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ON THE Ψ-CONDITIONAL ASYMPTOTIC STABILITY OF THE SOLUTIONS OF A NONLINEAR VOLTERRA INTEGRO-DIFFERENTIAL SYSTEM

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Abstract. We provide sufficient conditions for Ψ-conditional asymptotic stability of the solutions of a nonlinear Volterra integro-differential system.

1. Introduction

The purpose of this paper is to provide sufficient conditions for Ψ-conditional asymptotic stability of the solutions of the nonlinear Volterra integro-differential system

$$x' = A(t)x + \int_0^t F(t, s, x(s))ds \tag{1.1}$$

and for the linear system

$$x' = [A(t) + B(t)]x \tag{1.2}$$

as a perturbed systems of

$$y' = A(t)y. \tag{1.3}$$

We investigate conditions on a fundamental matrix $Y(t)$ of the linear equation (1.3) and on the functions $B(t)$ and $F(t, s, x)$ under which the solutions of (1.1), (1.2) or (1.3) are Ψ-conditionally asymptotically stable on $\mathbb{R}_+$. Here, Ψ is a continuous matrix function. The introduction of the matrix function Ψ permits to obtain a mixed asymptotic behavior of the solutions.

The problem of Ψ- stability for systems of ordinary differential equations has been studied by many authors, as e.g. Akinyele [1, 2], Constantin [4, 5], Hallam [13], Kuben [15], Morchalo [18]. In these papers, the function Ψ is a scalar continuous function (and monotone in [2], nondecreasing in [4]).

In our papers [8, 9, 10], we have proved sufficient conditions for various types of Ψ-stability of the trivial solution of the equations (1.1), (1.2) and (1.3). In these papers, the function Ψ is a continuous matrix function.

Recent works for stability of solutions of (1.1) have been by Avramescu [3], by Hara, Yoneyama and Itoh [14], by Lakshmikantham and Rama Mohana Rao [16], by Mahfoud [17] and others. Coppel's paper [6, Chapter III, Theorem 12], [7] deal with the instability and conditional asymptotic stability of the solutions

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of a systems of differential equations. Späth's paper [21] and Weyl's paper [22] deal with the conditional stability of solutions of systems of differential equations. In our papers [11, 12], we have proved a necessary and sufficient conditions for Ψ-instability and Ψ-conditional stability of the equation (1.3) and sufficient conditions for Ψ-instability and Ψ-conditional stability of trivial solution of the equations (1.1) and (1.2).

2. Definitions, notation and hypotheses

Let $\mathbb{R}^d$ denote the Euclidean d-space. For $x = (x_1, x_2, \dots x_d)^T \in R^d$, let $\|x\| = \max\{|x_1|, |x_2|, \dots |x_d|\}$ be the norm of x. For a $d \times d$ matrix $A = (a_{ij})$, we define the norm A by $|A| = \sup_{\|x\| \le 1} \|Ax\|$; it is well-known that $|A| = \max_{1 \le i \le d} \sum_{j=1}^{d} |a_{ij}|$.

In the equations (1.1)–(1.3) we assume that $A(t)$ is a continuous $d \times d$ matrix on $\mathbb{R}_+ = [0, \infty)$ and $F : D \times \mathbb{R}^d \to \mathbb{R}^d$, $D = \{(t, s) \in \mathbb{R}^2 : 0 \le s \le t < \infty\}$, is a continuous d-vector with respect to all variables.

Let $\Psi_i : \mathbb{R}_+ \to (0, \infty)$, $i = 1, 2, \dots d$, be a continuous functions and

$$\Psi = \operatorname{diag}[\Psi_1, \Psi_2, \dots \Psi_d].$$

A matrix P is said to be a projection matrix if $P^2 = P$. If P is a projection, then so is $I - P$. Two such projections, whose sum is I and whose product is 0, are said to be supplementary.

Definition 2.1. The solution $x(t)$ of (1.1) is said to be Ψ-stable on $\mathbb{R}_+$, if for every $\varepsilon > 0$ and any $t_0 \ge 0$, there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that any solution $\widetilde{x}(\mathrm{t})$ of (1.1) which satisfies the inequality $\|\Psi(t_0)(\widetilde{x}(t_0) - x(t_0))\| < \delta(\varepsilon, t_0)$ exists and satisfies the inequality $\|\Psi(t)(\widetilde{x}(t) - x(t))\| < \varepsilon$ for all $t \ge t_0$.

Otherwise, is said that the solution x(t) is Ψ-unstable on $\mathbb{R}_+$.

Definition 2.2. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}^d$ is said to be Ψ-bounded on $\mathbb{R}_+$ if $\Psi(t)\varphi(t)$ is bounded on $\mathbb{R}_+$.

Remark 2.3. For $\Psi_i = 1$, $i = 1, 2, \dots d$, we obtain the notion of classical stability, instability and boundedness, respectively.

Definition 2.4. The solution $x(t)$ of (1.1) is said to be Ψ-conditionally stable on $\mathbb{R}_+$ if it is not Ψ-stable on $\mathbb{R}_+$ but there exists a sequence $(x_n(t))$ of solutions of (1.1) defined for all $t \ge 0$ such that

$$\lim_{n\to\infty} \Psi(t)x_n(t) = \Psi(t)x(t), \quad \text{uniformly on } R_+.$$

If the sequence $x_n(t)$ can be chosen so that

$$\lim_{t\to\infty} \Psi(t)(x_n(t) - x(t)) = 0, \quad \text{for } n = 1, 2, \dots$$

then $x(t)$ is said to be Ψ-conditionally asymptotically stable on R_+.

Remark 2.5. (1) It is easy to see that if $|\Psi(t)|$ and $|\Psi^{-1}(t)|$ are bounded on $\mathbb{R}_+$, then the Ψ-conditional asymptotic stability is equivalent with the classical conditional asymptotic stability.

(2) In the same manner as in classical conditional asymptotic stability, we can speak about Ψ-conditional asymptotic stability of a linear equation. Indeed, let $x(t)$, $y(t)$ be two solutions of the linear equation (1.3). We suppose that $x(t)$ is

Ψ-conditionally asymptotically stable on $\mathbb{R}_+$. Then $y(t)$ is Ψ-unstable on $\mathbb{R}_+$ (see [11, Theorem 1]) and

$$\lim_{n\to\infty} \Psi(t)y_n(t) = \Psi(t)y(t), \quad \text{uniformly on } \mathbb{R}_+,$$
$$\lim_{t\to\infty} \Psi(t)(y_n(t) - y(t)) = 0, \quad \text{for } n = 1, 2, \dots$$

where $\mathrm{y}_n(t) = x_n(t) - x(t) + y(t)$, $n \in N$ are solutions of the linear equation (1.3). Thus, all solutions of (1.3) are Ψ-conditionally asymptotically stable on $\mathbb{R}_+$.

3. Ψ-conditional asymptotic stability of linear equations

In this section we give necessary and sufficient conditions for the Ψ-conditional asymptotic stability of the linear equation (1.3) and sufficient conditions for the Ψ-conditional asymptotic stability of the linear equations (1.3) and (1.2).

Theorem 3.1. *The linear equation* (1.3) *is* Ψ*-conditionally asymptotically stable on* $\mathbb{R}_+$ *if and only if it has a* Ψ*-unbounded solution on* $\mathbb{R}_+$ *and a non-trivial solution* $y_0(t)$ *such that* $\lim_{t\to\infty} \Psi(t)y_0(t) = 0$.

Proof. Let $Y(t)$ be a fundamental matrix for (1.3). Suppose that the linear equation (1.3) is Ψ-conditionally asymptotically stable on $\mathbb{R}_+$. From Definition 2.4 and [8, Theorem 3.1], it follows that $|\Psi(\mathrm{t})\mathrm{Y}(\mathrm{t})|$ is unbounded on $\mathbb{R}_+$. Thus, the linear equation (1.3) has at least one Ψ-unbounded solution on $\mathbb{R}_+$. In addition, there exists a sequence $(y_n(t))$ of non-trivial solutions of (1.3) such that $\lim_{n\to\infty} \Psi(t)y_n(t) = 0$, uniformly on $\mathbb{R}_+$ and $\lim_{t\to\infty} \Psi(t)y_n(t) = 0$ for $n = 1, 2, \dots$. The proof of the "only if" part is complete.

Suppose, conversely, that (1.3) has at least one Ψ-unbounded solution on $\mathbb{R}_+$ and at least one non-trivial solution $y_0(t)$ such that $\lim_{t\to\infty} \Psi(t)y_0(t) = 0$. It follows that the matrix $\Psi(t)Y(t)$ is unbounded on $\mathbb{R}_+$. Consequently, the linear equation (1.3) is Ψ-unstable on $\mathbb{R}_+$ (See [11, Theorem 1]). On the other hand, $(\frac{1}{n}y_0(t))$ is a sequence of solutions of (1.3) such that $\lim_{n\to\infty} \frac{1}{n}\Psi(t)y_0(t) = 0$, uniformly on $\mathbb{R}_+$ and $\lim_{t\to\infty} \frac{1}{n}\Psi(t)y_0(t) = 0$ for $n \in \mathbb{N}$. Thus, the linear equation (1.3) is Ψ-conditionally asymptotically stable on $\mathbb{R}_+$. The proof is complete. □

We remark that Theorem 3.1 generalizes a similar result in connection with the classical conditional asymptotic stability in [6].

The conditions for Ψ-conditional asymptotic stability of the linear equation (1.3) can be expressed in terms of a fundamental matrix for (1.3).

Theorem 3.2. *Let* $Y(t)$ *be a fundamental matrix for* (1.3)*. Then, the linear equation* (1.3) *is* Ψ*-conditionally asymptotically stable on* $\mathbb{R}_+$ *if and only if there are satisfied two following conditions:*

(a) *There exists a projection* P_1 *such that* $\Psi(t)Y(t)P_1$ *is unbounded on* $\mathbb{R}_+$*;*
(b) *there exists a projection* $P_2 \neq 0$ *such that* $\lim_{t\to\infty} \Psi(t)Y(t)P_2 = 0$*.*

Proof. First, we shall prove the sufficiency. From the hypoyhesis (a) and [11, Theorem 1], it follows that the linear equation (1.3) is Ψ-unstable on $\mathbb{R}_+$.

Let $y(t)$ be a non-trivial solution on $\mathbb{R}_+$ of the linear equation (1.3). Let (λ_n) be such that $\lambda_n \in \mathbb{R} \setminus \{1\}$, $\lim_{n\to\infty} \lambda_n = 1$ and let (y_n) be defined by

$$y_n(t) = Y(t)P_2Y^{-1}(0)(\lambda_n y(0)) + Y(t)(I - P_2)Y^{-1}(0)y(0), t \geq 0.$$

It is easy to see that $y_n(t)$, $n \in N$, are solutions of the linear equation (1.3).

For $n \in N$ and $t \geq 0$, we have

$$\begin{aligned} \|\Psi(t)y_n(t) - \Psi(t)y(t)\| &= \|\Psi(t)Y(t)P_2Y^{-1}(0)((\lambda_n - 1)y(0))\| \\ &\leq |\lambda_n - 1| |\Psi(t)Y(t)P_2| \|Y^{-1}(0)y(0)\| \end{aligned}$$

Thus,

$$\begin{gathered} \lim_{n\to\infty} \Psi(t)y_n(t) = \Psi(t)y(t), \quad \text{uniformly on } \mathbb{R}_+, \\ \lim_{t\to\infty} \Psi(t)(y_n(t) - y(t)) = 0, \quad \text{for } n = 1, 2, \dots . \end{gathered}$$

It follows that the linear equation (1.3) is Ψ-conditionally asymptotically stable on $\mathbb{R}_+$.

Now, we shall prove the necessity. From Ψ-conditional asymptotic stability on $\mathbb{R}_+$ of (1.3), it follows that $\Psi(t)Y(t)$ is unbounded on $\mathbb{R}_+$ (see [11, Theorem 1].

In addition, there exists a non-trivial solution $y_0(t)$ on $\mathbb{R}_+$ of (1.3) such that $\lim_{t\to\infty} \Psi(t)y_0(t) = 0$. Thus, there exists a constant vector $c \neq 0$ such that $\Psi(t)Y(t)c$ is such that $\lim_{t\to\infty} \Psi(t)Y(t)c = 0$. Let $c_s = \|c\|$. Let P_2 be the null matrix in which the s-th column is replaced with $\|c\|^{-1}c$. Thus, P_2 is a projection and $\lim_{t\to\infty} \Psi(t)Y(t)P_2 = 0$.

The proof is now complete. □

A sufficient condition for Ψ-conditional asymptotic stability is given by the following theorem.

Theorem 3.3. *If there exist two supplementary projections P_1, P_2, $P_i \neq 0$, and a positive constant K such that the fundamental matrix $Y(t)$ of the equation* (1.3) *satisfies the condition*

$$\int_0^t |\Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s)|ds + \int_t^\infty |\Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s)|ds \leq K$$

for all $t \geq 0$, then, the linear equation (1.3) *is Ψ-conditionally asymptotically stable on $\mathbb{R}_+$.*

The proof of the above theorem follows from [11, Theorem 2 and Lemmas 1, 2].

Theorem 3.4. *Suppose that:*

(1) *There exist supplementary projections P_1, P_2, $P_i \neq 0$, and a constant $K > 0$ such that the fundamental matrix $Y(t)$ of* (1.3) *satisfies the conditions*

$$\begin{gathered} |\Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s)| \leq K, \quad \text{for } 0 \leq s \leq t, \\ |\Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s)| \leq K, \quad \text{for } 0 \leq t \leq s. \end{gathered}$$

(2) $\lim_{t\to\infty} \Psi(t)Y(t)P_1 = 0$.

(3) *$B(t)$ is a $d \times d$ continuous matrix function on $\mathbb{R}_+$ such that*

$$\int_0^\infty |\Psi(t)B(t)\Psi^{-1}(t)|dt \quad \text{is convergent.}$$

(4) *The linear equations* (1.2) *and* (1.3) *are Ψ-unstable on $\mathbb{R}_+$.*

Then (1.2) *is Ψ-conditionally asymptotically stable on $\mathbb{R}_+$.*

Proof. We choose $t_0 \geq 0$ sufficiently large so that

$$q = K \int_{t_0}^\infty |\Psi(t)B(t)\Psi^{-1}(t)|dt < 1.$$

We put

$$S = \{x : t_0, \infty) \to \mathbb{R}^d : x \text{ is continuous and } \Psi\text{-bounded on } [t_0, \infty)\}.$$

Define on the set S a norm by

$$|||x||| = \sup_{t \geq t_0} \|\Psi(t)x(t)\|.$$

It is well known that $(S, |||\cdot|||)$ is a Banach real space.

For $x \in S$, we define

$$(Tx)(t) = \int_{t_0}^{t} Y(t)P_1Y^{-1}(s)B(s)x(s)ds - \int_{t}^{\infty} Y(t)P_2Y^{-1}(s)B(s)x(s)ds, \quad t \geq t_0.$$

It is easy to see that $(Tx)(t)$ exists and is continuous for $t \geq t_0$ (see the Proof of [12, Theorem 3]). We have

$$\begin{aligned}
\|\Psi(t)(Tx)(t)\| \leq&\ K \int_{t_0}^{\infty} |\Psi(s)B(s)\Psi^{-1}(s)| \|\Psi(s)x(s)\| ds \\
\leq&\ q \sup_{t \geq t_0} \|\Psi(t)x(t)\| = q|||x|||, \quad \text{for } t \geq t_0.
\end{aligned}$$

This shows that $TS \subseteq S$.

On the other hand, T is linear and

$$|||Tx_1 - Tx_2||| = |||T(x_1 - x_2)||| \leq\ q|||x_1 - x_2|||.$$

Thus, T is a contraction on the Banach space $(S, |||\cdot|||)$.

Now, for every fixed Ψ- bounded solution y of (1.3) we define an operator $S_y : S \to S$, by the relation

$$S_y x(t) = y(t) + Tx(t), \quad t \in [t_0, \infty). \tag{3.1}$$

It follows by the Banach contraction principle that S_y has a unique fixed point in S. An easy computation shows that the fixed point $x(t) = S_y x(t)$, $t \in [t_0, \infty)$, is a Ψ-bounded solution of (1.2).

Let S_2, S_3 be the spaces of Ψ-bounded solutions of equations (1.2) and (1.3) respectively. We define the mapping $C : S_3 \to S_2$ in the following way: For every $y \in S_3$, Cy will be the fixed point of the contraction S_y.

Now, from $x = Cy$ and $x_0 = Cy_0$, we have that $x = y + Tx$, $x_0 = y_0 + Tx_0$ respectively. We obtain

$$\begin{aligned}
|||\ x - x_0||| &\leq |||\ y - y_0||| + |||\ Tx - Tx_0||| \\
&\leq |||\ y - y_0||| + q|||\ x - x_0|||.
\end{aligned}$$

Thus

$$|||\ x - x_0||| \leq (1 - q)^{-1} |||\ y - y_0|||. \tag{3.2}$$

On the other hand,

$$\begin{aligned}
|||\ y - y_0||| &= |||\ x - Tx - x_0 + Tx_0||| \\
&\leq |||\ x - x_0\ |||\ +\ |||\ Tx - Tx_0\ ||| \\
&\leq\ (1 + q)|||\ x - x_0\ |||.
\end{aligned}$$

Thus, C is homeomorfism.

Now, we prove that if x, $y \in S$ are Ψ-bounded solutions of (1.2) and (1.3) respectively such that $x = Cy$, then

$$\lim_{t \to \infty} \|\Psi(t)(x(t) - y(t))\| = 0.$$

Indeed, for a given $\varepsilon > 0$, we choose $t_1 \geq t_0$ so that
$$K \sup_{t\geq t_0} \|\Psi(t)x(t)\| \int_{t_1}^{\infty} |\Psi(s)B(s)\Psi^{-1}(s)|ds < \frac{\varepsilon}{3}.$$
Thus, for $t \geq t_1$, we have
$$\begin{aligned}
&\|\Psi(t)(x(t) - y(t))\| \\
&= \|\Psi(t)(Tx)(t)\| \\
&\leq \int_{t_0}^{t} \|\Psi(t)Y(t)P_1Y^{-1}\ (s)B(s)x(s)\|ds \\
&\quad + \int_{t}^{\infty} \|\Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s)\Psi(s)B(s)\Psi^{-1}(s)\Psi(s)x(s)\|ds \\
&\leq |\Psi(t)Y(t)P_1| \int_{t_0}^{t_1} \|\ Y^{-1}(s)B(s)x(s)\|ds \\
&\quad + K \sup_{t\geq t_0} \|\Psi(t)x(t)\| \int_{t_1}^{\infty} |\Psi(s)B(s)\Psi^{-1}(s)|ds \\
&\quad + K \sup_{t\geq t_0} \|\Psi(t)x(t)\| \int_{t}^{\infty} |\Psi(s)B(s)\Psi^{-1}(s)|ds \\
&< |\Psi(t)Y(t)P_1| \int_{t_0}^{t_1} \|Y^{-1}(s)B(s)x(s)\|ds + 2\frac{\varepsilon}{3}.
\end{aligned}$$
Thus and assumption 3,
$$\lim_{t\to\infty} \|\Psi(t)(x(t) - y(t))\| = 0. \tag{3.3}$$
¿From the hypotheses, [11, Theorem1 and 2] it follows that the linear equation (1.3) is Ψ-conditionally asymptotically stable on $\mathbb{R}_+$.

Let $x(t)$ be a Ψ-bounded solution on $\mathbb{R}_+$ of (1.2). ¿From the assumption 4, this solution is Ψ-unstable on $\mathbb{R}_+$. Let $y = C^{-1}x$. From Definition 2.4, it follows that there exists a sequence (y_n) of solutions of (1.3) defined on $\mathbb{R}_+$ such that
$$\begin{gathered}
\lim_{n\to\infty} \Psi(t)y_n(t) = \Psi(t)y(t), \quad \text{uniformly on } \mathbb{R}_+, \\
\lim_{t\to\infty} \Psi(t)(y_n(t) - y(t)) = 0, \quad \text{for } n = 1, 2, \dots.
\end{gathered}$$
Let $x_n = Cy_n$. From (3.2) it follows that the sequence (x_n) of solutions of (1.2) defined on $[t_0, \infty)$ (in fact, defined on $\mathbb{R}_+$) satisfies the condition
$$\lim_{n\to\infty} \Psi(t)x_n(t) = \Psi(t)x(t), \quad \text{uniformly on } [t_0, \infty).$$
Clearly,
$$\lim_{n\to\infty} x_n(t_0) = x(t_0).$$
By the Dependence on initial conditions Theorem (see [6, Chapter I, Theorem 3]), it follows that
$$\lim_{n\to\infty} x_n(t) = x(t), \quad \text{uniformly on } [0, t_0].$$
Hence,
$$\lim_{n\to\infty} \Psi(t)x_n(t) = \Psi(t)x(t), \quad \text{uniformly on } [0, t_0].$$
Thus,
$$\lim_{n\to\infty} \Psi(t)x_n(t) = \Psi(t)x(t), \quad \text{uniformly on } \mathbb{R}_+.$$

This shows that the linear equation (1.2) is Ψ-conditionally stable on $\mathbb{R}_+$. From (3.3) and

$$\Psi(t)(x_n(t)-x(t)) = \Psi(t)(x_n(t)-y_n(t)) + \Psi(t)(y_n(t)-y(t)) + \Psi(t)(y(t)-x(t)),$$

it follows that

$$\lim_{t\to\infty} \Psi(t)(x_n(t)-x(t)) = 0, \quad \text{for } n = 1,2,\dots.$$

This shows that the linear equation (1.2) is Ψ-conditionally asymptotically stable on $\mathbb{R}_+$. The proof is complete. □

Theorem 3.5. *Suppose that:*

(1) *There exist two supplementary projections P_1, P_2, $P_i \neq 0$, and a positive constant K such that the fundamental matrix $Y(t)$ of the equation* (1.3) *satisfies the condition*

$$\int_0^t |\Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s)|ds + \int_t^\infty |\Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s)|ds \le K$$

for all $t \ge 0$.

(2) *$B(t)$ is a $d\times d$ continuous matrix function on $\mathbb{R}_+$ such that*

$$\lim_{t\to\infty} |\Psi(t)B(t)\Psi^{-1}(t)| = 0.$$

Then, the linear equation (1.2) *is Ψ-conditionally asymptotically stable on $\mathbb{R}_+$.*

The proof of the above theorem is similar to the proof of Theorem 3.4.

Remark 3.6. The first condition of the above Theorems can certainly be satisfied if A(t) = A is a d×d real constant matrix which has characteristic roots with different real parts. In this case, e.g., there exists an interval $(\alpha,\beta)\subset\mathbb{R}$ such that for $\lambda\in(\alpha,\beta)$, $\Psi(t) = e^{-\lambda t}I_d$ and $Y(t)$ can satisfy the first hypotheses of Theorems.

We have a similar situation if $A(t)$ is a $d\times d$ real continuous periodic matrix (See [12, Examples 1, 2]).

Thus, the above results can be considered as a generalization of a well-known result in conection with the classical conditional asymptotic stability.

Remark 3.7. If in the above Theorems, the linear equation (1.3) is only Ψ-conditionally asymptotically stable on $\mathbb{R}_+$, then the perturbed equation (1.2) can not be Ψ-conditionally asymptotically stable on $\mathbb{R}_+$.

This is shown by the next example transformed after an equation due to Perron [19].

Example 3.8. Let $a, b\in\mathbb{R}$ such that $0 < 4a < 1$, $b \neq 0$ and

$$A(t) = \begin{pmatrix} \sin\ln(t+1) + \cos\ln(t+1) - 4a & 0 \\ 0 & -2a \end{pmatrix}.$$

Then, a fundamental matrix for the homogeneous equation (1.3) is

$$Y(t) = \begin{pmatrix} e^{(t+1)[\sin\ln(t+1)-4a]} & 0 \\ 0 & e^{-2a(t+1)} \end{pmatrix}.$$

Let

$$\Psi(t) = \begin{pmatrix} 1 & 0 \\ 0 & e^{a(t+1)} \end{pmatrix}.$$

We have
$$\Psi(t)Y(t) = \begin{pmatrix} e^{(t+1)[\sin\ln(t+1)-4a]} & 0 \\ 0 & e^{-a(t+1)} \end{pmatrix}.$$
Let $t'_n = e^{(2n+\frac{1}{2})\pi} - 1$ for $n = 1, 2 \dots$. Since $\lim_{n\to\infty} |\Psi(t'_n)Y(t'_n)| = \infty$, it follows that the linear equation (1.3) is Ψ-unstable on $\mathbb{R}_+$ (see [11, Theorem 1])

From Theorem 3.1 it follows that the linear equation (1.3) is Ψ-conditionally asymptotically stable on $\mathbb{R}_+$. If we take
$$B(t) = \begin{pmatrix} 0 & be^{-2a(t+1)} \\ 0 & 0 \end{pmatrix},$$
then, a fundamental matrix for the perturbed equation (1.2) is
$$\widetilde{Y}(t) = \begin{pmatrix} be^{(t+1)[\sin\ln(t+1)-4a]}\int_1^{t+1} e^{-s\sin\ln s}ds & e^{(t+1)[\sin\ln(t+1)-4a]} \\ e^{-2a(t+1)} & 0 \end{pmatrix}.$$
We have
$$\Psi(t)\widetilde{Y}(t) = \begin{pmatrix} be^{(t+1)[\sin\ln(t+1)-4a]}\int_1^{t+1} e^{-s\sin\ln s}ds & e^{(t+1)[\sin\ln(t+1)-4a]} \\ e^{-a(t+1)} & 0 \end{pmatrix}.$$
Since $\lim_{n\to\infty} |\Psi(t'_n)\widetilde{Y}(t'_n)| = \infty$, it follows that the perturbed equation (1.2) is Ψ-unstable on $\mathbb{R}_+$ (see [11, Theorem 1]).

Let $\alpha \in (0, \frac{\pi}{2})$. Let $\mathrm{t}_n = e^{(2n-\frac{1}{2})\pi}$ for $n = 1, 2, \dots$. For $t_n \le s \le t_n e^{\alpha}$ we have $s\cos\alpha \le -s\sin\ln s \le s$ and hence
$$\begin{aligned} e^{t_n e^{\pi}(\sin\ln t_n e^{\pi} - 4a)} \int_1^{t_n e^{\pi}} e^{-s\sin\ln s}ds &> e^{t_n e^{\pi}(\sin\ln t_n e^{\pi} - 4a)} \int_{t_n}^{t_n e^{\alpha}} e^{-s\sin\ln s}ds \\ &\ge e^{t_n e^{\pi}(1-4a)} \int_{t_n}^{t_n e^{\alpha}} e^{s\cos\alpha}ds \\ &= e^{t_n[(1-4a)e^{\pi}+\cos\alpha]} \frac{e^{t_n(e^{\alpha}-1)\cos\alpha} - 1}{\cos\alpha} \to \infty. \end{aligned}$$
Thus, the columns of $\Psi(t)\widetilde{Y}(t)$ are unbounded on $\mathbb{R}_+$. It follows that the perturbed equation (1.2) is not Ψ-conditionally asymptotically stable on $\mathbb{R}_+$ (see Theorem 3.1).

Finally, we have $|\Psi(t)B(t)\Psi^{-1}(t) = be^{-3a(t+1)}$. Thus, $B(t)$ satisfies the conditions:
$$\lim_{t\to\infty} |\Psi(t)B(t)\Psi^{-1}(t)| = 0;$$
and $\int_0^{\infty} |\Psi(t)B(t)\Psi^{-1}(t)|dt$ can be a sufficiently small number.

4. Ψ-conditional asymptotic stability of the nonlinear equation (1.1)

In this section we give sufficient conditions for the Ψ-conditional asymptotic stability of Ψ-bounded solutions of the nonlinear Volterra integro-differential system (1.1).

Theorem 4.1. *Suppose that:*

(1) *There exist supplementary projections P_1, P_2, $P_i \neq 0$ and a constant $K > 0$ such that the fundamental matrix $Y(t)$ of (1.3) satisfies the condition*
$$\int_0^t |\Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s)|ds + \int_t^{\infty} |\Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s)|ds \le K$$

for all $t \geq 0$.

(2) *The function* $F(t,s,x)$ *satisfies the inequality*
$$\|\Psi(t)\left(F(t,s,x(s)) - F(t,s,y(s))\right)\| \leq f(t,s)\|\Psi(s)\left(x(s)-y(s)\right)\|,$$
for $0 \leq s \leq t < \infty$ *and for all continuous and* Ψ*-bounded functions* $x, y : \mathbb{R}_+ \to \mathbb{R}^d$*, where* $f(t,s)$ *is a continuous nonnegative function on* D *such that*
$$F(t,s,0) = 0, \quad \lim_{t\to\infty} \int_0^t f(t,s)ds = 0, \quad \sup_{t\geq 0} \int_0^t f(t,s)ds < K^{-1}.$$

Then, all Ψ*-bounded solutions of* (1.1) *are* Ψ*-conditionally asymptotically stable on* $\mathbb{R}_+$*.*

Proof. Let
$$q = K \sup_{t\geq 0} \int_0^t f(t,s)ds < 1.$$
We put
$$S = \{x : \mathbb{R}_+ \to R^d : x \text{ is continuous and } \Psi\text{-bounded on } \mathbb{R}_+\}.$$
Define on the set S a norm by
$$|||x||| = \sup_{t\geq 0} \|\Psi(t)x(t)\|.$$
It is well-known that $(S, |||\cdot|||)$ is a Banach space. For $x \in S$, we define
$$\begin{aligned}(Tx)(t) &= \int_0^t Y(t)P_1Y^{-1}(s) \int_o^s F(s,u,x(u))\,du\,ds \\ &\quad - \int_t^\infty Y(t)P_2Y^{-1}(s) \int_o^s \ F(s,u,x(u))\,du\,ds, t \geq 0.\end{aligned}$$
For $0 \leq t \leq v$, we have
$$\begin{aligned}&\|\Psi(t) \int_t^v Y(t)P_2Y^{-1}(s) \int_o^s F(s,u,x(u))\,du\,ds\| \\ &= \| \int_t^v \Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s) \int_o^s \Psi(s)F(s,u,x(u))du\,ds\| \\ &\leq \int_t^v |\Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s)| \int_o^s \|\Psi(s)F(s,u,x(u))\|\,du\,ds \\ &\leq \int_t^v |\Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s)| \int_0^s f(s,u)\|\Psi\ (u)x(u)\|\,du\,ds \\ &\leq \sup_{u\geq 0}\|\Psi(u)x(u)\| \int_t^v |\Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s)| \int_0^s f(s,u)\,du\,ds \\ &\leq \ qK^{-1} \sup_{u\geq 0}\|\Psi(u)x(u)\| \int_t^v |\Psi(t)Y(t)P_2Y^{-1}\ (s)\Psi^{-1}(s)|ds.\end{aligned}$$
¿From assumption 1, it follows that the integral
$$\int_t^\infty Y(t)P_2Y^{-1}(s) \int_o^s F(s,u,x(u))\,du\,ds$$

is convergent. Thus, $(Tx)(t)$ exists and is continuous for $t \geq 0$. For $x \in S$ and $t \geq 0$, we have

$$\begin{aligned}
\|\Psi(t)(Tx)(t)\| &= \| \int_0^t \Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s) \int_o^s \Psi(s)F(s,u,x(u))\,du\,ds \\
&\quad - \int_t^\infty \Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s) \int_o^s \Psi(s)F(s,u,x(u))\,du\,ds\| \\
&\leq \int_0^t |\Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s)| \int_o^s \|\Psi(s)F(s,u,x(u))\|\,du\,ds \\
&\quad + \int_t^\infty |\Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s)| \int_o^s \|\Psi(s)F(s,u,x(u))\|\,du\,ds \\
&\leq \int_0^t |\Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s)| \int_0^s f(s,u)\|\Psi(u)x(u)\|\,du\,ds \\
&\quad + \int_t^\infty |\Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s)| \int_o^s f(s,u)\|\Psi(u)x(u)\|\,du\,ds \\
&\leq q \sup_{u\geq 0} \|\Psi(u)x(u)\|.
\end{aligned}$$

This shows that $TS \subseteq S$. On the other hand, for $x, y \in S$ and $t \geq 0$, we have

$$\begin{aligned}
&\|\Psi(t)\left((Tx)(t) - (Ty)(t)\right)\| \\
&= \| \int_0^t \Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s) \int_o^s \Psi(s)\left(F(s,u,x(u)) - F(s,u,y(u))\right)\,du\,ds \\
&\quad - \int_t^\infty \Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s) \int_o^s \Psi(s)\left(F(s,u,x(u)) - F(s,u,y(u))\right)\,du\,ds\| \\
&\leq \int_0^t |\Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s)| \int_o^s \|\Psi(s)\left(F(s,u,x(u)) - F(s,u,y(u))\right)\|\,du\,ds \\
&\quad + \int_t^\infty |\Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s)| \int_o^s \|\Psi(s)\left(F(s,u,x(u)) - F(s,u,y(u))\right)\|\,du\,ds \\
&\leq \int_0^t |\Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s)| \int_0^s f(s,u)\|\Psi(u)(x(u) - y(u))\|\,du\,ds \\
&\quad + \int_t^\infty |\Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s)| \int_0^s f(s,u)\|\Psi(u)(x(u) - y(u))\|\,du\,ds \\
&\leq q \sup_{u\geq 0} \|\Psi(u)(x(u) - y(u))\|.
\end{aligned}$$

It follows that

$$\sup_{t\geq 0} \|\Psi(t)\left((Tx)(t) - (Ty)(t)\right)\| \leq q \sup_{t\geq 0} \|\Psi(t)(x(t) - y(t))\|.$$

Thus, we have

$$|||Tx - Ty||| \leq q|||x - y|||.$$

This shows that T is a contraction of the Banach space $(S, |||\cdot|||)$.

As in the Proof of Theorem 3.4, it follows by the Banach contraction principle that for any function $y \in S$, the integral equation

$$x = y + Tx \tag{4.1}$$

has a unique solution $x \in S$. Furthermore, by the definition of T, $x(t) - y(t)$ is differentiable and

$$(x(t) - y(t))' = A(t)\,(x(t) - y(t)) + \int_0^t F(t, s, x(s))ds, t \geq 0.$$

Hence, if y(t) is a Ψ-bounded solution of (1.3), $x(t)$ is a Ψ-bounded solution of (1.1). Conversely, if $x(t)$ is a Ψ-bounded solution of (1.1), the function $y(t)$ defined by (4.1) is a Ψ-bounded solution of (1.3).

Thus, (4.1) establishes a one-to-one correspondence C between the Ψ-bounded solutions of (1.1) and (1.3): $x = Cy$.

Now, we consider the analogous equation

$$x_0 = y_0 + Tx_0.$$

We get

$$(1 - q)|||\, x - x_0||| \leq |||\, y - y_0|||. \tag{4.2}$$

Now, we prove that if $x, y \in S$ are Ψ-bounded solutions of (1.1) and (1.3) respectively such that $x = Cy$, then

$$\lim_{t\to\infty} \|\Psi(t)(x(t) - y(t))\| = 0. \tag{4.3}$$

For a given $\varepsilon > 0$, we can choose $t_1 \geq 0$ such that

$$K|||x||| \int_0^t f(t, s)ds < \frac{\varepsilon}{2},$$

for $t \geq t_1$. Moreover, since $\lim_{t\to\infty} |\Psi(t)Y(t)P_1| = 0$ (see [11, Lemma 1]), there exists a number $t_2 \geq t_1$ such that

$$qK^{-1}|\Psi(t)Y(t)P_1||||x||| \int_0^{t_1} |P_1Y^{-1}(s)\Psi^{-1}(s)|ds < \frac{\varepsilon}{2}$$

for $t \geq t_2$. We have, for $t \geq t_2$,

$$\begin{aligned}
&\|\Psi(t)(x(t) - y(t))\| \\
&\leq \int_0^t |\Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s)| \int_o^s \|\Psi(s)F(s, u, x(u))\|\, du\, ds+ \\
&\quad + \int_t^\infty |\Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s)| \int_o^s \|\Psi(s)F(s, u, x(u))\|\, du\, ds \\
&\leq \int_0^t |\Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s)| \int_0^s f(s, u)\|\Psi(u)x(u)\|\, du\, ds \\
&\quad + \int_t^\infty |\Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s)| \int_o^s f(s, u)\|\Psi(u)x(u)\|\, du\, ds \\
&\leq qK^{-1}|\Psi(t)Y(t)P_1|\, |||x||| \int_0^{t_1} |P_1Y^{-1}(s)\Psi^{-1}(s)|ds \\
&\quad + |||x||| \int_{t_1}^t |\Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s)| \Big(\int_0^s f(s, u)du\Big)ds \\
&\quad + |||x||| \int_t^\infty |\Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s)| \Big(\int_0^s f(s, u)du\Big)ds < \varepsilon.
\end{aligned}$$

Now, let $x(t)$ be a Ψ-bounded solution of (1.1). This solution is Ψ-unstable on $\mathbb{R}_+$.

Indeed, if not, for every ε ¿ 0 and any $t_0 \geq 0$, there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that any solution $\widetilde{x}(t)$ of (1.1) which satisfies the inequality $\|\Psi(t_0)(\widetilde{x}(t_0) - x(t_0))\| < \delta(\varepsilon, t_0)$ exists and satisfies the inequality $\|\Psi(t)(\widetilde{x}(t) - x(t))\| < \varepsilon$ for all $t \geq t_0$.

Let $z_0 \in R^d$ be such that $P_1 z_0 = 0$ and $0 < \|\Psi(0)z_0\| < \delta(\varepsilon, 0)$ and let $\widetilde{x}(t)$ the solution of (1.1) with the initial condition $\widetilde{x}(0) = x(0) + z_0$. Then $\|\Psi(t)z(t)\| < \varepsilon$ for all $t \geq 0$, where $z(t) = \widetilde{x}(t) - x(t)$.

Now we consider the function $y(t) = z(t) - (Tz)(t)$, $t \geq 0$.

Clearly, $y(t)$ is a Ψ-bounded solution on $\mathbb{R}_+$ of (1.3). Without loss of generality, we can suppose that $Y(0) = I$. It is easy to see that $P_1 y(0) = 0$. If $P_2 y(0) \neq 0$, from [11, Lemma 2], it follows that $\limsup_{t\to\infty} \|\Psi(t)y(t)\| = \infty$, which is contradictory. Thus, $P_2 y(0) = 0$ and then $y(t) = 0$ for $t \geq 0$.

It follows that $z = Tz$ and then $z = 0$, which is a contradiction. This shows that the solution $x(t)$ is Ψ-unstable on $\mathbb{R}_+$.

Let $y = x - Tx$. From Theorem 3.3, it follows that there exists a sequence (y_n) of solutions of (1.3) defined on $\mathbb{R}_+$ such that

$$\lim_{n\to\infty} \Psi(t)y_n(t) = \Psi(t)y(t), \quad \text{uniformly on } \mathbb{R}_+,$$
$$\lim_{t\to\infty} \Psi(t)(y_n(t) - y(t)) = 0, \quad n = 1, 2, \dots .$$

Let $x_n = Cy_n$. From (4.2) it follows that the sequence (x_n) of solutions of (1.1) defined on $\mathbb{R}_+$ is such that

$$\lim_{n\to\infty} \Psi(t)x_n(t) = \Psi(t)x(t), \quad \text{uniformly on } \mathbb{R}_+.$$

This shows that the solution $x(t)$ is Ψ-conditionally stable on $\mathbb{R}_+$. From (4.3) and

$$\Psi(t)(x_n(t) - x(t)) = \Psi(t)(x_n(t) - y_n(t)) + \Psi(t)(y_n(t) - y(t)) + \Psi(t)(y(t) - x(t)),$$

it follows that

$$\lim_{t\to\infty} \Psi(t)(x_n(t) - x(t)) = 0, \quad \text{for } n = 1, 2, \dots .$$

This shows that the solution $x(t)$ is Ψ-conditionally asymptotically stable on $\mathbb{R}_+$. The proof is now complete. □

Corollary 4.2. *If in Theorem 4.1 we assume that $f(t.s) = g(t)h(s)$, where g and h are nonnegative continuous functions on $\mathbb{R}_+$ such that*

$$\sup_{t\geq 0} g(t) \int_0^t h(s)ds < K^{-1},$$
$$\lim_{t\to\infty} g(t) \int_0^t h(s)ds = 0,$$

then the conclusion of the Theorem remains valid.

Corollary 4.3. *If in Theorem 4.1 we assume that $f(t.s) = g(t)h(s)$, where g and h are nonnegative continuous functions on $\mathbb{R}_+$ such that*

$$I = \int_0^\infty h(s)\, ds \quad \textit{is convergent},$$
$$\lim_{t\to\infty} g(t) = 0, \quad \sup_{t\geq 0} g(t) < \frac{1}{KI},$$

then the conclusion of the Theorem remains valid.

References

[1] Akinyele, O. *On partial stability and boundedness of degree k;* Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., (8), 65(1978), 259 - 264.

[2] Akinyele, O. *On the φ-stability for differential systems;* Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.,(4), 68(1980), 287 - 293.

[3] Avramescu, C. *Asupra comportãrii asimptotice a solutiilor unor ecuatii functionale;* Analele Universitãtii din Timisoara, Seria Stiinte Matematice-Fizice, vol. VI, 1968, 41 - 55.

[4] Constantin, A. *Asymptotic properties of solutions of differential equations;* Analele Universitãtii din Timisoara, Seria Stiinte Matematice, vol. XXX, fasc. 2-3, 1992, 183 - 225.

[5] Constantin, A. *A note on the φ-stability for differential systems;* Boll. Un. Math. Ital., 6A(1992), 233 - 243.

[6] Coppel, W.A. *Stability and Asymptotic Behavior of Differential Equations,* Heath, Boston, 1965.

[7] Coppel, W.A. *On the stability of ordinary differential equations,* J. London Math. Soc. 38(1963), 255 - 260.

[8] Diamandescu, A. *On the Ψ-stability of a nonlinear Volterra integro-differential system;* Electron. J. Diff. Eqns., Vol. 2005(2005), No. 56, pp. 1 - 14; URL http://ejde.math.txstate.edu

[9] Diamandescu, A. *On the Ψ-asymptotic stability of a nonlinear Volterra integro-differential system;* Bull. Math. Sc. Math. Roumanie, Tome 46(94), No. 1-2, 2003, pp. 39 - 60.

[10] Diamandescu, A. *On the Ψ-uniform asymptotic stability of a nonlinear Volterra integro-differential system;* Analele Universităţii de Vest Timişoara, Seria Matematică - Informatică, Vol. XXXIX, fasc. 2, 2001, pp. 35 - 62.

[11] Diamandescu, A. *On the Ψ-instability of a nonlinear Volterra integro-differential system;* Bull. Math. Sc. Math. Roumanie, Tome 46(94), No. 3 - 4, 2003, pp. 103 - 119..

[12] Diamandescu, A. *On the Ψ-conditional stability of the solutions of a nonlinear Volterra integro-differential system;* Proceedings of the National Conference on Mathematical Analysis and Applications, Timişoara, 12-13 Dec. 2000, pp. 89 - 106.

[13] Halam, T.G. *On asymptotic equivalence of the bounded solutions of two systems of differential equations;* Mich. Math. Journal., Vol. 16(1969), 353 - 363.

[14] Hara, T., Yoneyama, T. and Ytoh, T. *Asymptotic Stability Criteria for Nonlinear Volterra Integro - Differential Equations;* Funkcialaj Ecvacioj, 33(1990), 39 - 57.

[15] Kuben, J. *Asymptotic equivalence of second order differential equations;* Czech. Math. J. 34(109), (1984), 189 - 202.

[16] Lakshmikantham, V. and Rama Mohana Rao, M. *Stability in variation for nonlinear integro-differential equations;* Appl. Anal. 24(1987), 165 - 173.

[17] Mahfoud, W.E. *Boundedness properties in Volterra integro-differential systems;* Proc. Amer. Math. Soc., 100(1987), 37 - 45.

[18] Morchalo, J. *On $\Psi - L_p$-stability of nonlinear systems of differential equations;* Analele Stiintifice ale Universitãtii quotedblright Al. I. Cuzaquotedblright Iasi, Tomul XXXVI, s. I - a, Matematicã, 1990, f. 4, 353 - 360.

[19] Perron, O. *Die Stabilitätsfrage bei Differentialgleichungen,* Math. Z., 32(1930), 703 - 728.

[20] Sansone, G. and Conti, R. *Non-linear differential equations,* Pergamon Press, 1964.

[21] Späth, H. *Uber das asymptotische Verhalten der Lö sungenlinearer Differentialgleichungen,* Math. Z., 30(1929), 487 - 513.

[22] Weyl, H. *Comment on the preceding paper,* Amer. J. Math., 68(1946), 7 - 12.

[23] Yoshizawa, T. *Stability Theory by Liapunov's Second Method*, The Mathematical Society of Japan, 1966.

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