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Weak Solutions to the One-dimensional Non-Isentropic Gas Dynamics by the Vanishing Viscosity Method *

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Abstract

In this paper we consider the non-isentropic equations of gas dynamics with the entropy preserved. Equations are formulated so that the problem is reduced into the 2×2 system of conservation laws with a forcing term in momentum equation. The method of compensated compactness is then applied to prove the existence of weak solution in the vanishing viscosity method.

1 Introduction

Consider the one-dimensional gas dynamics equation in the Eulerian coordinate

$$\begin{aligned} &\rho_t + (\rho u)_x = 0 \\ &(\rho u)_t + (\rho u^2 + p)_x = 0 \\ &s_t + u s_x = 0. \end{aligned} \tag{1.1}$$

where ρ, u, p and s denote the density, velocity, pressure and entropy. Other relevant quantities are the internal energy e and the temperature T. We assume that the gas is ideal, so that the equation of state is given by

$$p = R\rho T$$

and that it is polytropic, so that $e = c_v T$ and

$$p = (\gamma - 1) e^{s/c_v} \rho^\gamma \tag{1.2}$$

where $\gamma = c_p/c_v > 1$ and $R = c_p - c_v$. Define ϕ by

$$\phi^{1-\gamma} = \gamma(\gamma - 1)\, e^{s/c_v}. \tag{1.3}$$

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Then, ϕ satisfies

$$\phi_t + u\,\phi_x = 0.$$

Thus, we consider the Cauchy problem (equivalent to (1.1))

$$\begin{aligned} &\rho_t + m_x = 0 \\ &m_t + (m^2/\rho + p)_x = 0 \\ &\phi_t + u\,\phi_x = 0\,. \end{aligned} \tag{1.4}$$

where

$$m = \rho u \quad \text{and} \quad p = \frac{1}{\gamma}\,\phi^{1-\gamma}\rho^{\gamma}, \tag{1.5}$$

with smooth initial data (ρ_0, m_0) in $L^\infty(R^2)$ that approaches a constant state $(\bar\rho, \bar m)$ at infinity and satisfies $\rho_0(x) \ge \delta_1 > 0$, and ϕ_0 in $W^{1,\infty}(R)$ that satisfies $(\phi_0)_x$ converges to 0 at infinity and

$$\phi_0(x) \ge \delta_2 > 0 \quad \text{and} \quad (\phi_0)_x(x) \ge 0 \ \ (\text{or} \ \ (\phi_0)_x(x) \le 0). \tag{1.6}$$

Consider the conservation form of the gas dynamics

$$\begin{aligned} &\rho_t + (\rho u)_x = 0 \\ &(\rho u)_t + (\rho u^2 + p)_x = 0 \\ &[\rho\,(\frac{1}{2}\,u^2 + e)]_t + (\rho u\,[\frac{1}{2}\,u^2 + e] + pu)_x = 0. \end{aligned} \tag{1.7}$$

System (1.7) can be written as the hyperbolic system of conservation laws

$$y_t + f(y)_x = 0$$

where $y = y(t,x) = (\rho, \rho u, \rho\,(\frac{1}{2}\,u^2 + e)) \in R^3$ and f is a smooth nonlinear mapping from R^3 to R^3. System (1.4) is equivalent to system (1.7) when solutions are smooth but not necessarily when solutions are weak (e.g.,[Sm, Chapters 16-17]). It is proved in Corollary 3.6 that the viscosity limit of solutions to (1.10) satisfies $\eta_t + q_x \le 0$ in the sense of distributions, i.e., the third equation of (1.7), the conservation of energy $\eta_t + q_x = 0$ is replaced by the non-energy production. We also note that the isentropic solution ($\phi = \text{const}$) [Di1] is a weak solution of (1.4) but not necessarily of (1.7).

In this paper we show the existence of weak solutions to (1.4)-(1.5) using the vanishing viscosity method. The function $(\rho, m, \phi) \in L^\infty(\Omega) \times L^\infty(\Omega) \times W^{1,\infty}(\Omega)$ with $\Omega = [0, \tau] \times R$ is a weak solution of (1.4)-(1.5) if ϕ satisfies the third equation of (1.4) a.e in Ω and

$$\int_0^\tau \int_{-\infty}^\infty (v \cdot \psi_t + F(\rho, m, \phi) \cdot \psi_x)\, dx\, dt = 0 \tag{1.8}$$

for all $\psi \in C_c^\infty(\Omega; R^2)$ where $v = (\rho, m)$ and

$$F(\rho, m, \phi) = (m, m^2/\rho + p) \tag{1.9}$$

We consider the viscous equation of (1.4) with equal diffusion rates

$$\begin{aligned} &\rho_t + m_x = \epsilon\, \rho_{xx} \\ &m_t + (m^2/\rho + p)_x = \epsilon\, m_{xx} \\ &\phi_t + u\, \phi_x = \epsilon \phi_{xx}. \end{aligned} \tag{1.10}$$

It will be shown in Theorem 3.5 that the solutions $(\rho^\epsilon, m^\epsilon, \phi^\epsilon)$ to (1.10) converge to a locally defined (in time) weak solution (ρ, m, ϕ) of (1.4).

Our approach is based on the following observation. Suppose ϕ is a constant. Then equation (1.4) reduces to the isentropic gas dynamics. For the isentropic equation it is shown in DiPerna [Di1] that (1.4) has a weak solution by the vanishing viscosity method and using the theory of compensated compactness. The key steps in [Di1] are given as follows. First, if ϕ is a constant and $\theta = (\gamma - 1)/2$ then

(1.11)

$$w = G_1(\rho, m, \phi) = \frac{m}{\rho} + \frac{1}{\theta}\, \phi^{-\theta} \rho^\theta \quad \text{and} \quad -z = G_2(\rho, m, \phi) = -\frac{m}{\rho} + \frac{1}{\theta}\, \phi^{-\theta} \rho^\theta$$

are the Riemann invariants so that $\nabla_v G_1$ and $\nabla_v G_2$ are the two left eigenvectors of the 2×2 matrix

$$\nabla_v F = \begin{pmatrix} 0 & 1 \\ -\dfrac{m^2}{\rho^2} + \rho^{2\theta} \phi^{-2\theta} & \dfrac{2m}{\rho} \end{pmatrix}.$$

where ϕ is assumed to be a positive constant. The method of invariant regions ([CCS],[Sm]) is applied to $G_1,\ G_2$ to obtain that $0 \le \rho^\epsilon \le \text{const}$, $|m^\epsilon/\rho^\epsilon| \le \text{const}$. Then, there exist a subsequence of $v^\epsilon = (\rho^\epsilon, m^\epsilon)$, still denoted by v^ϵ and a Young measure $\nu_{t,x}$ such that for each $\Phi \in C(R^2)$ we have $\Phi(v^\epsilon)$ converges weak star to $\bar{\Phi}$ in $L^\infty(\Omega)$ where

$$\bar{\Phi}(t, x) = \langle \nu, \Phi \rangle = \int_\Omega \Phi(y)\, d\nu_{t,x}(y), \text{ a.e. } (t, x) \in \Omega.$$

Using the entropy fields [La] and the div-curl theorem of Murat [Mu] and Tatar [Ta] for bilinear maps in the weak topology,

$$\langle \nu, \eta_1 q_2 - \eta_2 q_1 \rangle = \langle \nu, \eta_1 \rangle \langle \nu, q_2 \rangle - \langle \nu, \eta_2 \rangle \langle \nu, q_1 \rangle \tag{1.12}$$

for all entropy/entropy flux pairs (η_i, q_i) so that

$$\nabla_v \eta \nabla_v F = \nabla_v q.$$

Then, using the weak entropy pairs (i.e., $\eta(0,\cdot)=0$) it is shown that ν reduces to a point mass, i.e., v_ϵ converges to v, a.e. in Ω.

We will apply the method described above for the non-isentropic equation (1.10). We need to overcome the two major difficulties. First, $G_1,\ G_2$ are no longer the Riemann invariants of the 3×3 matrix M:

$$M=\nabla F=\begin{pmatrix} 0 & 1 & 0 \\ -\dfrac{m^2}{\rho^2}+\rho^{2\theta}\phi^{-2\theta} & \dfrac{2m}{\rho} & -\dfrac{2\theta}{\gamma}\rho^{\gamma}\phi^{-2\theta-1} \\ 0 & 0 & \dfrac{m}{\rho} \end{pmatrix}.$$

Next, equation (1.12) should be extended to the 3×3 system. We resolve these difficulties by the following steps. Note (see Lemma 2.4) that $G_i = G_i(\rho,m,\phi),\ i=1,2$ satisfies

(1.13)
$$(G_i)_t+\lambda_i\nabla G_i\cdot y_x+\frac{1}{\gamma}\begin{cases} \rho^{2\theta}\phi^{-2\theta-1}\phi_x, & i=1 \\ -\rho^{2\theta}\phi^{-2\theta-1}\phi_x, & i=2 \end{cases} = \epsilon\left((G_i)_{xx}-\nabla^2 G_i(y_x,y_x)\right)$$

where $y=(\rho,m,\phi)$ is a solution to (1.10) and $\lambda_1=u+\rho^\theta\phi^{-\theta},\ \lambda_2=u-\rho^\theta\phi^{-\theta}$ are the eigenvalues of the 2×2 matrix $\nabla_v F$. Here, $G_i,\ i=1,2$ are quasi-convex functions of (ρ,m,ϕ) (see Lemma 2.5), i.e.,

$$r\cdot\nabla G_i=0 \ \text{ implies } \ \nabla^2 G_i(r,r)\ge 0.$$

Note that from the third equation of (1.10) that $\phi^\epsilon\ge\delta_2$ and $|\phi^\epsilon|_\infty\le|\phi_0|_\infty$ (see Lemma 2.1) and moreover ϕ^ϵ_x satisfies

$$(\phi^\epsilon_x)_t+(u^\epsilon\,\phi^\epsilon_x)_x=\epsilon\,(\phi^\epsilon_x)_{xx}$$

Observing that if (ρ,m,ϕ) is a solution to (1.10) then $\xi=\log(\frac{\phi_x}{\rho})$ satisfies

$$\xi_t+u\,\xi_x=\epsilon\,(\xi_{xx}+|(\log\phi_x)_x|^2-|(\log\rho)_x|^2), \tag{1.14}$$

we show that if $(\phi_0)_x\ge 0$ (resp. ≤ 0) then $\phi^\epsilon_x\ge 0$ (resp. ≤ 0) and $|\phi^\epsilon_x|\le c\,\rho^\epsilon$ in Ω provided that $|(\phi_0)_x|\le c\,\rho_0$ in R (see Lemma 2.3). It thus follows from (1.13) and the quasi-convexity of $G_i,\ i=1,2$ that $\max_x G_2(t,x)\le\max_x G_2(0,x)$ (resp. $\max_x G_1(t,x)\le\max_x G_1(0,x)$) and

$$|\rho^{2\theta}\phi^{-2\theta-1}\phi_x|\le c\left(\frac{\rho}{\phi}\right)^{2\theta+1}.$$

By the maximum principle (see Theorem 2.6), there exists a $\tau=\tau_c>0$ with $c\to\tau_c$ monotonically decreasing and $\tau_0=\infty$ such that $\max_{t\in[0,\tau],\,x\in R} G_1(t,x)$ is less than a constant independent of $\epsilon>0$. Hence, we obtain $0\le\rho^\epsilon\le$ const, $|m^\epsilon/\rho^\epsilon|\le$ const and $|\phi_x|\le$ const in Ω.

Second, in contrast to the isentropic case, the system (1.10) is not endowed with a rich family of entropy-entropy flux pairs. Thus, in order to prove that the Young measures $\nu_{t,x}$ of a weakly star convergent subsequence of $(\rho^\epsilon, m^\epsilon, \phi^\epsilon)$ reduce to a point mass, we first note that $\{\phi^\epsilon(t,x)\}$ is precompact in $L^2_{loc}(\Omega)$ and thus ϕ^ϵ converges ϕ a.e. in Ω (see Lemma 3.4). Also, we note that dividing the first two equations (1.10) by ϕ, we obtain

$$\begin{aligned} &\hat{\rho}_t + \hat{m}_x = \epsilon\,(\hat{\rho}_{xx} + 2\frac{\rho_x\phi_x}{\phi^2}) \\ &\hat{m}_t + (\hat{m}^2/\hat{\rho} + \frac{1}{\gamma}\,\hat{\rho}^\gamma)_x + \frac{p\phi_x}{\phi^2} = \epsilon\,(\hat{m}_{xx} + 2\,\frac{m_x\phi_x}{\phi^2}). \end{aligned} \tag{1.15}$$

where $\hat{\rho} = \dfrac{\rho}{\phi}$ and $\hat{m} = \dfrac{m}{\phi}$ (see Lemma 3.2). This implies that $\hat{v} = (\hat{\rho}, \hat{m})$ satisfies the (viscous) isentropic gas-dynamics with the forcing term $-p\phi_x/\phi^2$ in the momentum equation. Since $\dfrac{p^\epsilon\phi^\epsilon_x}{(\phi^\epsilon)^2} \in L^\infty(\Omega)$ uniformly in $\epsilon > 0$, thus $\{\dfrac{p^\epsilon\phi^\epsilon_x}{(\phi^\epsilon)^2}\}_{\epsilon>0}$ is precompact in $H^{-1,q}_{loc}(\Omega)$, $1 \le q < 2$. Hence, the method of compensated compactness in [Di1],[Ch] can be applied to the functions $(\hat{\rho}^\epsilon, \hat{m}^\epsilon)$ to show that $\nu_{t,x}$ is a point mass provided that $1 < \gamma \le 5/3$.

Regarding work on existence of weak solutions for conservation laws, we refer the reader to an excellent treatise by DiPerna [Di3] and references therein. Concerning basic framework on conservation laws, we refer the reader to [La],[Sm] and for the functional analytic framework of compensated compactness we refer [Mu],[Ta1],[Ev] and [Di2]. For scalar conservation laws the vanishing viscosity method is employed (e.g., in [Ol],[Kr] and references in [Sm]) to define the unique entropy solution. Also, the vanishing viscosity method is used to develop the viscosity solution to the Hamilton-Jacobi equation in [CL]. The finite-difference methods (e.g., Lax-Friedrichs and Gudunov schemes) are also used to construct weak solutions to a scalar and 2×2 system of conservation laws (e.g., see [Di2],[Ch] and [Sm]).

In the case where the initial data have small total variation, Glimm [Gl] proved the global existence of BV-solutions for a general class of hyperbolic systems as the strong limit of random choice approximations. However, the problem of existence of solutions to (1.7) with large initial data is still unsolved. In [CD] the vanishing viscosity method is applied to the system (1.7) under a special class of constitutive relations in Lagrangian coordinates.

2 The Viscosity Method

In this section we establish the uniform L^∞ bound of $y^\epsilon = (\rho^\epsilon, m^\epsilon, \phi^\epsilon)$.

Lemma 2.1 *If $\phi \in C^{1,2}([0,\tau] \times R)$ satisfies $\phi_t + u\,\phi_x = \epsilon\,\phi_{xx}$ then*

$$\min_x\,\phi_0(x) \le \phi(t,x) \le \max_x\,\phi_0(x).$$

Proof: Using the same arguments as in the proof of Theorem 2.6, we can show that $\max_x\, \phi(t,x) \le \max_x\, \phi_0(x)$ and $\min_x\, \phi(t,x) \ge \min_x\, \phi_0(x)$. $\Box$

It will be shown in Section 3 (see (3.4) and Lemma 3.1) that the normalized mechanical energy

$$E(\rho,u,\phi) = \frac{1}{2}\,\rho(u-\bar{u})^2 + \frac{1}{\gamma}\,(\rho^\gamma - \gamma\,\bar{\rho}^{\gamma-1}(\gamma-\bar{\gamma}) - \bar{\rho}^\gamma)\phi^{1-\gamma} \tag{2.1}$$

satisfies

$$\int_{-\infty}^{\infty} E(\rho(t,x),u(t,x),\phi(t,x))\,dx \le \int_{-\infty}^{\infty} E(\rho_0(x),u_0(x),\phi_0(x))\,dx. \tag{2.2}$$

The following lemma shows the lower bound of ρ_ϵ.

Lemma 2.2 *If $\rho \in C^{1,2}([0,\tau]\times R)$ satisfies*

$$\rho_t + (u\,\rho)_x = \epsilon\,\rho_{xx} \tag{2.3}$$

with $\rho(0,\cdot) \ge 0$ and $u \in C^1(\Omega)$, then $\rho(t,\cdot) \ge 0$. Moreover, if $\rho(0,\cdot) \ge \delta > 0$ and

$$\int_0^\tau \int_{-\infty}^{\infty} \rho\,|u-u_0|^2\,dx\,dt \le const, \tag{2.4}$$

then $\rho(t,\cdot) \ge \delta(\epsilon,\tau) > 0$ on $(0,\tau)$.

Proof: Choose $\psi = \min(\rho(t,x),0)$. Then we have

$$\int_{-\infty}^{\infty} \frac{1}{2}\,|\psi(t,x)|^2 + \int_0^t \int_{-\infty}^{\infty} (\epsilon\,|\psi_x|^2 - \psi u\,\psi_x)\,dx\,ds = 0.$$

By the Hölder inequality, we obtain

$$\int_{-\infty}^{\infty} |\psi(t,0)|^2 \le \frac{|u|_\infty}{2\epsilon} \int_0^t \int_{-\infty}^{\infty} |\psi|^2\,dx\,ds$$

where $|u|_\infty = \sup_{(t,x)\in(0,\tau)\times R} |u(t,x)|$, and the Gronwall's inequality implies $\psi = 0$. Thus, $\rho \ge 0$.

Next, we prove $\rho(t,\cdot) \ge \delta = \delta(\epsilon,\tau) > 0$ if $\varphi(0,\cdot) \ge \delta > 0$ by using the Stampacchia's lemma (e.g., see [FI],[Tr]), i.e., suppose $\chi(c)$ is a nonnegative, non-increasing function on $[c_0,\infty)$, and there exist positive constants K, s and t such that

$$\chi(\hat{c}) \le K c^s (\hat{c}-c)^{-s}\,\chi(c)^{1+t} \quad \text{for all } \hat{c} > c \ge c_0,$$

then

$$\chi(c^*) = 0 \quad \text{for } c^* = 2c_0\,(1 + 2^{\frac{1+2t}{t^2}} K^{\frac{1+t}{st}}\,\chi(c_0)^{\frac{1+t}{s}}).$$

First, we establish *a priori* bound. We consider the class K [Di1] of strictly convex C^2 functions h with following properties:

$$h(\bar\rho) = h'(\bar\rho) = 0, \quad h(\rho) = \rho^{-\alpha} \ \text{ on } \ (0, \bar\rho/2) \text{ for some } \ 0 < \alpha < 1.$$

Premultiplying the first equation of (1.10) by $h'(\rho)$ we obtain

$$h(\rho)_t + (h'(\rho)\rho u)_x - h''(\rho)\rho_x \rho u = \epsilon\,(h(\rho)_{xx} - h''(\rho)\rho_x^2)$$

Integration of this over $(0,t)\times R$ yields

$$\begin{aligned}\int_{-\infty}^{\infty} h(\rho(t,x)) - h(\rho_0(x))\,dx + \epsilon \int_0^t \int_{-\infty}^{\infty} h''(\rho)\rho_x^2\,dx\,dt \\ = \int_0^t \int_{-\infty}^{\infty} h''(\rho)\rho_x\rho(u-\bar u)\,dx\,dt\,.\end{aligned}$$

Note that

$$h''(\rho)\rho_x\rho(u-\bar u) \le \frac{\epsilon}{2}\,h''(\rho)\rho_x^2 + \frac{1}{2\epsilon}\,h''(\rho)\rho^2(u-\bar u)^2.$$

Since there exists some constant $\beta > 0$ such that

$$\begin{aligned}&\rho^2\,h''(\rho) \le \beta\,\rho \quad \text{for } \ \bar\rho/2 \le \rho \le M \\ &\rho^2\,h''(\rho) \le \beta\,h(\rho) \quad \text{for } \ 0 < \rho < \bar\rho/2\end{aligned}$$

it follows that $\rho^2\,h''(\rho)(u-\bar u)^2 \le \beta\,(\rho\,(u-\bar u)^2 + h(\rho))$. Hence,

$$\begin{aligned}\int_{-\infty}^{\infty} h(\rho(t,x)) - h(\rho_0(x))\,dx + \frac{\epsilon}{2}\int_0^t \int_{-\infty}^{\infty} h''(\rho)\rho_x^2\,dx\,dt \\ \le \frac{\beta}{2\epsilon}\int_0^t \int_{-\infty}^{\infty} \rho(u-\bar u)^2 + h(\rho)\,dx\,dt\,.\end{aligned}$$

and it follows from (2.4) and Gronwall's inequality that

$$\int_{-\infty}^{\infty} h(\rho(t,x))\,dx \le \text{const on } \ [0,\tau]. \tag{2.5}$$

Set $\eta = 1/\rho$. Then from (2.3) η satisfies

$$\eta_t + u\,\eta_x - u_x\eta = \epsilon\,(\eta_{xx} - \frac{2|\eta_x|^2}{\eta}).$$

We further introduce $\hat\eta = e^{\omega\,t}\eta$ with $\omega > 0$ to be determined later. The equation for $\hat\eta$ becomes

$$\hat\eta_t + \omega\,\hat\eta + u\,\hat\eta_x - u_x\hat\eta = \epsilon\,(\hat\eta_{xx} - \frac{2|\hat\eta_x|^2}{\hat\eta}). \tag{2.6}$$

Define $\xi = \xi_c = \max(0, \hat{\eta} - c)$ with $c \geq c_0 = \delta^{-1}$. Pre-multiplication of (2.6) by ξ^3 and integration over R yields

$$\int_{-\infty}^{\infty} (\frac{1}{4}(\xi^4)_t + \omega(\xi^4 + c\,\xi^3) + \frac{3\epsilon}{4}|(\xi^2)_x|^2) \leq -\int_{-\infty}^{\infty} u\,(5\,\xi^3 + 3c\,\xi^2)\xi_x\,dx. \tag{2.7}$$

Note that

$$-\int_{-\infty}^{\infty} 5u\,\xi^3\xi_x \leq -\int_{-\infty}^{\infty} \frac{5}{2}u\xi^2(\xi^2)_x\,dx \leq \frac{\epsilon}{4}\int_{-\infty}^{\infty} |(\xi^2)_x|^2\,dx + \frac{25}{4\epsilon}|u|_\infty^2 \int_{-\infty}^{\infty} |\xi|^4\,dx$$

To estimate the second term on the right hand side of (2.7), we define

$$I_c(t) = \{x \in R : \xi(t,x) > 0\} = \{x \in R : \eta(t,x) > c\,e^{\omega t}\}.$$

Then, using the Hölder inequality, we have

$$\begin{aligned}
-\int_{-\infty}^{\infty} 3c\,\xi^2\xi_x\,dx &= -\int_{-\infty}^{\infty} \frac{3c}{2}\,u\xi(\xi^2)_x\,dx \\
&\leq \frac{3c}{2}|u|_\infty \left(\int_{-\infty}^{\infty} |(\xi^2)_x|^2\,dx\right)^{1/2} \left(\int_{-\infty}^{\infty} |\xi|^6\,dx\right)^{1/6} |I_c(t)|^{1/3} \\
&\leq \frac{\epsilon}{4}\int_{-\infty}^{\infty} |(\xi^2)_x|^2\,dx + \frac{9c^2}{4\epsilon}\,|u|_\infty^2 \left(\int_{-\infty}^{\infty} |\xi|^6\,dx\right)^{1/3} |I_c(t)|^{2/3}.
\end{aligned}$$

Substituting these estimates into (2.7), choosing $\omega = \dfrac{\epsilon}{4} + \dfrac{25}{4\epsilon}\,|u|_\infty^2$, and integrating on the interval $[0,t]$, we obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty} |\xi|^4\,dx + \epsilon\int_0^t \int_{-\infty}^{\infty} (|\xi^2|^2 + |(\xi^2)_x|^2)\,dx\,ds \\
&\qquad \leq \frac{9c^2}{\epsilon}\,|u|_\infty^2 \int_0^t \left(\int_{-\infty}^{\infty} |\xi|^6\,dx\right)^{1/3} |I_c(s)|^{2/3}\,ds.
\end{aligned} \tag{2.8}$$

Since

$$|\xi^2|_\infty \leq \sqrt{2}\left(\int_{-\infty}^{\infty} (|\xi^2|^2 + |(\xi^2)_x|^2)\,dx\right)^{1/2},$$

we have

$$\left(\int_{-\infty}^{\infty} |\xi|^6\,dx\right)^{1/3} \leq 2^{1/6}\left(\int_{-\infty}^{\infty} (|\xi^2|^2 + |(\xi^2)_x|^2)\,dx\right)^{1/2}.$$

Thus, from (2.8),

$$\begin{aligned}
&\int_{-\infty}^{\infty} |\xi|^4\,dx + \epsilon\int_0^t \int_{-\infty}^{\infty} (|\xi^2|^2 + |(\xi^2)_x|^2)\,dx\,ds \\
&\qquad \leq 2^{1/3}\frac{81c^4}{4\epsilon}\,|u|_\infty^4 \int_0^t |I_c(s)|^{4/3}\,ds \leq Kc^4\,\chi(c)^{4/3}
\end{aligned}$$

where we define

$$\chi(c)=\sup_{t\in[0,\tau]}|I_c(t)| \quad\text{and}\quad K=2^{1/3}\frac{81\tau}{4\epsilon}.$$

Clearly $\chi(c)$ is nonnegative, non-increasing on $[c_0,\infty)$. Moreover, for $\hat{c}>c$,

$$\int_{-\infty}^{\infty}|\xi|^4\,dx\ge\int_{I_{\hat{c}}}|\xi|^4\,dx\ge(\hat{c}-c)^4\,I_{\hat{c}}(t).$$

Hence,

$$I_{\hat{c}}(t)\le K\,c^4(\hat{c}-c)^{-4}\chi(c)^{4/3}$$

and by taking the sup over t we obtain

$$\chi(\hat{c})\le Kc^4(\hat{c}-c)^{-4}\,\chi(c)^{4/3}.$$

It follows from (2.5) that $\chi(2/\bar{\rho})<\infty$ and thus from Stampacchia's lemma that $\chi(c^*)=0$ for some $c^*\ge c_0$ and hence

$$\eta(t,x)\le c^*e^{\omega t}\quad\text{and}\quad\rho(t,x)\ge(c^*)^{-1}\,e^{-\omega t}\,,$$

where c^* depends on ϵ and τ. $\Box$

The following lemma shows that $|\phi_x^\epsilon(t,\cdot)|$ is uniformly bounded by $\rho^\epsilon(t,\cdot)$ for every $t\in[0,\tau]$ under assumption (1.7).

Lemma 2.3 *Assume that ϕ_0 satisfies (1.7) and $|(\phi_0)_x|\le c\,\rho_0$ in R. Then $|\phi_x(t,\cdot)|\le c\,\rho(t,x)$ in R for every $t\in[0,\tau]$.*

Proof: If the initial condition (ρ_0,m_0,ϕ_0) is sufficiently smooth and $(\phi_0)_x\ge\delta_3$ then the solution to (1.10) satisfies $\rho,\ u,\ \phi\in C^3(\Omega)$ and $\phi_x(t,\cdot)>0$. Note that ϕ_x satisfies

$$(\phi_x)_t+(u\,\phi_x)_x=\epsilon\,(\phi_x)_{xx}. \tag{2.9}$$

Then, it is not difficult to show that if we define $\xi=\log(\frac{\phi_x}{\rho})$ then ξ satisfies

$$\xi_t+u\,\xi_x=\epsilon\,(\xi_{xx}+|(\log\phi_x)_x|^2-|(\log\rho)_x|^2)$$

Suppose $\xi(t,x_0)=\max_x\ \xi(t,x)$. Then

$$\xi_x(t,x_0)=(\log\phi_x)_x(t,x_0)-(\log\rho)_x(t,x_0)=0$$

and $\xi_{xx}(t,x_0)\le 0$. Thus, $\partial_t(\max_x\ \xi(t,x))\le 0$, which implies the lemma. Since the solution to (2.9) continuously depends on the initial data $(\phi_0)_x$ the estimate holds for when $(\phi_0)_x\ge 0$. $\Box$

The following lemmas provide the technical properties of the functions $G_i(t)$, $i=1,2$ defined by (1.11).

Lemma 2.4 *If $y = (\rho, m, \phi) \in C^{1,2}((0,\tau)\times R)^3$ is a solution to (1.10), then (1.13) holds.*

Proof: First, note that the 3×3 matrix $M = \nabla F$ has the eigenvalues $\lambda_1 = \dfrac{m}{\rho} + \rho^\theta \phi^{-\theta}$, $\lambda_2 = \dfrac{m}{\rho} - \rho^\theta\phi^{-\theta}$ and $\dfrac{m}{\rho}$ and that $\nabla_v G_i$, $i = 1, 2$ are the left-eigenvectors of the sub-matrix $\nabla_v F$ corresponding to λ_i. Thus,

$$(G_1)_t + \lambda_1 \left(\nabla G_i \cdot y_x + \rho^\theta \phi^{-\theta-1}\phi_x\right) - \left(\frac{2\theta}{\gamma}\rho^{2\theta-1}\phi^{-2\theta-1} + u\rho^\theta\phi^{-\theta-1}\right)\phi_x = \epsilon \nabla G_1 \cdot y_{xx}\,.$$

Since $\nabla G_1 \cdot y_{xx} = (G_1)_{xx} - \nabla^2 G_1(y_x, y_x)$ we obtain (1.13) for G_1. The same calculation applies to G_2. $\square$

Lemma 2.5 *If $\rho > 0$, $\phi > 0$ then G_i, $i = 1, 2$, are quasi-convex.*

Proof: We prove $G_1 = \dfrac{m}{\rho} + \dfrac{1}{\theta}\rho^\theta\phi^{-\theta}$ is quai-convex. The same proof applies to G_2. Note that

$$\nabla G_1 = \begin{pmatrix} -\dfrac{m}{\rho^2} + \rho^{\theta-1}\phi^{-\theta} \\ \dfrac{1}{\rho} \\ -\rho^\theta \phi^{-\theta-1} \end{pmatrix}$$

$$\nabla^2 G_1 = \begin{pmatrix} -\dfrac{2m}{\rho^3} + (\theta-1)\,\rho^{\theta-2}\phi^{-\theta} & -\dfrac{1}{\rho^2} & -\theta\,\rho^{\theta-1}\phi^{-\theta-1} \\ -\dfrac{1}{\rho^2} & 0 & 0 \\ -\theta\,\rho^{\theta-1}\phi^{-\theta-1} & 0 & (\theta+1)\,\rho^\theta\phi^{-\theta-2} \end{pmatrix}.$$

If $r = (X, Y, Z)$ satisfies $r \cdot \nabla G_1$ then $Y = -\dfrac{m}{\rho} + \rho^\theta\phi^{-\theta}\,X + \rho^{\theta+1}\phi^{-\theta-1}\,Z$. Thus,

$$\begin{aligned} \nabla^2 G_i(r,r) &= (\theta+1)\left(\rho^{\theta-2}\phi^{-\theta}\,X^2 - 2\,\rho^{\theta-1}\phi^{-\theta-1}\,XZ + \rho^\theta\phi^{-\theta-2}\,Z^2\right) \\ &= (\theta+1)\rho^{\theta-2}\phi^{-\theta-2}\,(\phi\,X - \rho\,Z)^2 \ge 0. \quad \square \end{aligned}$$

We now state the main result of this section that establishes the uniform L^∞-bound of $(\rho^\epsilon, m^\epsilon, \phi^\epsilon_x)$ in $\epsilon > 0$.

Theorem 2.6 *Suppose ϕ_0 satisfies (1.7) and $|(\phi_0)_x| \le c\rho_0$ in R. Then, there exists a $\tau = \tau_c > 0$ with $c \to \tau_c$ monotonically decreasing and $\tau_0 = \infty$ such that $0 \le \rho^\epsilon \le const$, $|\frac{m^\epsilon}{\rho^\epsilon}| \le const$ and $|\phi_x| \le const$ in $\Omega = [0,\tau]\times R$.*

Proof: Suppose that $(\phi_0)_x \le 0$. Then, it follows from Lemmas 2.3-2.6 that $\max_x G_2(t,x) \le \max_x G_2(0,x) = A$. Hence, $\dfrac{m}{\rho} + A \ge \dfrac{1}{\theta}\rho^\theta\phi^{-\theta} \ge 0$. Set $G = A + G_1$. It then follows from Lemma 2.4 that

$$G_t + \lambda_1 \nabla G \cdot y_x + \frac{1}{\gamma}\rho^{2\theta}\phi^{-2\theta-1}\phi_x = \epsilon\left(G_{xx} - \nabla^2 G(y_x, y_x)\right)$$

Let $G^k = G(kh),\ h > 0$. Then,

$$G^k - G^{k-1} + \lambda_1 \nabla G^k \cdot y_x^k + \frac{1}{\gamma}\,(\rho^k)^{2\theta}(\phi^k)^{-2\theta-1}(\phi^k)_x = \epsilon\,(G^k_{xx} - \nabla^2 G^k(y_x^k, y_x^k)) + \varepsilon(h)$$

where $\varepsilon(h)/h \to 0$ as $h \to 0^+$. Suppose $G^k(x_0) = \max_x\ G^k(x)$. Then, $G^k_x(x_0) = (\nabla G^k \cdot y^k_x)(x_0) = 0$ and $G^k_{xx}(x_0) \le 0$. It follows from Lemmas 2.3 and 2.5 that if $\psi(t) = \max_x\ G(t,x)$ then

$$\psi(kh) - \psi((k-1)h) - \varepsilon(h) \le \frac{c}{\gamma}\,(\frac{\rho}{\phi}(x_0))^{2\theta+1} \le \frac{c}{\gamma}\theta^{(2\theta+1)/\theta}\psi(kh)^{(2\theta+1)/\theta}.$$

Taking the limit $h \to 0^+$, we obtain

$$\psi(t) - \psi(0) \le \int_0^t \frac{c}{\gamma}\,\theta^{(2\theta+1)/\theta}\,\psi(\tau)^{(2\theta+1)/\theta}\,d\tau$$

and thus

$$\psi(t) \le \left(\frac{\psi(0)^{(\theta+1)/\theta}}{1 - \frac{c}{\gamma}(\frac{\theta+1}{\theta})\theta^{(2\theta+1)/\theta}\psi(0)^{(\theta+1)/\theta}\,t}\right)^{\theta/(\theta+1)}. \tag{2.10}$$

In fact, if

$$s(t) = \psi(0) + \int_0^t \frac{c}{\gamma}\,\theta^{(2\theta+1)/\theta}\,\psi(\tau)^{(2\theta+1)/\theta}\,d\tau$$

then $\psi(t) \le s(t)$ and $\dot{s} \le \frac{c}{\gamma}\,\theta^{(2\theta+1)/\theta}s^{(2\theta+1)/\theta}$, which implies (2.10). Since

$$0 \le \frac{1}{\theta}\,\rho^\theta\phi^{-\theta} \le \frac{1}{2}\,(G_1 + G_2) \quad \text{and} \quad -G_2 \le \frac{m}{\rho} \le G_1\,,$$

the lemma follows from (2.9). □

3 Compensated Compactness

In this section we show that the sequence $\{(\rho^\epsilon, m^\epsilon, \phi^\epsilon)\}_{\epsilon>0}$ has a subsequence that converges to a weak solution of (1.10) a.e in Ω using the method of compensated compactness. First note that the mechanical energy

$$\eta = \frac{1}{2}\,\frac{m^2}{\rho} + \frac{1}{\gamma(\gamma-1)}\,\rho^\gamma\phi^{-\gamma+1} \tag{3.1}$$

and the corresponding entropy-flux

$$q = \frac{\rho}{2}\,(\frac{m}{\rho})^3 + \frac{1}{\gamma-1}\,\frac{m}{\rho}\rho^\gamma\phi^{-\gamma+1} \tag{3.2}$$

form an entropy pair, i.e.,

$$\nabla\eta\, M = \nabla q \tag{3.3}$$

In order to treat solutions approaching a nonzero state at infinity, we consider a normalized entropy pair

$$\begin{aligned}\tilde\eta &= \eta(y) - \eta((\bar v,\phi)) - \nabla_v\eta((\bar v,\phi))(v-\bar v),\\ \tilde q &= q(y) - q((\bar v,\phi)) - \nabla_v\eta((\bar v,\phi))F(y)\end{aligned}$$

where $v = (\rho, m)$, $\bar v = (\bar\rho, \bar m)$ and $y = (v, \phi)$. Premultiplying (1.10) by $\nabla\tilde\eta$, we obtain

$$\tilde\eta_t + \tilde q_x = \epsilon\,(\tilde\eta_{xx} - \nabla^2\eta(y_x, y_x)).$$

Integration over Ω yields an energy estimate

$$\int_{-\infty}^{\infty}\tilde\eta(t,x)\,dx + \epsilon\int_0^t\int_{-\infty}^{\infty}\nabla^2\eta(y_x,y_x)\,dx\,dt = \int_{-\infty}^{\infty}\tilde\eta(0,x)\,dx. \tag{3.4}$$

The following lemma implies the energy estimate (2.2) where $\tilde\eta(y) = E(\rho, u, \phi)$.

Lemma 3.1 *For $\rho > 0$, $\phi > 0$, $\nabla^2\eta$ is non-negative.*

Proof: Note that

$$\nabla^2\eta = \begin{pmatrix} \dfrac{m^2}{\rho^3} + \rho^{\gamma-2}\phi^{-\gamma+1} & -\dfrac{m}{\rho^2} & -\rho^{\gamma-1}\phi^{-\gamma} \\ -\dfrac{m}{\rho^2} & \dfrac{1}{\rho} & 0 \\ -\rho^{\gamma-1}\phi^{-\gamma} & 0 & \rho^\gamma\phi^{-\gamma-1} \end{pmatrix}.$$

Thus,

$$\nabla^2\eta\,(y_x, y_x) = \frac{1}{\rho}\,(\frac{m}{\rho}\,\rho_x - m_x)^2 + \rho^{\gamma-2}\phi^{-\gamma-1}\,(\phi\,\rho_x - \rho\,\phi_x)^2 \ge 0 \tag{3.5}$$

for $y_x = (\rho_x, m_x, \phi_x)$. $\square$

The following lemma establishes the viscosity estimate which is essential for the method of compensated compactness.

Lemma 3.2 *Assume that $1 < \gamma \le 2$ and $\int_{-\infty}^{\infty}\tilde\eta(0,x)\,dx < \infty$. Then, if (ρ, m, ϕ) is a solution of (1.10)*

$$\epsilon\int_0^\tau\int_{-\infty}^{\infty}(|\rho_x(t,x)|^2 + |m_x(t,x)|^2)\,dx\,dt \le const$$

where $\tau > 0$ is defined in Theorem 2.6

Proof: From (2.9) and Lemma 2.2 we have

$$\int_{-\infty}^{\infty} |\phi_x(t,x)|\,dx = \int_{-\infty}^{\infty} |\phi_x(0,x)|\,dx\,, \quad t \in [0,\tau].$$

It thus follows from Theorem 2.6 that

$$\int_0^\tau \int_{-\infty}^{\infty} |\phi_x(t,x)|^2\,dx \le \text{const.}$$

Since $0 < \rho(t,x),\ \phi(t,x) \le \text{const}$ in Ω it follows from (3.5) that

$$\nabla^2\,\eta(y_x(t,x), y_x(t,x)) + |\phi_x(t,x)|^2 \ge c_1\,|y_x(t,x)|^2$$

for some $c_1 > 0$. Hence, the lemma follows from (3.4). $\square$

We apply the method of compensated compactness for the function $\hat{v}^\epsilon$ defined by

$$\hat{v}^\epsilon = (\hat{\rho}^\epsilon, \hat{m}^\epsilon) = (\frac{\rho^\epsilon}{\phi^\epsilon}, \frac{m^\epsilon}{\phi^\epsilon})$$

The function $\hat{v}^\epsilon$ satisfies the 2×2 viscous conservation law (1.15) with the forcing term which is in $L^\infty(\Omega)$. Based on this observation we have

Lemma 3.3 *Assume that the conditions in Theorem 2.6 are satisfied and that* $\int_{-\infty}^{\infty} \tilde{\eta}(0,x)\,dx < \infty$. *Then, for* $1 < \gamma \le 2$, *the measure set*

$$\eta(\hat{v}^\epsilon)_t + q(\hat{v}^\epsilon)_x$$

lies in a compact subset of $H^{-1}_{loc}(\Omega)$ *for all weak entropy/entropy flux pair* (η, q) *of* $\nabla_v F$, *where* $\hat{v}^\epsilon = (\dfrac{\rho^\epsilon}{\phi^\epsilon}, \dfrac{m^\epsilon}{\phi^\epsilon})$.

Proof: Suppose (ρ, m, ϕ) is a solution to (1.10). Then, dividing the first two equations of (1.10) by ϕ, we obtain (1.15) for $\hat{\rho} = \dfrac{\rho}{\phi}$ and $\hat{m} = \dfrac{m}{\phi}$. Let (η, q) be a weak entropy/entropy flux pair, i.e.,

$$\nabla\eta\nabla_v F = \nabla q \quad \text{and} \quad \eta(0,\cdot) = 0. \tag{3.6}$$

It can be shown that for $0 < \rho \le \text{const}$, $|\frac{m}{\rho}| \le \text{const}$

$$|\nabla\eta| \le \text{const} \quad \text{and} \quad |\nabla^2\eta(r,r)| \le \text{const}\,\nabla^2\eta^*(r,r) \tag{3.7}$$

where

$$\eta^* = \frac{1}{2}\rho\,(\frac{m}{\rho})^2 + \frac{1}{\gamma(\gamma-1)}\,\rho^\gamma$$

is the mechanical energy, r is any vector in R^2 and constant is independent of r. Premultiplying (1.15) by $\nabla\eta$, we obtain

$$\eta(\hat{v})_t + q(\hat{v})_x = \epsilon\,(\eta(\hat{v})_{xx} - \nabla^2\eta(\hat{v}_x, \hat{v}_x)) + \nabla\eta(\hat{v})A$$

where
$$A = 2\epsilon\left(\frac{\rho_x\phi_x}{\phi^2}, \frac{m_x\phi_x}{\phi^2}\right) - \left(0, \frac{p\phi_x}{\phi^2}\right)$$

It follows from Theorem 2.6 that $\dfrac{p^\epsilon\phi^\epsilon_x}{(\phi^\epsilon)^2} \in L^\infty(\Omega)$ uniformly in $\epsilon > 0$. It follows from Lemma 3.2 and Theorem 2.6 that
$$\epsilon^{1/2}\left(\frac{\rho^\epsilon_x\phi^\epsilon_x}{(\phi^\epsilon)^2}, \frac{m^\epsilon_x\phi^\epsilon_x}{(\phi^\epsilon)^2}\right) \in L^2(\Omega)$$

uniformly in $\epsilon > 0$. Thus, $\{\nabla\eta(v^\epsilon)A^\epsilon\}_{\epsilon>0}$ is precompact in $W^{-1,q}_{loc}(\Omega)$, $1 \le q < 2$. Since
$$\int_0^\tau\int_{-\infty}^\infty \epsilon\,|\hat{v}^\epsilon_x(t,x)|^2\,dx\,dt \le \text{const}$$

The set $\{\epsilon\nabla\eta\hat{v}^\epsilon_x\}_{\epsilon>0}$ is precompact in $L^2(\Omega)$ and so is $\{\epsilon\eta(\hat{v}^\epsilon)_{xx}\}_{\epsilon>0}$ in $H^{-1}(\Omega)$. Hence, the lemma follows from the fact that if set S is compact in $W^{-1,q}(U)$ and bounded in $W^{-1,r}(U)$ then S is compact in $H^{-1}(U)$ for $1 \le q < 2 < r$ and any bounded and open set U in R^2. [Ev] □

In the next lemma we prove that the sequence $\{\phi^\epsilon\}_{\epsilon>0}$ is precompact in $L^2_{loc}(\Omega)$.

Lemma 3.4 *For $\epsilon > 0$ and $\tau > 0$ defined in Theorem 2.6*
$$\int_0^\tau\int_{-\infty}^\infty (|\phi^\epsilon_t|^2 + |\phi^\epsilon_x|^2)\,dx\,dt \le const.$$

Thus, the family $\{\phi^\epsilon(t,x)\}_{\epsilon>0}$ is compact in $L^2(U)$ for any bounded rectangle $U = (0,\tau)\times(-L,L)$.

Proof: Premultiplying (1.10) by ϕ_{xx} and integrating in $(0,\tau)\times R$, we obtain
$$\begin{aligned}\frac{1}{2}\int_{-\infty}^\infty |\phi_x(\tau,x)|^2\,dx + \frac{\epsilon}{2}\int_0^\tau\int_{-\infty}^\infty |\phi_{xx}|^2\,dx\,dt \\ \le \frac{1}{2}\int_{-\infty}^\infty |\phi_x(0,x)|^2\,dx + \frac{1}{2\epsilon}|u|^2_\infty\int_0^\tau\int_{-\infty}^\infty |\phi_x|^2\,dx\,dt.\end{aligned}$$

where $|u|_\infty = \sup_{(t,x)\in(0,\tau)\times R}|u(t,x)|$. Thus,
$$\int_0^\tau\int_{-\infty}^\infty |\epsilon\,\phi_{xx}|^2\,dx\,dt \le |u|^2_\infty\int_0^t\int_{-\infty}^\infty |\phi_x|^2\,dx\,dt + \epsilon\int_{-\infty}^\infty |\phi_x(0,x)|^2\,dx$$

and
$$\int_0^\tau\int_{-\infty}^\infty |\phi_t|^2\,dx\,dt \le 4|u|^2_\infty\int_0^\tau\int_{-\infty}^\infty |\phi_x|^2\,dx\,dt + 2\epsilon\int_{-\infty}^\infty |\phi_x(0,x)|^2\,dx$$

which proves the lemma.

Now, we state the main result of the paper.

Theorem 3.5 *Assume that the conditions in Theorem 2.6 are satisfied and* $\int \tilde{\eta}(0,x)\,dx < \infty$. *Then, for* $1 < \gamma \leq 5/3$, *there exists a subsequence of* $(\rho^\epsilon, m^\epsilon, \phi^\epsilon)$ *such that*

(3.8)

$$(\rho^\epsilon(t,x), m^\epsilon(t,x), \phi^\epsilon(t,x)) \to (\rho(t,x), m(t,x), \phi(t,x)) \quad \textit{a.e. in } \Omega = [0,\tau] \times R.$$

where the triple $(\rho, m, \phi) \in L^\infty_+(\Omega) \times L^\infty(\Omega) \times W^{1,\infty}(\Omega)$ *is a weak solution to (1.4).*

Proof: It follows from Lemma 3.3 that there exists a subsequence of $(\hat{\rho}^\epsilon, \hat{m}^\epsilon)$ such that

$$(\hat{\rho}^\epsilon(t,x), \hat{m}^\epsilon(t,x)) \to (\hat{\rho}(t,x), \hat{m}(t,x)) \quad \text{a.e. in } \Omega.$$

by applying the results of [Di1] and [Ch]. It follows from Lemma 3.4 that using a standard diagonal process, there is a subsequence of $\phi^\epsilon(t,x)$ that converges a.e. in Ω, weakly in $H^1(\Omega)$ and weakly-star in $W^{1,\infty}(\Omega)$ to ϕ. Define $\rho(t,x) = \hat{\rho}(t,x)\phi(t,x)$, $m(t,x) = \hat{m}(t,x)\phi(t,x)$ a.e. $(t,x) \in \Omega$. Then, the statement (3.8) holds. It follows from the first two equations of (1.10) that

$$\int_0^\tau \int_{-\infty}^\infty ((\rho^\epsilon, m^\epsilon) \cdot (\psi_t - \epsilon\,\psi_{xx}) + F(\rho^\epsilon, m^\epsilon, \phi^\epsilon) \cdot \psi_x)\; dx\, dt = 0$$

for all $\psi \in C_c^\infty(\Omega; R^2)$. It thus follows from (3.8) and the dominated convergence theorem that (1.8) is satisfied. It follows from the third equation of (1.10) that

$$\int_0^\tau \int_{-\infty}^\infty ((\phi_t^\epsilon + u^\epsilon \phi_x^\epsilon)\,\xi + \epsilon\,\phi_x \xi_x)\; dx\, dt = 0$$

for all $\xi \in C_c^\infty(\Omega; R)$. Since $u^\epsilon \to u$ in $L^2(U)$ for any bounded rectangle $U = [0,\tau] \times [-L, L]$ and $\phi^\epsilon \to \phi$ weakly in $H^1(\Omega)$ it follows that

$$\int_0^\tau \int_{-\infty}^\infty (\phi_t + u\phi_x)\,\xi\, dx\, dt = 0$$

for all $\xi \in C_c^\infty(\Omega; R)$. Hence ϕ satisfies (1.4) a.e. in Ω. □

Corollary 3.6 *Suppose the entropy pair* (η, q) *is defined by (3.1)-(3.2). Then*

$$\int_0^\tau \int_{-\infty}^\infty (\eta\,\xi_t + q\,\xi_x)\; dx\, dt \geq 0 \tag{3.9}$$

for all $\xi \in C_c^\infty(\Omega; R)$ *satisfying* $\xi \geq 0$. *That is, the third equation of (1.1) is replaced by the inequality* $\eta_t + q_x \leq 0$ *in the sense of distributions.*

Proof: It follows from (3.3) that

$$\int_0^\tau \int_{-\infty}^\infty (\eta^\epsilon\,(\xi_t - \epsilon\,\xi_{xx}) + q^\epsilon\,\xi_x)\;dx\,dt = \epsilon \int_0^\tau \int_{-\infty}^\infty \nabla^2\eta(y^\epsilon, y^\epsilon)\,\xi\,dx\,dt$$

for all $\psi \in C_c^\infty(\Omega; R^2)$ satisfying $\xi \geq 0$. It follows from Lemma 3.1 that the right hand side of this equality is nonnegative. Thus, by taking the limit as $\epsilon \to 0^+$ we obtain (3.9) □

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