Electronic Journal of Differential Equations, Vol. 2006(2006), No. 65, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

ENERGY DECAY FOR SOLUTIONS TO SEMILINEAR SYSTEMS OF ELASTIC WAVES IN EXTERIOR DOMAINS

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Abstract. We consider the dynamical system of elasticity in the exterior of a bounded open domain in 3-D with smooth boundary. We prove that under the effect of "weak" dissipation, the total energy decays at a uniform rate as $t \to +\infty$, provided the initial data is "small" at infinity. No assumptions on the geometry of the obstacle are required. The results are then applied to a semilinear problem proving global existence and decay for small initial data.

1. Introduction

We study the uniform stabilization of the solutions of a hyperbolic system of equations in an exterior domain, as $t \to +\infty$. A classical example of this class is the system of elastic waves. Let us describe the model: Let $\mathcal{O}$ be an open bounded region of $\mathbb{R}^3$ with smooth boundary and $\Omega = \mathbb{R}^3 \setminus \overline{\mathcal{O}}$. We consider the system

$$\begin{gathered} u_{tt} - \sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}\big(A_{ij}\frac{\partial u}{\partial x_j}\big) + u_t = f(u_t) \quad \text{in } \Omega \times \mathbb{R} \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \times \mathbb{R} \end{gathered} \tag{1.1}$$

Here $x = (x_1, x_2, x_3) \in \Omega$, t is the time variable, $u(x,t) = \big(u_1(x,t), u_2(x,t), u_3(x,t)\big)$ denotes the displacement vector, $A_{ij} = [C^{ij}_{kh}]$ are 3×3 symmetric matrices and $f = (f_1, f_2, f_3)$ is a nonhomogeneous vector-valued function. Both A_{ij} and f will satisfy suitable assumptions. Associated to the initial boundary valued problem (1.1) we have the total energy

$$E(t) = \frac{1}{2}\int_\Omega \big\{|u_t|^2 + \sum_{i,j=1}^{3} A_{ij}\frac{\partial u}{\partial x_j} \cdot \frac{\partial u}{\partial x_i}\big\}dx \tag{1.2}$$

where $|u_t|^2 = u_t \cdot u_t = \sum_{j=1}^3 |\frac{\partial}{\partial t}u_j|^2$ and the dot $\cdot$ denotes the usual inner product in $\mathbb{R}^3$. Let u be the solution of problem (1.1) in a suitable function space and assume

2000 *Mathematics Subject Classification.* 35Q99, 35L99.
Key words and phrases. Uniform stabilization; exterior domain; system of elastic waves; semilinear problem.

Submitted March 20, 2006. Published May 22, 2006.

for a moment that $f \equiv 0$. Then, a formal calculation give us that the derivative of $E(t)$:

$$\frac{d}{dt}E(t) = -\int_{\Omega} |u_t|^2 \, dx \leq 0. \tag{1.3}$$

Thus, we may ask: Does $E(t)$ decays at a uniform rate as $t \to +\infty$? Furthermore, in case the answer is affirmative then we can ask if the same result would still hold for a class of functions f and initial data (u_0, u_1) satisfying suitable assumptions. Both questions above are by now not very difficult to answer in case Ω is a bounded domain (see for instance Racke [11] and the references therein). In our case, since Ω is an exterior domain, the uniform stabilization requires a more detailed discussion which is our main objective in this article. There is a large literature concerning the decay of solutions of hyperbolic problems in exterior domains. In a pioneering work, Morawetz [7, 8] studied the asymptotic behavior of the local energy for the scalar wave equation in exterior domains. Assuming geometric conditions on the obstacle and initial data with compact support she obtained uniform rates of decay. B. Kapitonov got similar results for the system of elastic waves and the Maxwell equations, Zuazua [13], Nakao [10] and Ikehata [4] obtained also stabilization results for scalar wave equations with localized damping (being effective only near "infinity"). As far as we know the results we present in this article for system (1.1) are the first of the kind for the system of elasticity. We do not assume geometric conditions on the obstacle nor special restrictions on the Lamé's coefficients in the isotropic case. Our strategy relies on recent work due to Ikehata [2] for the scalar wave equation adapted conveniently to system (1.1).

Let us make precise our assumptions on the matrices A_{ij} and the nonlinearity f in (1.1):

(H1) (a) Given a set of real numbers $\{a_{ijkh}\}$ with $i, j, k, h \in \{1, 2, 3\}$ satisfying the symmetric properties $a_{ijkh} = a_{jikh} = a_{khij}$, we consider

$$C^{ij}_{kh} = (1 - \delta_{ih}\delta_{jk})a_{ikjh} + \delta_{ik}\delta_{jh}a_{ihjk}$$

with $\delta_{\ell k} = \begin{cases} 1 \text{ if } \ell = k \\ 0 \text{ if } \ell \neq k \end{cases}$ and "build" the 3×3 matrices $A_{ij} = [C^{ij}_{kh}]$.

(b) We assume that there exist a constant $C_0 > 0$ such that

$$\sum_{i,j=1}^{3} A_{ij} v_j \cdot v_i \geq C_0 \sum_{i=1}^{3} |v_i|^2 \tag{1.4}$$

for any vector $v_i = (v_i^1, v_i^2, v_i^3) \in \mathbb{R}^3$ where $|v_i|^2 = v_i \cdot v_i$.

(H2) Let $f = (f_1, f_2, f_3)$ with $f_j \colon \mathbb{R}^3 \to \mathbb{R}$ satisfying the following assumptions: Each $f_j \in C^2(\mathbb{R}^3)$ and

(a) $|f(y)| \leq C_1 |y|^p$ for every $y \in \mathbb{R}^3$

(b) $|\nabla f(y)| \leq C_2 |y|^{p-1}$ for every $y \in \mathbb{R}^3$

(c) $\sum_{i,j=1}^{3} |\nabla \frac{\partial f_i(y)}{\partial y_j}| \leq C_3 |y|^{p-2}$ for every $y \in \mathbb{R}^3$

where C_j are positive constants ($1 \leq j \leq 3$), $\frac{7}{3} < p \leq 3$ and $|\nabla f(y)|^2 = \sum_{i=1}^{3} |\nabla f_i(y)|^2$.

Remark 1.1. In the simplest case, that is, when the medium is isotropic, the constants a_{ijkh} are

$$a_{ijkh} = \lambda \delta_{ij}\delta_{kh} + \mu\big(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}\big)$$

where λ and μ are Lamé's constants ($\mu > 0, \lambda + \mu > 0$). Furthermore, (1.4) holds with $C_0 = \mu > 0$ and $\sum_{i,j=1}^3 \frac{\partial}{\partial x_i}\big(A_{ij}\frac{\partial u}{\partial x_j}\big)$ reduces to $\mu\Delta u + (\lambda+\mu)\nabla \operatorname{div} u$.

Remark 1.2. Due to the symmetry conditions on the numbers a_{ijkh} it follows that $A_{ij}^* = A_{ji}$.

2. The linear case

In this section we consider the linear problem

$$\begin{gathered} u_{tt} - \sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}\Big(A_{ij}\frac{\partial u}{\partial x_j}\Big) + u_t = 0 \quad \text{in } \Omega\times\mathbb{R} \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega\times\mathbb{R} \end{gathered} \tag{2.1}$$

Using standard semigroup theory we can easily prove the following result.

Theorem 2.1. *Let $(u_0, u_1) \in [H_0^1(\Omega)]^3 \times [L^2(\Omega)]^3$ and A_{ij} satisfy assumption (H1). Then, there exist a unique (weak) solution u of problem* (2.1) *such that $u \in C\big(\mathbb{R}; [H_0^1(\Omega)]^3\big) \cap C^1\big(\mathbb{R}; [L^2(\Omega)]^3\big)$. If $(u_0,u_1) \in [H^2(\Omega)\cap H_0^1(\Omega)]^3 \times [H_0^1(\Omega)]^3$, then, there exist a unique (strong) solution u of problem* (2.1) *such that*

$$u \in C\big(\mathbb{R}; [H^2(\Omega)\cap H_0^1(\Omega)]^3\big) \cap C^1\big(\mathbb{R}; [H_0^1(\Omega)]^3\big) \cap C^2\big(\mathbb{R}; [L^2(\Omega)]^3\big).$$

Here $H^m(\Omega)$ denotes the usual Sobolev space of order m in Ω and $H_0^1(\Omega) = \big\{u \in H^1(\Omega), u\big|_{\partial\Omega} = 0\big\}$. Now, we want to devote our attention to the asymptotic behavior of the total energy $E(t)$ given by (1.2). Our result in this case is as follows.

Theorem 2.2. *Let $(u_0,u_1) \in [H_0^1(\Omega)]^3 \times [L^2(\Omega)]^3$ and assume that the initial data satisfy the condition*

$$\int_\Omega |x|^2 |u_0 + u_1|^2 \, dx < +\infty. \tag{2.2}$$

Then, there exist a positive constant C such that

$$\begin{gathered} E(t) \le C I_0 \big(1+|t|\big)^2 \quad \text{for every } t \in \mathbb{R}, \\ \int_\Omega |u(x,t)|^2\,dx \le C I_0\big(1+|t|\big)^{-1} \quad \text{for every } t\in\mathbb{R} \end{gathered}$$

where $I_0 = \|u_0\|^2_{[H^1(\Omega)]^3} + \|u_1\|^2 + \||\cdot|(u_0+u_1)\|^2$ and $\|g\|^2 = \sum_{j=1}^3 \int_\Omega |g_j|^2\,dx$ whenever $g = (g_1,g_2,g_3) \in [L^2(\Omega)]^3$.

As far as we know, results of this type for exterior domains are known only for scalar wave equations and most of them require geometrical conditions on the obstacle (like star-shaped condition). We need some preliminary lemmas. Obviously, is sufficient to prove Theorem 2.2 for $t \ge 0$.

Lemma 2.3. *Let $(u_0, u_1) \in [H^2(\Omega) \cap H_0^1(\Omega)]^3 \times [H_0^1(\Omega)]^3$. Then, the solution of (2.1) satisfies, for any $t \geq 0$,*

$$E(t) + \int_0^t \int_\Omega |u_s(x,s)|^2 \, dx \, ds = E(0), \tag{2.3}$$

$$\int_0^t \int_\Omega (1+s)|u_s(x,s)|^2 \, dx \, ds + (1+t)E(t) = E(0) + \int_0^t E(s) \, ds, \tag{2.4}$$

$$\begin{aligned} &\int_0^t \int_\Omega \sum_{i,j=1}^3 A_{ij} \frac{\partial u}{\partial x_j} \cdot \frac{\partial u}{\partial x_i} \, dx \, ds + \frac{1}{2} \int_\Omega |u(x,t)|^2 \, ds \\ &= \frac{1}{2}\|u_0\|^2 + \int_\Omega u_1 \cdot u_0 \, dx - \int_\Omega u_t \cdot u \, dx + \int_0^t \int_\Omega |u_s|^2 \, dx \, ds, \end{aligned} \tag{2.5}$$

$$\begin{aligned} &\int_0^t \int_\Omega (1+s) \sum_{i,j=1}^3 A_{ij} \frac{\partial u}{\partial x_j} \cdot \frac{\partial u}{\partial x_i} \, dx \, ds + (1+t) \int_\Omega |u|^2 \, dx \\ &\leq C + \frac{1}{2} \int_0^t \int_\Omega |u|^2 \, dx \, ds, \end{aligned} \tag{2.6}$$

where C is a positive constant which depends only on $E(0)$ and $\|u_0\|$.

Proof. Equality (2.3) follows directly from (1.3) by integration over $[0, t]$. Also, from (1.3) it follows that

$$(1+t)\frac{dE}{dt} = -\int_\Omega (1+t)|u_t|^2 \, dx$$

that is,

$$\int_\Omega (1+t)|u_t|^2 \, dx = -\frac{d}{dt}\big\{(1+t)E(t)\big\} + E(t). \tag{2.7}$$

Integration of this equality over $[0, t]$ proves (2.4). Next, we take the inner product in $[L^2(\Omega)]^3$ of system (2.1) with u to obtain

$$\frac{d}{dt} \int_\Omega u_t \cdot u \, dx - \int_\Omega \sum_{i,j=1}^3 \frac{\partial}{\partial x_i}\Big(A_{ij} \frac{\partial u}{\partial x_j}\Big) \cdot u \, dx + \frac{1}{2}\frac{d}{dt} \int_\Omega |u|^2 \, dx = \int_\Omega |u_t|^2 \, dx. \tag{2.8}$$

Using the divergence theorem and the boundary conditions we know that

$$\int_\Omega \sum_{i,j=1}^3 \frac{\partial}{\partial x_i}\Big(A_{ij} \frac{\partial u}{\partial x_j}\Big) \cdot u \, dx = -\int_\Omega \sum_{i,j=1}^3 A_{ij} \frac{\partial u}{\partial x_i} \cdot \frac{\partial u}{\partial x_j} \, dx.$$

Substitution of the above identity into (2.8) and integration over $[0, t]$ proves (2.5). To prove (2.6), we proceed as above: Let us take the inner product in $[L^2(\Omega)]^3$ of system (2.1) with $(1+t)u$ and use the divergence theorem to obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt} \int_\Omega t|u|^2 \, dx + (1+t) \int_\Omega \sum_{i,j=1}^3 A_{ij} \frac{\partial u}{\partial x_j} \cdot \frac{\partial u}{\partial x_i} \, dx \\ &= (1+t) \int_\Omega |u_t|^2 \, dx + \frac{1}{2} \int_\Omega |u|^2 \, dx - \frac{d}{dt} \int_\Omega (1+t) u_t \cdot u \, dx. \end{aligned}$$

Integration of this equality over $[0,t]$ and using Holder's inequality implies

$$\begin{aligned} &\int_0^t \int_\Omega (1+s) \sum_{i,j=1}^3 A_{ij} \frac{\partial u}{\partial x_j} \cdot \frac{\partial u}{\partial x_i} \, dx \, ds + \frac{t}{2} \int_\Omega |u|^2 \, dx \\ &\leq \int_\Omega u_1 \cdot u_0 \, dx + \int_0^t \int_\Omega (1+s)|u_s|^2 \, dx \, ds + \frac{1}{2} \int_0^t \int_\Omega |u|^2 \, dx \, ds \\ &\quad + \frac{1+t}{4} \int_\Omega |u|^2 \, dx + (1+t) \int_\Omega |u_t|^2 \, dx \, . \end{aligned} \tag{2.9}$$

From (2.4) and (2.5) in Lemma 2.3, we know that

$$\int_0^t \int_\Omega (1+s)|u_s|^2 \, dx \, ds \leq E(0) + \int_0^t E(s) \, ds \tag{2.10}$$

and

$$\begin{aligned} &\int_0^t \int_\Omega \sum_{i,j=1}^3 A_{ij} \frac{\partial u}{\partial x_j} \cdot \frac{\partial u}{\partial x_i} \, dx \, ds + \frac{1}{4} \int_\Omega |u(x,t)|^2 \, dx \\ &\leq \frac{1}{2} \|u_0\|^2 + \int_\Omega u_1 \cdot u_0 \, dx + \int_\Omega |u_t|^2 \, dx + E(0) - E(t). \end{aligned} \tag{2.11}$$

From the above inequality, and using again (2.3), we deduce that

$$2 \int_0^t E(s) \, ds + \frac{1}{4} \int_\Omega |u|^2 \, dx \leq 2E(0) + \frac{1}{2} \|u_0\|^2 + \int_\Omega u_1 \cdot u_0 \, dx. \tag{2.12}$$

Using the estimates (2.10), (2.11) and (2.12) we obtain from (2.9) the inequality

$$\begin{aligned} &\int_0^t \int_\Omega (1+s) \sum_{i,j=1}^3 A_{ij} \frac{\partial u}{\partial x_j} \cdot \frac{\partial u}{\partial x_i} \, dx \, ds + \frac{(1+t)}{4} \int_\Omega |u|^2 \, dx \\ &\leq 3 \int_\Omega u_1 \cdot u_0 \, dx + 5E(0) + \|u_0\|^2 + \frac{1}{2} \int_0^t \int_\Omega |u|^2 \, dx \, ds + 2(1+t)E(t). \end{aligned} \tag{2.13}$$

It remains to estimate $2(1+t)E(t)$. Observing that

$$\frac{d}{dt} \big\{ (1+t)E(t) \big\} = E(t) + (1+t) \frac{dE}{dt} \leq E(t).$$

Consequently

$$2(1+t)E(t) \leq 2E(0) + 2 \int_0^t E(s) \, ds \leq 4E(0) + \frac{1}{2} \|u_0\|^2 + \int_\Omega u_1 \cdot u_0 \, dx.$$

Substitution of this inequalit into (2.13) completes the proof □

Lemma 2.4. *Let* $(u_0, u_1) \in [H^2(\Omega) \cap H_0^1(\Omega)]^3 \times [H_0^1(\Omega)]^3$ *and* (u_0, u_1) *satisfy* (2.2). *Then the solution* u *of problem* (2.1) *satisfies*

$$\int_\Omega |u|^2 \, dx + \int_0^t \int_\Omega |u|^2 \, dx \, ds \leq \|u_0\|^2 + \frac{4}{C_0} \int_\Omega |x|^2 |u_0 + u_1|^2 \, dx$$

where C_0 *is the positive constant which appears in* (1.4).

Proof. First, let us observe that whenever $u_j \in H_0^1(\Omega)$ then Hardy's inequality states that

$$\int_\Omega \frac{|u_j|^2}{|x|^2} \, dx \leq 4 \int_\Omega |\nabla u_j|^2 \, dx.$$

Therefore, $u=(u_1,u_2,u_3)$ satisfies

$$\int_\Omega \frac{|u|^2}{|x|^2}\,dx \le 4\int_\Omega \sum_{i,j=1}^{3}|\frac{\partial u_j}{\partial x_i}|^2 dx \le \frac{4}{C_0}\int_\Omega \sum_{i,j=1}^{3} A_{ij}\frac{\partial u}{\partial x_j}\cdot\frac{\partial u}{\partial x_i}\,dx \tag{2.14}$$

due to (1.4). Let $w(x,t)=\int_0^t u(x,s)\,ds$. It follows that $w(x,t)$ satisfies the equation

$$\begin{gathered} w_{tt}-\sum_{i,j=1}^{3}\frac{\partial}{\partial x_i}\Big(A_{ij}\frac{\partial w}{\partial x_j}\Big)+w_t=u_0+u_1 \quad \text{in } \Omega\times\mathbb{R}^+ \\ w(x,0)=0,\quad w_t(x,0)=u_0(x) \quad \text{in } \Omega \\ w=0 \quad \text{on } \partial\Omega\times\mathbb{R}^+ \end{gathered} \tag{2.15}$$

Let us consider the inner product in $[L^2(\Omega)]^3$ of the above equation with w_t and use the divergence theorem to obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \big\{|w_t|^2+\sum_{i,j=1}^{3}A_{ij}\frac{\partial w}{\partial x_j}\cdot\frac{\partial w}{\partial x_i}\big\}dx+\int_\Omega |w_t|^2\,dx=\frac{d}{dt}\int_\Omega (u_0+u_1)\cdot w\,dx.$$

Integrating this equality over $[0,t]$, using Hölder's inequality and (2.14) implies that

$$\begin{aligned} &\frac{1}{2}\int_\Omega \big\{|w_t|^2+\sum_{i,j=1}^{3}A_{ij}\frac{\partial w}{\partial x_j}\cdot\frac{\partial w}{\partial x_i}\big\}dx+\int_0^t\int_\Omega |w_s|^2\,dx\,ds \\ &=\int_\Omega (u_0+u_1)\cdot w\,dx+\frac{1}{2}\|u_0\|^2 \\ &\le \Big(\int_\Omega |x|^2|u_0+u_1|^2\,dx\Big)^{1/2}\Big(\int_\Omega \frac{|w|^2}{|x|^2}\,dx\Big)^{1/2}+\frac{1}{2}\|u_0\|^2 \\ &\le \big(\frac{4}{C_0}\big)^{1/2}\Big(\int_\Omega \sum_{i,j=1}^{3}A_{ij}\frac{\partial w}{\partial x_j}\cdot\frac{\partial w}{\partial x_i}\,dx\Big)^{1/2}\Big(\int_\Omega |x|^2|u_0+u_1|^2\,dx\Big)^{1/2}+\frac{1}{2}\|u_0\|^2 \\ &\le \frac{1}{4}\int_\Omega \sum_{i,j=1}^{3}A_{ij}\frac{\partial w}{\partial x_j}\cdot\frac{\partial w}{\partial x_i}\,dx+\frac{4}{C_0}\int_\Omega |x|^2|u_0+u_1|^2\,dx+\frac{1}{2}\|u_0\|^2. \end{aligned}$$

This inequality proves Lemma 2.4 because $w_t=u$. □

Proof of Theorem 2.2. It follows from Lemmas 2.3 and 2.4 that

$$\int_0^t\int_\Omega (1+s)\sum^{3} A_{ij}\frac{\partial u}{\partial x_j}\cdot\frac{\partial u}{\partial x_i}\,dx\,ds+(1+t)\int_\Omega |u|^2\,dx \le CI_0 \tag{2.16}$$

for any $t\ge 0$. Observing that

$$\frac{d}{dt}\big\{(1+t)^2E(t)\big\}=2(t+1)E(t)+(1+t)^2\frac{dE}{dt}\le 2(1+t)E(t)$$

it follows that

$$\begin{aligned}(1+t)^2E(t) &\le E(0)+2\int_0^t (1+s)E(s)\,ds\\ &\le E(0)+CI_0+\int_0^T\int_\Omega (1+s)|u_s|^2\,dx\,ds\\ &\le 2E(0)+CI_0+\int_0^t E(s)\,ds\le \widetilde{C}I_0\,.\end{aligned}$$

Here we used Lemma 2.3 and (2.12), with $\widetilde{C}$ a positive constant. This completes the proof of Theorem 2.2. □

Remark 2.5. It is quite interesting to mention here that a similar procedure to the one presented above was done by the first author (M.F) in [1] for the Maxwell equations in exterior domains and the requirement (2.2) was not needed in order to obtain uniform decay rates.

Remark 2.6. The above procedure could be extended to include the anisotropic case, that is, when the coefficients a_{ijkh} do depend on each $x\in\Omega$. In that case $A_{ij}=A_{ij}(x)$ and assumptions (a) and (b) would be required to be valid for each $x\in\Omega$ with $C_0>0$ independent of $x\in\Omega$. As it is clear in the proof of Lemma 2.3 additional assumptions on the behavior of partial derivatives $\frac{\partial}{\partial x_i}A_{ij}(x)$ would be required to arrive to the conclusion of Theorem 2.2.

3. The semilinear problem

This section, we apply the results obtained in Section 2 to study the asymptotic behavior of the solutions of the semilinear model. We will sketch the proof that for small enough initial data the solution of problem (1.1) exists globally and enjoys the same rate of decay as $t\to+\infty$ as the solution of the linear model (2.1). We will assume that f satisfies all conditions given in (H2). Local existence will be done via contraction arguments and the global existence as well as the asymptotic behavior using the decay rates for the linear part obtained in Section 2. Due to the character of the nonlinearity in problem (1.1) we will require more regular solutions. First, let us rewrite problem (1.1) as a first order evolution system:

$$\frac{dU}{dt}=AU+F(U),\quad U(0)=U_0 \tag{3.1}$$

where $U=(u,u_t)$, $U_0=(u_0,u_1)$, $F(U)=(0,f(u_t)+u)$ and A with domain $\mathcal{D}(A)=[H^2(\Omega)\cap H_0^1(\Omega)]^3\times[H_0^1(\Omega)]^3$ given by

$$A(u,v)=\Big(v,\sum_{i,j=1}^3\frac{\partial}{\partial x_i}\Big(A_{ij}\frac{\partial u}{\partial x_j}\Big)-u-v\Big)$$

for every $(u,v)\in\mathcal{D}(A)$. The operator A is the infinitesimal generator of a C_0 group of operators $\{T(t)\}_{t\in\mathbb{R}}$ in the Hilbert space $X=[H_0^1(\Omega)]^3\times[L^2(\Omega)]^3$. The main result of this section for the solution of problem (1.1) will be present with initial data with compact support. However, it seems to us that using recent work due to Todorova and Yordanov [12] and Ikehata and Matsuyana [3] for the scalar wave equation then our result may be improved for initial data satisfying only (2.2). We want to prove the following result.

Theorem 3.1. *Assume condition (H1) and (H2). Let $(u_0, u_1) \in \mathcal{D}(A^2)$ with compact support. Then, there exist $\delta > 0$ such that if $\widetilde{I} < \delta$ then problem* (1.1) *has a unique global solution (u, u_t) such that*

$$(u, u_t) \in C(\mathbb{R}; \mathcal{D}(A^2)) \cap C^1(\mathbb{R}; \mathcal{D}(A)) \cap C^2(\mathbb{R}; X)$$

and satisfies

$$\int_\Omega |u|^2\, dx \le C\widetilde{I}\big(1 + |t|\big)^{-1} \quad \forall t \in \mathbb{R}$$

$$E(t) + E_1(t) + E_2(t) \le C\widetilde{I}\big(1 + |t|\big)^{-2} \quad \forall t \in \mathbb{R},$$

where $E(t)$ is given by (1.2) *and E_1 and E_2 will be given by* (3.7) *and* (3.9) *and $C > 0$ is a positive constant. Here $\widetilde{I}$ depends only on the Sobolev norms (up to order three) of the initial data.*

First, we sketch the proof of existence of a local solution. Let $T > 0$ and consider the space

$$Y(T) = C\big([0, T]; \mathcal{D}(A^2)\big) \cap C^1\big([0, T]; \mathcal{D}(A)\big) \cap C^2\big([0, T]; X\big)$$

with norm

$$\|U\|_{Y(T)} = \sup_{[0,T]} \|U(t)\|_{\mathcal{D}(A^2)} + \sup_{[0,T]} \|U_t(t)\|_{\mathcal{D}(A)} + \sup_{[0,T]} \|U_{tt}(t)\|_X. \tag{3.2}$$

Clearly $Y(T)$ is a Banach space. Let $U = (u, v) \in Y(T)$. Using our assumptions (H2) on f and the embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ for $2 \le q \le 6$ and $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ we obtain the estimates

$$\|f(v)\| \le C\|v\|^p_{[L^{2p}(\Omega)]^3} \le C\|v\|^p_{[H_0^1(\Omega)]^3},$$

$$\|\nabla f(v)\| \le C\|v\|^{p-1}_{[H^2(\Omega)]^3}\|v\|_{[H_0^1(\Omega)]^3},$$

$$\|\frac{\partial^2 f(v)}{\partial x_i \partial x_j}\| \le C\|v\|^p_{[H^2(\Omega)]^3}, \quad i, j = 1, 2, 3.$$

We recall that $\|g\|^2 = \sum_{j=1}^3 \int_\Omega |g_j|^2\, dx$ whenever $g = (g_1, g_2, g_3) \in [L^2(\Omega)]^3$. The above estimates imply

$$f(v) \in C\big([0, T]; [H^2(\Omega) \cap H_0^1(\Omega)]^3\big).$$

Now, we claim that $f(v) \in C^1\big([0, T].[H_0^1(\Omega)]^3\big) \cap C^2\big([0, T]; [L^2(\Omega)]^3\big)$. In fact,

$$\frac{d}{dt} f(v) = \big(\nabla f_1(v) \cdot v_t, \nabla f_2(v) \cdot v_t, \nabla f_3(v) \cdot v_t\big).$$

Therefore, using assumption (H2) and Hölder's inequality we obtain

$$\begin{aligned}
\|\frac{d}{dt} f(v)\|^2 &\le \int_\Omega \sum_{j=1}^3 |\nabla f_j(v) \cdot v_t|^2\, dx \\
&\le C \int_\Omega |v|^{2(p-1)} |v_t|^2\, dx \\
&\le C\|v_t\|^2_{[L^{2p}]^3} \|v\|^{2(p-1)}_{[L^{2p}]^3} \\
&\le C\|v_t\|^2_{[H_0^1]^3} \|v\|^{2(p-1)}_{[H_0^1]^3}.
\end{aligned}$$

Similarly, we can estimate

$$|\frac{\partial}{\partial x_j}\Big(\frac{d}{dt}f(v)\Big)| \leq C|v_t||\frac{\partial v}{\partial x_j}||v|^{p-2} + C|\frac{\partial v_t}{\partial x_j}||v|^{p-1}$$

for some positive constant C. Consequently,

$$\|\frac{\partial}{\partial x_j}\Big(\frac{d}{dt}f(v)\Big)\| \leq C\|v\|^{p-1}_{[H^2]^3}\|v_t\|_{[H_0^1]^3}$$

for $j = 1, 2, 3$. It follows from the above discussion that

$$f(v) \in C^1\big([0,T].[H_0^1(\Omega)]^3\big).$$

By a similar procedure we can prove that $f(v) \in C^2\big([0,T];[L^2(\Omega)]^3\big)$ which proves our claim. Thus, whenever we consider an element $\widetilde{U} = (\tilde{u}, \tilde{v}) \in Y(T)$ then, the nonlinearity $F(\widetilde{U}) = \big(0, f(\tilde{v}) + \tilde{u}\big)$ belongs to

$$C^1\big([0,T];\mathcal{D}(A)\big) \cap C^2\big([0,T];X\big).$$

It follows by semigroup theory that the nonhomogeneous problem

$$\frac{dU}{dt} = AU + F(\widetilde{U}), \quad U(0) = U_0 = (u_0, u_1) \tag{3.3}$$

has a unique (local) solution $U = (u, v) \in Y(T)$ provided $U_0 \in \mathcal{D}(A^2)$.

Lemma 3.2. *Assume (H1) and (H2). Let $U_0 = (u_0, u_1) \in \mathcal{D}(A^2)$. Then, there exist $T_0 > 0$ such that problem (1.1) has a unique solution $U = (u, u_t)$ belonging to the space*

$$C\big([0,T_0];\mathcal{D}(A^2)\big) \cap C^1\big([0,T_0];\mathcal{D}(A)\big) \cap C^2\big([0,T_0];X\big).$$

Sketch of proof. We consider the map $\Phi\colon Y(T) \mapsto Y(T)$ given by $\Phi(\widetilde{U}) = U$ where U is the solution of (3.3) and we will prove that Φ has a unique fixed point in $Y(T)$ as long as we choose T sufficiently small. We achieve this in the following way: Using the formula of variation of parameters and our assumptions of f we can prove that the solution U of (3.3) satisfies

$$\|U\|_{Y(T)} \leq C(U_0) + CT\Big\{\|\widetilde{U}\|^p_{Y(T)} + \|\widetilde{U}\|_{Y(T)}\Big\} \tag{3.4}$$

where $C(U_0)$ depends only on the norm $\|A^2U_0\|_X$ and the Sobolev norms (up to order three) of $U(0)$ and $U_t(0)$. Next, we choose $K \geq 1$ and consider the set

$$B_K = \big\{\widetilde{U} \in Y(t); \widetilde{U}(0) = U_0, \widetilde{U}_t(0) = U_1, \|\widetilde{U}\|_{Y(T)} \leq K\big\}$$

where

$$U_1 = \Big(u_1, \sum_{i,j=1}^{3}\frac{\partial}{\partial x_i}\big(A_{ij}\frac{\partial u_0}{\partial x_j}\big) - u_0 - u_1\Big).$$

We claim that $\Phi(B_K) \subseteq B_K$, if we choose T small and K large. In fact, let $\widetilde{U} \in B_K$ then, from (3.4) we obtain

$$\|U\|_{Y(T)} \leq C(U_0) + CT\big\{K^p + K\big\}.$$

Now, we choose K such that $C(U_0) \leq K/2$ and $T > 0$ such that $T < [2C(K^{p-1} + 1)]^{-1}$. Thus $\|U\|_{Y(T)} \leq K$. Obviously $U(0) = U_0$ and $U_t(0) = U_1$. Using the semigroup properties and the formula of variation of parameters we can prove that Φ is a contraction map, that is for any $\widetilde{U}$ and $\widetilde{W}$ belonging to B_K we have

$$\|\Phi(\widetilde{U}) - \Phi(\widetilde{W})\|_{Y(T)} \leq \alpha\|\widetilde{U} - \widetilde{W}\|_{Y(T)}$$

where $0 < \alpha = \alpha(K,T) < 1$ as long as we choose K large and $T > 0$ sufficiently small. This proves Lemma 3.2. $\square$

Next we prove Theorem 3.1. First, we extend the local solution we found in Lemma 3.2 to the maximal interval of existence $[0, T_{\max})$. Technically it will be more convenient to rewrite problem (1.1) as

$$\frac{dU}{dt} = \widetilde{A}U + \widetilde{F}(U), \quad U(0) = U_0 = (u_0, u_1) \tag{3.5}$$

with

$$\widetilde{A}(u,v) = \Big(v, \sum_{i,j=1}^{3} \frac{\partial}{\partial x_j}\big(A_{ij}\frac{\partial u}{\partial x_i}\big) - v\Big)$$

and $\widetilde{F}(U) = (0, f(u_t))$ where $U = (u,v)$, $v = u_t$. Let $\{S(t)\}$ be the semigroup associated to the generator $\widetilde{A}$. Then Theorem 2.1 tell us that the solution of the linear equation satisfies

$$E(t) \le CI_0\big(1+t\big)^{-2} \quad \forall t \ge 0. \tag{3.6}$$

In this article, we denote by C various positive constants which may vary from line to line. Let $v = u_t$. Taking the derivative in time of equation (2.1) we deduce that v satisfies

$$\begin{gathered} v_{tt} - \sum^{3} \frac{\partial}{\partial x_i}\Big(A_{ij}\frac{\partial v}{\partial x_i}\Big) + v_t = 0 \quad \text{in } \Omega \times [0,\infty) \\ v(x,0) = u_1(x), \quad v_t(x,0) = \sum^{3} \frac{\partial}{\partial x_j}\Big(A_{ij}\frac{\partial u_0}{\partial x_i}\Big) - u_1(x) \\ v = 0 \quad \text{on } \partial\Omega \times [0,+\infty) \end{gathered}$$

Applying the same reasoning as in the proof of Theorem 2.2,

$$E_1(t) = \frac{1}{2}\int_\Omega \{|v_1|^2 + \sum_{i,j=1}^{3} A_{ij}\frac{\partial v}{\partial x_j}\cdot\frac{\partial v}{\partial x_i}\}dx \le CI_1(1+t)^{-2} \tag{3.7}$$

with $v = u_t$, where I_1 depends on the Sobolev norms (up to order two) of the initial data and the quantity $\int_\Omega |x|^2\big|\sum_{i,j=1}^{3}\frac{\partial}{\partial x_j}\Big(A_{ij}\frac{\partial u_0}{\partial x_i}\Big)\big|^2dx$. Thus, from the equation (2.1) we also obtain

$$\|\sum_{i,j=1}^{3}\frac{\partial}{\partial x_j}\Big(A_{ij}\frac{\partial u}{\partial x_i}\Big)\|^2 \le C(I_0 + I_1)(1+t)^{-2} \tag{3.8}$$

Similarly, if $w = v_t = u_{tt}$ we obtain

$$E_2(t) = \frac{1}{2}\int_\Omega \{|w_t|^2 + \sum_{i,j=1}^{3}\sum_{i,j=1}^{3} A_{ij}\frac{\partial w}{\partial x_j}\cdot\frac{\partial w}{\partial x_i}\}dx \le CI_2(1+t)^{-2} \tag{3.9}$$

where I_2 depends on the Sobolev norm (up to order three) of the initial data and the quantity $\int_\Omega |x|^2\big|\sum_{i,j=1}^{3}\frac{\partial}{\partial x_j}\big(A_{ij}\frac{\partial u_1}{\partial x_i}\big)\big|^2dx$. Let $\widetilde{I} = I_0 + I_1 + I_2$ and $K > 1$ such that

$$\|u_0\|^2 < K\widetilde{I}, \tag{3.10}$$

$$E(0) + E_1(0) + E_2(0) + \|Lu_0\|^2 + \|Lu_1\|^2 < K\widetilde{I}, \tag{3.11}$$

where $L=\sum_{i,j=1}^3 \frac{\partial}{\partial x_j}\big(A_{ij}\frac{\partial}{\partial x_i}\big)$.

We proceed to prove Theorem 3.1: Let (u,u_t) be the local solution for the semilinear model (1.1) obtained in Lemma 3.2. Clearly, by continuity of the quantities on the left hand side of (3.6), (3.7) and (3.9) then in an small interval $[0,t)$ we will have that

$$(1+t)\|u(\cdot,t)\|^2 < K\widetilde{I}, \tag{3.12}$$

$$(1+t)^2\big\{E(t)+E_1(t)+E_2(t)+\|Lu(\cdot,t)\|^2+\|Lu_t\|^2\big\} < K\widetilde{I} \tag{3.13}$$

are valid. We want to prove that (3.12) and (3.13) hold for any $t\ge 0$. To do this we will choose K large and after $\widetilde{I}$ small. Suppose that (3.12) and (3.13) are not valid for any $\widetilde{T}$ "near" $T_{\max}$. Therefore, there must exist $T\in[0,\widetilde{T}]$ such that (3.12) and (3.13) hold in $[0,T)$ but

$$(1+T)\|u(\cdot,T)\|^2 = K\widetilde{I} \tag{3.14}$$

and/or

$$(1+T)^2\big\{E(T)+E_1(T)+E_2(T)+\|Lu(\cdot,T)\|^2+\|Lu_t(\cdot,T)\|^2\big\} = K\widetilde{I} \tag{3.15}$$

From (3.5) it follows that

$$U(t)=S(t)U_0+\int_0^t S(t-r)\widetilde{F}(r)\,dr.$$

Consequently, from Theorem 2.2 we deduce

$$E(t)\le C\widetilde{I}(1+t)^{-1}+C\int_0^t (1+t+r)^{-1}J(r)\,dr \tag{3.16}$$

where $J(r)=\|f(u_r)\|+\||\cdot|f(u_r)\|$. Using assumptions (H2) and Gagliardo-Nirenberg's inequality we obtain

$$\|f(u_r)\|\le C\|u_r\|_{L^{2p}}^p\le C\|u_r\|^{(1-\theta)p}\Big(\int_\Omega\sum_{i,j=1}^3 A_{ij}\frac{\partial u_r}{\partial x_j}\cdot\frac{\partial u_r}{\partial x_i}\,dx\Big)^{\theta p/2}$$

where $0<\theta=\frac{3(p-1)}{2p}\le 1$ because $\frac{7}{3}<p\le 3$. Due to (3.12)-(3.15) it follows that

$$\|f(u_r)\|\le C\big\{K\widetilde{I}(1+r)^{-1}\big\}^{(1-\theta)p}\big\{K\widetilde{I}(1+r)^{-1}\big\}^{\theta p}=CK^p\widetilde{I}^p(1+r)^{-p} \tag{3.17}$$

for any $r\in[0,T]$. Now we use finite propagation speed valid for the solution of problem (1.1): If $\operatorname{supp}u_0\cup\operatorname{supp}u_1\subseteq\{x\in\mathbb{R}^3,|x|\le R\}$ then in the interval of existence $(u,u_t)=(0,0)$, if $|x|\ge C_1t+R$ where $C_1=\|A\|/\sqrt{C_0}$, $\|A\|^2=\sum_{i,j=1}^3\|A_{ij}\|^2$ and C_0 is as in (1.4). We estimate

$$\begin{aligned}\||\cdot|f(u_r)\|^2 &\le C\int_\Omega |x|^2|u_r(x,r)|^{2p}\,dx\\ &= C\int_{\Omega\cap\{|x|\le C_1r+R\}}|x|^2|u_r(x,r)|^{2p}\,dx\\ &\le (C_1r+R)^2C\|u_r(\cdot,r)\|_{L^{2p}}^{2p}\end{aligned}$$

and by Gagliardo-Nirenberg it follows that

$$\||\cdot|f(u_r)\|\le C(C_1r+R)K^p\widetilde{I}^p(1+r)^{-p} \tag{3.18}$$

From (3.16), (3.17) and (3.18) we deduce

$$\begin{aligned} E(t) &\le C\widetilde{I}(1+t)^{-1} + CK^p\widetilde{I}^p \int_0^t (1+t-r)^{-1}(1+r)^{-p+1}\,dr \\ &\le (C\widetilde{I} + CK^p\widetilde{I}^p))(1+t)^{-1} \end{aligned} \tag{3.19}$$

for any $t \in [0,T]$. Here we used a calculus type lemma (see [11, Lemma 7.4]). Using the formula of variation of parameters we also obtain

$$\|u(\cdot,t)\| \le CI_0(1+t)^{-1/2} + C\int_0^t (1+t-r)^{-1/2}J(r)\,dr$$

where $J(r)$ is as in (3.16). Due to our above calculation we get

$$\begin{aligned} \|u(\cdot,t)\| &\le CI_0(1+t)^{-1/2} + CK^p\widetilde{I}^p \int_0^t (1+t-r)^{-1/2}(1+r)^{-p+1}\,dr \\ &\le (CI_0 + CK^p\widetilde{I}^p)(1+t)^{-1/2}. \end{aligned}$$

Next, we differentiate in time equation (1.1) and use the same sequence of ideas given above to obtain that $v = u_t$ satisfies

$$E_1(t) \le C(\widetilde{I} + \widetilde{I}^p + K^p\widetilde{I}^p)(1+t)^{-2}$$

where $E_1(t)$ is given as in (3.7). Using the equation it follows that

$$\|Lu(\cdot,t)\| \le C(\widetilde{I} + \widetilde{I}^p + K^p\widetilde{I}^p)(1+t)^{-2}$$

for any $t \in [0,T]$. Finally, we differentiate twice in time equation (1.1) and repeat the above reasoning to obtain that $w = v_t = u_{tt}$ satisfies

$$E_2(t) \le C(\widetilde{I} + \widetilde{I}^p + \widetilde{I}^{2p-1)} + K^p\widetilde{I}^p)(1+t)^{-2}, \tag{3.20}$$

$$\|Lu(\cdot,t)\| \le C(\widetilde{I} + \widetilde{I}^p + \widetilde{I}^{2p-1)} + K^p\widetilde{I}^p)(1+t)^{-2}\,. \tag{3.21}$$

Collecting information from (3.19) up to (3.21), we have

$$(1+t)\|u(\cdot,t)\|^2 \le C(1 + K^p\widetilde{I}^{p-1})\widetilde{I}, \tag{3.22}$$

and

$$\begin{aligned} &(1+t)^2\big\{E(t) + E_1(t) + E_2(t) + \|Lu(\cdot,t)\|^2 + \|Lu_t\|^2\big\} \\ &\le C\big(1 + \widetilde{I}^{p-1} + \widetilde{I}^{2p-2} + K^p\widetilde{I}^{p-1}\big)\widetilde{I} \end{aligned} \tag{3.23}$$

for any $t \in [0,T]$ and some positive constant C. Now we choose K large so that $K > C$ and

$$\widetilde{I} < \min\Big\{\big(\frac{K-C}{3C}\big)^{1/p-1}, \big(\frac{K-C}{3C}\big)^{1/2p-2}, \big(\frac{K-C}{3CK^p}\big)^{1/p-1}\Big\}\,.$$

With this choice, we clearly have that

$$C\big(1 + \widetilde{I}^{p-1} + \widetilde{I}^{2p-2} + K^p\widetilde{I}^{p-1}\big) < K.$$

Consequently from (3.22) and (3.23), we deduce that

$$(1+t)\|u(\cdot,t)\|^2 < K\widetilde{I},$$

$$(1+t)^2\big\{E(t) + E_1(t) + E_2(t) + \|Lu(\cdot,t)\|^2 + \|Lu_t\|^2\big\} < K\widetilde{I}$$

for any $t \in [0,T]$ which is a contradiction with (3.14) and (3.15). It follows that (3.12) and (3.13) should be valid for any $t \in [0, T_{\max})$; therefore, the solution of (1.1) exists globally and decays at the desired rate.

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