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LOCAL EXTREMA OF POSITIVE SOLUTIONS OF NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. We study the positive solutions of a general class of second-order functional differential equations, which includes delay, advanced, and delay-advanced equations. We establish integral conditions on the coefficients on a given bounded interval J such that every positive solution has a local maximum in J. Then, we use the connection between that integral condition and Rayleigh quotient to get a sufficient condition that is easier to be applied. Several examples are provided to demonstrate the importance of our results.

1. Introduction

We consider the functional differential equations of the second-order,

$$\big(r(t)x'(t)\big)' + \sum_{i=1}^{n} p_i(t) f(x(h_i(t)) + \sum_{j=1}^{m} q_j(t)|x(\tau_j(t))|^{\alpha_j - 1} x(\tau_j(t)) = e(t), \tag{1.1}$$

where $r(t) \in C^1(\mathbb{R})$, $n, m \in \mathbb{N}$ and $p_i(t), h_i(t), q_j(t), \tau_j(t), e(t) \in C(\mathbb{R})$.

Two auxiliary functions $M_{\min}(t)$ and $M_{\max}(t)$ are associated to the functional terms $h_i(t)$ and $\tau_j(t)$:

$$M_{\min}(t) = \min\{h_1(t), \dots, h_n(t), \tau_1(t), \dots, \tau_m(t)\},$$
$$M_{\max}(t) = \max\{h_1(t), \dots, h_n(t), \tau_1(t), \dots, \tau_m(t)\}.$$

For $a < b$, let $J_{a,b} \subseteq \mathbb{R}$ denote the open interval,

$$J_{a,b} = \big(\min\{a, M_{\min}(a)\}, \max\{b, M_{\max}(b)\}\big).$$

In particular,

$$J_{a,b} = \begin{cases} \big(M_{\min}(a), b\big), & \text{if } h_i(t) \le t \text{ and } \tau_j(t) \le t \text{ on } [a,b], \\ \big(a, M_{\max}(b)\big), & \text{if } h_i(t) \ge t \text{ and } \tau_j(t) \ge t \text{ on } [a,b], \end{cases} \tag{1.2}$$

for all $i \in [1,n]_{\mathbb{N}} := \{1, \dots, n\}$ and $j \in [1,m]_{\mathbb{N}} := \{1, \dots, m\}$.

The main coefficients in (1.1) satisfy

$$\begin{gathered} r(t) > 0, \quad t \in \mathbb{R} \text{ and } e(t) \le 0,\ t \in J_{a,b}, \\ p_i(t) \ge 0 \text{ and } q_j(t) \ge 0, \quad t \in J_{a,b},\ i \in [1,n]_{\mathbb{N}},\ j \in [1,m]_{\mathbb{N}}, \\ \exists i_0, j_0 \text{ such that } p_{i_0}(t) > 0 \text{ if } e(t) \equiv 0, \text{ otherwise } q_{j_0}(t) > 0. \end{gathered} \tag{1.3}$$

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For each $i \in [1,n]_{\mathbb{N}}$ and $j \in [1,m]_{\mathbb{N}}$ let exist functions $R_{h_i}(t)$ and $R_{\tau_j}(t)$ (depending on $h_i(t)$ and $\tau_j(t)$, respectively) such that for any $x \in C^2(J_{a,b})$ and $x(t) > 0$, $t \in J_{a,b}$, we have

$$\text{if } \big(r(t)x'(t)\big)' \le 0 \text{ in } J_{a,b}, \text{ then } \begin{cases} \frac{x(h_i(t))}{x(t)} \ge R_{h_i}(t) \text{ in } (a,b),\ i \in [1,n]_{\mathbb{N}}, \\ \frac{x(\tau_j(t))}{x(t)} \ge R_{\tau_j}(t) \text{ in } (a,b),\ j \in [1,m]_{\mathbb{N}}. \end{cases} \tag{1.4}$$

The so-called *generalized concave condition* (1.4) is more natural than restrictive, because it is fulfilled in the two most important functional cases, delay and advance:

$$R_g(t) = \begin{cases} \frac{g(t)-g(a)}{t-g(a)}, & \text{if } g(t) \le t \text{ and } r(t) \text{ is non-decreasing}, \\ \frac{g(b)-g(t)}{g(b)-t}, & \text{if } g(t) \ge t \text{ and } r(t) \text{ is non-increasing}, \end{cases} \tag{1.5}$$

where $g(t)$ is an arbitrary continuous functional term (see Proposition 5.1 in the appendix).

The nonlinear terms in (1.1) satisfy

$$\exists f_0 > 0,\ f(x) \ge f_0 x \quad \text{for all } x \ge 0, \tag{1.6}$$

and

$$\begin{gathered} \alpha_j \ge 0, \quad j \in [1,m]_{\mathbb{N}}, \\ \text{there exists } (\eta_0, \eta_1, \eta_2, \dots, \eta_m),\ \eta_j > 0,\ j \in [1,m]_{\mathbb{N}} \\ \text{such that } \sum_{j=0}^{m} \eta_j = 1 \text{ and } \sum_{j=1}^{m} \alpha_j \eta_j = 1. \end{gathered} \tag{1.7}$$

If $q_j(t) \equiv 0$ for all $j \in [1,m]_{\mathbb{N}}$, then the assumption (1.7) is not required. As to the existence of an $(m+1)$-tuple $(\eta_0, \eta_1, \eta_2, \dots, \eta_m)$ satisfying (1.7) with respect to a given m-tuple $(\alpha_1, \alpha_2, \dots, \alpha_m)$ such that $\alpha_1 > \dots > \alpha_{j_0} > 1 > \alpha_{j_0+1} > \dots. > \alpha_m > 0$ for some j_0, we refer the reader to [18]. Also, if $m = 1$ and $\alpha_1 > 1$, then $\eta_1 = 1/\alpha_1$ and $\eta_0 = 1 - 1/\alpha_1$ satisfy the required conditions in (1.7): $\eta_0 + \eta_1 = 1$, $\alpha_1\eta_1 = 1$ and $\eta_j > 0$.

Note that (1.1) contains several types of nonlinear functional differential equations. Here we consider several special cases.

(i) If $q_j(t) \equiv 0$ for all $j \in [1,m]_{\mathbb{N}}$, $f(x) = x$ and $e(t) \equiv 0$, then (1.1) is a linear differential equation with several functional arguments.

(ii) If $h_i(t) \le t$ and $\tau_j(t) \le t$ (resp. $h_i(t) \ge t$ and $\tau_j(t) \ge t$) for all $i \in [1,n]_{\mathbb{N}}$, $j \in [1,m]_{\mathbb{N}}$, then (1.1) is a nonlinear delay (resp. advance) differential equation with several arguments.

(iii) If $h_i(t) \le t$ and $\tau_j(t) \le t$ for all $i \in [1,i_0]_{\mathbb{N}}$, $j \in [1,j_0]_{\mathbb{N}}$ as well as $h_i(t) \ge t$ and $\tau_j(t) \ge t$ for all $i \in [i_0+1,n]_{\mathbb{N}}$, $j \in [j_0+1,m]_{\mathbb{N}}$ and $t \in \mathbb{R}$, then (1.1) is a nonlinear delayed-advanced differential equation with several arguments.

(iv) If $f(x) \equiv 0$ and $\alpha_1 > \dots > \alpha_{j_0} > 1 > \alpha_{j_0+1} > \dots > \alpha_m > 0$ for some j_0, then (1.1) is a functional differential equation with mixed nonlinearities.

As can be seen from the preceding comments, (1.1) includes several types of functional differential equations.

Definition 1.1. A function $x \in C^1(a,b)$ is non-monotonic in (a,b), if there exists $t_* \in (a,b)$ such that $x'(t_*) = 0$ and $x'(t)$ changes sign at $t = t_*$.

Recently, in [8], the authors have studied the non-monotonicity of the solutions of the delay differential equation

$$(r(t)x'(t))' + f(x(\tau(t)) = 0,$$

which is a special case of (1.1). That study used the zero-point analysis of the corresponding dual equation. In the present paper, we follow a different approach. Moreover, we obtain an integral criterion, for the non-monotonicity of solutions, which is a different type of conditions than the point-wise criterion presented in [8]. This aspect will be explained in more detail, in the following sections. For some results on the classic oscillations as a particular case of the non-monotonic behaviour of the functional differential equations, we refer the reader to [4, 5, 9, 12, 19, 20]. About the importance of non-monotonic behaviour of some modern mathematical models in applied science, see for instance in [13, 14, 21]. The global non-monotonicity (case where $(a, b) = (a, \infty)$) of the second-order differential equation

$$(r(t)x'(t))' + q(t)f(x(t)) = e(t),$$

with possible non-homogeneous term and without functional terms, has been recently considered in [15, 16]. Furthermore, in [7], for a nonlinear functional differential equation

$$(r(t)h(x)x'(t))' + q(t)f(x(g(t)) = 0,$$

the global non-monotonicity of solutions was considered in the form of weakly oscillatory solutions.

2. Main results

Theorem 2.1. *Let $a < b$ and (1.3), (1.4), (1.6) and (1.7) hold. Let $\Theta(t)$ be a function defined as*

$$\Theta(t) = f_0 \sum_{i=1}^{n} p_i(t) R_{h_i}(t) + \Big(\frac{|e(t)|}{\eta_0}\Big)^{\eta_0} \prod_{j=1}^{m} \Big(\frac{q_j(t)}{\eta_j}\Big)^{\eta_j} [R_{\tau_j}(t)]^{\eta_j \alpha_j}, \tag{2.1}$$

where $R_{h_i}(t)$, $R_{\tau_j}(t)$, f_0, and η_j are defined in (1.4) (1.6) and (1.7), respectively. If $a < a' < b' < b$ and there exists a test function $\varphi \in C([a', b']) \cap C^1(a', b')$, $\varphi(a') = \varphi(b') = 0$ such that

$$\int_{a'}^{b'} \frac{\varphi^2(t)}{r(t)} dt > \int_{a'}^{b'} \frac{1}{\Theta(t)} \Big(\frac{d\varphi}{dt}\Big)^2 dt, \tag{2.2}$$

then every positive solution $x(t)$ of (1.1) has a local maximum in (a, b) and is non-monotonic in (a, b).

Remark 2.2. (i) The restriction from (a, b) to (a', b') in (2.2) is necessary, because often $R_{h_i}(a) = 0$ or $R_{h_i}(b) = 0$ (resp., $R_{\tau_j}(a) = 0$ or $R_{\tau_j}(b) = 0$), see for instance (1.5). In such a case, $\Theta(a) = 0$ or $\Theta(b) = 0$; hence, to avoid any singular behaviour in the right integral in (2.2), we use $[a', b']$ for the domain of integration, where $a < a' < b' < b$.

(ii) If $e(t) \equiv 0$, then $\Theta(t)$ is reduced to the first sum. Hence in (1.3), we assume the existence of a number k such that $p_k(t) > 0$, in order to avoid $\Theta(t) = 0$, for some $t \in (a, b)$.

In the Sturm-Liouville theory and the variational characterization of lowest eigenvalues, the next *Rayleigh quotient* plays a crucial role,

$$\mathcal{R}(\varphi)=\frac{\int_{a'}^{b'}\left(\frac{d\varphi}{dt}\right)^2dt}{\int_{a'}^{b'}\varphi^2(t)dt},\quad \varphi\in C^1(a',b'),\ \varphi\not\equiv 0.$$

It is not difficult to see that if $a<a'<b'<b$ and there exists a test function $\varphi\in C([a',b'])\cap C^1(a',b')$, $\varphi(a')=\varphi(b')=0$ such that

$$\frac{\min_{t\in[a',b']}\Theta(t)}{\max_{t\in[a',b']}r(t)}>\mathcal{R}(\varphi), \tag{2.3}$$

then (2.2) holds. Hence, Theorem 2.1 takes the simple form:

Theorem 2.3. *Let $a<b$ and (1.3), (1.4), (1.6) and (1.7) hold. Let $\Theta(t)$ be the function defined by (2.1). If $a<a'<b'<b$ and there exists a test function $\varphi\in C([a',b'])\cap C^1(a',b')$, $\varphi(a')=\varphi(b')=0$ such that (2.3) holds, then every positive solution $x(t)$ of (1.1) is non-monotonic in (a,b), having a local maximum in (a,b).*

The well-known variational principle (see [2, 11, 17]), which has been formulated in a higher dimensional case and is also known as the Courant-Fisher formula (see [3]) or the Rayleigh-Ritz variational formula (see [6]), says that for a set of eigenvalues λ of the second-order Dirichlet problem $\varphi''+\lambda\varphi=0$, $\varphi(a')=\varphi(b')=0$ which consists of a sequence $(\lambda_n)_{n\in\mathbb{N}}$ satisfying $0<\lambda_1<\lambda_2\dots<\lambda_n<\dots$ and $\lim_{n\to\infty}\lambda_n=\infty$, we have that

$$\lambda_1=\min\left\{\mathcal{R}(\varphi):\varphi\in C_0^2(a',b'),\varphi\not\equiv 0\right\}=\mathcal{R}(\varphi_1),$$

where $\varphi_1(t)=\sin\left(\pi(t-a')/(b'-a')\right)$ is the eigenvector which corresponds to eigenvalue λ_1. Hence, $\lambda_1=\left(\pi/(b'-a')\right)^2$. Now we can use this formula to simplify inequality (2.3) which lead us to a more applicable result, stated in the following theorem.

Theorem 2.4. *Let $a<b$ and (1.3), (1.4), (1.6) and (1.7) hold. Let $\Theta(t)$ be the function defined by (2.1) and let $a<a'<b'<b$. If the inequality*

$$\frac{\min_{t\in[a',b']}\Theta(t)}{\max_{t\in[a',b']}r(t)}>\left(\frac{\pi}{b'-a'}\right)^2 \tag{2.4}$$

holds, then every positive solution $x(t)$ of (1.1) is non-monotonic in (a,b), having a local maximum in (a,b).

Furthermore, it is known that the previous observation can be generalized to the Rayleigh quotient and the corresponding eigenvalue problem, using the weight $\omega(t)=1/r(t)$. In that case, we have

$$\mathcal{R}_{1/r}(\varphi)=\frac{\int_{a'}^{b'}\left(\frac{d\varphi}{dt}\right)^2dt}{\int_{a'}^{b'}\frac{\varphi^2(t)}{r(t)}dt},\quad \varphi\in C^1(a',b'),\ \varphi\not\equiv 0,$$

$$\varphi''+\frac{\lambda}{r(t)}\varphi=0,\quad \varphi(a')=\varphi(b')=0.$$

Then the set of all eigenvalues λ is represented by the sequence $(\lambda_n)_{n\in\mathbb{N}}$ satisfying: $0<\lambda_1<\lambda_2\dots<\lambda_n<\dots$, $\lim_{n\to\infty}\lambda_n=\infty$ and

$$\lambda_1=\min\left\{\mathcal{R}_{1/r}(\varphi):\varphi\in C_0^2(a',b'),\ \varphi\not\equiv 0\right\}=\mathcal{R}_{1/r}(\varphi_1), \tag{2.5}$$

where $\varphi_1(t)$ is the eigenvector which corresponds to the eigenvalue λ_1. Hence, each of the following two conditions (C1) and (C2) implies the inequality which constitutes the main assumption (2.2), in Theorem 2.1.

- (C1) There exists a test function $\varphi \in C([a',b']) \cap C^1(a',b')$, $\varphi(a') = \varphi(b') = 0$ such that $\min_{t\in[a',b']} \Theta(t) > \mathcal{R}_{1/r}(\varphi)$.
- (C2) It holds that $\min_{t\in[a',b']} \Theta(t) > \lambda_1$, where λ_1 is given by (2.5).

Consequently, we have the following theorem.

Theorem 2.5. *Let $a < b$ and (1.3), (1.4), (1.6) and (1.7) hold. Let $\Theta(t)$ be the function defined by (2.1) and $a < a' < b' < b$. If either (C1) or (C2) holds, then every positive solution $x(t)$ of (1.1) is non-monotonic in (a,b), having a local maximum in (a,b).*

3. Proofs of the main results

First, we postulate three lemmas which we will use to prove Theorem 2.1.

Lemma 3.1. *If (1.3) and (2.2) hold, then the differential inequality*

$$\frac{dw}{dt} \geq \frac{1}{r(t)} + \Theta(t)w^2, \ t \in (a,b), \tag{3.1}$$

does not allow any solution $w \in C^1(a,b)$, where $\Theta(t)$ is defined by (2.1).

Proof. Assume the opposite of the lemma's conclusion, namely that there exists a function $w \in C^1(a,b)$ satisfying the differential inequality (3.1). Multiplying (3.1) with $\varphi^2(t)$ where φ is a test function $\varphi \in C_0([a',b']) \cap C^1(a',b')$ and $a < a' < b' < b$ and then integrating the resulting inequality on $[a',b']$, we obtain

$$\begin{aligned}
\int_{a'}^{b'} \frac{\varphi^2(t)}{r(t)} dt &\leq -\int_{a'}^{b'} \Theta(t)w^2(t)\varphi^2(t)dt + \int_{a'}^{b'} \frac{dw}{dt}\varphi^2(t)dt \\
&= -\int_{a'}^{b'} \Theta(t)w^2(t)\varphi^2(t)dt - 2\int_{a'}^{b'} w(t)\varphi(t)\varphi'(t)dt \\
&= -\int_{a'}^{b'} \Big[\big(w(t)\varphi(t)\sqrt{\Theta(t)}\big)^2 + 2w(t)\varphi(t)\sqrt{\Theta(t)}\frac{\varphi'(t)}{\sqrt{\Theta(t)}}\Big]dt \\
&= -\int_{a'}^{b'} \Big(w(t)\varphi(t)\sqrt{\Theta(t)} + \frac{\varphi'(t)}{\sqrt{\Theta(t)}}\Big)^2 dt + \int_{a'}^{b'} \frac{\varphi'^2(t)}{\Theta(t)} dt \\
&\leq \int_{a'}^{b'} \frac{\varphi'^2(t)}{\Theta(t)} dt,
\end{aligned}$$

which is a contradiction to assumption (2.2). Thus, (3.1) does not allow any solution $w \in C^1(a,b)$, which proves this lemma. □

Lemma 3.2. *Let (1.3), (1.4), (1.6) and (1.7) hold. If the differential inequality*

$$\frac{dw}{dt} \geq \frac{1}{r(t)} + \Theta(t)w^2, \ t \in (a,b), \tag{3.2}$$

does not allow any solution $w \in C^1(a,b)$, then every positive solution $x(t)$ of (1.1) has a stationary point in (a,b).

Proof. Suppose to the contrary that $x(t)$ is a positive solution of (1.1), having no stationary point on (a,b), that is,

$$x'(t) \neq 0 \quad \text{on } (a,b). \tag{3.3}$$

According to (3.3), the function

$$w(x) = \frac{x(t)}{r(t)x'(t)},\ t \in (a,b), \tag{3.4}$$

is well defined and $w \in C^1(a,b)$. Now, we recall the well-known arithmetic-geometric mean inequality (see [10]),

$$\text{if } A_j \geq 0,\ \eta_j > 0 \text{ and } \sum_{j=0}^{m} \eta_j = 1, \text{ then } \sum_{j=0}^{m} \eta_j A_j \geq \prod_{j=0}^{m} A_j^{\eta_j}$$

and use that inequality, taking

$$A_0 = \frac{|e(t)|}{\eta_0} \quad \text{and} \quad A_j = \frac{q_j(t)|x(\tau_j(t))|^{\alpha_j}}{\eta_j}, \quad j \in [1,m]_{\mathbb{N}}.$$

Note that we can use (1.4), because from (1.1), (1.3) and $x(t) \geq 0$, we have $(r(t)x'(t))' \leq 0$ in $J_{a,b}$. Then by means of (1.3), (1.4), (1.6) and (1.7), we have

$$\begin{aligned}
&\frac{d\omega}{dt} \\
&= \frac{1}{r(t)} - \frac{x(t)}{(r(t)x'(t))^2}(r(t)x'(t))' \\
&= \frac{1}{r(t)} + \frac{x(t)}{(r(t)x'(t))^2}\Big[\sum_{i=1}^{n} p_i(t) f(x(h_i(t)) + \sum_{j=1}^{m} q_j(t)|x(\tau_j(t))|^{\alpha_j - 1}x(\tau_j(t)) - e(t)\Big] \\
&= \frac{1}{r(t)} + \omega^2(t)\Big\{\sum_{i=1}^{n} p_i(t)\frac{f(x(h_i(t))}{x(t)} + \frac{1}{x(t)}\Big[\sum_{j=1}^{m} q_j(t)|x(\tau_j(t))|^{\alpha_j} + e(t)\Big]\Big\} \\
&\geq \frac{1}{r(t)} + \omega^2(t)\Big\{f_0\sum_{i=1}^{n} p_i(t)R_{h_i}(t) + \frac{1}{x(t)}\Big[\sum_{j=1}^{m} \eta_j\Big(\frac{q_j(t)|x(\tau_j(t))|^{\alpha_j}}{\eta_j}\Big) + \eta_0\Big(\frac{|e(t)|}{\eta_0}\Big)\Big]\Big\} \\
&\geq \frac{1}{r(t)} + \omega^2(t)\Big\{f_0\sum_{i=1}^{n} p_i(t)R_{h_i}(t) + \frac{1}{x(t)}\Big(\frac{|e(t)|}{\eta_0}\Big)^{\eta_0}\prod_{j=1}^{m}\Big(\frac{q_j(t)}{\eta_j}\Big)^{\eta_j}|x(\tau_j(t))|^{\eta_j\alpha_j}\Big\} \\
&= \frac{1}{r(t)} + \omega^2(t)\Big\{f_0\sum_{i=1}^{n} p_i(t)R_{h_i}(t) + \Big(\frac{|e(t)|}{\eta_0}\Big)^{\eta_0}\prod_{j=1}^{m}\Big(\frac{q_j(t)}{\eta_j}\Big)^{\eta_j}\Big(\frac{|x(\tau_j(t))|}{x(t)}\Big)^{\eta_j\alpha_j}\Big\} \\
&\geq \frac{1}{r(t)} + \omega^2(t)\Big\{f_0\sum_{i=1}^{n} p_i(t)R_{h_i}(t) + \Big(\frac{|e(t)|}{\eta_0}\Big)^{\eta_0}\prod_{j=1}^{m}\Big(\frac{q_j(t)}{\eta_j}\Big)^{\eta_j}[R_{\tau_j}(t)]^{\eta_j\alpha_j}\Big\} \\
&= \frac{1}{r(t)} + \Theta(t)w^2,\ t \in (a,b).
\end{aligned}$$

Thus, the function w defined in (3.4) satisfies the differential inequality (3.1), which contradicts the main assumption of this lemma. Therefore, every positive solution $x(t)$ of (1.1) has a stationary point in (a,b). $\square$

Lemma 3.3. *Let* (1.3) *and* (1.6) *hold and* $x(t)$ *be a solution of* (1.1) *such that* $x(t) > 0$ *on* $[a, b]$. *If* $t_* \in (a, b)$ *is a stationary point of* $x(t)$, *then* $x(t)$ *attains a local maximum at* t_*.

Proof. Let $t_* \in (a, b)$ be a point such that $x'(t_*) = 0$. Integrating (1.1) over $[t_*, t]$ for all $t \in (a, b)$, we obtain

$$x'(t) = \frac{-1}{r(t)} \int_{t_*}^{t} \Big[\sum_{i=1}^{n} p_i(t) f(x(h_i(t)) + \sum_{i=j}^{m} q_j(t) |x(\tau_j(t))|^{\alpha_j - 1} x(\tau_j(t)) - e(t) \Big]. \quad (3.5)$$

According to (1.3), (1.6) and $x(t) > 0$, the integral function in (3.5) is positive in (a, b) and hence, the right hand-side in (3.5) is negative for $t > t_*$ and positive for $t < t_*$. That shows that t_* is a point of local maximum of $x(t)$. □

Note that the statements in lemmas 3.2 and 3.3 are mutually independent.

Proof of Theorem 2.1. It follows the assumptions of theorem and Lemma 3.1 that the differential inequality (3.1) does not have any solution. Now, from Lemma 3.2 we get that every positive solution has a stationary point and by Lemma 3.3 we have that this stationary point has to be a maximum. □

Proof of Theorem 2.3. It can be shown by a straightforward calculation that inequality (2.2) follows from inequality (2.3). □

Proof of Theorem 2.4. We can construct a test function $\varphi \in C([a', b']) \cap C^1(a', b')$, $\varphi(a') = \varphi(b') = 0$ such that

$$\frac{\int_{a'}^{b'} \varphi'^2(t) dt}{\int_{a'}^{b'} \varphi^2(t) dt} = \Big(\frac{\pi}{b' - a'} \Big)^2. \quad (3.6)$$

It is easy to show that the function $\varphi(t) = A \sin\big(\pi \frac{t - a'}{b' - a'}\big)$ is such a test function. Now, the statement follows from Theorem 2.3. □

4. Examples

In this section, we illustrate our main results, trough two simple examples.

Example 4.1. Consider the differential equation

$$x'' + A \sin(\omega t) x(t - \tau) = e(t), \quad (4.1)$$

where $A > 0$, $\omega > 0$, $\tau \geq 0$ and $e(t)$ is an arbitrary continuous function. The above equation is of the type of (1.1) with $r(t) \equiv 1$, $n = 1$, $p_1(t) = A \sin(\omega t)$, $f(x) = x$, $h_1(t) = t - \tau$, $m = 1$ and $q_1(t) \equiv 0$. Let $(a, b) \subset (\tau, \pi/\omega)$ be an open interval such that

$$\tau < a < \frac{\pi}{6\omega}, \quad A > \frac{9\omega^2 \big(\frac{\pi}{6\omega} - a + \tau \big)}{2 \big(\frac{\pi}{6\omega} - a \big)}, \quad \frac{5\pi}{6\omega} < b < \frac{\pi}{\omega}, \quad e(t) \leq 0 \quad \text{in } (a, b). \quad (4.2)$$

If especially, $\tau < a \leq \frac{\pi}{12\omega}$ and $A \geq 9\omega^2$, we can easily see the first two inequalities in (4.2) are satisfied, because

$$A \geq 9\omega^2 = \frac{9\omega^2}{2} 2 \geq \frac{9\omega^2 \big(\frac{\pi}{6\omega} \big)}{2 \big(\frac{\pi}{6\omega} - a \big)} > \frac{9\omega^2 \big(\frac{\pi}{6\omega} - a + \tau \big)}{2 \big(\frac{\pi}{6\omega} - a \big)}.$$

We claim that *every positive solution of equation* (4.1) *has a local maximum in* (a, b), *provided* (4.2) *holds.* Note that this statement cannot easily be derived, even in the homogeneous case ($e(t) \equiv 0$). In that case, from (4.1), we have $x'' = -A\sin(\omega t)x(t-\tau)$. If $x(t)$ is a positive solution, the preceding equality implies that $x''(t)$ is sign-changing, but in general, does not imply that $x'(t)$ is sign-changing.

To show the above statement, we use Theorem 2.4. At first, we see that the set $J_{a,b}$ defined in (1.2) satisfies $J_{a,b} = (a-\tau, b) \subset (0, \pi/\omega)$, which implies

$$p_1(t) = A\sin(\omega t) > 0, \quad t \in J_{a,b}.$$

Hence, (1.3) is satisfied. Since $h_1(t) = t - \tau \le t$ and $r(t) = 1$ is non-decreasing, it follows that (1.4) holds because of (1.5), where

$$R_{h_1}(t) = \frac{h_1(t) - h_1(a)}{t - h_1(a)} = \frac{t-a}{t-a+\tau}, \quad t > a.$$

From $R_{h_1}(t)$ being an increasing function, we have that in any $[a', b'] \subset (0, \infty)$,

$$\min_{t\in[a',b']} R_{h_1}(t) = R_{h_1}(a'). \tag{4.3}$$

Since $f(x) = x$, we have that (1.6) holds with $f_0 = 1$. Since $q_j(t) \equiv 0$ for all $j \in [1, m]_{\mathbb{N}}$, we do not need assumption (1.7).

Now, let $[a', b'] = [\frac{\pi}{6\omega}, \frac{5\pi}{6\omega}]$. Since $r(t) = 1$, from (2.1), (4.2) and (4.3), we derive that $a < a' < b' < b$ and

$$\frac{\min_{t\in[a',b']}\Theta(t)}{\max_{t\in[a',b']} r(t)} = \min_{t\in[a',b']}\big[A\sin(t)R_{h_1}(t)\big] = \frac{A}{2}\,\frac{\frac{\pi}{6\omega} - a}{\frac{\pi}{6\omega} - a + \tau} > \frac{9\omega^2}{4} = \Big(\frac{\pi}{b'-a'}\Big)^2.$$

Therefore, (2.4) is satisfied. Consequently, all conditions of Theorem 2.4 are fulfilled, thus establishing the main statement of this example.

Example 4.2. Consider the special case of the Duffing equation with time delay feedback,

$$x'' + \omega_0 x + \beta x^3 + \lambda\sin(t)x(t-\tau) = -\cos(t/2), \tag{4.4}$$

where $\omega_0 > 0$ is natural frequency, $\beta > 0$, $\lambda > 0$ is the gain parameter and $\tau \ge 0$. Equation (4.4) is a particular case of the main equation (1.1) with

$$\begin{gathered} r(t) = 1, \quad n = 2, \quad p_1(t) = \omega_0, \quad h_1(t) = t, \quad p_2(t) = \lambda\sin(t), \quad h_2(t) = t - \tau, \\ f(x) = x, \quad m = 1, \quad q_1(t) = \beta, \quad \tau_1(t) = t, \quad \alpha_1 = 3, \quad e(t) = -\cos(t/2). \end{gathered} \tag{4.5}$$

Note that the cubic term βx^3, introducing a strong nonlinearity into the equation, cannot be considered as part of the linear term $\sum_{i=1}^n p_i(t)f(x(h_i(t)))$, because $f(x) = \beta x^3$ does not satisfy the required condition (1.6). Let (a, b) be an open interval such that

$$\tau < a < \frac{\pi}{3}, \quad \frac{2\pi}{3} < b < \pi \quad \text{and} \quad [a', b'] = \Big[\frac{\pi}{3}, \frac{2\pi}{3}\Big].$$

Since $m = 1$, condition (1.7) is always satisfied, because for $\alpha_1 = 3$, the system

$$\eta_0 + \eta_1 = 1 \quad \text{and} \quad \eta_1\alpha_1 = 1, \quad \eta_j > 0,$$

imply $\eta_0 = 2/3$ and $\eta_1 = 1/3$. Hence, from (1.5), (2.1) and (4.5), we have $R_{h_1}(t) = 1$, $R_{\tau_1}(t) = 1$,

$$R_{h_2}(t) = \frac{t-a}{t-a+\tau} \quad \Theta(t) = \omega_0 + \lambda\sin(t)R_{h_2}(t) + \frac{3}{2^{2/3}}\beta^{1/3}|\cos(t/2)|^{2/3}. \tag{4.6}$$

Note that $\sin(t) > 0$ and $-\cos(t/2) \le 0$ on $[a-\tau, b]$ as well as $R_{h_2}(a') \le R_{h_2}(t)$ for all $t \in [a', b']$. Hence from (4.6), we obtain

$$\begin{aligned}
\frac{\min_{t\in[a',b']}\Theta(t)}{\max_{t\in[a',b']} r(t)} &= \min_{t\in[a',b']}\Theta(t) = \omega_0 + \frac{\sqrt{3}\lambda}{2}\frac{\pi-3a}{\pi-3a+3\tau} + \frac{3}{2^{2/3}}\beta^{1/3}(1/2)^{1/3} \\
&\ge \min\{\omega_0, \lambda, \beta^{1/3}\}\Big(\frac{5}{2} + \frac{\sqrt{3}}{2}\frac{\pi-3a}{\pi-3a+3\tau}\Big) \\
&> 9 = \Big(\frac{\pi}{b'-a'}\Big)^2,
\end{aligned}$$

provided

$$\min\{\omega_0, \lambda, \beta^{1/3}\} > \frac{18(\pi-3a+3\tau)}{(5+\sqrt{3})(\pi-3a)+15\tau}. \tag{4.7}$$

Now, by Theorem 2.4, if (4.7) is true, then every positive solution of equation (4.4) has a local maximum in (a, b).

5. Appendix

In this section, we state a proposition that justifies why the generalized condition (1.4) is fulfilled both in the delay and the advanced cases, for any functional term $g(t)$, satisfying (1.5). Below, we show this proposition, for the delay case where $R_g(t)$ is defined by the upper branch of (1.5), i.e.,

$$g(a) < g(t) \le t,\ \ t \in (a, b). \tag{5.1}$$

Note that condition (5.1) holds especially, for the standard delay term $g(t) = t - \tau$, $\tau > 0$. The proposition can be stated in a corresponding manner, for the advanced case and has a similar proof, for that case.

Proposition 5.1. *Let the functional term $g(t)$ satisfy (5.1), $J_{a,b} := (g(a), b)$ and $r(t)$ be a non-decreasing positive function on $J_{a,b}$. If $x \in C^2(J_{a,b})$, $x(s) > 0$, $s \in J_{a,b}$ and*

$$\big(r(s)x'(s)\big)' \le 0,\ \ s \in J_{a,b}, \tag{5.2}$$

then

$$\frac{x(g(t))}{x(t)} \ge \frac{g(t)-g(a)}{t-g(a)},\ \ t \in (a, b). \tag{5.3}$$

Proof. We will proceed by showing that assumption (5.2) implies

$$\frac{x'(s)}{x(s)} \le \frac{1}{s-g(a)}, \quad s \in J_{a,b}. \tag{5.4}$$

Since $(g(t), t) \subseteq J_{a,b}$ for any $t \in (a, b)$, integrating (5.4) over $[g(t), t]$, we obtain

$$\ln\frac{x(t)}{x(g(t))} \le \ln\frac{t-g(a)}{g(t)-g(a)},$$

which proves the desired inequality (5.3). Thus, the proof of the proposition reduces to establishing that assumption (5.2) implies (5.4).

Since $x(s) > 0$ on $J_{a,b}$, let us remark that (5.4) is trivially satisfied for all $s \in J_{a,b}$ such that $x'(s) \le 0$. Hence, let $s \in J_{a,b}$ be such that $x'(s) \ge 0$. Now, integrating (5.2) over (σ, s) for every $\sigma \in J_{a,b}$ such that $\sigma < s$, we have

$$0 \le r(s)x'(s) \le r(\sigma)x'(\sigma). \tag{5.5}$$

Since $r(t)$ is non-decreasing, we have $r(\sigma) \le r(s)$, which together with (5.5), imply

$$x'(s) \le \frac{r(\sigma)}{r(s)} x'(\sigma) \le x'(\sigma), \quad \text{for all } \sigma \in J_{a,b} \text{ such that } \sigma < s.$$

Now, by the Lagrange mean value theorem on $(g(a), s)$, there exists a $\sigma \in (g(a), s)$ such that $x(s) - x(g(a)) = x'(\sigma)(s - g(a))$. Since $x(g(a)) \ge 0$, we have that

$$x(s) \ge x'(\sigma)(s - g(a)) \ge x'(s)(s - g(a)),$$

which proves the required inequality (5.4). □

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