{u}^k\Big|_{\Gamma_k\times]0,T[}, \quad k = 1,\dots,n$$

where $\nu = \nu(\mathbf{x})$ (for $x \in \Gamma_k$) is the unit normal vector pointing into the exterior of B_k; P^k, $\mathbf{u}^k$ are the restrictions of P and $\mathbf{u}$ on Ω_k.

5. Appendix

We shall show here the details in the proof of the identity used in Theorem 3.1. We use the following notation

$$\mathbf{u} = \big(\mathbf{u}^1,\dots,\mathbf{u}^m\big)\,, \quad \partial_t\mathbf{u} = \big(\partial_t\mathbf{u}^1,\dots,\partial_t\mathbf{u}^m\big)\,, \quad \nabla = (\partial_{x_i},\dots,\partial_{x_i})\,,$$
$$\partial_t^2\mathbf{u} = \big(\partial_t^2\mathbf{u}^1,\dots,\partial_t^2\mathbf{u}^m\big)\,, \quad \partial_{x_i}\mathbf{u} = \big(\partial_{x_i}\mathbf{u}^1,\dots,\partial_{x_i}\mathbf{u}^m\big)\,,$$

the matrix $P(\mathbf{x}) = \big(a_{pq}(\mathbf{x})\big)_{m\times m}$ and

$$(\nabla\varphi, \nabla)\,\mathbf{u} = \Big(\sum_{i=1}^n \partial_{x_i}\varphi\,\partial_{x_i}\mathbf{u}^q\Big)_{1\le q\le m}$$

The identity (3.2) can be verified by direct computations as follows

$$\begin{aligned}
&2\left(tu_t^q + \frac{\partial\varphi}{\partial x_i}\frac{\partial u^q}{\partial x_i} + \frac{n-1}{2}u^q\right)\left(u_{tt}^q - \frac{\partial}{\partial x_i}\Big(a_{pq}\frac{\partial u^q}{\partial x_i}\Big)\right)\\
&= 2tu_{tt}^q u_t^q - 2tu_t^p\frac{\partial}{\partial x_i}\Big(a_{pq}\frac{\partial u^q}{\partial x_i}\Big) + 2\frac{\partial\varphi}{\partial x_i}\frac{\partial u^q}{\partial x_i}u_{tt}^q - 2\frac{\partial\varphi}{\partial x_i}\frac{\partial u^p}{\partial x_i}\frac{\partial}{\partial x_i}\Big(a_{pq}\frac{\partial u^q}{\partial x_i}\Big)\\
&\quad + (n-1)u^q u_{tt}^q - (n-1)u^p\frac{\partial}{\partial x_i}\Big(a_{pq}\frac{\partial u^q}{\partial x_i}\Big)\\
&= t\frac{\partial|u_t^q|^2}{\partial t} - 2t\frac{\partial}{\partial x_i}\left(a_{pq}u_t^p\frac{\partial u^q}{\partial x_i}\right) + 2ta_{pq}\frac{\partial u_t^p}{\partial x_i}\frac{\partial u^q}{\partial x_i} + 2\frac{\partial\varphi}{\partial x_i}\frac{\partial u^q}{\partial x_i}u_{tt}^q
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\partial}{\partial x_i}\left(2\frac{\partial\varphi}{\partial x_i}a_{pq}\frac{\partial u^p}{\partial x_i}\frac{\partial u^q}{\partial x_i}\right) + \frac{\partial}{\partial x_i}\left(2\frac{\partial\varphi}{\partial x_i}\frac{\partial u^p}{\partial x_i}\right)a_{pq}\frac{\partial u^q}{\partial x_i} \\
&+ (n-1)\frac{\partial}{\partial t}(u^q u_t^q) - (n-1)|u_t^q|^2 - (n-1)\frac{\partial}{\partial x_i}\left(a_{pq}u^p\frac{\partial u^q}{\partial x_i}\right) \\
&+ (n-1)a_{pq}\frac{\partial u^p}{\partial x_i}\frac{\partial u^q}{\partial x_i} \\
&= t\frac{\partial |u_t^q|^2}{\partial t} - 2t\frac{\partial}{\partial x_i}\left(a_{pq}u_t^p\frac{\partial u^q}{\partial x_i}\right) + 2ta_{pq}\frac{\partial u_t^p}{\partial x_i}\frac{\partial u^q}{\partial x_i} + \frac{\partial}{\partial t}\left(2\frac{\partial\varphi}{\partial x_i}\frac{\partial u^q}{\partial x_i}u_t^q\right) \\
&- \frac{\partial}{\partial t}\left(2\frac{\partial\varphi}{\partial x_i}\frac{\partial u^q}{\partial x_i}\right)u_t^q - \frac{\partial}{\partial x_i}\left(2a_{pq}\frac{\partial\varphi}{\partial x_i}\frac{\partial u^p}{\partial x_i}\frac{\partial u^q}{\partial x_i}\right) + \frac{\partial}{\partial x_i}\left(2\frac{\partial\varphi}{\partial x_i}\frac{\partial u^p}{\partial x_i}\right)a_{pq}\frac{\partial u^q}{\partial x_i} \\
&+ (n-1)\frac{\partial}{\partial t}\Big(u^q u_t^q\Big) - (n-1)|u_t^q|^2 - (n-1)\frac{\partial}{\partial x_i}\left(a_{pq}u^p\frac{\partial u^q}{\partial x_i}\right) \\
&+ (n-1)a_{pq}\frac{\partial u^p}{\partial x_i}\frac{\partial u^q}{\partial x_i} \\
&= t\frac{\partial |u_t^q|^2}{\partial t} + t\frac{\partial}{\partial t}\left(a_{pq}\frac{\partial u^p}{\partial x_i}\frac{\partial u^q}{\partial x_i}\right) + |u_t^q|^2 + a_{pq}\frac{\partial u^p}{\partial x_i}\frac{\partial u^q}{\partial x_i} + \frac{\partial}{\partial t}\left(2\frac{\partial\varphi}{\partial x_i}\frac{\partial u^q}{\partial x_i}u_t^q\right) \\
&+ \frac{\partial}{\partial t}\Big((n-1)u^q u_t^q\Big) - 2t\frac{\partial}{\partial x_i}\left(a_{pq}\frac{\partial u^q}{\partial x_i}u_t^p\right) - \frac{\partial}{\partial x_i}\left(2a_{pq}\frac{\partial\varphi}{\partial x_i}\frac{\partial u^p}{\partial x_i}\frac{\partial u^q}{\partial x_i}\right) \\
&+ \frac{\partial}{\partial x_i}\left(2\frac{\partial\varphi}{\partial x_i}\frac{\partial u^p}{\partial x_i}\right)a_{pq}\frac{\partial u^q}{\partial x_i} - 2\frac{\partial}{\partial t}\left(\frac{\partial\varphi}{\partial x_i}\frac{\partial u^q}{\partial x_i}\right)u_t^q - (n-1)\frac{\partial}{\partial x_i}\left(a_{pq}u^p\frac{\partial u^q}{\partial x_i}\right) \\
&+ (n-2)a_{pq}\frac{\partial u^p}{\partial x_i}\frac{\partial u^q}{\partial x_i} - n|u_t^q|^2 \\
&= \frac{\partial}{\partial t}\left[t\Big(|u_t^q|^2 + a_{pq}\frac{\partial u^p}{\partial x_i}\frac{\partial u^q}{\partial x_i}\Big) + 2\frac{\partial\varphi}{\partial x_i}\frac{\partial u^q}{\partial x_i}u_t^q + (n-1)u^q u_t^q\right] \\
&- \frac{\partial}{\partial x_i}\Big[a_{pq}\frac{\partial u^p}{\partial x_i}\left(2tu_t^q + 2\frac{\partial\varphi}{\partial x_i}\frac{\partial u^q}{\partial x_i} + (n-1)u^q\right) \\
&+ \frac{\partial\varphi}{\partial x_i}\Big(|u^q|^2 - a_{pq}\frac{\partial u^p}{\partial x_i}\frac{\partial u^q}{\partial x_i}\Big)\Big] + \frac{\partial^2\varphi}{\partial x_i}a_{pq}\frac{\partial u^p}{\partial x_i}\frac{\partial u^q}{\partial x_i} \\
&+ (n-2)a_{pq}\frac{\partial u^p}{\partial x_i}\frac{\partial u^q}{\partial x_i} - n|u_t^q|^2 + \frac{\partial^2\varphi}{\partial x_i^2} + 2\frac{\partial\varphi}{\partial x_i}\frac{\partial^2 u^p}{\partial x_i^2}a_{pq}\frac{\partial u^q}{\partial x_i}.
\end{aligned}
$$

The computation is now complete.

To obtain (3.4) it's enough to add to the above identity for equation (3.3) after multiplication by a parameter fixed γ, and finally apply Green's formula and interface conditions (2.1) and (2.2).

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