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OSCILLATION OF THIRD-ORDER NEUTRAL DAMPED DIFFERENTIAL EQUATIONS

MIROSLAV BARTUŠEK

ABSTRACT. We study a third-order damped neutral sublinear differential equation whose differential operator is non-oscillatory. Specifically, we obtain sufficient conditions for all solutions to be oscillatory.

1. INTRODUCTION

Consider the third-order differential equation

$$z''' + q(t)z' + r(t)f\big(x(\sigma(t))\big) = 0\,, \quad t \geq 0, \tag{1.1}$$

$$z(t) = x(t) + a(t)x\big(\tau(t)\big)\,. \tag{1.2}$$

In this article we impose er the following assumptions:

(H1) $q \in C(\mathbb{R}_+)$, $q(t) \geq 0$ for large t, $r \in C(\mathbb{R}_+)$, $r(t) > 0$ for large t, $\mathbb{R}_+ = [0, \infty)$;

(H2) $\sigma \in C(\mathbb{R})$, $\mathbb{R} = (-\infty, \infty)$, $\sigma(t) \leq t$ for $t \in \mathbb{R}$, $\lim_{t\to\infty} \sigma(t) = \infty$, there exists a constant σ_1 such that $0 < \sigma'(t) \leq \sigma_1$ for all $t \in \mathbb{R}$;

(H3) $\tau \in C^3(\mathbb{R})$, $\sigma(t) \leq \tau(t) \leq t$ for all $t \in \mathbb{R}$, $\lim_{t\to\infty} \tau(t) = \infty$, and there exists a τ_0 exists such that $0 < \tau_0 \leq \tau'(t)$ for all $t \in \mathbb{R}$;

(H4) $a \in C^3(\mathbb{R}_+)$, there exists a number a_1 such that $0 \leq a(t) \leq a_1$ for all $t \in \mathbb{R}_+$;

(H5) $f \in C(\mathbb{R})$, $f(u)u > 0$ for $u \neq 0$ and there exists a $\lambda \in (0, 1]$ such that

$$|f(u)| \geq |u|^\lambda \quad \forall u \in \mathbb{R}\,;$$

(H6) The associated second-order linear equation

$$h'' + q(t)h = 0\,, \quad t \geq 0 \tag{1.3}$$

has a solution $h(t) > 0$ for all t large enough.

Definition 1.1. Let $T \in \mathbb{R}_+$ and $T_0 = \sigma(T)$. A function x is said to be *a solution* of (1.1) on $[T, \infty)$ if x is defined and continuous on $[T_0, \infty)$, $z \in C^3[T, \infty)$, and x satisfies (1.1) on $[T, \infty)$.

A solution is said to be *non-oscillatory* if $x(t) \neq 0$ for all large t, otherwise it is said to be *oscillatory*. Equation (1.1) is oscillatory if all its solutions are oscillatory.

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In recent years, a great attention has been paid to qualitative theory of third-order neutral differential equations. Such equations have applications in mathematical modeling in biology and physics, see for example [10, 11, 12, 15]. A great effort has been devoted to oscillation theory of the damped equations of the forms

$$x''' + q(t)x' + r(t)f\big(x(\sigma(t))\big) = 0, \tag{1.4}$$

$$\big(r_2(t)(r_1(t)x')'\big)' + q(t)x'(t) + r(t)f\big(x(\sigma(t))\big) = 0 \tag{1.5}$$

with $r_i \in C(\mathbb{R}_+)$, $r_i(t) > 0$ for $t \in \mathbb{R}_+$ and $i = 1, 2$.

An equation is said to have Property A if every solution is either oscillatory or $x(t)x'(t) < 0$ for all large t. Sufficient (and or necessary) conditions have been studied under which equation either (1.4) or (1.5) has Property A. Equation (1.4) has been studied in [8] (where there is a nice review of the results.), in [2], and the references therein. For studies of (1.5), see for example [1, 3, 14].

Property A has been generalized for the neutral differential equation

$$z''' + r(t)f\big(x(\sigma(t))\big) = 0 \tag{1.6}$$

in [13], and for the equation

$$\big(r_2(t)(r_1(t)z')'\big)' + R(t)x\big(\sigma(t)\big) = 0 \tag{1.7}$$

in [5, 6], where $r_i \in C(\mathbb{R}_+)$, $R \in C(\mathbb{R}_+)$, $r_i > 0$ for $i = 1, 2$, $R > 0$, and z is given by (1.2). An interesting question was solved in [6] for (1.7) in the canonical case, i.e. when

$$\int_0^\infty \frac{1}{r_i(t)}\,dt = \infty \quad \text{for } i = 1, 2. \tag{1.8}$$

Reference [5] shows sufficient conditions for (1.5) (with $q \equiv 0$) no having a solution x such that $z(t)z'(t) < 0$ for large t.

Since (1.3) is non-oscillatory and $q \ge 0$, every eventually positive solution of (1.3) is nondecreasing for large t, and the following holds, see [9].

Lemma 1.2. *Equation* (1.3) *has a solution h which is positive and nondecreasing for $t \ge t_0 \ge 0$ and*

$$\int_{t_0}^\infty \frac{dt}{h^2(t)} = \infty\,, \quad \int_{t_0}^\infty h(t)\,dt = \infty\,. \tag{1.9}$$

If $\int_0^\infty tq(t)\,dt < \infty$ then $\lim_{t\to\infty} h(t) \in (0, \infty)$. Also if $\int_0^\infty tq(t)\,dt = \infty$, then $\lim_{t\to\infty} h(t) = \infty$.

Note that if a solution h satisfies (1.9), then a positive constant times h also satisfies (1.9). This solution is called a principal solution.

Definition 1.3. Let h be a principal solution of (1.3) such that $h(t) > 0$ on $[t^*, \infty) \subset \mathbb{R}_+$. In the case $\int_0^\infty tq(t)\,dt < \infty$, h is chosen such that $\lim_{t\to\infty} h(t) = 1$.

It is easy to see that for for $t \ge t^*$, (1.1) can be rewritten as

$$\Big(h^2(t)\Big(\frac{z'}{h(t)}\Big)'\Big)' + h(t)r(t)f\big(x(\sigma(t))\big) = 0\,. \tag{1.10}$$

For $t \ge t^*$, we denote the quasiderivatives of z as follows:

$$z^{[1]}(t) = \frac{z'(t)}{h(t)}\,, \quad z^{[2]}(t) = h^2(t)\big(z^{[1]}(t)\big)' \quad z^{[3]}(t) = \big(z^{[2]}(t)\big)'\,. \tag{1.11}$$

Then we rewrite (1.1) as (1.10) and using (1.11),

$$z^{[3]}(t) + h(t)r(t)f\big(x(\sigma(t))\big) = 0\,. \tag{1.12}$$

Note, that For $t \geq t^*$, (1.10) is a special case of the equation

$$\big(r_2(t)\big(r_1(t)z'\big)'\big)' + R(t)f\big(x(\sigma(t))\big) = 0, \tag{1.13}$$

where

$$r_1(t) = \frac{1}{h(t)}\,, \quad r_2(t) = h^2(t)\,, \quad R(t) = h(t)r(t)\,. \tag{1.14}$$

Because of (1.9), equation (1.13) is in canonical form, i.e. (1.8) holds.

Our goal is to find sufficient conditions for (1.1) to be oscillatory. A crucial problem is to prove nonexistence of non-oscillatory solutions such that $z(t)z'(t) < 0$ for large t. So, if $f(u) = u$ on $\mathbb{R}$ it is possible to use results from [6] for equation (1.7) with (1.14). However, a very restrictive assumption $\tau(\sigma(t)) \equiv \sigma(\tau(t))$ is used in [6]. We give sufficient conditions for the nonexistence of such solutions without this assumption and without the assumption $f(u) \equiv u$. Note, that our assumption $0 < \sigma'(t) \leq \sigma_1$ is not assumed in [6].

Let $\mathcal{N}$ be the set of all non-oscillatory solutions of (1.1) which are defined on subintervals of $\mathbb{R}_+$ and which are positive for large t. We shall study only the set $\mathcal{N}$. Non-oscillatory solutions which are negative for large t can be study by a similar way.

It is known (see, e.g., [6, Lemma]) that $\mathcal{N}$ can be divided into two subsets $\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_1$ where z is given by (1.2) and

$$\begin{gathered}
\mathcal{N}_0 = \big\{x \in \mathcal{N} : z(t) > 0, z^{[1]}(t) < 0, z^{[2}(t) > 0, z^{[3]}(t) < 0 \text{ for large } t\big\}\,, \\
\mathcal{N}_1 = \big\{x \in \mathcal{N} : z^{[i]}(t) > 0, i = 0,1,2, z^{[3]}(t) < 0 \text{ for large } t\big\}]\,.
\end{gathered}$$

In this article, τ^{-1} and σ^{-1} denote the inverse functions of τ and σ, respectively. Also we define

$$\begin{gathered}
\mathcal{N}_{00} = \big\{x \in \mathcal{N}_0 : \lim_{t\to\infty} z(t) = 0\big\}\,, \\
\mathcal{N}_{01} = \big\{x \in \mathcal{N}_0 : \lim_{t\to\infty} z(t) \in (0,\infty)\big\}\,.
\end{gathered}$$

For simplicity, for $t \geq 0$, we define

$$r^*(t) = \min\big\{r(\sigma^{-1}(t)), r(\sigma^{-1}(\tau(t)))\big\}\,. \tag{1.15}$$

Note, that by (H3),

$$\sigma^{-1}\big(\tau(t)\big) \geq t\,, \tag{1.16}$$

where e denotes the Euler number.

2. Preliminaries

Here we state some auxiliary results which will be needed later.

Lemma 2.1. *Let $x \in \mathcal{N}$ be defined on $[T,\infty)$ and $T_0 = \sigma(T)$. Let $A \in C[T,\infty)$ be positive and*

$$\int_T^\infty A(t)\,|x\big(\sigma(t)\big)|^\lambda\,dt < \infty\,. \tag{2.1}$$

Then

$$\int_T^\infty A^*(t)\,|z(t)|^\lambda\,dt < \infty \tag{2.2}$$

where

$$A^*(t) = \min\left(A(\sigma^{-1}(t)), A(\sigma^{-1}(\tau(t)))\right).$$

Proof. Let $x \in \mathcal{N}$ and $t_1 \ge T$ be such that $x(t) > 0$ for $t \ge \sigma(t_1)$. The substitution $s = \sigma(t)$ and (2.1) yield

$$\begin{aligned} \frac{1}{\sigma_1}\int_{\sigma(t_1)}^{\infty} A\big(\sigma^{-1}(s)\big)x^\lambda(s)\,ds &\le \int_{\sigma(t_1)}^{\infty} A\big(\sigma^{-1}(s)\big)x^\lambda(s)\frac{ds}{\sigma'(\sigma^{-1}(s))} \\ &= \int_{t_1}^{\infty} A(t)x^\lambda\big(\sigma(t)\,dt < \infty\,. \end{aligned} \tag{2.3}$$

From this, applying substitution $s = \tau(t)$, for $t_0 = \tau^{-1}(\sigma(t_1))$, we obtain

$$\begin{aligned} \frac{\tau_0}{\sigma_1}\int_{t_0}^{\infty} A\big(\sigma^{-1}(\tau(t))\big)x^\lambda\big(\tau(t)\big)\,dt &\le \frac{1}{\sigma_1}\int_{t_0}^{\infty} A\big(\sigma^{-1}(\tau(t))\big)x^\lambda\big(\tau(t)\big)\tau'(t)\,dt \\ &= \frac{1}{\sigma_1}\int_{\sigma(t_1)}^{\infty} A\big(\sigma^{-1}(s)\big)x^\lambda(s)\,ds < \infty\,. \end{aligned} \tag{2.4}$$

We have

$$z^\lambda(t) \le \big(x(t) + a_1 x\big(\tau(t)\big)\big)^\lambda \le M\big(x^\lambda(t) + x^\lambda\big(\tau(t)\big) \tag{2.5}$$

with $M = 2^\lambda(1 + a_1^\lambda)$. As τ is increasing and $\sigma(t_1) \le t_0$, (2.3), (2.4), (2.5) imply

$$\begin{aligned} &\min\Big\{\frac{1}{\sigma_1}, \frac{\tau_0}{\sigma_1}\Big\}\int_{t_0}^{\infty} A^*(t)z^\lambda(t)\,dt \\ &\le M\Big\{\frac{1}{\sigma_1}\int_{\sigma(t_1)}^{\infty} A\big(\sigma^{-1}(t)\big)x^\lambda(t)\,dt\Big\} + \frac{\tau_0}{\sigma_1}\int_{t_0}^{\infty} A\big(\sigma^{-1}(\tau(t))\big)x^\lambda\big(\tau(t)\big)\,dt < \infty\,. \end{aligned}$$

Hence, (2.2) is valid. □

Lemma 2.2. *There exist $k_0 \ge k > 0$ such that*

$$k_0 t \ge h(t) \ge k\ \exp\Big\{\int_0^t sq(s)\,ds\Big\} \quad \textit{for } t \ge t^* \tag{2.6}$$

where t^* and h are given by Definition 1.3. Moreover, if $\varepsilon > 0$, then

$$\frac{h(u)}{h(v)} \le (1+\varepsilon)\frac{u}{v} \quad \textit{for } u \ge v \ge \frac{1+\varepsilon}{\varepsilon}\,t^* > t^*\,. \tag{2.7}$$

Proof. As for (2.6), see [4, Lemma 2] and 1.3. Now we prove (2.7). We have $\frac{\varepsilon}{1+\varepsilon}v \ge t^*$ which is equivalent to $v - t^* \ge \frac{v}{1+\varepsilon}$. From this we have

$$\frac{u - t^*}{v - t^*} \le \frac{u}{v - t^*} \le (1+\varepsilon)\frac{u}{v} \quad \text{for } u \ge v \ge \frac{1+\varepsilon}{\varepsilon}t^*\,. \tag{2.8}$$

As $h'(t) > 0$ and h' is non-increasing for $t \ge t^*$, we obtain

$$h(t) = h(t^*) + \int_{t^*}^t h'(s)\,ds \ge h'(t)(t - t^*)\,.$$

This inequality and (2.8) imply

$$\frac{h(u)}{h(v)} = \exp\Big\{\int_v^u \frac{h'(s)}{h(s)}\,ds\Big\} \le \exp\Big\{\int_v^u \frac{ds}{s - t^*}\Big\} = \frac{u - t^*}{v - t^*} \le (1+\varepsilon)\frac{u}{v}$$

for $u \ge v \ge \frac{1+\varepsilon}{\varepsilon}t^*$; hence, (2.7) holds. □

Lemma 2.3. *Let $x \in \mathcal{N}$ and $T \ge 0$ be such that x is positive on $[\sigma(T), \infty)$.*

(i) *If* $x \in \mathcal{N}_0$ *and* $\int_0^\infty t\, q(t)\,dt < \infty$, *then*

$$\int_T^\infty t^2 r^*(t) z^\lambda(t)\,dt < \infty\,. \tag{2.9}$$

(ii) *If* $x \in \mathcal{N}$, *then*

$$\int_T^\infty \exp\Big\{\int_0^t sq(s)\,ds\Big\} r^*(t)\, z^\lambda(t)\,dt < \infty\,. \tag{2.10}$$

Proof. (i) Let $x \in \mathcal{N}_0$ and $t_0 \ge \max(T, t^*)$ be such that $x(t) > 0$ for $t \ge \sigma(t_0)$, $z^{[i]}(t) \ne 0$ for $t \ge t_0$, $i = 1, 2$. Then $\lim_{t\to\infty} z(t) = C \ge 0$. It is easy to see that (1.9), (1.11) and $x \in \mathcal{N}_0$ imply $\lim_{t\to\infty} z^{[i]}(t) = 0$ for $i = 1, 2$. Hence (1.11) and (H5) yield

$$\begin{gathered} z^{[1]}(t) = -\int_t^\infty h^{-2}(s) z^{[2]}(s)\,ds, \\ z^{[2]}(t) = -\int_t^\infty z^{[3]}(s)\,ds = \int_t^\infty h(s) r(s) x^\lambda\big(\sigma(s)\big)\,ds \end{gathered} \tag{2.11}$$

for $t \ge t_0$.

As $t_0 \ge t^*$, by Definition 1.3 there exist positive constants C_1 and C_2 such that $C_1 \le h(t) \le C_2$ for $t \ge t_0$. From this, (1.11), (2.11), and Fubini's theorem, we have

$$\begin{aligned} \infty > z(t_0) - C &= -\int_{t_0}^\infty h(s) z^{[1]}(s)\,ds \\ &\ge \int_{t_0}^\infty h(s) \int_s^\infty \frac{1}{h^2(v)} \int_v^\infty h(w) r(w) x^\lambda\big(\sigma(w)\big)\,dw\,dv\,ds \\ &\ge \Big(\frac{C_1}{C_2}\Big)^2 \int_{t_0}^\infty \int_s^\infty \int_v^\infty r(w) x^\lambda\big(\sigma(w)\big)\,dw\,dv\,ds \\ &= C_3 \int_{t_0}^\infty \int_s^\infty (w-s) r(w) x^\lambda\big(\sigma(w)\big)\,dw\,ds \\ &= \frac{1}{2} C_3 \int_{t_0}^\infty (w-t_0)^2 r(w) x^\lambda\big(\sigma(w)\big)\,dw \\ &\ge \frac{C_3}{8} \int_{2t_0}^\infty w^2 r(w) x^\lambda\big(\sigma(w)\big)\,dw \end{aligned}$$

with $C_3 = (C_1/C_2)^2$. From this and Lemma 2.1 (with $A(t) = t^2 r(t)$, $T = 2t_0$),

$$I := \int_{2t_0}^\infty \min\Big\{\big(\sigma^{-1}(t)\big)^2 r\big(\sigma^{-1}(t)\big), \big(\sigma^{-1}(\tau(t))\big)^2 r\big(\sigma^{-1}(\tau(t))\big)\Big\} z^\lambda(t) dt < \infty\,.$$

Using (1.15) and (1.16) we obtain (2.9).

(ii) Let $x \in \mathcal{N}$ be defined on $[T, \infty)$. Then there exists $t_0 \ge \max(T, t^*)$ such that

$$x(t) > 0 \quad \text{for } t \ge \sigma(t_0), \quad z^{[2]}(t) > 0 \quad \text{for } t \ge t_0\,.$$

From this, (1.11), (1.12), (H5), and Lemma 2.2, we have

$$\begin{aligned} \infty > z^{[2]}(t_0) &\ge z^{[2]}(t_0) - z^{[2]}(\infty) \\ &= -\int_{t_0}^\infty z^{[3]}(s)\,ds \\ &= \int_{t_0}^\infty h(t) r(t) f\big(x(\sigma(t))\big)\,dt \end{aligned}$$

$$\geq k \int_{t_0}^{\infty} \exp\Big\{ \int_{t_0}^{t} sq(s)\,ds \Big\} r(t) x^{\lambda}(\sigma(t))\,dt\,.$$

Therefore, (2.10) follows Lemma 2.1 (with $A(t) = \exp\big\{ \int_0^t sq(s)\,ds \big\} r(t)$). $\square$

3. Main results

We begin with the following lemma which states sufficient conditions for $\mathcal{N}_0$ to be empty in case $f(u) = u$.

Lemma 3.1. *Let $f(u) \equiv u$ on $\mathbb{R}$ and let one of the following assumptions hold.*

(i) *There exists a function $\xi \in C(\mathbb{R}_+)$ such that $t < \xi(t) < \sigma^{-1}(\tau(t))$ for large t and either*

$$I = \infty \quad or \quad \frac{2\sigma_1(\tau_0 + a_1)}{\tau_0 e} < I < \infty \tag{3.1}$$

where

$$I := \liminf_{t\to\infty} \int_{\tau^{-1}(\sigma(\xi(t)))}^{t} r^*(s) \frac{h(s)}{h(\xi(s))} \big(\xi(s) - s\big)^2\,ds\,;$$

(ii) *there exists a function $\eta \in C(\mathbb{R}_+)$ such that $\tau^{-1}(\sigma(t)) \leq \eta(t) \leq t$ for large t and either*

$$J = \infty \quad or \quad 2\sigma_1\Big(1 + \frac{a_1}{\tau_0}\Big) < J < \infty \tag{3.2}$$

where

$$J := \limsup_{t\to\infty} \frac{h(t)}{h(\sigma^{-1}(\tau(\eta(t))))} \big(\sigma^{-1}(\tau(\eta(t))) - t\big)^2 \int_{\eta(t)}^{t} r^*(s)\,ds\,.$$

Then $\mathcal{N}_0 = \emptyset$.

Proof. Let $x \in \mathcal{N}_0$. Then there exists $T \geq t^*$ (see Definition 1.3) such that for $t \geq T$ and $i = 0, 1, 2$,

$$h(t) > 0\,, \quad x\big(\sigma(t)\big) > 0\,, \quad (-1)^i z^{[i]}(t) > 0, \tag{3.3}$$

and $t < \xi(t) < \sigma^{-1}(\tau(t))$ (resp. $\tau^{-1}(\sigma(t)) \leq \eta(t) \leq t$) in case (i) (resp. (ii)).

From this, (H2), and (H3), we obtain

$$\begin{aligned}
&\frac{\sigma_1}{\tau_0}\big(z^{[2]}(\sigma^{-1}(\tau(t)))\big)' + h\big(\sigma^{-1}(\tau(t))\big) r\big(\sigma^{-1}(\tau(t))\big) x\big(\tau(t)\big) \\
&\leq \frac{1}{(\sigma^{-1}(\tau(t)))'}\big(z^{[2]}(\sigma^{-1}(\tau(t)))\big)' + h\big(\sigma^{-1}(\tau(t))\big) r\big(\sigma^{-1}(\tau(t))\big) x\big(\tau(t)\big) = 0\,,
\end{aligned}$$

where $' = \frac{d}{dt}$. Similarly,

$$\begin{aligned}
&\sigma_1\big(z^{[2]}(\sigma^{-1}(t))\big)' + h\big(\sigma^{-1}(t)\big) r\big(\sigma^{-1}(t)\big) x(t) \\
&\leq \frac{1}{(\sigma^{-1}(t))'}\big(z^{[2]}(\sigma^{-1}(t))\big)' + h\big(\sigma^{-1}(t)\big) r\big(\sigma^{-1}(t)\big) x(t) = 0\,.
\end{aligned}$$

Hence, using (H4) for $t \geq T$, we have

$$\begin{aligned}
&\Big[\sigma_1 z^{[2]}\big(\sigma^{-1}(t)\big) + \frac{a_1\sigma_1}{\tau_0} z^{[2]}\big(\sigma^{-1}(\tau(t))\big)\Big]' + h\big(\sigma^{-1}(\tau(t))\big) r^*(t) z(t) \\
&\leq \Big[\sigma_1 z^{[2]}\big(\sigma^{-1}(t)\big) + \frac{a_1\sigma_1}{\tau_0} z^{[2]}\big(\sigma^{-1}(\tau(t))\big)\Big]' + h\big(\sigma^{-1}(t)\big) r\big(\sigma^{-1}(t)\big) x(t) \\
&\quad + a_1 h\big(\sigma^{-1}(\tau(t))\big) r\big(\sigma^{-1}(\tau(t))\big) x\big(\tau(t)\big) \leq 0\,.
\end{aligned} \tag{3.4}$$

Furthermore, for $v \geq t \geq T$, we have

$$-z^{[1]}(t) \geq z^{[1]}(v) - z^{[1]}(t) = \int_t^v \frac{z^{[2]}(s)}{h^2(s)}\,ds \geq \frac{z^{[2]}(v)}{h^2(v)}\,(v-t)$$

and thus using (1.11), and integration from u to v, with $v \geq u$, imply

$$z(u) \geq \frac{z^{[2]}(v)}{h^2(v)} \int_u^v h(s)(v-s)\,ds \geq \frac{h(u)}{2h^2(v)}(v-u)^2 z^{[2]}(v)\,. \tag{3.5}$$

Assuming Case (i), we define

$$v(t) = \sigma_1 z^{[2]}\big(\sigma^{-1}(t)\big) + \frac{a_1\sigma_1}{\tau_0} z^{[2]}\big(\sigma^{-1}(\tau(t))\big) \tag{3.6}$$

for $t \geq T$. Then (3.4) and (3.5) with $u = t$, $v = \xi(t)$ imply

$$v'(t) + \frac{h(t)h(\sigma^{-1}(\tau(t)))}{2h^2(\xi(t))}\big(\xi(t)-t\big)^2 r^*(t) z^{[2]}\big(\xi(t)\big) \leq 0\,.$$

As $\xi(t) < \sigma^{-1}(\tau(t))$ and h is nondecreasing, we obtain

$$v'(t) + \frac{h(t)}{2h(\xi(t))}\big(\xi(t)-t\big)^2 r^*(t) z^{[2]}\big(\xi(t)\big) \leq 0\,. \tag{3.7}$$

As $z^{[2]} > 0$ is non-increasing, (3.6) implies

$$v(t) \leq \Big[\sigma_1 + \frac{a_1\sigma_1}{\tau_0}\Big] z^{[2]}\big(\sigma^{-1}(\tau(t))\big)\,,$$

and, hence,

$$z^{[2]}\big(\xi(t)\big) \geq \frac{\tau_0}{\sigma_1(\tau_0+a_1)} v\big(\tau^{-1}(\sigma(\xi(t)))\big)\,.$$

Substituting this into (3.7) yields

$$v'(t) + \frac{\tau_0}{2\sigma_1(\tau_0+a_1)} \frac{h(t)}{h(\xi(t))}\big(\xi(t)-t\big)^2 r^*(t) v\big(\tau^{-1}(\sigma(\xi(t)))\big) \leq 0\,. \tag{3.8}$$

Using (3.1), $\tau^{-1}(\sigma(\xi(t))) < t$, and the well-known criterion for (3.8) to be oscillatory (see [7, Theorem 2.1.1]) implies a contradiction.

Now assume Case (ii). According to (3.5) for $u = t$, $v = \sigma^{-1}(\tau(\eta(t))) \geq u$ we have

$$z(t) \geq \frac{h(t)}{2h^2(\sigma^{-1}(\tau(\eta(t))))}\big(\sigma^{-1}(\tau(\eta(t)))-t\big)^2 z^{[2]}\big(\sigma^{-1}(\tau(\eta(t)))\big)\,. \tag{3.9}$$

Integrating (3.4) from $\eta(t)$ to t, we have

$$\begin{aligned}
&\sigma_1 z^{[2]}\big(\sigma^{-1}(\eta(t))\big) + \frac{a_1\sigma_1}{\tau_0} z^{[2]}\big(\sigma^{-1}(\tau(\eta(t)))\big)\\
&\geq \sigma_1 z^{[2]}\big(\sigma^{-1}(t)\big) + \frac{a_1\sigma_1}{\tau_0} z^{[2]}\big(\sigma^{-1}(\tau(t))\big) + \int_{\eta(t)}^t h\big(\sigma^{-1}(\tau(s))\big) r^*(s) z(s)\,ds\\
&\geq h\big(\sigma^{-1}(\tau(\eta(t)))\big) z(t) \int_{\eta(t)}^t r^*(s)\,ds\,.
\end{aligned}$$

From this, (3.3), (3.9), and $z^{[2]} > 0$ and decreasing, we have

$$\begin{aligned}
&\sigma_1\Big(1+\frac{a_1}{\tau_0}\Big) z^{[2]}\big(\sigma^{-1}(\tau(\eta(t)))\big)\\
&\geq h\big(\sigma^{-1}(\tau(\eta(t)))\big) z(t) \int_{\eta(t)}^t r^*(s)\,ds
\end{aligned}$$

$$\geq \frac{h(t)}{2h(\sigma^{-1}(\tau(\eta(t))))}\big(\sigma^{-1}(\tau(\eta(t)))-t\big)^2\int_{\eta(t)}^t r^*(s)\,ds z^{[2]}\big(\sigma^{-1}(\tau(\eta(t)))\big).$$

This contradicts (3.2) and proves the statement. □

Note, that some ideas from [6] are used in the second part of the proof of Lemma 3.1.

Theorem 3.2. *(i) Let either*

$$\int_0^\infty tq(t)\,dt<\infty \quad \textit{and} \quad \int_0^\infty t^2 r^*(t)\,dt=\infty \tag{3.10}$$

or

$$\int_0^\infty tq(t)\,dt=\infty \quad \textit{and} \quad \int_0^\infty \exp\Big\{\int_0^t sq(s)\,ds\Big\} r^*(t)\,dt=\infty\,. \tag{3.11}$$

Then the set $\mathcal{N}_{01}$ is empty.

(ii) If

$$\int_0^\infty \exp\Big\{\int_0^t sq(s)\,ds\Big\} t^\lambda r^*(t)\,dt=\infty \tag{3.12}$$

then the set $\mathcal{N}_1$ is empty.

Proof. (i) Let $x\in\mathcal{N}_{01}$ be such that $x(t)>0$ for $t\in[\sigma(T),\infty)$. Then $\lim_{t\to\infty} z(t)=C\in(0,\infty)$ and (3.10), (resp. (3.12)) contradicts (2.9) (resp. (2.10)).

(ii) Let $x\in\mathcal{N}_1$. From this and from (1.11), positive constants $T_0\geq T$ and M exist such that $z(t)\geq Mt$ for $t\geq T_0$. Now, this fact and (3.12) contradict (2.10). □

Now we can formulate the main results. For $\xi\in C(\mathbb{R}_+)$ and $\eta\in C(\mathbb{R}_+)$, we denote

$$I_1=\liminf_{t\to\infty}\int_{\tau^{-1}(\sigma(\xi(t)))}^t r^*(s)\big(\xi(s)-s\big)^2\,ds\,, \tag{3.13}$$

$$J_1=\limsup_{t\to\infty}\big(\sigma^{-1}(\tau(\eta(t)))-t\big)^2\int_{\eta(t)}^t r^*(s)\,ds\,, \tag{3.14}$$

$$I_2=\liminf_{t\to\infty}\int_{\tau^{-1}(\sigma(\xi(t)))}^t \frac{s}{\xi(s)} r^*(s)\big(\xi(s)-s\big)^2\,ds\,, \tag{3.15}$$

$$J_2=\limsup_{t\to\infty}\frac{t}{\sigma^{-1}(\tau(\eta(t)))}\big(\sigma^{-1}(\tau(\eta(t)))-t\big)^2\int_{\eta(t)}^t r^*(s)\,ds\,. \tag{3.16}$$

Lemma 3.3. *Suppose $K>0$, $C>0$, $\int_0^\infty tq(t)\,dt<\infty$, $f(u)\geq Ku$ for $u\in[0,C]$ and one of the following assumptions holds.*

(i) *There exists a function $\xi(t)\in C(\mathbb{R}_+)$ such that $t\leq\xi(t)<\sigma^{-1}(\tau(t))$ for large t, and either $I_1=\infty$ or*

$$M:=\frac{2\sigma_1(\tau_0+a_1)}{K\,\tau_0 e}<I_1<\infty\,;$$

(ii) *There exists a function $\eta\in C(\mathbb{R}_+)$ such that $\tau^{-1}(\sigma(t))\leq\eta(t))\leq t$ for large t, and either $J_1=\infty$ or*

$$\frac{2\sigma_1}{K}\Big(1+\frac{a_1}{\tau_0}\Big)<J_1<\infty\,.$$

Then (1.1) *has no solution* $x \in \mathcal{N}_0$ *such that* $z(t) \le C$ *for large* t.

Proof. (i) Let $x \in \mathcal{N}_0$ and $T \ge 0$ be such that

$$\begin{gathered} 0 < x\big(\sigma(t)\big) \le C\,, \quad 0 < z(t) \le C \quad \text{for } t \ge T\,,\\ t \le \xi(t) < \sigma^{-1}\big(\tau(t)\big) \quad \text{for } t \ge T\,,\\ 1-\varepsilon \le h(t) \le 1 \quad \text{for } t \ge T\,, \end{gathered} \tag{3.17}$$

where

$$\varepsilon = \begin{cases} \frac{1}{2} - \frac{M}{2I_1} & \text{if } I_1 < \infty,\\ \frac{1}{2} & \text{if } I_1 = \infty. \end{cases} \tag{3.18}$$

Note, that (3.17) and (3.18) imply

$$\frac{h(t)}{h(\xi(t))} \ge 1-\varepsilon = \frac{1}{2} + \frac{M}{2I_1} > \frac{I_1+3M}{4I_1} = \frac{NM}{I_1} \tag{3.19}$$

with $N = \frac{I_1}{4M} + \frac{3}{4}$ for $t \ge T$ in case $I_1 < \infty$. Then x is the solution of the equation

$$z''' + q(t)z' + r_0(t)x\big(\sigma(t)\big) = 0 \tag{3.20}$$

for $t \ge T$ with

$$r_0(t) = \frac{f(x(\sigma(t)))}{x(\sigma(t))} r(t) \ge Kr(t)\,. \tag{3.21}$$

Now we apply Lemma 3.1 to (3.20), considering the assumption posed in I. If $I_1 = \infty$, then using (3.19) and (3.21), $I = \infty$. Let $I_1 < \infty$. Then (3.19) and (3.21) imply

$$\begin{aligned} &\liminf_{t\to\infty} \int_{\tau^{-1}(\sigma(\xi(t)))}^{t} \min\big\{r_0(\sigma^{-1}(s)), r_0(\sigma^{-1}(\tau(s)))\big\} \frac{h(s)}{h(\xi(s))}\big(\xi(s)-s\big)^2\,ds\\ &\ge \liminf_{t\to\infty} \frac{KMN}{I_1} \int_{\tau^{-1}(\sigma(\xi(t)))}^{t} r^*(s)\big(\xi(s)-s\big)^2\,ds > \frac{2\sigma_1(\tau_0+a_1)}{\tau_0 e}\,; \end{aligned}$$

hence, all assumptions of Lemma 3.1 applied on (3.20) are satisfied and $\mathcal{N}_0$ is empty. The contradiction proves the statement.

Statement (ii) can be proved similarly. $\square$

Lemma 3.4. *Suppose* $K > 0$, $C > 0$, $\int_0^\infty tq(t)\,dt = \infty$, $f(u) \ge Ku$ *for* $u \in [0, C]$ *and one of the following assumptions holds*

(i) *There exists a function* $\xi(t) \in C(\mathbb{R}_+)$ *such that* $t \le \xi(t) < \sigma^{-1}(\tau(t))$ *for large* t, *and either* $I_2 = \infty$ *or*
$$M = \frac{2\sigma_1(\tau_o + a_1)}{K\tau_0 e} < I_2 < \infty\,;$$

(ii) *There exists a function* $\eta \in C(\mathbb{R}_+)$ *such that* $\tau^{-1}(\sigma(t) \le \eta(t)) \le t$ *for large* t, *and either* $J_2 = \infty$ *or*
$$\frac{2\sigma_1(\tau_0 + a_1)}{K\tau_0} < J_2 < \infty\,.$$

Then (1.1) *has no solution* $x \in \mathcal{N}_0$ *such that* $z(t) \le C$ *for large* t.

Proof. It is similar as the one of Lemma 3.3; instead of (3.19), we apply (2.7) with $0 < \varepsilon \le \frac{I_2 - M}{I_2 + M}$ in case (i). We obtain
$$\frac{h(t)}{h(\xi(t))} \ge \Big(\frac{1}{2} + \frac{M}{2I_2}\Big)\frac{t}{\xi(t)}\,.$$
Case (ii) is similar. □

The following results solve our problem for $\lambda = 1$. Recall, that I_1, J_1, I_2 and J_2 are given by (3.13)–(3.16), respectively.

Theorem 3.5. *Suppose $\lambda = 1$, $\int_0^\infty tq(t)\,dt < \infty$ and one of the following assumptions holds.*

(i) *There exists a function $\xi(t) \in C(\mathbb{R}_+)$ such that $t \le \xi(t) < \sigma^{-1}(\tau(t))$ for large t, and either $I_1 = \infty$ or*
$$\frac{2\sigma_1(\tau_0 + a_1)}{\tau_0 e} < I_1 < \infty\,;$$
(ii) *there exists a function $\eta \in C(\mathbb{R}_+)$ such that $\tau^{-1}(\sigma(t)) \le \eta(t) \le t$ for large t, and either $J_1 = \infty$ or*
$$2\sigma_1\Big(1 + \frac{a_1}{\tau_0}\Big) < J_1 < \infty\,.$$

Then the set $\mathcal{N}_0$ is empty. If, moreover,
$$\int_0^\infty tr^*(t)\,dt = \infty\,,$$
then (1.1) is oscillatory.

Proof. Let $x \in \mathcal{N}_0$. As $\lambda = 1$, we can put $K = 1$ and $C = 1 + 2\lim_{t\to\infty} z(t)$. Then a contradiction follows from Lemma 3.3. The nonexistence of $x \in \mathcal{N}_1$ follows from Theorem 3.2(ii), (3.12) and $\int_0^\infty tq(t)\,dt < \infty$. □

Theorem 3.6. *Suppose $\lambda = 1$, $\int_0^\infty tq(t)\,dt = \infty$, and one of the following assumptions holds:*

(i) *There exists a function $\xi(t) \in C(\mathbb{R}_+)$ such that $t \le \xi(t) < \sigma^{-1}(\tau(t))$ for large t, and either $I_2 = \infty$ or*
$$\frac{2\sigma_1(\tau_0 + a_1)}{\tau_0 e} < I_2 < \infty\,;$$
(ii) *there exists a function $\eta \in C(\mathbb{R}_+)$ such that $\tau^{-1}(\sigma(t)) \le \eta(t)) \le t$ for large t, and either $J_2 = \infty$ or*
$$2\sigma_1\Big(1 + \frac{a_1}{\tau_0}\Big) < J_2 < \infty\,.$$

Then the set $\mathcal{N}_0$ is empty. If, moreover,
$$\int_0^\infty \exp\Big\{\int_0^t sq(s)\,ds\Big\} tr^*(t)\,dt = \infty$$
then (1.1) is oscillatory.

The proof of the above theorem is similar to that of Theorem 3.5.

Theorem 3.7. *Let $\lambda \in (0,1)$, $\int_0^\infty tq(t)\,dt < \infty$ and one of the following assumptions hold.*

(i) *Thee exists a function* $\xi(t) \in C(\mathbb{R}_+)$ *such that* $t \le \xi(t) < \sigma^{-1}(\tau(t))$ *for large* t*, and either* $I_1 = \infty$ *or* $I_1 > 0$*;*

(ii) *there exists a function* $\eta \in C(\mathbb{R}_+)$ *such that* $\tau^{-1}(\sigma(t)) \le \eta(t)) \le t$ *for large* t*, and either* $J_1 = \infty$ *or* $J_1 > 0$.

Then set $\mathcal{N}_{00}$ *is empty. If, moreover,*

$$\int_0^\infty t^\lambda r^*(t)\,dt = \infty \tag{3.22}$$

then (1.1) *is oscillatory.*

Proof. (i) Let $x \in \mathcal{N}_{00}$. Put

$$K = \frac{3\sigma_1(\tau_0 + a_1)}{\tau_0 e I_1} \quad \text{and} \quad C = K^{-\frac{1}{1-\lambda}}$$

in case $I_1 < \infty$ and $K = C = 1$ if $I_1 = \infty$. Then $f(u) \ge u^\lambda \ge Ku$ on $[0, C]$. Hence, all assumptions of Lemma 3.3 are satisfied and Lemma 3.3 contradicts $x \in \mathcal{N}_{00}$. The nonexistence of $x \in \mathcal{N}_{01} \cup \mathcal{N}_1$ follows from (3.22) and Theorem 3.2.

Statement (ii) can be proved similarly. □

Theorem 3.8. *Let* $\lambda \in (0,1)$, $\int_0^\infty tq(t)\,dt = \infty$ *and one of the following assumptions hold.*

(i) *There exists a function* $\xi(t) \in C(\mathbb{R}_+)$ *such that* $t \le \xi(t) < \sigma^{-1}(\tau(t))$ *for large* t*, and either*

$$I_2 = \infty \quad \text{and} \quad \int_0^\infty \exp\Big\{\int_0^t sq(s)\,ds\Big\} t^\lambda r^*(t)\,dt = \infty \tag{3.23}$$

or

$$0 < I_2 < \infty \quad \text{and} \quad \int_0^\infty \exp\Big\{\int_0^t sq(s)\,ds\Big\} r^*(t)\,dt = \infty\,; \tag{3.24}$$

(ii) *there is a function* $\eta \in C(\mathbb{R}_+)$ *such that* $\tau^{-1}(\sigma(t)) \le \eta(t)) \le t$ *for large* t*, and either*

$$J_2 = \infty \quad \text{and} \quad \int_0^\infty \exp\Big\{\int_0^t sq(s)\,ds\Big\} t^\lambda r^*(t)\,dt = \infty$$

or

$$0 < J_2 < \infty \quad \text{and} \quad \int_0^\infty \exp\Big\{\int_0^t sq(s)\,ds\Big\} r^*(t)\,dt = \infty\,.$$

Then (1.1) *is oscillatory.*

Proof. (i) Suppose (3.23) holds. Then Theorem 3.2(ii) implies $\mathcal{N}_1 = \emptyset$. Let $x \in \mathcal{N}_0$. Then $\lim_{t\to\infty} z(t) = C_0 \in [0, \infty)$ and $z(t) \le C = 2C_0 + 1$ for large t. Moreover, $f(u) \ge K\,u$ for $[0, C]$ where $K = C^{\lambda-1}$. Then all assumptions of Lemma 3.4(i) are satisfied whose statement contradicts $x \in \mathcal{N}_0$.

Suppose (3.24) holds. Then Theorem 3.2 implies $\mathcal{N}_{01} \cup \mathcal{N}_1 = \emptyset$. The nonexistence of $x \in \mathcal{N}_{00}$ can be proved as in Theorem 3.7(i).

The proof of (ii) is similar. □

4. Examples

Remark 4.1. (i) It follows from the assumptions of Theorems 3.5–3.8 that $\sigma(t) \le \tau(t)$ for large t as it is supposed in (H3).

(ii) In Theorems 3.5–3.8, it is possible to choose e.g. $\xi(t) = \frac{1}{2}(t + \sigma^{-1}(\tau(t)))$; similarly, we can choose either $\tau(t) \le t$, $\tau(t) \not\equiv t$ in any neighborhood of ∞ and $\eta(t) \equiv \tau(t)$, or $\tau(t) \equiv t$ for large t and $\eta(t) = \frac{1}{2}(t + \sigma(t))$, $\sigma(t) < t$.

Example 4.2. Consider the equation

$$z''' + q(t)z' + r(t)|x(C_1 t)|^\lambda \operatorname{sgn} x(C_1 t) = 0 \tag{4.1}$$

with $z(t) = x(t) + a(t)x(C_0 t)$ where $0 < \lambda \le 1$, $0 < C_1 < C_0 \le 1$, $r(t) \ge \frac{r_0}{t^v}$ for large t with $r_0 > 0$ and $v \ge 0$, $0 \le a(t) \le a_1 < \infty$ and (H6) holds. Put $\xi(t) = C_2 t$, $1 < C_2 < \frac{C_0}{C_1}$. Let $\lambda = 1$. Then $\mathcal{N}_0$ is empty for (4.1) if either $v < 3$ or $v = 3$ and

$$(C_2 - 1)^2 \log \frac{C_0}{C_1 C_2} > m \frac{C_0 + a_1}{r_0 e C_0 C_1^2}$$

where $m = 1$ for $\int_0^\infty t q(t)\, dt < \infty$ and $m = C_2$ for $\int_0^\infty t q(t)\, dt = \infty$ (see Theorems 3.5 and 3.6). Moreover, (4.1) is oscillatory if $v \le 2$.

Let $0 < \lambda < 1$. Equation (4.1) is oscillatory if $v \le \lambda + 1$ (Theorems 3.7 and 3.8).

Example 4.3. Consider the equation

$$z''' + q(t)z' + r(t)|x(t - C_1)|^\lambda \operatorname{sgn} x(t - C_1) = 0 \tag{4.2}$$

with $z(t) = x(t) + a(t)x(t - C_0)$ where $0 \le C_0 < C_1$, $0 \le a(t) \le a_1 < \infty$ on $\mathbb{R}_+$, $r(t) \ge r_0 t^v$, $v \ge 0$ and (H6) holds. Put $C_2 \in (0, C_1 - C_0)$, $\xi(t) = t + C_2$. If $\lambda = 1$, then Theorems 3.5 and 3.6 imply that (4.2) is oscillatory if either $v > 0$ or

$$v = 0 \quad \text{and} \quad C_2^2[C_1 - C_0 - C_2] > \frac{2\sigma_1(\tau_0 + a_1)}{r_0 \tau_0 e}\,.$$

If $\lambda \in (0,1)$, then Theorems 3.7 and 3.8 imply that (4.2) is oscillatory if $v \ge 0$.

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Miroslav Bartušek
Department of Mathematics and Statistics, Faculty of Science, Masaryk University, Kotlářská 2, 611 37 Brno, Czech Republic
Email address: bartusek@math.muni.cz