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COMPOSITION AND CONVOLUTION THEOREMS FOR μ-STEPANOV PSEUDO ALMOST PERIODIC FUNCTIONS AND APPLICATIONS TO FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this article we establish new convolution and composition theorems for μ-Stepanov pseudo almost periodic functions. We prove that the space of vector-valued μ-Stepanov pseudo almost periodic functions is a Banach space. As an application, we prove the existence and uniqueness of μ-pseudo almost periodic mild solutions for the fractional integro-differential equation

$$D^{\alpha}u(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)\,ds + f(t,u(t)),$$

where A generates an α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ on a Banach space X, $a \in L^1_{\rm loc}(\mathbb{R}_+)$, $\alpha > 0$, the fractional derivative is understood in the sense of Weyl and the nonlinearity f is a μ-Stepanov pseudo almost periodic function.

1. Introduction

Ezzinbi et al. [1] defined the space of μ-S^p-pseudo almost periodic functions. This space contains the space of Stepanov-like weighted pseudo almost periodic functions (see [8, 11]) and the space of μ-pseudo almost periodic functions (see [5]). Several composition theorems and their applications in the context of Stepanov-like almost periodic, Stepanov-like pseudo almost periodic and Stepanov-like weighted pseudo almost periodic functions appear for example in [2, 9, 10, 12, 14]. Here we generalize the composition theorem given by Zhao et al. for the space of Stepanov-like weighted pseudo almost periodic functions (see [14, Th. 2.15]). Also, we recover the composition result given by Ezzinbi et al. for μ-S^p-pseudo almost periodic functions (see [1, Th. 2.29]). Moreover, we establish another composition theorem that does not require Lipschitzian nonlinearities (Theorem 3.5 and Theorem 3.8).

In Theorem 3.10 we prove that the convolution of a strongly continuous family $\{S(t)\}_{t\geq 0}$ with a μ-S^p-pseudo almost periodic function F, $(S*f)(t) = \int_{-\infty}^{t} S(t-s)F(s)ds$, is a μ-pseudo almost periodic function. We prove that the collection of μ-S^p-pseudo almost periodic functions is a Banach space with a natural norm (Theorem 3.3), and combine our results to prove the existence and uniqueness

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of μ-pseudo almost periodic solutions to a class of abstract fractional differential equations

$$D^\alpha u(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)\,ds + f(t,u(t)), \tag{1.1}$$

where A generates an α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ on a Banach space X, $a \in L^1_{\rm loc}(\mathbb{R}_+)$, $\alpha > 0$, the fractional derivative is understood in the sense of Weyl and provided that the nonlinear term f is μ-Stepanov pseudo almost periodic.

2. Preliminaries

Throughout this article $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ denote complex Banach spaces and $B(X,Y)$ the Banach space of bounded linear operators from X to Y; when $X = Y$ we write $B(X)$.

We denote by $BC(\mathbb{R}, X)$ the Banach space of X-valued bounded and continuous defined functions on $\mathbb{R}$, with norm

$$\|f\| = \sup\{\|f(t)\|_X : t \in \mathbb{R}\}. \tag{2.1}$$

Definition 2.1 ([6])**.** A function $f \in C(\mathbb{R}, X)$ is called (Bohr) almost periodic if for each $\epsilon > 0$ there exists $l = l(\epsilon) > 0$ such that every interval of length l contains a number τ with the property that

$$\|f(t+\tau) - f(t)\| < \epsilon \quad (t \in \mathbb{R}).$$

The collection of all such functions will be denoted by $AP(\mathbb{R}, X)$.

This definition is equivalent to the so-called Bochner's criterion, namely, $f \in AP(\mathbb{R}, X)$ if and only if for every sequence of reals (s'_n) there exists a subsequence (s_n) such that $(f(\cdot + s_n))$ is uniformly convergent on $\mathbb{R}$.

Definition 2.2 ([6])**.** A function $f \in C(\mathbb{R}\times Y, X)$ is called (Bohr) almost periodic in $t \in \mathbb{R}$ uniformly in $y \in K$ where $K \subset Y$ is any compact subset if for each $\epsilon > 0$ there exists $l = l(\epsilon) > 0$ such that every interval of length l contains a number τ with the property that

$$\|f(t+\tau, y) - f(t,y)\| < \epsilon \quad (t \in \mathbb{R},\ y \in K).$$

The collection of such functions will be denoted by $AP(\mathbb{R}\times Y, X)$.

Let $\mathcal{B}$ denote the Lebesgue σ-field of $\mathbb{R}$, see [4]. Let $\mathcal{M}$ stand for the set of all positive measures ν on $\mathcal{B}$ satisfying $\mu(\mathbb{R}) = \infty$ and $\mu([a,b]) < \infty$ for all $a, b \in \mathbb{R}$. Throughout this paper will consider the following hypotheses:

- (H1) For all a, b and $c \in \mathbb{R}$, such that $0 \leq a < b \leq c$, there exist $\tau_0 \geq 0$ and $\alpha_0 > 0$ such that
$$|\tau| \leq \tau_0 \Rightarrow \mu((a+\tau, b+\tau)) \geq \alpha_0 \mu([\tau, c+\tau]).$$
- (H2) For all $\tau \in \mathbb{R}$, there exist $\beta > 0$ and a bounded interval I such that $\mu(\{a+\tau, a \in A\}) \leq \beta\mu(A)$ if $A \in \mathcal{B}$ satisfies $A \cap I = \emptyset$.

Note that Hypothesis (H2) implies (H1), see [5, Lemma 2.1].

Definition 2.3 ([4])**.** Let $\mu \in \mathcal{M}$. A function $f \in BC(\mathbb{R}, X)$ is said to be μ-ergodic if

$$\lim_{T\to+\infty} \frac{1}{\mu([-T,T])} \int_{[-T,T]} \|f(t)\| d\mu(t) = 0.$$

We denote by $\mathcal{E}(\mathbb{R}, X, \mu)$ the set of such functions. A function $f \in BC(\mathbb{R} \times X, X)$ is said to be μ-ergodic if

$$\lim_{T\to+\infty} \frac{1}{\mu([-T,T])} \int_{[-T,T]} \|f(t,x)\| d\mu(t) = 0,$$

uniformly in $x \in X$. Denote by $\mathcal{E}(\mathbb{R} \times X, X, \mu)$ the set of such functions.

Definition 2.4 ([5])**.** Let $\mu \in \mathcal{M}$. A function $f \in C(\mathbb{R}, X)$ is said to be μ-pseudo almost periodic if it can be decomposed as $f = g + \varphi$, where $g \in AP(\mathbb{R}, X)$ and $\varphi \in \mathcal{E}(\mathbb{R}, X, \mu)$. Denote by $PAP(\mathbb{R}, X, \mu)$ the collection of such functions.

Definition 2.5 ([11])**.** The Bochner transform $f^b(t, s)$ with $t \in \mathbb{R}, s \in [0, 1]$ of a function $f : \mathbb{R} \to X$ is defined by

$$f^b(t, s) := f(t + s).$$

Definition 2.6 ([11])**.** The Bochner transform $f^b(t, s, u)$ with $t \in \mathbb{R}$, $s \in [0, 1]$, $u \in X$ of a function $f : \mathbb{R} \times X \to X$ is defined by

$$f^b(t, s, u) := f(t + s, u) \quad \text{for all } u \in X.$$

Definition 2.7 ([11])**.** Let $p \in [1, \infty)$. The space $BS^p(\mathbb{R}, X)$ of all Stepanov bounded functions, with exponent p, consist of all measurable functions $f : \mathbb{R} \to X$ such that $f^b \in L^\infty(\mathbb{R}, L^p(0, 1; X))$. This is a Banach space with the norm

$$\|f\|_{BS^p(\mathbb{R},X)} := \|f^b\|_{L^\infty(\mathbb{R},L^p)} = \sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} \|f(\tau)\|^p \, d\tau\Big)^{1/p}.$$

Definition 2.8 ([8])**.** A function $f \in BS^p(\mathbb{R}, X)$ is called Stepanov almost periodic if $f^b \in AP(\mathbb{R}, L^p(0, 1; X))$. We denote the set of all functions by $APS^p(\mathbb{R}, X)$.

Definition 2.9 ([8])**.** A function $f : \mathbb{R} \times X \to Y$ with $f(\cdot, u) \in BS^p(\mathbb{R}, Y)$, for each $u \in X$, is called Stepanov almost periodic function in $t \in \mathbb{R}$ uniformly for $u \in X$ if, for each $\epsilon > 0$ and each compact set $K \subset X$ there exists a relatively dense set $P = P(\epsilon, f, K) \subset \mathbb{R}$ such that

$$\sup_{t\in\mathbb{R}} \Big(\int_0^1 \|f(t + s + \tau, u) - f(t + s, u)\| \, ds\Big)^{1/p} < \epsilon,$$

for each $\tau \in P$ and each $u \in K$. We denote by $APS^p(\mathbb{R} \times X, Y)$ the set of such functions.

Definition 2.10 ([1])**.** Let $\mu \in \mathcal{M}$. A function $f \in BS^p(\mathbb{R}, X)$ is said μ-Stepanov-like pseudo almost periodic (or μ-S^p-pseudo almost periodic) if it can be expressed as $f = g + \phi$, where $g \in APS^p(\mathbb{R}, X)$ and $\phi^b \in \mathcal{E}(\mathbb{R}, L^p(0, 1; X), \mu)$. In other words, a function $f \in L^p_{\rm loc}(\mathbb{R}, X)$ is said μ-S^p-pseudo almost periodic relatively to measure μ, if its Bochner transform $f^b : \mathbb{R} \to L^p(0, 1; X)$ is μ-pseudo almost periodic in the sense that there exist two functions $g, \phi : \mathbb{R} \to X$ such that $f = g + \phi$, where $g \in APS^p(\mathbb{R}, X)$ and $\phi^b \in \mathcal{E}(\mathbb{R}, L^p(0, 1; X), \mu)$, that is $\phi^b \in BC(\mathbb{R}, L^p(0, 1; X))$ and

$$\lim_{T\to+\infty} \frac{1}{\mu([-T,T])} \int_{[-T,T]} \Big(\int_t^{t+1} \|\phi(s)\|^p ds\Big)^{1/p} d\mu(t) = 0.$$

We denote by $PAPS^p(\mathbb{R}, X, \mu)$ the set of all such functions.

Definition 2.11 ([1])**.** Let $\mu \in \mathcal{M}$. A function $f : \mathbb{R} \times Y \to X$ with $f(\cdot, u) \in L^p_{\rm loc}(\mathbb{R}, X)$ for each $u \in Y$, is said to be μ-Stepanov-like pseudo almost periodic (or μ-S^p-pseudo almost periodic) if it can be expressed as $f = g + \phi$, where $g \in APS^p(\mathbb{R} \times Y, X)$ and $\phi^b \in \mathcal{E}(\mathbb{R} \times Y, L^p(0, 1; X), \mu)$. We denote by $PAPS^p(\mathbb{R} \times Y, X, \mu)$ the set of all such functions.

3. Main results

For $1 \le p < \infty$, we define $\mathcal{B} : BS^p(\mathbb{R}, X) \to L^\infty(\mathbb{R}, L^p(0, 1; X))$ by

$$f \mapsto (\mathcal{B}f)(t)(s) = f^b(t, s) = f(t + s) \quad (t \in \mathbb{R},\ s \in [0, 1]),$$

see [2].

Remark 3.1. It follows from its definition that the operator $\mathcal{B}$ is a linear isometry between $BS^p(\mathbb{R}, X)$ and $L^\infty(\mathbb{R}, L^p(0, 1; X))$. More precisely,

$$\|\mathcal{B}f\|_{L^\infty(\mathbb{R},L^p)} = \|f\|_{BS^p(\mathbb{R},X)}.$$

Remark 3.2. The definition of μ-Stepanov-like pseudo almost periodic functions can be written using the preceding notation. Thus, for $\mu \in \mathcal{M}$, we say that a function f is said to be μ-Stepanov-like pseudo almost periodic (or μ-S^p-pseudo almost periodic) if and only if $f \in \mathcal{B}^{-1}(AP(\mathbb{R}, L^p(0, 1; X))) + \mathcal{B}^{-1}(\mathcal{E}(\mathbb{R}, L^p(0, 1; X), \mu))$. Thus,

$$PAPS^p(\mathbb{R}, X, \mu) = \mathcal{B}^{-1}(AP(\mathbb{R}, L^p(0, 1; X))) + \mathcal{B}^{-1}(\mathcal{E}(\mathbb{R}, L^p(0, 1; X), \mu)). \tag{3.1}$$

Also, assume that μ satisfies (H1). Since $\mathcal{B}$ is an isometry and $AP(\mathbb{R}, L^p(0, 1; X)) \cap \mathcal{E}(\mathbb{R}, L^p(0, 1; X), \mu) = \{0\}$ by [5, Cor. 2.29] we have that the sum is direct, that is,

$$PAPS^p(\mathbb{R}, X, \mu) = \mathcal{B}^{-1}(AP(\mathbb{R}, L^p(0, 1; X))) \oplus \mathcal{B}^{-1}(\mathcal{E}(\mathbb{R}, L^p(0, 1; X), \mu)).$$

Based on the definition of the operator $\mathcal{B}$, next we prove that $PAPS^p(\mathbb{R}, X, \mu)$ is a Banach space.

Theorem 3.3. *If $\mu \in \mathcal{M}$ satisfies* (H1)*, then $PAPS^p(\mathbb{R}, X, \mu)$ is a Banach space with the norm*

$$\|f\|_{PAPS^p(\mathbb{R},X,\mu)} = \|g\|_{BS^p(\mathbb{R},X)} + \|h\|_{BS^p(\mathbb{R},X)}$$

where $f = g + h$ with $g \in \mathcal{B}^{-1}(AP(\mathbb{R}, L^p(0, 1; X)))$, $h \in \mathcal{B}^{-1}(\mathcal{E}(\mathbb{R}, L^p(0, 1; X), \mu))$.

Proof. Let (f_n) be a Cauchy sequence in $PAPS^p(\mathbb{R}, X, \mu)$. Then

$$\|f_n - f_m\|_{PAPS^p(\mathbb{R},X,\mu)} \to 0 \quad \text{as } n, m \to \infty.$$

Let $f_n = g_n + h_n$ and $f_m = g_m + h_m$ with $g_n, g_m \in \mathcal{B}^{-1}(AP(\mathbb{R}, L^p(0, 1; X)))$ and $h_n, h_m \in \mathcal{B}^{-1}(\mathcal{E}(\mathbb{R}, L^p(0, 1; X), \mu))$. If $n, m \to \infty$, then

$$\begin{gathered}
\|\mathcal{B}g_n - \mathcal{B}g_m\|_{L^\infty(\mathbb{R},L^p)} = \|g_n - g_m\|_{BS^p(\mathbb{R},X)} \le \|f_n - f_m\|_{PAPS^p(\mathbb{R},X,\mu)} \to 0, \\
\|\mathcal{B}h_n - \mathcal{B}h_m\|_{L^\infty(\mathbb{R},L^p)} = \|h_n - h_m\|_{BS^p(\mathbb{R},X)} \le \|f_n - f_m\|_{PAPS^p(\mathbb{R},X,\mu)} \to 0.
\end{gathered}$$

This implies that $(\mathcal{B}g_n)$ and $(\mathcal{B}h_n)$ are Cauchy sequences in $AP(\mathbb{R}, L^p(0, 1; X))$ and $\mathcal{E}(\mathbb{R}, L^p(0, 1; X), \mu)$ respectively. Since $AP(\mathbb{R}, L^p(0, 1; X))$ is a closed subspace of $BC(\mathbb{R}, L^p(0, 1; X))$ then it is a Banach space. Also, it follows from [5, Cor. 2.31] that $\mathcal{E}(\mathbb{R}, L^p(0, 1; X), \mu)$ is a Banach space. Then there exist $g \in AP(\mathbb{R}, L^p(0, 1; X))$ and $h \in \mathcal{E}(\mathbb{R}, L^p(0, 1; X), \mu)$ such that

$$\|\mathcal{B}g_n - g\|_{L^\infty(\mathbb{R},L^p)} \to 0, \quad \|\mathcal{B}h_n - h\|_{L^\infty(\mathbb{R},L^p)} \to 0 \quad (n \to \infty).$$

Let

$$\begin{aligned} f_1 &:= \mathcal{B}^{-1}(\{g\}) \in \mathcal{B}^{-1}(AP(\mathbb{R}, L^p(0,1;X))) \\ f_2 &:= \mathcal{B}^{-1}(\{h\}) \in \mathcal{B}^{-1}(\mathcal{E}(\mathbb{R}, L^p(0,1;X),\mu)). \end{aligned}$$

Note that f_1 and f_2 are well defined because $\mathcal{B}$ is injective. Let $f := f_1 + f_2 \in PAPS^p(\mathbb{R}, X, \mu)$. Thus

$$\begin{aligned} \|f_n - f\|_{PAPS^p(\mathbb{R},X,\mu)} &= \|(g_n + h_n) - (f_1 + f_2)\|_{PAPS^p(\mathbb{R},X,\mu)} \\ &= \|g_n - f_1\|_{BS^p(\mathbb{R},X)} + \|h_n - f_2\|_{BS^p(\mathbb{R},X)} \\ &= \|\mathcal{B}g_n - \mathcal{B}f_1\|_{L^\infty(\mathbb{R},L^p)} + \|\mathcal{B}h_n - \mathcal{B}f_2\|_{L^\infty(\mathbb{R},L^p)} \\ &= \|\mathcal{B}g_n - g\|_{L^\infty(\mathbb{R},L^p)} + \|\mathcal{B}h_n - h\|_{L^\infty(\mathbb{R},L^p)} \to 0 \quad (n \to \infty). \end{aligned}$$

Hence $PAPS^p(\mathbb{R}, X, \mu)$ is a Banach space. □

The following theorem is taken from [7, Theorem 2.1].

Theorem 3.4. *Let $\mu \in \mathcal{M}$ and I be a bounded interval (eventually $\emptyset$). Assume that $f(\cdot) \in BS^p(\mathbb{R}, X)$. Then the following assertions are equivalent.*

(a) $f^b(\cdot) \in \mathcal{E}(\mathbb{R}, L^p(0,1;X)),\mu)$.

(b)

$$\lim_{T\to\infty} \frac{1}{\mu([-T,T]\setminus I)} \int_{\mu([-T,T]\setminus I)} \Big(\int_t^{t+1} \|f(s)\|^p ds\Big)^{1/p} d\mu(t) = 0.$$

(c) *For any $\epsilon > 0$,*

$$\lim_{T\to\infty} \frac{\mu\Big(t \in [-T,T]\setminus I : \big(\int_t^{t+1} \|f(s)\|^p ds\big)^{1/p} > \epsilon\Big)}{\mu([-T,T]\setminus I)} = 0.$$

The following theorem about composition of Stepanov-like type pseudo almost periodic functions generalizes [14, Theorem 2.15].

Theorem 3.5. *Let $\mu \in \mathcal{M}$ and let $f = g + \phi \in PAPS^p(\mathbb{R} \times X, X, \mu)$ with $g \in \mathcal{B}^{-1}(AP(\mathbb{R} \times X, L^p(0,1;X)))$ and $\phi \in \mathcal{B}^{-1}(\mathcal{E}(\mathbb{R} \times X, L^p(0,1;X)),\mu)$. Assume the following conditions.*

(a) *$f(t,x)$ is uniformly continuous in any bounded set $K' \subset X$ uniformly for $t \in \mathbb{R}$,*

(b) *$g(t,x)$ is uniformly continuous in any bounded set $K' \subset X$ uniformly for $t \in \mathbb{R}$,*

(c) *for every bounded subset $K' \subset X$, the set $\{f(\cdot,x) : x \in K'\}$ is bounded in $PAPS^p(\mathbb{R} \times X, X, \mu)$.*

If $x = \alpha + \beta \in PAPS^p(\mathbb{R}, X, \mu) \cap B(\mathbb{R}, X)$, with $\alpha \in \mathcal{B}^{-1}(AP(\mathbb{R}, L^p(0,1;X)))$, $\beta \in \mathcal{B}^{-1}(\mathcal{E}(\mathbb{R}, L^p(0,1;X),\mu))$ and $Q = \overline{\{x(t) : t \in \mathbb{R}\}}$, $Q_1 = \overline{\{\alpha(t) : t \in \mathbb{R}\}}$ are compact, then $f(\cdot, x(\cdot)) \in PAPS^p(\mathbb{R}, X, \mu)$.

Proof. Let

$$f(t, x(t)) = G(t) + H(t) + W(t),$$

where

$$G(t) = g(t, \alpha(t)), \quad H(t) = f(t, x(t)) - f(t, \alpha(t)), \quad W(t) = \phi(t, \alpha(t)).$$

Since g satisfies condition (b) and $Q_1 = \overline{\{\alpha(t) : t \in \mathbb{R}\}}$ is compact, by [3, Prop. 1] we have $G \in \mathcal{B}^{-1}(AP(\mathbb{R}, L^p(0,1;X)))$. To show that $f(\cdot, x(\cdot)) \in PAPS^p(\mathbb{R}, X, \mu)$ it is sufficient to show that $H, W \in \mathcal{B}^{-1}(\mathcal{E}(\mathbb{R}, L^p(0,1;X)))$.

First, we see that $H \in \mathcal{B}^{-1}(\mathcal{E}(\mathbb{R}, L^p(0,1;X)))$. Since $x(\cdot)$ and $\alpha(\cdot)$ are bounded, we can choose a bounded subset $K' \subset X$ such that $x(\mathbb{R}), \alpha(\mathbb{R}) \subset K'$. By assumption (c) we have that $H(\cdot) \in BS^p(\mathbb{R}, X)$ and by assumption (a) we obtain that f is uniformly continuous on the bounded set $K' \subset X$ uniformly $t \in \mathbb{R}$. Then, given $\epsilon > 0$, there exists $\delta > 0$, such that $u, v \in K'$ and $\|u - v\| < \delta$ imply that $\|f(t,u) - f(t,v)\| \leq \epsilon$ for all $t \in \mathbb{R}$. Then, we have

$$\Big(\int_t^{t+1} \|f(s,u) - f(s,v)\|^p ds\Big)^{1/p} \leq \epsilon.$$

Hence, for each $t \in \mathbb{R}$, $\|\beta(s)\|_{BS^p(\mathbb{R},X)} < \delta$, $s \in [t, t+1]$ implies that for all $t \in \mathbb{R}$,

$$\Big(\int_t^{t+1} \|H(s)\|^p ds\Big)^{1/p} = \Big(\int_t^{t+1} \|f(s,x(s)) - f(s,\alpha(s))\|^p ds\Big)^{1/p} \leq \epsilon.$$

Therefore,

$$\begin{aligned}
&\frac{\mu\Big(t \in [-T,T] : \big(\int_t^{t+1} \|f(s,x(s)) - f(s,\alpha(s))\|^p ds\big)^{1/p} > \epsilon\Big)}{\mu([-T,T])} \\
&\leq \frac{\mu\Big(t \in [-T,T] : \big(\int_t^{t+1} \|\beta(s)\|^p ds\big)^{1/p} > \delta\Big)}{\mu([-T,T])}.
\end{aligned}$$

Since $\beta \in \mathcal{B}^{-1}(\mathcal{E}(\mathbb{R}, L^p(0,1;X), \mu))$, then Theorem 3.4 implies that for the above mentioned δ we have

$$\lim_{T \to \infty} \frac{\mu\Big(t \in [-T,T] : \big(\int_t^{t+1} \|f(s,x(s)) - f(s,\alpha(s))\|^p ds\big)^{1/p} > \epsilon\Big)}{\mu([-T,T])} = 0.$$

By Theorem 3.4 we have that $H \in \mathcal{B}^{-1}(\mathcal{E}(\mathbb{R}, L^p(0,1;X)))$.

Now, we prove that $W \in \mathcal{B}^{-1}(\mathcal{E}(\mathbb{R}, L^p(0,1;X)))$. Since f and g satisfy (a) and (b) respectively, then, given $\epsilon > 0$, exists $\delta > 0$, such that $u, v \in Q_1$, $\|u - v\| < \delta$ imply that

$$\begin{gathered}
\Big(\int_t^{t+1} \|f(s,u) - f(s,v)\|^p ds\Big)^{1/p} \leq \frac{\epsilon}{16}, \quad t \in \mathbb{R}, \\
\Big(\int_t^{t+1} \|g(s,u) - g(s,v)\|^p ds\Big)^{1/p} \leq \frac{\epsilon}{16}, \quad t \in \mathbb{R}.
\end{gathered}$$

Let $\delta_0 := \min\{\epsilon, \delta\}$. Then

$$\begin{aligned}
&\Big(\int_t^{t+1} \|\phi(s,u) - \phi(s,v)\|^p ds\Big)^{1/p} \\
&\leq \Big(\int_t^{t+1} \|f(s,u) - f(s,v)\|^p ds\Big)^{1/p} + \Big(\int_t^{t+1} \|g(s,u) - g(s,v)\|^p ds\Big)^{1/p} \\
&\leq \frac{\epsilon}{8},
\end{aligned}$$

for all $t \in \mathbb{R}$, and $u, v \in Q_1$, $\|u - v\| < \delta_0$.

Since $Q_1 = \overline{\{\alpha(t) : t \in \mathbb{R}\}}$ is compact, there exist open balls O_k $(k = 1, 2, \dots, m)$ with center in $u_k \in Q_1$ and radius δ_0 given above, such that $\{\alpha(t) : t \in \mathbb{R}\} \subset$

$\cup_{k=1}^{m} O_k$. Define and choose B_k such that $B_k := \{t \in \mathbb{R} : \|\alpha(t) - u_k\| < \delta_0\}$, $k = 1, 2, \dots, m$, $\mathbb{R} = \cup_{k=1}^{m} B_k$ and set $C_1 = B_1$, $C_k = B_k \setminus (\cup_{j=1}^{k-1} B_j)$ $(k = 2, 3, \dots, m)$. Then $\mathbb{R} = \cup_{k=1}^{m} C_k$ where $C_i \cap C_j = \emptyset$, $i \neq j$, $1 \leq i, j \leq m$. Let us define the function $\overline{u} : \mathbb{R} \to X$ by $\overline{u}(t) = u_k$ for $t \in C_k$, $k = 1, \dots, m$. Then $\|\alpha(t) - \overline{u}\| < \delta_0$ for all $t \in \mathbb{R}$ and

$$
\begin{aligned}
&\Big(\sum_{k=1}^{m} \int_{C_k \cap [t,t+1]} \|\phi(s, \alpha(s)) - \phi(s, u_k)\|^p ds\Big)^{1/p} \\
&= \Big(\int_t^{t+1} \|\phi(s, \alpha(s)) - \phi(s, \overline{u}(s))\|^p ds\Big)^{1/p} < \frac{\epsilon}{8}.
\end{aligned}
$$

Since $\phi \in \mathcal{B}^{-1}(\mathcal{E}(\mathbb{R} \times X, L^p(0,1;X)), \mu)$, there exists $T_0 > 0$ such that

$$
\frac{1}{\mu([-T,T])} \int_{[-T,T]} \Big(\int_t^{t+1} \|\phi(s, u_k)\|^p \, d\sigma\Big)^{1/p} d\mu(t) < \frac{\epsilon}{8m^2},
$$

for all $T > T_0$ and $1 \leq k \leq m$. Therefore,

$$
\begin{aligned}
&\frac{1}{\mu([-T,T])} \int_{[-T,T]} \Big(\int_t^{t+1} \|W(s)\|^p \, ds\Big)^{1/p} d\mu(t) \\
&= \frac{1}{\mu([-T,T])} \int_{[-T,T]} \Big(\sum_{k=1}^{m} \int_{C_k \cap [t,t+1]} \|\phi(s, \alpha(s)) - \phi(s, u_k) \\
&\quad + \phi(s, u_k)\|^p ds\Big)^{1/p} d\mu(t) \\
&\leq \frac{2^{1+\frac{1}{p}}}{\mu([-T,T])} \int_{[-T,T]} \Big(\int_{C_k \cap [t,t+1]} \|\phi(s, \alpha(s)) - \phi(s, \overline{u}(s))\|^p ds\Big)^{1/p} d\mu(t) \\
&\quad + \frac{2^{1+\frac{1}{p}}}{\mu([-T,T])} \int_{[-T,T]} \Big(\sum_{k=1}^{m} \int_{C_k \cap [t,t+1]} \|\phi(s, u_k)\|^p ds\Big)^{1/p} d\mu(t) \\
&< \frac{\epsilon}{2} + m^{1/p} \frac{\epsilon}{2m} < \epsilon.
\end{aligned}
$$

Hence $W \in \mathcal{B}^{-1}(\mathcal{E}(\mathbb{R}, L^p(0,1;X)))$. The conclusion follows. ☐

From Theorem 3.5 we obtain the following result of [1].

Corollary 3.6. *Let $\mu \in \mathcal{M}$ and let $f = g + \phi \in PAPS^p(\mathbb{R} \times X, X, \mu)$ that satisfies a Lipschitz condition in $x \in X$ uniformly in $t \in \mathbb{R}$, that is, there is a constant $L \geq 0$ such that $\|f(t,x) - f(t,y)\| \leq L\|x - y\|$, for all $x, y \in X$ and $t \in \mathbb{R}$. If $x \in PAP(\mathbb{R}, X, \mu)$, then $f(\cdot, x(\cdot)) \in PAPS^p(\mathbb{R}, X, \mu)$.*

To prove the next composition theorem, we need the following lemma.

Lemma 3.7 ([9])**.** *Suppose that*

(a) *$f \in APS^p(\mathbb{R} \times X, X)$ with $p > 1$ and there exists a function $L_f \in BS^r(\mathbb{R}, \mathbb{R})$ $(r \geq \max\{p, p/p-1\})$ such that*
$$
\|f(t,u) - f(t,v)\| \leq L_f(t)\|u - v\| \quad t \in \mathbb{R}, \ u, v \in X.
$$
(b) *$x \in APS^p(\mathbb{R}, X)$, and there exist a set $E \subset \mathbb{R}$ with $\operatorname{meas}(E) = 0$ such that*
$$
K = \overline{\{x(t) : t \in \mathbb{R} \setminus E\}}
$$
is compact in X.

Then there exist $q \in [1,p)$ *such that* $f(\cdot, x(\cdot)) \in APS^q(\mathbb{R}, X)$.

The next result of composition is new.

Theorem 3.8. *Let* $\mu \in \mathcal{M}$, $p > 1$, $f = g + \phi \in PAPS^p(\mathbb{R} \times X, X, \mu)$ *with* $g \in \mathcal{B}^{-1}(AP(\mathbb{R} \times X, L^p(0,1;X)))$ *and* $\phi \in \mathcal{B}^{-1}(\mathcal{E}(\mathbb{R} \times X, L^p(0,1;X), \mu))$. *Assume that*

(i) *there exist nonnegative functions* L_f, L_g *in the space* $APS^r(\mathbb{R}, \mathbb{R})$, *with* $r \geq \max\{p, p/p-1\}$, *such that*
$$\|f(t,u) - f(t,v)\| \leq L_f(t)\|u - v\|, \quad \|g(t,u) - g(t,v)\| \leq L_g(t)\|u - v\|$$
for $t \in \mathbb{R}$ *and* $u, v \in X$.

(ii) $h = \alpha + \beta \in PAPS^p(\mathbb{R}, X, \mu)$ *with*
$$\alpha \in \mathcal{B}^{-1}(AP(\mathbb{R}, L^p(0,1;X))), \quad \beta \in \mathcal{B}^{-1}(\mathcal{E}(\mathbb{R}, L^p(0,1;X), \mu))$$
and there exist a set $E \subset \mathbb{R}$ *with* $\operatorname{meas}(E) = 0$ *such that the set* $K = \overline{\{\alpha(t) : t \in \mathbb{R} \setminus E\}}$ *is compact in* X.

Then there exist $q \in [1,p)$ *such that* $f(\cdot, h(\cdot)) \in PAPS^q(\mathbb{R}, X, \mu)$.

Proof. We can decompose
$$f(t,h(t)) = g(t,\alpha(t)) + f(t,h(t)) - f(t,\alpha(t)) + \phi(t,\alpha(t)).$$
Set
$$F(t) := g(t,\alpha(t)), \quad G(t) := f(t,h(t)) - f(t,\alpha(t)), \quad H(t) := \phi(t,\alpha(t)).$$
Since $r \geq \frac{p}{p-1}$ then there exists $q \in [1,p)$ such that $r = \frac{pq}{p-q}$. Let $p' = p/p-q$ and $q' = p/q$. Therefore $\frac{1}{p'} + \frac{1}{q'} = 1$. Since $\alpha \in APS^p(\mathbb{R}, X)$ and $g \in APS^p(\mathbb{R} \times X, X)$ then by assumptions and Lemma 3.7 we obtain that $F \in \mathcal{B}^{-1}(AP(\mathbb{R}, L^q(0,1;X)))$.

Next we show that $G \in \mathcal{B}^{-1}(\mathcal{E}(\mathbb{R}, L^q(0,1;X), \mu))$. By Hölder inequality we have
$$\begin{aligned}
\int_t^{t+1} \|G(\sigma)\|^q \, d\sigma &= \int_t^{t+1} \|f(\sigma, h(\sigma)) - f(\sigma, \alpha(\sigma))\|^q \, d\sigma \\
&\leq \int_t^{t+1} L_f^q(\sigma) \|h(\sigma) - \alpha(\sigma)\|^q \, d\sigma \\
&= \int_t^{t+1} L_f^q(\sigma) \|\beta(\sigma)\|^q \, d\sigma \\
&\leq \Big(\int_t^{t+1} L_f^{qp'}(\sigma) \, d\sigma \Big)^{1/p'} \Big(\int_t^{t+1} \|\beta(\sigma)\|^{qq'} \, d\sigma \Big)^{1/q'} \\
&= \Big[\Big(\int_t^{t+1} L_f^r(\sigma) \, d\sigma \Big)^{1/r} \Big]^{r/p'} \Big[\Big(\int_t^{t+1} \|\beta(\sigma)\|^p \, d\sigma \Big)^{1/p} \Big]^{p/q'} \\
&\leq \|L_f\|_{BS^r}^q \Big[\Big(\int_t^{t+1} \|\beta(\sigma)\|^p \, d\sigma \Big)^{1/p} \Big]^q.
\end{aligned}$$
Then
$$\begin{aligned}
&\frac{1}{\mu([-T,T])} \int_{[-T,T]} \Big(\int_t^{t+1} \|G(\sigma)\|^q \, d\sigma \Big)^{1/q} d\mu(t) \\
&\leq \frac{\|L_f\|_{BS^r}}{\mu([-T,T])} \int_{[-T,T]} \Big(\int_t^{t+1} \|\beta(\sigma)\|^p \, d\sigma \Big)^{1/p} d\mu(t).
\end{aligned}$$

Since $\beta \in \mathcal{B}^{-1}(\mathcal{E}(\mathbb{R}, L^p(0,1;X),\mu))$ we obtain that $G \in \mathcal{B}^{-1}(\mathcal{E}(\mathbb{R}, L^q(0,1;X),\mu))$.

Next, we prove that $H \in \mathcal{B}^{-1}(\mathcal{E}(\mathbb{R}, L^q(0,1;X),\mu))$.

Since $\phi \in \mathcal{B}^{-1}(\mathcal{E}(\mathbb{R}, L^p(0,1;X),\mu))$, for each $\epsilon > 0$ there exist $T_0 > 0$ such that $T > T_0$ implies that

$$\frac{1}{\mu([-T,T])}\int_{[-T,T]}\Big(\int_t^{t+1}\|\phi(\sigma,u)\|^p\,d\sigma\Big)^{1/p}\,d\mu(t) < \epsilon \quad (u \in X).$$

Since K is compact, we can find finite open balls O_k $(k = 1,2,3,\dots,m)$ with center x_k such that $K \subset \cup_{k=1}^m O_k$. Thus, for all $u \in K$ there exist x_k such that

$$\begin{aligned}
&\|\phi(t+\sigma,u)\| \\
&\le \|\phi(t+\sigma,u) - \phi(t+\sigma,x_k)\| + \|\phi(t+\sigma,x_k)\| \\
&\le \|f(t+\sigma,u) - f(t+\sigma,x_k)\| + \|g(t+\sigma,u) - g(t+\sigma,x_k)\| + \|\phi(t+\sigma,x_k)\| \\
&\le L_f(t+\sigma)\epsilon + L_g(t+\sigma)\epsilon + \|\phi(t+\sigma,x_k)\| \quad (t \in \mathbb{R},\ \sigma \in [0,1]).
\end{aligned}$$

Hence

$$\sup_{u\in K}\|\phi(t+\sigma,u)\| \le L_f(t+\sigma)\epsilon + L_g(t+\sigma)\epsilon + \sum_{k=1}^m \|\phi(t+\sigma,x_k)\|.$$

Since $r \ge p$ then $L_f, L_g \in APS^r(\mathbb{R},\mathbb{R}) \subset APS^p(\mathbb{R},\mathbb{R}) \subset BS^p(\mathbb{R},\mathbb{R})$.

By Minkowskii's inequality, we obtain

$$\begin{aligned}
&\Big[\int_0^1 (\sup_{u\in K}\|\phi(t+\sigma,u)\|)^p\,d\sigma\Big]^{1/p} \\
&\le (\|L_f\|_{BS^p} + \|L_g\|_{BS^p})\epsilon + \sum_{k=1}^m \Big(\int_0^1\Big(\sup_{u\in K}\|\phi(t+\sigma,u)\|\Big)^p d\sigma\Big)^{1/p}.
\end{aligned}$$

For $T > T_0$ we have

$$\begin{aligned}
&\frac{1}{\mu([-T,T])}\int_{[-T,T]}\Big(\int_0^1\Big(\sup_{u\in K}\|\phi(t+\sigma,u)\|\Big)^p d\sigma\Big)^{1/p} d\mu(t) \\
&\le (\|L_f\|_{BS^p} + \|L_g\|_{BS^p} + m)\epsilon.
\end{aligned}$$

Hence

$$\lim_{T\to\infty}\frac{1}{\mu([-T,T])}\int_{[-T,T]}\Big(\int_0^1\Big(\sup_{u\in K}\|\phi(t+\sigma,u)\|\Big)^p d\sigma\Big)^{1/p} d\mu(t) = 0.$$

On the other hand

$$\begin{aligned}
&\frac{1}{\mu([-T,T])}\int_{[-T,T]}\|H^b(t)\|_q\,d\mu(t) \\
&\le \frac{1}{\mu([-T,T])}\int_{[-T,T]}\|H^b(t)\|_p\,d\mu(t) \\
&= \frac{1}{\mu([-T,T])}\int_{[-T,T]}\Big(\int_0^1\|\phi(t+\sigma,\alpha(t+\sigma))\|^p\,d\sigma\Big)^{1/p} d\mu(t) \\
&\le \frac{1}{\mu([-T,T])}\int_{[-T,T]}\Big(\int_0^1(\sup_{u\in K}\|\phi(t+\sigma,u)\|)^p\,d\sigma\Big)^{1/p} d\mu(t) \to 0
\end{aligned}$$

as $T \to \infty$. Hence $H \in \mathcal{B}^{-1}(\mathcal{E}(\mathbb{R}, L^q(0,1;X),\mu))$. It proves that $f(\cdot,h(\cdot)) = F(\cdot) + [G(\cdot) + H(\cdot)] \in PAPS^q(\mathbb{R},X,\mu)$. $\square$

We recall the following convolution theorem.

Theorem 3.9 ([2, Theorem 3.1]). *Let $S : \mathbb{R} \to B(X)$ be strongly continuous. Suppose that there exists a function $\phi \in L^1(\mathbb{R})$ such that*

(a) $\|S(t)\| \le \phi(t), \quad t \in \mathbb{R}$;
(b) $\phi(t)$ *is nonincreasing;*
(c) $\sum_{n=1}^{\infty} \phi(n) < \infty$.

If $g \in APS^p(\mathbb{R}, X)$, then

$$(S * g)(t) := \int_{-\infty}^{t} S(t-s)g(s)\,ds \in AP(\mathbb{R}, X).$$

The next result is one of the original contributions of this work.

Theorem 3.10. *Let $\mu \in \mathcal{M}$ be given and let $S : \mathbb{R} \to B(X)$ be strongly continuous. Suppose that there exists a function $\phi \in L^1(\mathbb{R})$ such that*

(a) $\|S(t)\| \le \phi(t) \quad t \in \mathbb{R}$;
(b) $\phi(t)$ *is nonincreasing;*
(c) $\sum_{n=1}^{\infty} \phi(n) < \infty$.

If $f = g + h \in PAPS^p(\mathbb{R}, X, \mu)$ with $g \in \mathcal{B}^{-1}(AP(\mathbb{R}, L^p(0,1;X)))$ and $h \in \mathcal{B}^{-1}(\mathcal{E}(\mathbb{R}, L^p(0,1;X)))$, then

$$(S * f)(t) := \int_{-\infty}^{t} S(t-s)f(s)\,ds \in PAP(\mathbb{R}, X, \mu).$$

Proof. Since

$$(S * f)(t) := \int_{-\infty}^{t} S(t-s)f(s)\,ds = \int_{-\infty}^{t} S(t-s)g(s)\,ds + \int_{-\infty}^{t} S(t-s)h(s)\,ds,$$

and, from Theorem 3.9, $(S * g) \in AP(\mathbb{R}, X)$ it remains to show that $(S * h) \in \mathcal{E}(\mathbb{R}, X, \mu)$. Set

$$H(t) := \int_{-\infty}^{t} S(t-s)h(s)\,ds = \int_{-\infty}^{t} S(s)h(t-s)\,ds,$$

and

$$H_n(t) := \int_{t-n}^{t-n+1} S(t-\sigma)h(\sigma)\,d\sigma, \quad n = 1, 2, \dots .$$

Note that $H_n(t)$ is continuous and

$$\begin{aligned} \|H_n(t)\| &\le \int_{t-n}^{t-n+1} \|S(t-\sigma)\|\|h(\sigma)\|\,d\sigma \\ &= \int_{n-1}^{n} \|S(\sigma)\|\|h(t-\sigma)\|\,d\sigma \\ &\le \int_{n-1}^{n} \phi(s)\|h(t-\sigma)\|\,d\sigma \\ &\le \phi(n-1)\Big(\int_{n-1}^{n} \|h(t-\sigma)\|^p\,d\sigma\Big)^{1/p}. \end{aligned}$$

Hence, for $T > 0$,

$$\frac{1}{\mu([-T,T])}\int_{[-T,T]} \|H_n(t)\|\,d\mu(t)$$

$$\le \phi(n-1)\frac{1}{\mu([-T,T])}\int_{[-T,T]}\Big(\int_{n-1}^{n}\|h(t-\sigma)\|^p\,d\sigma\Big)^{1/p}\,d\mu(t).$$

Using the fact that the space $\mathcal{E}(\mathbb{R},X,\mu)$ is translation invariant, it follows that $t\to h(t-\sigma)$ belongs to $\mathcal{E}(\mathbb{R},X,\mu)$. The above inequality leads to $H_n\in\mathcal{E}(\mathbb{R},X,\mu)$ for each $n=1,2,\dots$. The above estimate implies

$$\|H_n(t)\|\le\phi(n-1)\|h\|_{BS^p(\mathbb{R},X)}.$$

By hypothesis we have

$$\sum_{n=1}^{\infty}\|H_n(t)\|\le\sum_{n=1}^{\infty}\phi(n-1)\|h\|_{BS^p(\mathbb{R},X)}<C\|h\|_{BS^p(\mathbb{R},X)}<\infty.$$

It follows from Weierstrass test that the series $\sum_{n=1}^{\infty}H_n(t)$ is uniformly convergent on $\mathbb{R}$. Moreover

$$H(t)=\int_{-\infty}^{t}S(t-s)h(s)\,ds=\sum_{n=1}^{\infty}H_n(t).$$

Since $H\in C(\mathbb{R},X)$ and

$$\|H(t)\|\le\sum_{n=1}^{\infty}\|H_n(t)\|\le C\|h\|_{BS^p(\mathbb{R},X)},$$

we have

$$\begin{aligned}\frac{1}{\mu([-T,T])}\int_{[-T,T]}\|H(t)\|\,d\mu(t)&\le\frac{1}{\mu([-T,T])}\int_{[-T,T]}\Big\|H(t)-\sum_{k=1}^{n}H_k(t)\Big\|\,d\mu(t)\\&\quad+\sum_{k=1}^{n}\frac{1}{\mu([-T,T])}\int_{[-T,T]}\|H_k(t)\|\,d\mu(t).\end{aligned}$$

Since $H_k(t)\in\mathcal{E}(\mathbb{R},X,\mu)$ and $\sum_{k=1}^{n}H_n(t)$ converges uniformly to $H(t)$, it follows that

$$\lim_{T\to\infty}\frac{1}{\mu([-T,T])}\int_{[-T,T]}\|H(t)\|\,d\mu(t)=0.$$

Hence $H(\cdot)=\sum_{n=1}^{\infty}H_n(t)\in\mathcal{E}(\mathbb{R},X,\mu)$. Therefore, $(S*f)(t)=\int_{-\infty}^{t}S(t-s)f(s)\,ds$ is μ-pseudo almost periodic. □

4. An application to fractional integro-differential equations

Given a function $g:\mathbb{R}\to X$, the *Weyl fractional integral* of order $\alpha>0$ is defined by

$$D^{-\alpha}g(t):=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{t}(t-s)^{\alpha-1}g(s)ds,\quad t\in\mathbb{R},$$

when this integral is convergent. The *Weyl fractional derivative* $D^\alpha g$ of order $\alpha>0$ is defined by

$$D^{\alpha}g(t):=\frac{d^n}{dt^n}D^{-(n-\alpha)}g(t),\quad t\in\mathbb{R},$$

where $n=[\alpha]+1$. It is known that $D^\alpha D^{-\alpha}g=g$ for any $\alpha>0$, and $D^n=\frac{d^n}{dt^n}$ holds with $n\in\mathbb{N}$.

Definition 4.1 ([13])**.** Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X, and $\alpha > 0$. Given $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$, we say that A is the generator of an α-resolvent family if there exist $\omega \geq 0$ and a strongly continuous family $S_\alpha : [0, \infty) \to \mathcal{B}(X)$ such that $\{\frac{\lambda^\alpha}{1+\hat{a}(\lambda)} : \operatorname{Re}\lambda > \omega\} \subset \rho(A)$ and for all $x \in X$,

$$\big(\lambda^\alpha - (1 + \hat{a}(\lambda))A\big)^{-1} x = \frac{1}{1 + \hat{a}(\lambda)} \Big(\frac{\lambda^\alpha}{1 + \hat{a}(\lambda)} - A\Big)^{-1} x = \int_0^\infty e^{-\lambda t} S_\alpha(t) x \, dt,$$

for $\operatorname{Re}\lambda > \omega$. In this case, $\{S_\alpha(t)\}_{t\geq 0}$ is called the *α-resolvent family* generated by A.

Next, we consider the existence and uniqueness of μ-pseudo almost periodic mild solutions for the fractional integro-differential equations

$$D^\alpha u(t) = Au(t) + \int_{-\infty}^t a(t-s)Au(s)\, ds + f(t, u(t)), \tag{4.1}$$

where A generates an α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ on a Banach space X, $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ and $f \in PAPS^p(\mathbb{R} \times X, X, \mu)$ satisfies the Lipschitz condition.

Definition 4.2. A function $u : \mathbb{R} \to X$ is said to be a mild solution of (4.1) if

$$u(t) = \int_{-\infty}^t S_\alpha(t-s) f(s, u(s))\, ds \quad (t \in \mathbb{R})$$

where $\{S_\alpha(t)\}_{t\geq 0}$ is the α-resolvent family generated by A.

Theorem 4.3. *Let $\mu \in \mathcal{M}$, and assume* (H2) *holds. Let $p > 1$ and $f \in PAPS^p(\mathbb{R}\times X, X, \mu)$ be given. Suppose that*

- (H3) *There exists $L_f \geq 0$ such that*
$$\|f(t,u) - f(t,v)\| \leq L_f \|u - v\|, \quad t \in \mathbb{R},\ u, v \in X.$$
- (H4) *Operator A generates an α-resolvent family $\{S_\alpha(t)\}_{t\geq 0}$ such that $\|S_\alpha(t)\| \leq \varphi_\alpha(t)$, for all $t \geq 0$, where $\varphi_\alpha(\cdot) \in L^1(\mathbb{R}_+)$ is nonincreasing such that $\varphi_0 := \sum_{n=0}^\infty \varphi_\alpha(n) < \infty$.*

If $L_f < \|\varphi_\alpha\|_1^{-1}$, then (4.1) *has a unique mild solution in $PAP(\mathbb{R}, X, \mu)$.*

Proof. Consider the operator $Q : PAP(\mathbb{R}, X, \mu) \to PAP(\mathbb{R}, X, \mu)$ defined by

$$(Qu)(t) := \int_{-\infty}^t S(t-s) f(s, u(s))\, ds, \quad t \in \mathbb{R}.$$

First, we show that $Q(PAP(\mathbb{R}, X, \mu)) \subset PAP(\mathbb{R}, X, \mu)$. Let $u \in PAP(\mathbb{R}, X, \mu)$. Since $f \in PAPS^p(\mathbb{R} \times X, X, \mu)$ and satisfy (H3) we have from Corollary 3.6 that $f(\cdot, u(\cdot)) \in PAPS^p(\mathbb{R}, X, \mu)$. Then, by assumption (h4) we obtain from Theorem 3.10 that $Qu \in PAP(\mathbb{R}, X, \mu)$.

Let $u, v \in PAP(\mathbb{R}, X, \mu)$. By conditions (H3) and (H4) we have

$$\begin{aligned}
\|Qu - Qv\|_\infty &= \sup_{t\in\mathbb{R}} \|(Qu)(t) - (Qv)(t)\| \\
&= \sup_{t\in\mathbb{R}} \Big\| \int_{-\infty}^t S(t-s)[f(s, u(s)) - f(s, v(s))]\, ds \Big\| \\
&\leq L_f \sup_{t\in\mathbb{R}} \int_0^\infty \|S(s)\| \|u(t-s) - v(t-s)\|\, ds
\end{aligned}$$

$$
\begin{aligned}
&\le L_f \|u-v\|_\infty \int_0^\infty \varphi_\alpha(s)\,ds \\
&= L_f \|\varphi_\alpha\|_1 \|u-v\|_\infty .
\end{aligned}
$$

This proves that Q is a contraction, so by the Banach Fixed Point Theorem we conclude that Q has unique fixed point. It follows that $Qu = u \in PAP(\mathbb{R}, X, \mu)$ and it is unique. Hence u is the unique mild solution of (4.1) which belongs to $PAP(\mathbb{R}, X, \mu)$. ∎

Theorem 4.4. *Let $\mu \in \mathcal{M}$. Assume that* (H2) *holds. Let $p > 1$ and $f = g + h \in PAPS^p(\mathbb{R} \times X, X, \mu)$ be given. Suppose that:*

- (H5) *There exist nonnegative functions $L_f(\cdot), L_g(\cdot) \in APS^r(\mathbb{R}, \mathbb{R})$ with $r \ge \max\{p, \frac{p}{p-1}\}$ such that*
$$
\|f(t,u) - f(t,v)\| \le L_f(t)\|u-v\|, \quad \|g(t,u) - g(t,v)\| \le L_g(t)\|u-v\|,
$$
for $t \in \mathbb{R}$ and $u, v \in X$.
- (H6) *Operator A generates an α-resolvent family $\{S_\alpha(t)\}_{t\ge 0}$ such that $\|S_\alpha(t)\| \le Me^{-\omega t}$, for all $t \ge 0$ and*
$$
\|L_f\|_{BS^r} < \frac{1-e^{-\omega}}{M}\Big(\frac{\omega r_0}{1-e^{-\omega r_0}}\Big)^{1/r_0}
$$
where $\frac{1}{r} + \frac{1}{r_0} = 1$.

Then (4.1) *has a unique mild solution in $PAP(\mathbb{R}, X, \mu)$.*

Proof. Let $u = u_1 + u_2 \in PAP(\mathbb{R}, X, \mu)$ where $u_1 \in AP(\mathbb{R}, X)$ and $u_2 \in \mathcal{E}(\mathbb{R}, X, \mu)$. Then $u \in PAPS^p(\mathbb{R}, X, \mu)$. Since the range of almost periodic functions is relatively compact set, then $K = \overline{\{u_1(t) : t \in \mathbb{R}\}}$ is compact in X. Thus, by conditions (H5) and (H6) we have that all the hypotheses of Theorem 3.8 fulfilled, then there exists $q \in [1, p)$ such that $f(\cdot, u(\cdot)) \in PAPS^q(\mathbb{R}, X, \mu)$.

Consider the operator $Q : PAP(\mathbb{R}, X, \mu) \to PAP(\mathbb{R}, X, \mu)$ such that

$$
(Qu)(t) := \int_{-\infty}^t S(t-s) f(s, u(s))\,ds, \quad (t \in \mathbb{R}).
$$

Since $f(\cdot, u(\cdot)) \in PAPS^q(\mathbb{R}, X, \mu)$ it follows from Theorem 3.10 that Q maps $PAP(\mathbb{R}, X, \mu)$ into $PAP(\mathbb{R}, X, \mu)$.

For any $u, v \in PAP(\mathbb{R}, X, \mu)$ we have

$$
\begin{aligned}
\|(Qu)(t) - (Qv)(t)\| &\le \int_{-\infty}^t \|S(t-s)\|\|f(s, u(s) - f(s, v(s)))\|\,ds \\
&\le \int_{-\infty}^t Me^{-\omega(t-s)} L_f(s)\|u(s) - v(s)\|\,ds \\
&\le \|u-v\| \sum_{k=1}^\infty \int_{t-k}^{t-k+1} Me^{-\omega(t-s)} L_f(s)\,ds \\
&\le \|u-v\| \sum_{k=1}^\infty \Big(\int_{t-k}^{t-k+1} M^{r_0} e^{-\omega r_0 (t-s)} \Big)^{1/r_0} ds \|L_f(s)\|_{BS^r} \\
&= \frac{M}{1-e^{-\omega}} \Big(\frac{1-e^{-\omega r_0}}{\omega r_0} \Big)^{1/r_0} \|u-v\| \|L_f(s)\|_{BS^r}.
\end{aligned}
$$

From Banach contraction mapping principle we have that Q has a unique fixed point in $PAP(\mathbb{R}, X, \mu)$ which is the unique mild solution of Equation (4.1). $\square$

Example 4.5. We put $A = -\varrho$ in $X = \mathbb{R}$, $a(t) = \frac{\varrho}{4} \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $\varrho > 0$, $0 < \alpha < 1$, and $f(t, u) = g(t, u) + h(t, u)$ where

$$g(t, u(t, x)) = [\sin t + \sin(\sqrt{2}\, t)] \sin(u(t, x)), \quad h(t, u(t, x)) = \phi(t) \sin(u(t, x)),$$

and $\phi(t)$ is such that $|\phi(t)e^t| \leq K$ with $K > 0$.

Consider the measure μ whose Radon-Nikodym derivative is $\rho(t) = e^t$. Then $\mu \in \mathcal{M}$ and satisfies the (H2) (see [5, Ex. 3.6]). Note that $g \in \mathcal{B}^{-1}(AP(\mathbb{R}, L^p(0, 1; X)))$ and $h \in \mathcal{B}^{-1}(\mathcal{E}(\mathbb{R}, L^p(0, 1; X), \mu))$. Hence $f \in PAPS^p(\mathbb{R} \times X, X, \mu)$. Furthermore,

$$|f(t, u) - f(t, v)| \leq L|u - v|,$$

where $L := \max\{2, K\}$. Therefore f satisfies $(C1)$.

Now, note that Equation (4.1) takes the form

$$D^\alpha u(t) = -\varrho u(t) - \frac{\varrho^2}{4} \int_{-\infty}^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s) ds + f(t, u(t)), \quad t \in \mathbb{R}. \tag{4.2}$$

It follows from [13, Example 4.17] that A generates an α-resolvent family $\{S_\alpha(t)\}_{t \geq 0}$ such that

$$\widehat{S}_\alpha(\lambda) = \frac{\lambda^\alpha}{(\lambda^\alpha + 2/\varrho)^2} \frac{\lambda^{\alpha-\alpha/2}}{(\lambda^\alpha + 2/\varrho)^2} \cdot \frac{\lambda^{\alpha-\alpha/2}}{(\lambda^\alpha + 2/\varrho)^2}\,.$$

Thus, we obtain explicitly

$$S_\alpha(t) = (r * r)(t) \quad t > 0,$$

with $r(t) = t^{\frac{\alpha}{2}-1} E_{\alpha, \frac{\alpha}{2}}(-\frac{\varrho}{2} t^\alpha)$, and where $E_{\alpha, \frac{\alpha}{2}}(\cdot)$ is the Mittag-Leffler function.

By properties of the Mittag-Leffler function we obtain that (H4) holds. Then, by Theorem 4.3, (4.2) has a unique mild solution $u \in PAP(\mathbb{R}, X, \mu)$ provided $\|S_\alpha\| < \frac{1}{2}$. Finally we note that, for $0 < \alpha < 1$, $\varrho > 0$ may be chosen so that $\|S_\alpha\| < \frac{1}{2}$ as in the proof of [13, Lemma 3.9].

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