SOME ASPECTS OF VECTOR-VALUED
FUNCTIONS OF A VECTOR

THESIS

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P R E F A C E

This thesis is concerned with the consideration of the theory of vector-valued functions of vectors. Limits and continuity of vector-valued functions of vectors, matrix theorems concerning vector-valued functions of vectors, differentials and derivatives of such functions, as well as line integrals are considered. A final chapter concerning vector fields is presented.

The vector-valued functions of vectors are especially important in the vector field, since other classes of vector functions are sub-classes of this class of functions. There is developed here, in the main, material of interest in scientific application, and it is in such fields that it is hoped that the material developed will prove most significant and useful.

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C H A P T E R I

INTRODUCTION

1.1 Glossary of Symbols.

The symbol:	means:
J	the set of counting numbers.
E^1	the set of all points on a straight line or the set of all numbers associated with these points.
E^2	the set of all points in a plane.
E^3	the set of all points in a three-dimensional space.
E^n	the set of all points in an n-dimensional space.
ε	belongs to or is of the class of.
iff	if and only if.
C^o	the set of all continuous functions.
$\Longrightarrow$	implies.

The symbol:	means:
$[a,b]$	the set of points such that a point $x \in [a,b]$, iff, x is a, x is b, or else x is between a and b, where $a, b \in E^1$.
(a,b)	the set of points such that a point $x \in (a,b)$, iff, x is between a and b, where $a, b \in E^1$.
Dom $\vec{f}$	domain of $\vec{f}$.
V	neighborhood.
V*	deleted neighborhood.
∇	the del operator.
D $\vec{f}$	the derivative of $\vec{f}$.

1.2 Definitions.

The statement that:	means:
1. a set A is a subset of the set B, denoted by $A \subset B$	A and B are each sets such that if $x \in A$, then $x \in B$.
2. $A = B$	A and B are each sets such that $A \subset B$ and $B \subset A$.

The statement that:	means:
3. $A \cup B$ is the union of two point sets A and B	$A \cup B$ is the set of all points x such that $x \varepsilon A$ or $x \varepsilon B$.
4. $A \cap B$ is the intersection of two point sets A and B	$A \cap B$ is the set of all points x such that $x \varepsilon A$ and $x \varepsilon B$.
5. $V(\vec{x};r)$ is a neighborhood of a point $\vec{x} \varepsilon E^n$	$V(\vec{x};r)$ is the set of all points $\vec{y} \varepsilon E^n$ such that if $r > 0$, then $\lvert\vec{y} - \vec{x}\rvert < r$.
6. $V^*(\vec{x};r)$ is a deleted neighborhood of a point $\vec{x} \varepsilon E^n$	$V^*(\vec{x}:r)$ is the neighborhood $V(\vec{x}:r)$ minus the point $\vec{x}$.
7. $\vec{x}$ is an accumulation point of a set $S \varepsilon E^n$	$\vec{x}$ is a point in E^n such that every deleted neighborhood $V^*(\vec{x};r)$ of $\vec{x}$ contains at least one point of S.
8. δ_{ij}, $i,j = 1,2,3,\ldots,n$, is the Kronecker delta	$\delta_{ij} = \begin{cases} 1, \text{ if } i = j \\ 0, \text{ if } i \neq j \end{cases}, \quad i,j = 1,2,3,\ldots,n.$
9. a function f from E^n to E^1 is differentiable at a point $\vec{x}$	f is defined in a neighborhood $V(\vec{x};r)$ of $\vec{x}$ and there exists a vector $\vec{a}$ (independent of $\vec{h}$) such that for any point $\vec{x} + \vec{h}$ of $V^*(\vec{x};r)$, $f(\vec{x} + \vec{h}) = f(\vec{x}) + \vec{a}\cdot\vec{h} + \vec{\Phi}(\vec{x};\vec{h})\cdot\vec{h}$, where $\lim_{\vec{h}\to\vec{0}} \vec{\Phi}(\vec{x};\vec{h}) = \vec{0}$.

Notation: The term $\vec{a}\cdot\vec{h}$ is called the differential of f at $\vec{x}$ and $\vec{h}$ and is denoted by $d\ f(\vec{x};\vec{h})$. The vector $\vec{a}$ is called the derivative of f at $\vec{x}$ and is denoted by $\vec{D}\ f(\vec{x})$.

1.3 Algebra of Vectors.

The statement that:	means:
V^n is an n-dimensional vector space	V^n is the set of all n-tuples of numbers which belong to E^1 specified by $\vec{x} = (x_1,x_2,\ldots,x_n)$, where $x_k \in E^1$, $k = 1,2,\ldots,n$, and are called vectors.
$\vec{x} + \vec{y}$ is the sum of two vectors, $\vec{x} = (x_1,x_2,\ldots,x_n)$ and $\vec{y} = (y_1,y_2,\ldots,y_n)$, each of which belongs to V^n, $n \in J$,	$\vec{x} + \vec{y} = (x_1+y_1,x_2+y_2,\ldots,x_n+y_n)$.
$r\ \vec{x}$ is the product of r and $\vec{x}$, where $r \in E^1$ and $\vec{x} = (x_1,x_2,\ldots,x_n) \in V^n$, $n \in J$,	$r\ \vec{x} = (rx_1,rx_2,\ldots,rx_n)$.
$\vec{x} = \vec{y}$, where $\vec{x} = (x_1,x_2,\ldots,x_n)$ and $\vec{y} = (y_1,y_2,\ldots,y_n)$, each of which belongs to V^n, $n \in J$,	$x_k = y_k$, for all $k = 1,2,\ldots,n$.

The statement that:	means:
$\vec{x} - \vec{y}$ is the difference of two vectors $\vec{x} = (x_1, x_2, \ldots, x_n)$ and $\vec{y} = (y_1, y_2, \ldots, y_n)$, each of which belongs to V^n, $n \in J$,	$\vec{x} - \vec{y} = (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n)$.
$\lvert\vec{x}\rvert$ is the absolute value of the vector $\vec{x} = (x_1, x_2, \ldots, x_n) \in V^n$, $n \in J$,	$\lvert\vec{x}\rvert = \left[\sum_{k=1}^{n} x_k^{\,2} \right]^{\frac{1}{2}}$.
$\vec{0}$ is the zero vector	$\vec{0}$ is the vector such that $\lvert\vec{0}\rvert = 0$.
$\vec{u}$ is a unit vector	$\vec{u}$ is the vector such that $\lvert\vec{u}\rvert = 1$.
a set of vectors $\vec{u}_1$, $\vec{u}_2$, $\vec{u}_3$, . . . $\vec{u}_k$ constitutes a basis in V^n	1) $\vec{u}_i$, $i = 1,2,3,\ldots,k$, are linearly independent, and 2) each vector of V^n can be expressed as a linear combination of the $\vec{u}_i$, $i = 1,2,3,\ldots,k$.
∇ is the del operator in E^3	∇ is the operator $\vec{u}_1 D_{x_1} + \vec{u}_2 D_{x_2} + \vec{u}_3 D_{x_3}$, where $\vec{u}_1$, $\vec{u}_2$, $\vec{u}_3$ is a basis in E^3.

1.4 Assumed Properties.

Property 1. If each of $\vec{x}$ and $\vec{y} \in V^n$, then $\vec{x} + \vec{y} \in V^n$.

Property 2. If each of $\vec{x}$ and $\vec{y} \in V^n$, then $\vec{x} + \vec{y} = \vec{y} + \vec{x}$.

Property 3. If each of $\vec{x}$, $\vec{y}$, and $\vec{z} \in V^n$, then
$\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$.

Property 4. If $\vec{x} \in V^n$, and $\vec{0}$ is the zero vector, then $\vec{x} + \vec{0} = \vec{x}$.

Property 5. If $\vec{x} \in V^n$, and $r \in E^1$, then $r\,\vec{x} \in V^n$.

Property 6. If $\vec{x} \in V_n$, then $(1)\,\vec{x} = \vec{x}$.

Property 7. If r, $s \in E^1$, and $\vec{x} \in V^n$, then $(r + s)\,\vec{x} = r\,\vec{x} + s\,\vec{x}$.

Property 8. If $r \in E_1$, and $\vec{x}$, $\vec{y}$ each $\in V^n$, then
$r\,(\vec{x} + \vec{y}) = r\,\vec{x} + r\,\vec{y}$.

Property 9. If $\vec{x} \in V^n$, $n \in J$, then $|-\vec{x}| = |\vec{x}|$.

Property 10. If $\vec{x} \in V^n$, $n \in J$, and $c \in E^1$, then $|c\,\vec{x}| = |c|\;|\vec{x}|$.

Property 11. If $\vec{x}$ and $\vec{y} \in V^n$, $n \in J$, then $|\vec{x} + \vec{y}| \leq |\vec{x}| + |\vec{y}|$.

<u>Scalar</u> <u>Properties</u>.

Definition. The statement that $\vec{x}\cdot\vec{y}$ is the scalar (dot) product of two vectors $\vec{x} = (x_1,x_2,\ldots,x_n)$ and $\vec{y} = (y_1,y_2,\ldots,y_n)$, each of which $\in V^n$, $n \in J$, means $\vec{x}\cdot\vec{y} = \sum_{k=1}^{n} x_k\, y_k$.

Property 12. If $\vec{x}$ and $\vec{y}$ each ε V^n, n ε J, then $\vec{x}\cdot\vec{y} = \vec{y}\cdot\vec{x}$.

Property 13. If $\vec{x}$ and $\vec{y}$ each ε V^n, n ε J, and r ε E^1, then

$$(r\ \vec{x})\cdot\vec{y} = r\ (\vec{x}\cdot\vec{y}).$$

Property 14. If $\vec{x}$, $\vec{y}$, and $\vec{z}$ each ε V^n, n ε J, then

$$\vec{x}\cdot(\vec{y} + \vec{z}) = \vec{x}\cdot\vec{y} + \vec{x}\cdot\vec{z}.$$

Property 15. If $\vec{x}$ ε V^n, n ε J, then $\vec{x}\cdot\vec{x} \geqq 0$, and $\vec{x}\cdot\vec{x} = 0$, iff,

$$\vec{x} = \vec{0}.$$

Property 16. If $\vec{x}$ and $\vec{y}$ ε V^n, n ε J, then $|\vec{x}\cdot\vec{y}| \leqq |\vec{x}|\ |\vec{y}|$, and

$|\vec{x}\cdot\vec{y}| = |\vec{x}|\ |\vec{y}|$, iff, $\vec{x} = \vec{0}$, $\vec{y} = \vec{0}$, or $\vec{x} = \vec{y}$.

<u>Properties of Vector Cross Products</u>.

Definition. The statement that $\vec{x} \times \vec{y}$ is the vector cross product of two vectors $\vec{x} = (x_1,x_2,x_3)$ and $\vec{y} = (y_1,y_2,y_3)$, each of which ε V^3, means

$$\vec{x}\times\vec{y} = \begin{vmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} \vec{u}_1 + \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} \vec{u}_2 + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \vec{u}_3\ .$$

Property 17. If $\vec{x}$ and $\vec{y}$ each ε V^3, then $\vec{x} \times \vec{y} = -\ (\vec{y} \times \vec{x})$.

Property 18. If $\vec{x}$ and $\vec{y}$ each ε V^3, and r ε E^1, then

$$(r\ \vec{x}) \times \vec{y} = r\ (\vec{x} \times \vec{y}).$$

Property 19. If $\vec{x}$, $\vec{y}$, and $\vec{z}$ each ε V^3, then

$$\vec{x} \times (\vec{y} + \vec{z}) = (\vec{x} \times \vec{y}) + (\vec{x} \times \vec{z}).$$

Property 20. If $\vec{x}$ and $\vec{y}$ each ε V^3, then $|\vec{x} \times \vec{y}| \leqq |\vec{x}| \; |\vec{y}|$.

Property 21. If $\vec{x}$, $\vec{y}$, and $\vec{z}$ each ε V^3, then

$$\vec{x} \times (\vec{y} \times \vec{z}) = (\vec{x} \cdot \vec{z}) \; \vec{y} - (\vec{x} \cdot \vec{y}) \; \vec{z}.$$

1.5 Assumed Theorems.

1.5.1 Theorem. If a function f from E^n to E^1 is differentiable at $\vec{x}$, then f ε C^o at $\vec{x}$.

1.5.2 Theorem.

1. f is a function from E^n to E^1.
2. g is a function from E^n to E^1.
3. f is differentiable at a point $\vec{x}$ in E^n.
4. g is differentiable at $\vec{x}$.

$\Longrightarrow$ f + g and f g are differentiable at $\vec{x}$, and

$$d\; [f + g](\vec{x};\vec{h}) = d\; f(\vec{x};\vec{h}) + d\; g(\vec{x};\vec{h}),$$

$$\vec{D}\; [f + g](\vec{x}) = \vec{D}\; f(\vec{x}) = \vec{D}\; g(\vec{x}),$$

$$d\; [f\; g](\vec{x};\vec{h}) = f(\vec{x})\; d\; g(\vec{x};\vec{h}) + g(\vec{x})\; d\; f(\vec{x};\vec{h}); \text{ and}$$

$$\vec{D}\; [f\; g](\vec{x}) = f(\vec{x})\; \vec{D}\; g(\vec{x}) + g(\vec{x})\; \vec{D}\; f(\vec{x}).$$

1.5.3 Theorem.

1. f is a function from E^n to E^1.
2. f is differentiable at $\vec{x}$ ε E^n.
3. g is a function from E^1 to E^1.
4. g is differentiable at $f(\vec{x})$.

$\Longrightarrow$ $g \circ f$ is differentiable at $\vec{x}$, and

$$d\; [g \circ f](\vec{x};\vec{h}) = d\; g(f(\vec{x});df(\vec{x};\vec{h})), \text{ and}$$

$$\vec{D}\; [g \circ f](\vec{x}) = \vec{D}\; g(f(\vec{x}))\; \vec{D}\; f(\vec{x})\; .$$

1.5.4 Mean Value Theorem.

1. $f(x) \in C^0$, $a \leq x \leq b$.

2. $f'(x)$ exists, $a < x < b$.

$\Longrightarrow$ $f(b) - f(a) = (b - a)\, f'(a + \theta(b - a))$, where $0 < \theta < 1$.

1.5.5 First Fundamental Theorem of Integral Calculus.

If $f \in C^0$ on an interval I, and $a, t \in I$, then

$$D_t \int_a^t f(x)\, d\,x = f(t),\ t \in I.$$

1.5.6 Second Fundamental Theorem of Integral Calculus.

If $F' \in C^0$ on an interval I and $a, b \in I$, then

$$\int_a^b F'(x)\, d\,x = F(b) - F(a).$$

1.5.7 Theorem.

1. $f(x) \in C^0$, $a \leq x \leq b$.

$\Longrightarrow$ $f(x)$ is uniformly continuous on $[a,b]$.

1.5.8 Theorem.

1. $u(x) = g(x)$, $a \leq x \leq b$.

2. $y(x) = f(u)$, $u(a) \leq u \leq u(b)$.

3. $g'(x)$ exists, $a \leq x \leq b$.

4. $f'(u)$ exists, $u(a) \leq u \leq u(b)$.

$\Longrightarrow$ $y'(x) = f'(u)\, g'(x)$.

CHAPTER II

LIMITS AND CONTINUITY OF VECTOR-VALUED FUNCTIONS OF A VECTOR

2.1 Definitions.

2.1.1 Definition. The statement that $\vec{f}$ is a vector-valued function of a vector means $\vec{f}$ is a correspondence from a set A of vectors to a set B of vectors such that to each vector $\vec{a} \in A$, there corresponds only one vector $\vec{f}(\vec{a}) \in B$; i.e., $\vec{f}$ is a mapping of the set A into the set B. If A is a set in E^n and B is a set in E^m, then we call $\vec{f}$ a function from E^n to E^m. If $\vec{f}$ is a function from E^n to E^m, then $\vec{f}(\vec{x}) = (f_1(\vec{x}), f_2(\vec{x}), f_3(\vec{x}), \ldots, f_m(\vec{x}))$, where $f_k(\vec{x})$, $k = 1,2,3,\ldots,m$, is a function from E^n to E^1 with domain Dom $\vec{f}$ and rule of correspondence that $f_k(\vec{x})$ is the k^{th} component of the vector $\vec{f}(\vec{x})$.

2.1.2 Definition. The statement that $\vec{b}$ is the limit of the function $\vec{f}(\vec{x})$ at $\vec{a}$, written $\lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x}) = \vec{b}$, means $\vec{a}$ is an accumulation point of Dom $\vec{f}$ and if $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\vec{x} \in$ Dom $\vec{f}$ and $0 < |\vec{x} - \vec{a}| < \delta$, then $|\vec{f}(\vec{x}) - \vec{b}| < \varepsilon$.

2.1.3 Definition. If $\vec{f}$ and $\vec{g}$ are two vector-valued functions from E^n to E^m, and ϕ is a function from E^n to E^1, then $\vec{f} + \vec{g}$, $\vec{f} - \vec{g}$, $\vec{f}\cdot\vec{g}$, $\vec{f} \times \vec{g}$, and $\phi\,\vec{f}$ are defined as follows:

The domains of $\vec{f} \pm \vec{g}$, $\vec{f}\cdot\vec{g}$, and $\vec{f} \times \vec{g}$ are all Dom $\vec{f} \cap$ Dom $\vec{g}$, and Dom $[\phi\ \vec{f}]$ is Dom $\phi \cap$ Dom $\vec{f}$; and

if $\vec{x} \in$ Dom $[\vec{f} \pm \vec{g}]$, then $[\vec{f} \pm \vec{g}](\vec{x}) = \vec{f}(\vec{x}) \pm \vec{g}(\vec{x})$; and

if $\vec{x} \in$ Dom $[\vec{f}\cdot\vec{g}]$, then $[\vec{f}\cdot\vec{g}](\vec{x}) = \vec{f}(\vec{x})\cdot\vec{g}(\vec{x})$; and

if $\vec{f}$ and $\vec{g}$ are functions from E^n to E^3, and $\vec{x} \in$ Dom $[\vec{f} \times \vec{g}]$, then $[\vec{f} \times \vec{g}](\vec{x}) = \vec{f}(\vec{x}) \times \vec{g}(\vec{x})$; and

if $\vec{x} \in$ Dom $[\phi\ \vec{f}]$, then $[\phi\ \vec{f}](\vec{x}) = \phi(\vec{x})\ \vec{f}(\vec{x})$.

2.1.4 The statement that $[\vec{f} \circ \vec{g}](\vec{x})$ is a vector-valued function of a vector means if $\vec{g}(\vec{x})$ is a function from E^n to E^m and $\vec{f}$ is a function from E^m to E^p, then $[\vec{f} \circ \vec{g}](\vec{x})$ is a function from E^n to E^p, with rule of correspondence $[\vec{f} \circ \vec{g}](\vec{x}) = \vec{f}(\vec{g}(\vec{x}))$, where Dom $[\vec{f} \circ \vec{g}]$ is $\{\vec{x} \mid \vec{x} \in \text{Dom } \vec{g},\ \vec{g}(\vec{x}) \in \text{Dom } \vec{f}\}$.

From the definitions of the operations above, it is seen quite easily that if $\vec{f} = (f_1,f_2,f_3,\ldots,f_m)$ and $\vec{g} = (g_1,g_2,g_3,\ldots,g_m)$, then

$\vec{f} + \vec{g} = (f_1+g_1, g_2+g_2, f_3+g_3,\ldots,f_m+g_m)$,

$\vec{f} - \vec{g} = (f_1-g_1, f_2-g_2, f_3-g_3,\ldots,f_m-g_m)$,

$\phi\ \vec{f} = (\phi f_1, \phi f_2, \phi f_3,\ldots,\phi f_m)$, where $\phi\ f_k$ means the product of the functions $\phi(\vec{x})$ and $f_k(\vec{x})$,

$$\vec{f}\cdot\vec{g} = \sum_{k=1}^{m} f_k\ g_k,$$

$$\vec{f} \times \vec{g} = (f_2g_3-f_3g_2, f_3g_1-f_1g_3, f_1g_2-f_2g_1).$$

2.2 Limit Theorems.

2.2.1 Theorem.

1. $\vec{b} = (b_1, b_2, \ldots, b_m) \in E^m$.

2. $\vec{f}(\vec{x}) = (f_1(\vec{x}), f_2(\vec{x}), \ldots, f_m(\vec{x}))$ is a function from E^n to E^m.

3. $\vec{a}$ is a point of accumulation of Dom $\vec{f}$.

4. $\lim\limits_{\vec{x}\to\vec{a}} \vec{f}(\vec{x}) = \vec{b}$.

$\Longrightarrow \lim\limits_{\vec{x}\to\vec{a}} f_k(\vec{x}) = b_k$ for each $k = 1,2,3,\ldots,m$.

Proof:

Since $\lim\limits_{\vec{x}\to\vec{a}} \vec{f}(\vec{x}) = \vec{b}$, then given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\vec{x} \in$ Dom $\vec{f}$ and $0 < |\vec{x} - \vec{a}| < \delta$, then $|\vec{f}(\vec{x}) - \vec{b}| < \varepsilon$ or

$$\left[\sum_{k=1}^{n} (f_k(\vec{x}) - b_k)^2\right]^{\frac{1}{2}} = |\vec{f}(\vec{x}) - \vec{b}| < \varepsilon.$$

Then, $\sum\limits_{k=1}^{n} (f_k(\vec{x}) - b_k)^2 < \varepsilon^2$, if $0 < |\vec{x} - \vec{a}| < \delta$.

Hence, we can conclude that $(f_k(\vec{x}) - b_k)^2 < \varepsilon^2$, $k = 1,2,3,\ldots,m$, and $|f_k(\vec{x}) - b_k| < \varepsilon$, $k = 1,2,3,\ldots,m$, if $\vec{x} \in$ Dom f and $0 < |\vec{x} - \vec{a}| < \delta$.

Thus, $\lim\limits_{\vec{x}\to\vec{a}} f_k(\vec{x}) = b_k$, $k = 1,2,3,\ldots,m$.

2.2.2 Theorem.

1. $\vec{b} = (b_1, b_2, \ldots, b_m) \in E^m$.

2. $\vec{f}(\vec{x}) = (f_1(\vec{x}), f_2(\vec{x}), \ldots, f_m(\vec{x}))$ is a function from E^n to E^m.

3. $\vec{a}$ is a point of accumulation of Dom $\vec{f}$.

4. $\lim_{\vec{x}\to\vec{a}} f_k(\vec{x}) = b_k$, where $k = 1,2,3,\ldots,n$.

$\Longrightarrow \quad \lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x}) = \vec{b}$.

Proof:

Now, if $\lim_{\vec{x}\to\vec{a}} f_k(\vec{x}) = b_k$, $k = 1,2,3,\ldots,m$, then if $\varepsilon > 0$, then corresponding to $\frac{\varepsilon}{\sqrt{m}} > 0$, there exists $\delta_k > 0$ such that if $\vec{x} \varepsilon$ Dom $\vec{f}$ and $0 < |\vec{x} - \vec{a}| < \delta_k$, then $|f_k(\vec{x}) - b_k| < \frac{\varepsilon}{\sqrt{m}}$, $k = 1,2,3,\ldots,m$.

If we choose $\delta = \min \{\delta_k\}$, $k = 1,2,3,\ldots,m$, then, if $\vec{x} \varepsilon$ Dom $\vec{f}$ and $0 < |\vec{x} - \vec{a}| < \delta$, then

$$|\vec{f}(\vec{x}) - \vec{b}| = \left[\sum_{k=1}^{m} (f_k(\vec{x}) - b_k)^2\right]^{\frac{1}{2}} < \left[\sum_{k=1}^{m} \frac{\varepsilon^2}{m}\right]^{\frac{1}{2}} = \varepsilon.$$

Thus, $\lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x}) = \vec{b}$.

2.2.3 Theorem.

1. $\vec{a}$ and $\vec{b}$ are each vectors.

2. For each $\varepsilon > 0$, $|\vec{a} - \vec{b}| < \varepsilon$.

$\Longrightarrow \quad \vec{a} = \vec{b}$.

Proof:

Either $\vec{a} = \vec{b}$, or else $\vec{a} \neq \vec{b}$.

Let us assume that $\vec{a} \neq \vec{b}$, then $\vec{a} - \vec{b} \neq \vec{0}$, $|\vec{a} - \vec{b}| \neq 0$, and we have $|\vec{a} - \vec{b}| = d > 0$, $d \varepsilon E^1$.

Let $\varepsilon = d$, then from the hypothesis, $|\vec{a} - \vec{b}| < d$.

Hence, we have a contradiction, and thus we must reject our assumption and accept the only other possibility; i.e., that $\vec{a} = \vec{b}$.

2.2.4 Theorem.

1. $\vec{f}$ and $\vec{g}$ are functions from E^n to E^m.

2. $\vec{a}$ is an accumulation point of Dom $\vec{f} \cap$ Dom $\vec{g}$.

3. $\lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x}) = \vec{L}$.

4. $\lim_{\vec{x}\to\vec{a}} \vec{g}(\vec{x}) = \vec{T}$.

$$\Longrightarrow \quad \lim_{\vec{x}\to\vec{a}} \left(\vec{f} \pm \vec{g}\right)(\vec{x}) = \lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x}) \pm \lim_{\vec{x}\to\vec{a}} \vec{g}(\vec{x}) = \vec{L} \pm \vec{T}.$$

Proof:

Let $\vec{f}(\vec{x}) = (f_1(\vec{x}), f_2(\vec{x}), \ldots, f_m(\vec{x}))$,

$\vec{g}(\vec{x}) = (g_1(\vec{x}), g_2(\vec{x}), \ldots, g_m(\vec{x}))$.

$\vec{L} = (\vec{L}_1, \vec{L}_2, \ldots, \vec{L}_m)$, and

$\vec{T} = (\vec{T}_1, \vec{T}_2, \ldots, \vec{T}_m)$.

Now, $\vec{f}(\vec{x}) \pm \vec{g}(\vec{x}) = (f_1(\vec{x}) \pm g_1(\vec{x}), f_2(\vec{x}) \pm g_2(\vec{x}), \ldots, f_m(\vec{x}) \pm g_m(\vec{x}))$.

From hypothesis (1) and 2.2.1, we know that $\lim_{\vec{x}\to\vec{a}} f_k(\vec{x}) = L_k$,

$k = 1,2,3,\ldots,m$.

Now, from hypothesis (2) and 2.2.1, we know that $\lim_{\vec{x}\to\vec{a}} g_k(\vec{x}) = T_k$,

$k = 1,2,3,\ldots,m$.

Hence, $\lim_{\vec{x}\to\vec{a}} \left(f_k(\vec{x}) \pm g_k(\vec{x})\right) = L_k \pm T_k$, $k = 1,2,3,\ldots,m$.

Thus,

$$\lim_{\vec{x}\to\vec{a}} \left(\vec{f}(\vec{x}) \pm \vec{g}(\vec{x})\right)$$

$$= \left(\lim_{\vec{x}\to\vec{a}}(f_1(\vec{x}) \pm g_1(\vec{x})), \lim_{\vec{x}\to\vec{a}}(f_2(\vec{x}) \pm g_2(\vec{x})), \ldots, \lim_{\vec{x}\to\vec{a}}(f_m(\vec{x}) \pm g_m(\vec{x}))\right)$$

$$= (L_1 \pm T_1, L_2 \pm T_2, \ldots, L_m \pm T_m)$$

$= (L_1, L_2, \ldots, L_m) \pm (T_1, T_2, \ldots, T_m)$

$= \vec{L} \pm \vec{T}$.

Hence, $\lim_{\vec{x}\to\vec{a}} (\vec{f} \pm \vec{g})(\vec{x}) = \vec{L} \pm \vec{T}$

$$= \lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x}) \pm \lim_{\vec{x}\to\vec{a}} \vec{g}(\vec{x}) .$$

2.2.5 Theorem.

1. $\vec{f}(\vec{x})$ and $\vec{g}(\vec{x})$ are each functions from E^n to E^m.
2. $\vec{a}$ is an accumulation point of Dom $\vec{f} \cap$ Dom $\vec{g}$.
3. $\lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x}) = \vec{L}$.
4. $\lim_{\vec{x}\to\vec{a}} \vec{g}(\vec{x}) = \vec{T}$.

$$\Longrightarrow \quad \lim_{\vec{x}\to\vec{a}} (\vec{f}\cdot\vec{g})(\vec{x}) = \lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x}) \cdot \lim_{\vec{x}\to\vec{a}} \vec{g}(\vec{x}) = \vec{L} \cdot \vec{T}.$$

Proof:

Let $\vec{f}(\vec{x}) = (f_1(\vec{x}), f_2(\vec{x}), \ldots, f_m(\vec{x}))$,

$\vec{g}(\vec{x}) = (g_1(\vec{x}), g_2(\vec{x}), \ldots, g_m(\vec{x}))$,

$\vec{L} = (L_1, L_2, \ldots, L_m)$, and

$\vec{T} = (T_1, T_2, \ldots, T_m)$.

Now,

$(\vec{f}\cdot\vec{g})(\vec{x}) = f_1g_1 + f_2g_2 + \ldots + f_mg_m$, where f_kg_k, $k = 1,2,3,\ldots,m$,

means the products of the functions $f_k(\vec{x})$ and $g_k(\vec{x})$.

Hence,

$$\lim_{\vec{x}\to\vec{a}} (\vec{f}\cdot\vec{g})(\vec{x}) = \lim_{\vec{x}\to\vec{a}} (f_1g_1 + f_2g_2 + \ldots + f_mg_m)$$

$$= \lim_{\vec{x}\to\vec{a}} f_1g_1 + \lim_{\vec{x}\to\vec{a}} f_2g_2 + \ldots + \lim_{\vec{x}\to\vec{a}} f_mg_m$$

$$= L_1 T_1 + L_2 T_2 + \cdots + L_m T_m$$

$$= \vec{L} \cdot \vec{T} .$$

Hence,

$$\lim_{\vec{x} \to \vec{a}} \left(\vec{f} \cdot \vec{g}\right)(\vec{x}) = \vec{L} \cdot \vec{T} = \lim_{\vec{x} \to \vec{a}} \vec{f}(\vec{x}) \cdot \lim_{\vec{x} \to \vec{a}} \vec{g}(\vec{x}) .$$

2.2.6 Theorem.

1. $\vec{f}(\vec{x})$ and $\vec{g}(\vec{x})$ are each functions from E^n to E^3.
2. $\vec{a}$ is an accumulation point of Dom $\vec{f} \cap$ Dom $\vec{g}$.
3. $\lim_{\vec{x} \to \vec{a}} \vec{f}(\vec{x}) = \vec{L}$.
4. $\lim_{\vec{x} \to \vec{a}} \vec{g}(\vec{x}) = \vec{T}$.

$$\Longrightarrow \lim_{\vec{x} \to \vec{a}} \left(\vec{f} \times \vec{g}\right)(\vec{x}) = \vec{L} \times \vec{T}.$$

Proof:

Let $\vec{f}(\vec{x}) = (f_1(\vec{x}), f_2(\vec{x}), f_3(\vec{x}))$,

$\vec{g}(\vec{x}) = (g_1(\vec{x}), g_2(\vec{x}), g_3(\vec{x}))$,

$\vec{L} = (L_1, L_2, L_3)$, and

$\vec{T} = (T_1, T_2, T_3)$.

Now, from hypothesis (1) and 2.2.1, we know that $\lim_{\vec{x} \to \vec{a}} f_k(\vec{x}) = L_k$,

$k = 1,2,3$.

Also, by hypothesis (2) and 2.2.1, we know that $\lim_{\vec{x} \to \vec{a}} g_k(\vec{x}) = T_k$,

$k = 1,2,3$.

Now,

$$\left(\vec{f} \times \vec{g}\right)(\vec{x}) = (f_2 g_3 - f_3 g_2, f_3 g_1 - f_1 g_3, f_1 g_2 - f_2 g_1).$$

Thus,

$$\lim_{\vec{x}\to\vec{a}}(\vec{f}\times\vec{g})(\vec{x}) = \left(\lim_{\vec{x}\to\vec{a}}(f_2g_3-f_3g_2), \lim_{\vec{x}\to\vec{a}}(f_3g_1-f_1g_3), \lim_{\vec{x}\to\vec{a}}(f_1g_2-f_2g_1)\right).$$

Now, from the limit theorems regarding real-valued functions of vectors, we know that

$$\lim_{\vec{x}\to\vec{a}}(\vec{f}\times\vec{g})(\vec{x}) = (L_2\,T_3 - L_3\,T_2,\ L_3\,T_1 - L_1\,T_3,\ L_1\,T_2 - L_2\,T_1)$$
$$= \vec{L}\times\vec{T}.$$

2.2.7 Theorem.

1. $\vec{f}(\vec{x})$ is a vector-valued function from E^n to E^m.
2. $\vec{a}$ is an accumulation point of Dom $\vec{f}$.
3. $\lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x}) = \vec{b}$.

$\Longrightarrow$ $\vec{b}$ is unique.

Proof:

Either $\vec{b}$ is unique or else $\vec{b}$ is not unique.

Assume that $\vec{b}$ is not unique, then there exists a vector $\vec{b}' \neq \vec{b}$ such that $\lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x}) = \vec{b}'$, which means if $\varepsilon > 0$, then corresponding to $\frac{\varepsilon}{2}$, there exists a $\delta' > 0$ such that if $\vec{a}$ is an accumulation point of Dom $\vec{f}$, and $0 < |\vec{x} - \vec{a}| < \delta'$, then $|\vec{f}(\vec{x}) - \vec{b}'| < \frac{\varepsilon}{2}$.

Now, from hypothesis (3), we know that $\lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x}) = \vec{b}$, then corresponding to $\frac{\varepsilon}{2}$, there exists a $\delta'' > 0$ such that if $\vec{a}$ is an accumulation point of Dom $\vec{f}$ and $0 < |\vec{x} - \vec{a}| < \delta''$, then $|\vec{f}(\vec{x}) - \vec{b}| < \frac{\varepsilon}{2}$.

Choose $\delta = \min(\delta', \delta'')$, then if $0 < |\vec{x} - \vec{a}| < \delta$, then

$$|\vec{b} - \vec{b}'| = |\vec{b} - \vec{f}(\vec{x}) + \vec{f}(\vec{x}) - \vec{b}'|$$
$$\leq |\vec{f}(\vec{x}) - \vec{b}| + |\vec{f}(\vec{x}) - \vec{b}'|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, from this result, since $\vec{b}$ and $\vec{b}'$ are constant vectors, from 2.2.2, we know that $\vec{b}' = \vec{b}$, which contradicts our assumption, and thus $\vec{b}$ is unique.

2.2.8 Theorem.

1. $\vec{f}$ is a vector-valued function from E^n to E^m.
2. ϕ is a real-valued function from E^n to E^1.
3. $\vec{a}$ is an accumulation point of Dom $\vec{f} \cap$ Dom ϕ.
4. $\lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x}) = \vec{A}$.
5. $\lim_{\vec{x}\to\vec{a}} \phi(\vec{x}) = B$.

$\Longrightarrow \quad \lim_{\vec{x}\to\vec{a}} \phi(\vec{x})\ \vec{f}(\vec{x}) = B\ \vec{A}$.

Proof:

Now $\phi(\vec{x})\ \vec{f}(\vec{x}) = \left(\phi(\vec{x})f_1(\vec{x}),\phi(\vec{x})f_2(\vec{x}),\ldots,\phi(\vec{x})f_m(\vec{x})\right)$.

From hypothesis, $\lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x}) = \vec{A}$ and $\lim_{\vec{x}\to\vec{a}} \phi(\vec{x}) = B$.

Let $\vec{A} = (A_1, A_2, \ldots, A_m)$, then

$\lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x}) = (A_1, A_2, \ldots, A_m)$, and

$\lim_{\vec{x}\to\vec{a}} f_k(\vec{x}) = A_k$, $k = 1,2,3,\ldots,m$.

Hence,
$$\begin{aligned}\lim_{\vec{x}\to\vec{a}} \phi(\vec{x})\ \vec{f}(\vec{x}) &= \lim_{\vec{x}\to\vec{a}} \left(\phi(\vec{x})f_1(\vec{x}),\phi(\vec{x})f_2(\vec{x}),\ldots,\phi(\vec{x})f_m(\vec{x})\right) \\ &= \left(\lim_{\vec{x}\to\vec{a}}\phi(\vec{x})f_1(\vec{x}),\lim_{\vec{x}\to\vec{a}}\phi(\vec{x})f_2(\vec{x}),\ldots,\lim_{\vec{x}\to\vec{a}}\phi(\vec{x})f_m(\vec{x})\right) \\ &= (B\ A_1, B\ A_2, \ldots, B\ A_m) \\ &= B\ (A_1, A_2, \ldots, A_m) \\ &= B\ \vec{A}.\end{aligned}$$

Hence, the theorem is proved.

2.3 Continuity Definitions.

2.3.1 Definition. The statement that a vector-valued function $\vec{f}$ of a vector is continuous at the point $\vec{a}$ in Dom $\vec{f}$ means for each $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$, such that if $\vec{x} \in$ Dom $\vec{f}$ and $|\vec{x} - \vec{a}| < \delta_\varepsilon$, then $|\vec{f}(\vec{x}) - \vec{f}(\vec{a})| < \varepsilon$.

Remark. In the case $\vec{a}$ is not an accumulation point of Dom $\vec{f}$ and if $\vec{f}(\vec{a})$ exists, then $\vec{f}$ is continuous at $\vec{x} = \vec{a}$. When $\vec{a}$ is an accumulation point of Dom $\vec{f}$, then the definition is equivalent to $\lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x}) = \vec{f}(\vec{a})$.

2.3.2 Definition. The statement that a vector-valued function $\vec{f}$ is continuous on a set $S \in$ Dom $\vec{f}$ means the restricted function $\vec{f}_S$ (the set of values of $\vec{f}$ on the set S) is continuous, where $\vec{f}_S$ is the function with domain, Dom $\vec{f} \cap S$, such that $\vec{f}_S(\vec{x}) = \vec{f}(\vec{x})$, if $\vec{x} \in$ Dom $\vec{f} \cap S$.

2.4 Continuity Theorems.

2.4.1 Theorem.

1. $\vec{f} = (f_1(\vec{x}), f_2(\vec{x}), \ldots, f_m(\vec{x}))$ is a function from E^n to E^m.
2. $\vec{a} \in$ Dom $\vec{f}$.
3. $\vec{f} \in C^0$ at $\vec{x} = \vec{a}$.

$\Longrightarrow$ $f_k \in C^0$ at $\vec{x} = \vec{a}$, where $k = 1,2,3,\ldots,m$.

Proof:

Since $\vec{f} \in C^0$ at $\vec{x} = \vec{a}$, and $\vec{a} \in$ Dom $\vec{f}$, then for each $\varepsilon > 0$, there exists a $\delta_\varepsilon > 0$ such that if $\vec{x} \in$ Dom $\vec{f}$ and $|\vec{x} - \vec{a}| < \delta_\varepsilon$, then

$|\vec{f}(\vec{x}) - \vec{f}(\vec{a})| < \varepsilon.$

But, $\left[\sum_{k=1}^{m} (f_k(\vec{x}) - f_k(\vec{a}))^2\right]^{\frac{1}{2}} = |\vec{f}(\vec{x}) - \vec{f}(\vec{a})| < \varepsilon$, where $\vec{x} \in \mathrm{Dom}\, \vec{f}$

and $|\vec{x} - \vec{a}| < \delta_\varepsilon$, so,

$$\sum_{k=1}^{m} (f_k(\vec{x}) - f_k(\vec{a}))^2 < \varepsilon^2 ,$$

$$(f_k(\vec{x}) - f_k(\vec{a}))^2 < \varepsilon^2, \text{ or}$$

$|f_k(\vec{x}) - f_k(\vec{a})| < \varepsilon$, if $\vec{x} \in \mathrm{Dom}\, \vec{f}$ and $|\vec{x} - \vec{a}| < \delta_\varepsilon$.

Hence, $f_k \in C^0$ at $\vec{x} = \vec{a}$, $k = 1,2,3,\ldots,m$.

2.4.2 Theorem.

1. $\vec{f} = (f_1(\vec{x}), f_2(\vec{x}), \ldots, f_m(\vec{x}))$ is a function from E^n to E^m.
2. $\vec{a} \in \mathrm{Dom}\, \vec{f}$.
3. $f_k \in C^0$ at $\vec{x} = \vec{a}$, $k = 1,2,3,\ldots,m$.

$\Longrightarrow \quad \vec{f} \in C^0$ at $\vec{x} = \vec{a}$.

Proof:

Since $f_k \in C^0$ at $\vec{x} = \vec{a}$, $k = 1,2,3,\ldots,m$, then if $\varepsilon > 0$, then corresponding to $\frac{\varepsilon}{\sqrt{m}}$, there exists a $\delta_\varepsilon^{(k)} > 0$ such that if $\vec{x} \in \mathrm{Dom}\, \vec{f}$ and $|\vec{x} - \vec{a}| < \delta_\varepsilon^{(k)}$, then $|f_k(\vec{x}) - f_k(\vec{a})| < \frac{\varepsilon}{\sqrt{m}}$.

Let $\delta = \min\, \{\delta_\varepsilon^{(k)}\}$, then if $\vec{x} \in \mathrm{Dom}\, \vec{f}$ and $|\vec{x} - \vec{a}| < \delta$, then

$|f_k(\vec{x}) - f_k(\vec{a})| < \frac{\varepsilon}{\sqrt{m}}$, $k = 1,2,3,\ldots,m$, and

$$|\vec{f}(\vec{x}) - \vec{f}(\vec{a})| = \left[\sum_{k=1}^{m} (f_k(\vec{x}) - f_k(\vec{a}))^2\right]^{\frac{1}{2}}$$

$$< \left[\sum_{k=1}^{m} \frac{\varepsilon^2}{m}\right]^{\frac{1}{2}} = \varepsilon .$$

Therefore, $|\vec{f}(\vec{x}) - \vec{f}(\vec{a})| < \varepsilon$, if $|\vec{x} - \vec{a}| < \delta$, and hence,

$\vec{f} \in C^o$ at $\vec{x} = \vec{a}$.

2.4.3 Theorem.

If the functions $\vec{f}$, $\vec{g}$, from E^n to E^m, are each continuous at $\vec{x} = \vec{a}$, then $\vec{f} + \vec{g}$ is continuous at $\vec{x} = \vec{a}$.

Proof:

Let $\vec{f} = (f_1(\vec{x}), f_2(\vec{x}), \ldots, f_m(\vec{x}))$, and

$\vec{g} = (g_1(\vec{x}), g_2(\vec{x}), \ldots, g_m(\vec{x}))$

where $\vec{x} \in E^n$ and $\vec{x} \in \text{Dom } \vec{f} \cap \text{Dom } \vec{g}$.

Now,

$\vec{f} + \vec{g} = (f_1 + g_1,\ f_2 + g_2,\ \ldots,\ f_m + g_m)$.

Since $\vec{f}$ and $\vec{g} \in C^o$ at $\vec{x} = \vec{a}$, then f_k and g_k, $k = 1,2,3,\ldots,m$, $\in C^o$ at $\vec{x} = \vec{a}$, from 2.1.4.

Hence, the function $f_k + g_k$, $k = 1,2,3,\ldots,m$, $\in C^o$ at $\vec{x} = \vec{a}$.

Thus, $\vec{f} + \vec{g}$ is continuous at $\vec{x} = \vec{a}$.

2.4.4 Theorem.

If the functions $\vec{f}$, $\vec{g}$, from E^n to E^m, are each continuous at $\vec{x} = \vec{a}$, then $\vec{f} - \vec{g}$ is continuous at $\vec{x} = \vec{a}$.

Proof:

Let $\vec{f} = (f_1(\vec{x}), f_2(\vec{x}), \ldots, f_m(\vec{x}))$, and

$\vec{g} = (g_1(\vec{x}), g_2(\vec{x}), \ldots, g_m(\vec{x}))$,

where $\vec{x} \in E^n$ and $\vec{x} \in \text{Dom } \vec{f} \cap \text{Dom } \vec{g}$.

Now,

$\vec{f} - \vec{g} = (f_1 - g_1,\ f_2 - g_2,\ \ldots,\ f_m - g_m)$.

Since $\vec{f}$ and $\vec{g} \in C^o$ at $\vec{x} = \vec{a}$, then f_k and g_k, $k = 1,2,3,\ldots,m$, $\in C^o$ at $\vec{x} = \vec{a}$ from 2.4.1.

Thus, the $f_k - g_k$, $k = 1,2,3,\ldots,m$, $\in C^o$ at $\vec{x} = \vec{a}$.

Hence, $\vec{f} - \vec{g}$ is continuous at $\vec{x} = \vec{a}$.

2.4.5 Theorem.

If the functions $\vec{f}$, $\vec{g}$, from E^n to E^m, are each continuous at $\vec{x} = \vec{a}$, then $\vec{f} \cdot \vec{g}$ is continuous at $\vec{x} = \vec{a}$.

Proof:

Let $\vec{f} = (f_1(\vec{x}), f_2(\vec{x}), \ldots, f_m(\vec{x}))$, and

$\vec{g} = (g_1(\vec{x}), g_2(\vec{x}), \ldots, g_m(\vec{x}))$,

where $\vec{x} \in E^n$ and $\vec{x} \in \text{Dom } \vec{f} \cap \text{Dom } \vec{g}$.

Now, $\vec{f} \cdot \vec{g} = \sum_{k=1}^{m} f_k\, g_k$.

Since $\vec{f}$ and $\vec{g}$ are each continuous at $\vec{x} = \vec{a}$, then from 2.4.1, f_k and g_k, $k = 1,2,3,\ldots,m$, are continuous at $\vec{x} = \vec{a}$, and $f_k\, g_k \in C^o$ at $\vec{x} = \vec{a}$, $k = 1,2,3,\ldots,m$.

Hence, $\sum_{k=1}^{m} f_k\, g_k$ is continuous at $\vec{x} = \vec{a}$, and thus, $\vec{f} \cdot \vec{g}$ is continuous at $\vec{x} = \vec{a}$.

2.4.6 Theorem.

If the functions $\vec{f}$, $\vec{g}$, from E^n to E^3, are each continuous at $\vec{x} = \vec{a}$, then $\vec{f} \times \vec{g}$ is continuous at $\vec{x} = \vec{a}$.

Proof:

Let $\vec{f} = (f_1(\vec{x}), f_2(\vec{x}), \ldots, f_m(\vec{x}))$, and

$\vec{g} = (g_1(\vec{x}), g_2(\vec{x}), \ldots, g_m(\vec{x}))$, where $\vec{x} \in E^n$ and $\vec{x} \in \text{Dom } \vec{f} \cap \text{Dom } \vec{g}$.

Now,

$\vec{f} \times \vec{g} = (f_2g_3 - f_3g_2, f_3g_1 - f_1g_3, f_1g_2 - f_2g_1)$.

Since $\vec{f}$ and $\vec{g}$ are each continuous at $\vec{x} = \vec{a}$, then from 2.4.1, f_k and g_k are continuous at $\vec{x} = \vec{a}$, where $k = 1,2,3,\ldots,m$.

Hence, $f_2g_3 - f_3g_2$, $f_3g_1 - f_1g_3$, $f_1g_2 - f_2g_1$ are each continuous at $\vec{x} = \vec{a}$, and thus, $\vec{f} \times \vec{g}$ is continuous at $\vec{x} = \vec{a}$.

2.4.7 Theorem.

1. $\vec{f}$ is a function from E^n to E^m.
2. $\vec{f} \in C^0$ at $\vec{x} = \vec{a}$.
3. ϕ is a function from E^n to E^1.
4. $\phi \in C^0$ at $\vec{x} = \vec{a}$.

$\Longrightarrow \quad \phi \vec{f} \in C^0$ at $\vec{x} = \vec{a}$.

Proof:

Let $\vec{f} = (f_1, f_2, f_3, \ldots, f_m)$, then

$\phi \vec{f} = (\phi f_1, \phi f_2, \phi f_3, \ldots, \phi f_m)$.

Since ϕ and $\vec{f}$ are continuous at $\vec{x} = \vec{a}$, then $f_k(\vec{x}) \in C^0$ at $\vec{x} = \vec{a}$ by 2.4.1, $k = 1,2,3,\ldots,m$, and $\phi(\vec{x})\ f_k(\vec{x}) \in C^0$ at $\vec{x} = \vec{a}$, $k = 1,2,3,\ldots,m$.

Since we have a product of continuous real-valued functions of vectors at $\vec{x} = \vec{a}$, then by 2.4.2, $\phi \vec{f} \in C^0$ at $\vec{x} = \vec{a}$.

2.4.8 Lemma.

1. $\vec{f}$ is a continuous function from E^n to E^m with domain D.

2. A is open relative to $R = \vec{f}(D)$.

$\Longrightarrow$ $\vec{f}^*(A) = \{\vec{x} \mid \vec{f}(\vec{x}) \in A\}$ is open relative to D.

Proof:

Let $\vec{x}_o \in \vec{f}^*(A)$, and let $\vec{y}_o = \vec{f}(\vec{x}_o)$.

Since A is open relative to R and $\vec{y}_o \in A$, there exists a neighborhood $V(\vec{y}_o;\varepsilon)$ such that $V(\vec{y}_o;\varepsilon) \cap R \subset A$. Also, since $\vec{f} \in C^o$ at $\vec{x}_o$, corresponding to $\varepsilon > 0$, there exists $\delta > 0$ such that $\vec{x} \in V(\vec{x}_o;\delta) \cap D$ implies $\vec{f}(\vec{x}) \in V(\vec{y}_o;\varepsilon) \cap R \subset A$.

Thus, $V(\vec{x}_o;\delta) \cap D$ is contained in $\vec{f}^*(A)$; and hence, $\vec{f}^*(A)$ is open relative to D.

2.4.9 Theorem.

1. $\vec{f}$ is a continuous function from E^n to E^m.

2. E is any connected subset of Dom $\vec{f}$.

$\Longrightarrow$ $\vec{f}(E)$ is a connected set.

Proof:

Let us assume E is the domain of $\vec{f}$.

Suppose, then that $\vec{f}(E)$ is not a connected set. Then, there exist two disjoint sets A and B, both open relative to $\vec{f}(E)$, such that $\vec{f}(E) = A \cup B$.

By 2.4.8, the sets $\vec{f}^*(A)$ and $\vec{f}^*(B)$ are open relative to E. Also, these sets are disjoint, and $E = \vec{f}^*(A \cup B) = \vec{f}^*(A) \cup \vec{f}^*(B)$. This means that E is not connected. Thus, we have a contradiction, and must conclude that $\vec{f}(E)$ is a connected set.

2.4.10 Theorem.

1. $\vec{f}$ is a function from E^n to E^m.

2. $\vec{x}$, $\vec{y}$ ε Dom $\vec{f}$.

3. $|\vec{f}(\vec{x}) - \vec{f}(\vec{y})| \leq |\vec{x} - \vec{y}|$, for all $\vec{x}$, $\vec{y}$ ε Dom $\vec{f}$.

$\Longrightarrow$ $\vec{f}$ ε C^o in Dom $\vec{f}$.

Proof:

Suppose $\vec{y}$ ε Dom $\vec{f}$.

Let $\varepsilon > 0$, then if $\vec{x}$ ε Dom $\vec{f}$, and we take $\delta = \varepsilon$, then if $|\vec{x} - \vec{y}| < \delta = \varepsilon$, then we have $|\vec{f}(\vec{x}) - \vec{f}(\vec{y})| \leq |\vec{x} - \vec{y}| < \varepsilon$. Hence, $\vec{f}$ ε C^o at $\vec{y}$ ε Dom $\vec{f}$, and thus $\vec{f}$ ε C^o in Dom $\vec{f}$.

CHAPTER III

MATRICES

3.1 Definitions.

3.1.1 Definition. The statement that A is an $m \times n$ matrix of real numbers means A is a function with domain the set of pairs of integers $\{(i,j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ and with range in E^1, and a function value $A(i,j)$ is an entry of the matrix and will be denoted by a_{ij}, and where the matrix is described by displaying the entries in a rectangular array:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

3.1.2 Definition. The statement that two matrices A and B are equal means A and B are each $m \times n$ matrices, and $A(i,j) = B(i,j)$; i.e., their corresponding entries are equal.

Remark. We will write the matrix A in the abridged notation: $A = (a_{ij})$, $i = 1,2,3,\ldots,m$, $j = 1,2,3,\ldots,n$.

3.1.3 Definition. The statement that $A + B$ is the sum of two $m \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$, $i = 1,2,3,\ldots,m$,

$j = 1,2,3,\ldots,n$, means $A + B = (a_{ij} + b_{ij})$, $i = 1,2,3,\ldots,m$, $j = 1,2,3,\ldots,n$.

3.1.4 Definition. The statement that $r\,A$ is a matrix, where $A = (a_{ij})$, $i = 1,2,3,\ldots,m$, $j = 1,2,3,\ldots,n$, and $r \in E^1$ means $r\,A$ is the $m \times n$ matrix $[r\,A](i,j) = r\,A(i,j)$, $i = 1,2,3,\ldots,m$, $j = 1,2,3,\ldots,n$.

3.1.5 Definition. The statement that θ is the zero matrix means θ is the matrix (θ_{ij}) such that $\theta(i,j) = 0$, $i = 1,2,3,\ldots,m$, $j = 1,2,3,\ldots,n$.

3.1.6 Definition. The statement that $-A$ is the additive inverse of an $m \times n$ matrix A means $-A$ is the matrix such that $[-A](i,j) = -A(i,j)$, $i = 1,2,3,\ldots,m$, $j = 1,2,3,\ldots,n$.

Remark. We note that the vector space consisting of all $1 \times n$ matrices of numbers of E^1 is isomorphic to E^n: i.e., there is a one-to-one correspondence between the $1 \times n$ matrices and the vectors in E^n such that the operation of addition and multiplication by a number of E^1 are preserved under this correspondence.

Let $\vec{a} = (a_1, a_2, \ldots, a_n)$ correspond to $A = (a_{11} \;\; a_{12} \;\; \cdots \;\; a_{1n})$, iff, $a_j = a_{1j}$, for $j = 1,2,\ldots,n$. Then if $\vec{a}$ and $\vec{b}$ correspond to A and B, respectively, then

$\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$ corresponds to

$A + B = (a_{11}+b_{11} \quad a_{12}+b_{12} \quad \ldots \quad a_{1n}+b_{1n})$, and

$r\,\vec{a} = (ra_1, ra_2, \ldots, ra_n)$ corresponds to

$r\,A = (ra_{11} \quad ra_{12} \quad \ldots \; ra_{1n})$.

Since $1 \times n$ matrices have the characteristics of vectors in E^n, we will identify a $1 \times n$ matrix with the corresponding vector; $1 \times n$ matrices are called row vectors. In a similar manner, we can set up an isomorphism between the space of $n \times 1$ matrices and E^n. Thus an $n \times 1$ matrix may be identified with the corresponding vectors in E^n, and thus $n \times 1$ matrices are called column vectors. In general, the vector space consisting of all $m \times n$ matrices of numbers in E^1 is isompophic to E^{mn}.

3.1.7 Definition. The statement that A B is the product of an $m \times n$ matrix A and an $n \times p$ matrix B, m, n, p ε J, means AB is the $m \times p$ matrix $C = (c_{ij})$ such that $c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj}$, $i = 1,2,3,\ldots,m$, $j = 1,2,3,\ldots,p$.

Remark. If A is a 1×1 matrix, the following one-to-one correspondence follows: A corresponds to its single entry a_{11}. We can see that the operations of addition and multiplication are preserved under the correspondence between 1×1 matrices and the set of numbers of E^1. Then, we can identify a 1×1 matrix with the numbers of E^1 which is its single entry.

If A is an $1 \times n$ matrix and B is an $n \times 1$ matrix, and $\vec{a}$ and $\vec{b}$ are corresponding vectors in E^n, then the scalar product $\vec{a} \cdot \vec{b}$ corresponds to A B.

3.1.8 Definition. The statement that a real-valued function defined on a set Ω of all matrices with entries from E^1, denoted by $\| \ \|$, is a matrix norm means for all matrices A, B which belong to Ω and numbers r of E^1, then

(1) $\|A\| > 0$, if $A \neq \theta$, and $\|\theta\| = 0$;

(2) $\|r\ A\| = |r|\ \|A\|$;

(3) $\|A + B\| \leqq \|A\| + \|B\|$, where A and B are each $m \times n$ matrices; and

(4) $\|A\ B\| \leqq \|A\|\ \|B\|$, where A is an $m \times n$ matrix and B is an $n \times p$ matrix, m, n, $p \in J$.

Remark. If we identify vectors in E^n with $n \times 1$ matrices (or $1 \times n$ matrices), then the matrix norm could be the Euclidean distance:

$|\vec{x}| > 0$, if $\vec{x} \neq \vec{0}$, and $|\vec{0}| = 0$;

$|r\ \vec{x}| = |r|\ |\vec{x}|$;

$|\vec{x} + \vec{y}| \leqq |\vec{x}| + |\vec{y}|$ (Triangle Inequality); and

$|\vec{x} \cdot \vec{y}| \leqq |\vec{x}|\ |\vec{y}|$ (Schwarz Inequality).

Also, for vectors in E^n, we may define a vector (matrix) norm by the rule $\|\vec{x}\| = |x_1| + |x_2| + \ldots + |x_n|$.

We see very readily that this definition satisfies (1) - (4) of 3.1.8. For example,

$$\begin{aligned}\|\vec{x} + \vec{y}\| &= |x_1 + y_1| + |x_2 + y_2| + \ldots + |x_n + y_n| \\ &\leqq |x_1| + |y_1| + |x_2| + |y_2| + \ldots + |x_n| + |y_n| \\ &\leqq \|\vec{x}\| + \|\vec{y}\|.\end{aligned}$$

3.1.9 Definition. The statement that a matrix A is the limit of the matrix-valued function F at $\vec{x}$, written $\lim\limits_{\vec{x}} F = A$ or $\lim\limits_{\vec{y} \to \vec{x}} f(\vec{y}) = A$ means, corresponding to each $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $\vec{y} \in \text{Dom } F$ and $0 < |\vec{y} - \vec{x}| < \delta$, then $\|F(\vec{y}) - A\| < \varepsilon$.

Remark. It should be observed that if $F = (f_{ij})$, $i = 1,2,3,\ldots,m$, $j = 1,2,3,\ldots,n$, is an $m \times n$ matrix-valued function, then A is an $m \times n$ matrix.

3.2 Matrix Theorems.

3.2.1 Theorem.

1. A is an $m \times n$ matrix.
2. B is an $n \times p$ matrix.
3. C is a $p \times q$ matrix.

$$\Longrightarrow \quad A\,(B\,C) = (A\,B)\,C\ .$$

Proof:

$A\,(B\,C)$ and $(A\,B)\,C$ are each $m \times q$ matrices.

Also, for $i = 1,2,3,\ldots,m$, and $j = 1,2,3,\ldots,q$,

$$
\begin{aligned}
[A\,(B\,C)](i,j) &= \sum_{k=1}^{n} A(i,k)\ [B\,C](k,j) \\
&= \sum_{k=1}^{n} a_{ik} \sum_{r=1}^{p} b_{kr}\, c_{rj} \\
&= \sum_{k=1}^{n} \sum_{r=1}^{p} a_{ik}\, b_{kr}\, c_{rj} \\
&= \sum_{r=1}^{p} \sum_{k=1}^{n} a_{ik}\, b_{kr}\, c_{rj} \\
&= \sum_{r=1}^{p} \left[\sum_{k=1}^{n} a_{ik}\, b_{kr} \right] c_{rj} \\
&= \sum_{r=1}^{p} [A\,B](i,r)\ C(r,j) \\
&= [(A\,B)\,C](i,j)\ .
\end{aligned}
$$

Thus, $A(B\,C) = (A\,B)\,C$.

3.2.2 Theorem.

1. A is an $m \times n$ matrix.

2. B is an $n \times p$ matrix.

3. C is an $n \times p$ matrix.

$\Longrightarrow$ $A\,(B + C) = A\,B + A\,C$.

Proof:

$A\,(B + C)$ and $A\,B + A\,C$ are each $m \times p$ matrices.

For each (i,j) such that $i = 1,2,3,\ldots,m$ and $j = 1,2,3,\ldots,p$, we have

$$[A(B + C)](i,j) = \sum_{k=1}^{n} A(i,k)\ [B + C](k,j)$$

$$= \sum_{k=1}^{n} a_{ik}\,(b_{kj} + c_{kj})$$

$$= \sum_{k=1}^{n} (a_{ik}\,b_{kj} + a_{ik}\,c_{kj})$$

$$= \sum_{k=1}^{n} a_{ik}\,b_{kj} + \sum_{k=1}^{n} a_{ik}\,c_{kj}$$

$$= [A\,B + A\,C](i,j)\ .$$

Thus, $A\,(B + C) = A\,B + A\,C$.

3.2.3 Theorem.

1. A is an $m \times n$ matrix.

2. B is an $m \times n$ matrix.

3. C is an $n \times p$ matrix.

$\Longrightarrow$ $(A + B)\,C = A\,C + B\,C$.

Proof:

$(A + B)\,C$ and $A\,C + B\,C$ are each $m \times p$ matrices.

For each (i,j) such that $i = 1,2,3,\ldots,m$ and $j = 1,2,3,\ldots,p$ we have

$$[(A + B)\ C](i,j) = \sum_{k=1}^{n} [A + B](i,k)\ C(k,j)$$

$$= \sum_{k=1}^{n} (a_{ik} + b_{ik})\ c_{kj}$$

$$= \sum_{k=1}^{n} (a_{ik}\ c_{kj} + b_{ik}\ c_{kj})$$

$$= \sum_{k=1}^{n} a_{ik}\ c_{kj} + \sum_{k=1}^{n} b_{ik}\ c_{kj}$$

$$= [A\ C + B\ C](i,j)\ .$$

Hence, $(A + B)\ C = A\ C + B\ C$.

3.2.4 Theorem.

1. A and B are each $m \times n$ matrices.
2. C and D are each $n \times p$ matrices.

$\Longrightarrow \quad (A + B)\ (C + D) = A\ C + B\ C + A\ D + B\ D$.

Proof:

Using the two preceding distributive laws, we have

$$(A + B)\ (C + D) = (A + B)\ C + (A + B)\ D$$

$$= A\ C + B\ C + A\ D + B\ D\ .$$

Hence, $(A + B)\ (C + D) = A\ C + B\ C + A\ D + B\ D$.

3.2.5 Theorem.

The real-valued function defined on the set Ω of all matrices with real entries by the rule $\|A\| = \left[\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^{2}\right]^{\frac{1}{2}}$, where A is an $m \times n$ matrix, is a matrix norm.

Proof:

If we identify the $m \times n$ matrix $A \in \Omega$ with a vector in E^{mn}, then $\|A\|$ is just the Euclidean length of this vector and properties (1), (2), and (3) are fundamental properties of length of a vector. To prove property (4) holds, if $C = A\,B$, where $A = (a_{ij})$, $B = (b_{jk})$, then

$$\|A\|^2\,\|B\|^2 = \left[\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^{\,2}\right]\left[\sum_{k=1}^{n}\sum_{r=1}^{p} b_{kr}^{\,2}\right]$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{r=1}^{p} a_{ij}^{\,2}\, b_{kr}^{\,2}\,,$$

and $\|A\,B\|^2 = \|C\|^2 = \sum_{i=1}^{m}\sum_{r=1}^{p} c_{ir}^{\,2}$

$$= \sum_{i=1}^{m}\sum_{r=1}^{p}\left[\sum_{j=1}^{n} a_{ij}\, b_{jr}\right]^2$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{r=1}^{p} a_{ij}\, b_{jr}\, a_{ik}\, b_{kr}\,.$$

Then, $\|A\|^2\,\|B\|^2 - \|A\,B\|^2 = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{r=1}^{p} a_{ij}^{\,2}\, b_{kr}^{\,2}$

$$- \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{r=1}^{p} a_{ij}\, b_{jr}\, a_{ik}\, b_{kr}$$

$$= \tfrac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{r=1}^{p} (a_{ij}\, b_{kr} - a_{ik}\, b_{jr})^2$$

$$\geqq 0.$$

Hence, $\|A\|^2\,\|B\|^2 \geqq \|A\,B\|^2$ and $\|A\,B\| \leqq \|A\|\,\|B\|$, so property (4) holds.

Remark. The matrix norm in 3.2.5 is called the Euclidean Matrix Norm.

3.2.6 Theorem.

1. A is an $m \times n$ matrix, $m, n \in J$.

2. F is an $m \times n$ matrix-valued function of a vector.

3. $\vec{x}$ is a point of accumulation of Dom F.

$\Longrightarrow$ $\lim_{\vec{x}} F = A$, iff, $\lim_{\vec{x}} f_{ij} = a_{ij}$ for each

$i = 1,2,3,\ldots,m$, and $j = 1,2,3,\ldots,n$.

Proof:

If $\lim_{\vec{x}} F = A$, then for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|F(\vec{y}) - A\| = \left[\sum_{i=1}^{m} \sum_{j=1}^{n} (f_{ij}(\vec{x}) - a_{ij})^2\right]^{\frac{1}{2}} < \varepsilon, \text{ whenever } \vec{x} \in \text{Dom } F$$

and $0 < |\vec{y} - \vec{x}| < \delta$.

Hence, $|f_{ij}(\vec{y}) - a_{ij}| < \varepsilon$, for each $i = 1,2,3,\ldots,m$, $j = 1,2,3,\ldots,n$, if $0 < |\vec{y} - \vec{x}| < \delta$.

This shows that

$\lim_{\vec{y}\to\vec{x}} f_{ij}(\vec{y}) = a_{ij}$, where $i = 1,2,3,\ldots,m$ and $j = 1,2,3,\ldots,n$.

If $\lim_{\vec{y}\to\vec{x}} f_{ij}(\vec{y}) = a_{ij}$, $i = 1,2,3,\ldots,m$, and $j = 1,2,3,\ldots,n$, then for $\varepsilon > 0$, there exists $\delta_{ij} > 0$ such that $|f_{ij}(\vec{y}) - a_{ij}| < \frac{\varepsilon}{\sqrt{mn}}$, $i = 1,2,3,\ldots,m$, $j = 1,2,3,\ldots,n$, whenever $\vec{y} \in$ Dom F and $0 < |\vec{y} - \vec{x}| < \delta_{ij}$.

Let $\delta = \min\{\delta_{ij}\}$, $i = 1,2,3,\ldots,m$ and $j = 1,2,3,\ldots,n$.

Then, whenever $\vec{y} \in$ Dom F, and $0 < |\vec{y} - \vec{x}| < \delta$, we have

$$\|F(\vec{y}) - A\| = \left[\sum_{i=1}^{m} \sum_{j=1}^{n} (f_{ij}(\vec{y}) - a_{ij})^2\right]^{\frac{1}{2}}$$

$$< \left[\sum_{i=1}^{m} \sum_{j=1}^{n} \frac{\varepsilon^2}{mn}\right]^{\frac{1}{2}}$$

$$< \varepsilon.$$

So, $\lim\limits_{\vec{y}\to\vec{x}} F(\vec{y}) = A.$

CHAPTER IV

THE DIFFERENTIAL AND DERIVATIVE

4.1 Definitions and Theorems.

4.1.1 Definition. The statement that a function $\vec{f}$ from E^n to E^m is differentiable at the point $\vec{x}$ means $\vec{f}$ is defined in a neighborhood $V(\vec{x};r)$ of $\vec{x}$ and there exists a matrix A (independent of $\vec{h}$) such that for any point $\vec{x} + \vec{h}$ of $V^*(\vec{x};r)$,

(1) $\quad \vec{f}(\vec{x} + \vec{h}) = \vec{f}(\vec{x}) + A\,\vec{h} + \Phi(\vec{x};\vec{h})\,\vec{h}$, where $\lim_{\vec{h}\to\vec{0}} \Phi(\vec{x};\vec{h}) = \theta$.

The term $A\,\vec{h}$ is called the differential of f at $\vec{x}$ and $\vec{h}$ and is denoted by $d\,\vec{f}(\vec{x};\vec{h})$. The matrix A is called the derivative of $\vec{f}$ at $\vec{x}$ and is denoted by $D\,\vec{f}(\vec{x})$.

In (1) all the vectors are column vectors, A and $\Phi(\vec{x};\vec{h})$ are $m \times n$ matrices, and θ is the $m \times n$ zero matrix.

Equation (1) can be written

$$\begin{pmatrix} f_1(\vec{x}+\vec{h}) \\ f_2(\vec{x}+\vec{h}) \\ \vdots \\ f_m(\vec{x}+\vec{h}) \end{pmatrix} = \begin{pmatrix} f_1(\vec{x}) \\ f_2(\vec{x}) \\ \vdots \\ f_m(\vec{x}) \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix}$$

$$+ \begin{pmatrix} \phi_{11}(\vec{x};\vec{h}) & \phi_{12}(\vec{x};\vec{h}) & \cdots & \phi_{1n}(\vec{x};\vec{h}) \\ \phi_{21}(\vec{x};\vec{h}) & \phi_{22}(\vec{x};\vec{h}) & \cdots & \phi_{2n}(\vec{x};\vec{h}) \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ \phi_{m1}(\vec{x};\vec{h}) & \phi_{m2}(\vec{x};\vec{h}) & \cdots & \phi_{mn}(\vec{x};\vec{h}) \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix}$$

$$= \begin{pmatrix} f_1(\vec{x}) \\ f_2(\vec{x}) \\ \vdots \\ f_m(\vec{x}) \end{pmatrix} + \begin{pmatrix} \vec{a}_1 \cdot \vec{h} \\ \vec{a}_2 \cdot \vec{h} \\ \vdots \\ \vec{a}_m \cdot \vec{h} \end{pmatrix} + \begin{pmatrix} \vec{\phi}_1(\vec{x};\vec{h}) \cdot \vec{h} \\ \vec{\phi}_2(\vec{x};\vec{h}) \cdot \vec{h} \\ \vdots \\ \vec{\phi}_m(\vec{x};\vec{h}) \cdot \vec{h} \end{pmatrix}$$

$$= \begin{pmatrix} f_1(\vec{x}) + \vec{a}_1 \cdot \vec{h} + \vec{\phi}_1(\vec{x};\vec{h}) \cdot \vec{h} \\ f_2(\vec{x}) + \vec{a}_2 \cdot \vec{h} + \vec{\phi}_2(\vec{x};\vec{h}) \cdot \vec{h} \\ \vdots \\ f_m(\vec{x}) + \vec{a}_m \cdot \vec{h} + \vec{\phi}_m(\vec{x};\vec{h}) \cdot \vec{h} \end{pmatrix} .$$

Thus, equation (1) is equivalent to

(2) $$f_k(\vec{x} + \vec{h}) = f_k(\vec{x}) + \vec{a}_k \cdot \vec{h} + \phi_k(\vec{x};\vec{h}) \cdot \vec{h} \ , \ k = 1,2,3,\ldots,m,$$

where $\vec{a}_k = (a_{k1}, a_{k2}, \ . \ . \ . \ , a_{kn})$, and

$\vec{\phi}_k(\vec{x};\vec{h}) = (\phi_{k1}(\vec{x};\vec{h}) \quad \phi_{k2}(\vec{x};\vec{h}) \quad . \ . \ . \quad \phi_{kn}(\vec{x};\vec{h}))$, $k = 1,2,3,\ldots,m$.

If $\lim_{\vec{h}\to\vec{0}} \Phi(\vec{x};\vec{h}) = \theta_{mn}$, then for each k, $k = 1,2,3,\ldots,m$,

$\lim_{\vec{h}\to\vec{0}} \vec{\phi}_k(\vec{x};\vec{h}) = \vec{0}$.

Thus, if $\vec{f}$ is differentiable at $\vec{x}$, then each of the component functions f_k is differentiable at $\vec{x}$, and similarly, if

$\lim_{\vec{h}\to\vec{0}} \vec{\phi}_k(\vec{x};\vec{h}) = \vec{0}$ for each $k = 1,2,3,\ldots,m$, then $\lim_{\vec{h}\to\vec{0}} \Phi(\vec{x};\vec{h}) = \theta_{mn}$.

This shows that $\vec{f}$ is differentiable at $\vec{x}$ if each component function f_k is differentiable at $\vec{x}$. Thus, we have:

4.1.2 Theorem.

1. $\vec{f} = (f_1,f_2,\ldots,f_m)$ is a function from E^n to E^m.

2. $\vec{f}$ is differentiable at $\vec{x}$.

$\Longrightarrow$ Each component function f_k, $k = 1,2,3,\ldots,m$, is differentiable at $\vec{x}$.

4.1.3 Theorem.

1. $\vec{f} = (f_1, f_2, \ldots, f_m)$ is a function from E^n to E^m.

2. Each component function $f_k(\vec{x})$ is differentiable at $\vec{x}$.

$\Longrightarrow$ $\vec{f}$ is differentiable at $\vec{x}$.

Remark: If $\vec{f}$ is differentiable at $\vec{x}$, then each component function $f_k(\vec{x})$ is differentiable at $\vec{x}$ and the vector $\vec{a}_k$ is $D\, f_k(\vec{x})$, $k = 1,2,3,\ldots,m$.

Hence,

$$(3) \qquad D\,\vec{f}(\vec{x}) = \begin{pmatrix} D_1\, f_1(\vec{x}) & D_2\, f_1(\vec{x}) & \cdots & D_n\, f_1(\vec{x}) \\ D_1\, f_2(\vec{x}) & D_2\, f_2(\vec{x}) & \cdots & D_n\, f_2(\vec{x}) \\ \vdots & \vdots & \vdots\,\vdots\,\vdots & \vdots \\ D_1\, f_m(\vec{x}) & D_2\, f_m(\vec{x}) & \cdots & D_n\, f_m(\vec{x}) \end{pmatrix}$$

and

$$d\,\vec{f}(\vec{x};\vec{h}) = D\,\vec{f}(\vec{x})\,\vec{h} = \begin{pmatrix} \vec{D}\, f_1(\vec{x}) \cdot \vec{h} \\ \vec{D}\, f_2(\vec{x}) \cdot \vec{h} \\ \vdots \\ \vec{D}\, f_m(\vec{x}) \cdot \vec{h} \end{pmatrix} = \begin{pmatrix} d\, f_1(\vec{x};\vec{h}) \\ d\, f_2(\vec{x};\vec{h}) \\ \vdots \\ d\, f_m(\vec{x};\vec{h}) \end{pmatrix}.$$

The matrix-valued function defined by

$$\begin{pmatrix} D_1\, f_1 & D_2\, f_2 & \cdots & D_n\, f_1 \\ D_1\, f_2 & D_2\, f_2 & \cdots & D_n\, f_2 \\ \vdots & \vdots & \vdots\,\vdots\,\vdots & \vdots \\ D_1\, f_m & D_2\, f_m & \cdots & D_n\, f_m \end{pmatrix}$$

is known as the Jacobian matrix of the function $\vec{f}$ from E^n to E^m. We have shown, then, that if $\vec{f}$ is differentiable at $\vec{x}$, then the derivative of $\vec{f}$ at $\vec{x}$ is the value of the Jacobian matrix of $\vec{f}$ at $\vec{x}$.

Let us suppose that F is a matrix-valued function defined on an open set E in E^n. Then we have:

4.1.4 Definition. The statement that a matrix-valued function F, defined on an open set E on E^n, is continuous at the point $\vec{x}_o$ of E means $\lim_{\vec{x}\to\vec{x}_o} F(\vec{x}) = F(\vec{x}_o)$.

We see, then, that F is continuous at $\vec{x}_o$ in E, iff, each entry f_{ij} is continuous at $\vec{x}_o$. Then, from 4.1.3, we see that the following theorem is obtained.

4.1.5 Theorem.

1. $\vec{f}$ is a function from E^n to E^m.
2. The Jacobian matrix of $\vec{f}$ is continuous on an open set E.

$\Longrightarrow$ $\vec{f}$ is differentiable on E; i.e., corresponding to each $\vec{x} \in E$, there is a neighborhood $V(\vec{x};r) \subset E$ such that for any $\vec{x} + \vec{h} \in V^*(\vec{x};r)$,

$$\vec{f}(\vec{x} + \vec{h}) = \vec{f}(\vec{x}) + D\,\vec{f}(\vec{x})\vec{h} + \Phi(\vec{x};\vec{h})\,\vec{h},$$

where $\lim_{\vec{h}\to\vec{0}} \Phi(\vec{x};\vec{h}) = \theta$. Moreover, for any closed set $F \subset E$ and any $\varepsilon > 0$, there exists a $\delta > 0$ such that $\|\Phi(\vec{x};\vec{h})\| < \varepsilon$ whenever $\vec{x}$, $\vec{x} + \vec{h} \in F$ and $0 < |\vec{h}| < \delta$.

Proof:

Let $\vec{x} \in E$, and let $V(\vec{x};r)$ be a neighborhood of $\vec{x}$ contained in E. Take $\vec{h}$ such that $|\vec{h}| < r$. We know, from the theory of real-valued functions of a vector, that if f is a function from E^n to E^1, and if $D_{\vec{u}}\, f$ exists on an open set containing the closed line segment from $\vec{x}$ to $\vec{x} + h\vec{u}$, where $\vec{u}$ is a unit vector, then there exists a number $\theta \in (0,1)$ such that $f(\vec{x} + h\vec{u}) - f(\vec{x}) = h\, D_{\vec{u}}\, f(\vec{x} + \theta h\vec{u})$.

Applying this mean value theorem for $i = 1,2,3,\ldots,m$, we have

$$\begin{aligned}
f_i(\vec{x} + \vec{h}) - f_i(\vec{x}) &= f_i(\vec{x} + h_1\vec{u}_1 + \ldots + h_n\vec{u}_n) - f_i(\vec{x}) \\
&= f_i(\vec{x} + h_1\vec{u}_1 + \ldots + h_n\vec{u}_n) \\
&\qquad - f_i(\vec{x} + h_2\vec{u}_2 + \ldots + h_n\vec{u}_n) \\
&\qquad + f_i(\vec{x} + h_2\vec{u}_2 + \ldots + h_n\vec{u}_n) \\
&\qquad - f_i(\vec{x} + h_3\vec{u}_3 + \ldots + h_n\vec{u}_n) \\
&\qquad + \ldots + f_i(\vec{x} + h_n\vec{u}_n) + f_i(\vec{x}) \\
&= h_1\, D_1\, f_i(\vec{x} + \theta_{i1}h_1\vec{u}_1 + h_2\vec{u}_2 + \ldots + h_n\vec{u}_n) \\
&\qquad + h_2\, D_2\, f_i(\vec{x} + \theta_{i2}h_2\vec{u}_2 + h_3\vec{u}_3 + \ldots + h_n\vec{u}_n) \\
&\qquad + \ldots + h_n\, D_n\, f_i(\vec{x} + \theta_{in}h_n\vec{u}_n) \ ,
\end{aligned}$$

where $\theta_{ij} \in (0,1)$, $i = 1,2,3,\ldots,m$, $j = 1,2,3,\ldots,n$, and $\vec{u}_j$ is the unit vector with j^{th} component 1 and all other components 0.

Since the partial derivatives $D_j\, f_i$ are continuous on E, then

$D_j\, f_i(\vec{x} + \theta_{ij}h_j\vec{u}_j + \ldots + h_n\vec{u}_n) = D_j\, f_i(\vec{x}) + \phi_{ij}(\vec{x};\vec{h})$,

where $\lim_{\vec{h}\to\vec{0}} \phi_{ij}(\vec{x};\vec{h}) = 0$.

Then,

$$\begin{aligned}
f_i(\vec{x} + \vec{h}) - f_i(\vec{x}) &= h_1\, \{D_1\, f_i(\vec{x}) + \phi_{i1}(\vec{x};\vec{h})\} + \ldots \\
&\qquad + h_n\, \{D_n\, f_i(\vec{x}) + \phi_{in}(\vec{x};\vec{h})\} \\
&= \vec{D}\, f_i(\vec{x}) \cdot \vec{h} + \vec{\phi}_i(\vec{x};\vec{h}) \cdot \vec{h},
\end{aligned}$$

and

$$\vec{f}(\vec{x} + \vec{h}) - \vec{f}(\vec{x}) = D\, \vec{f}(\vec{x})\, \vec{h} + \Phi(\vec{x};\vec{h})\, \vec{h} \ .$$

Since each partial derivative $D_j f_i$ is continuous on any closed set $F \subset E$, then it is uniformly continuous on F. Hence, corresponding to $\varepsilon > 0$, there is a $\delta_{ij} > 0$ such that $\vec{x}$, $\vec{x} + \vec{h} \varepsilon F$ and $|\vec{h}| < \delta_{ij}$ imply that

$$|D_j f_i(\vec{x} + \theta_{ij}h_j\vec{u}_j + \ldots + h_n\vec{u}_n) - D_jf_i(\vec{x})| = |\phi_{ij}(\vec{x};\vec{h})| < \frac{\varepsilon}{\sqrt{mn}}.$$

Let $\delta = \min \{\delta_{ij}\}$. Then $\vec{x}$, $\vec{x} + \vec{h} \varepsilon F$ and $0 < |\vec{h}| < \delta$ imply

$$\|\Phi(\vec{x};\vec{h})\| = \left[\sum_{i=1}^{m} \sum_{j=1}^{n} \phi_{ij}{}^2(\vec{x};\vec{h})\right]^{\frac{1}{2}} < \left[\sum_{i=1}^{m} \sum_{j=1}^{n} \frac{\varepsilon^2}{mn}\right]^{\frac{1}{2}} = \varepsilon, \text{ and thus,}$$

$\lim_{\vec{h}\to\vec{0}} \Phi(\vec{x};\vec{h}) = \theta$, and our proof is complete.

4.1.6 Definition. The statement that a function $\vec{f}$ from E^n to E^m is of class C^k on an open set E, written $\vec{f} \varepsilon C^k$ on E, means each of the components f_i, $i = 1,2,3,\ldots,m$, is of class C^k on E; i.e., all the k^{th} order partial derivatives of f_i are continuous on E for each $i = 1,2,3,\ldots,m$.

4.1.7 Theorem.

If $\vec{f}$ is differentiable at $\vec{x}$, then $\vec{f}$ is continuous at $\vec{x}$.

Proof:

If $\vec{f}$ is differentiable at $\vec{x}$, then for any point $\vec{x} + \vec{h}$ in some deleted neighborhood of $\vec{x}$,

$\vec{f}(\vec{x} + \vec{h}) = \vec{f}(\vec{x}) + A\,\vec{h} + \Phi(\vec{x};\vec{h})\,\vec{h}$, where $\lim_{\vec{h}\to\vec{0}} \Phi(\vec{x};\vec{h}) = \theta$.

Thus, $\lim_{\vec{h}\to\vec{0}} \vec{f}(\vec{x} + \vec{h}) = \vec{f}(\vec{x})$, and hence, $\vec{f}$ is continuous at $\vec{x}$.

4.1.8 Theorem.

1. $\vec{f}$ is differentiable at $\vec{x}$.

2. $\vec{g}$ is differentiable at $\vec{x}$.

$\Longrightarrow$ $\vec{f} + \vec{g}$ is differentiable at $\vec{x}$ and

$D\ (\vec{f} + \vec{g})(\vec{x}) = D\ \vec{f}(\vec{x}) + D\ \vec{g}(\vec{x})$, and

$d\ (\vec{f} + \vec{g})(\vec{x};\vec{h}) = d\ \vec{f}(\vec{x};\vec{h}) + d\ \vec{g}(\vec{x};\vec{h})$.

Proof:

Since $\vec{f}$ and $\vec{g}$ are each differentiable at $\vec{x}$, there exists some neighborhood $V(\vec{x};r)$ of $\vec{x}$ such that for any point $\vec{x} + \vec{h}$ in $V^*(\vec{x};r)$,

$\vec{f}(\vec{x} + \vec{h}) = \vec{f}(\vec{x}) + D\ \vec{f}(\vec{x})\ \vec{h} + \Phi(\vec{x};\vec{h})\ \vec{h}$, where $\lim_{\vec{h}\to\vec{0}} \Phi(\vec{x};\vec{h}) = \theta$, and

$\vec{g}(\vec{x} + \vec{h}) = \vec{g}(\vec{x}) + D\ \vec{g}(\vec{x})\ \vec{h} + \Psi(\vec{x};\vec{h})\ \vec{h}$, where $\lim_{\vec{h}\to\vec{0}} \Psi(\vec{x};\vec{h}) = \theta$.

Then, if $\vec{x} + \vec{h} \in V^*(\vec{x};r)$,

$$\begin{aligned}(\vec{f} + \vec{g})(\vec{x} + \vec{h}) &= \vec{f}(\vec{x}) + D\ \vec{f}(\vec{x})\ \vec{h} + \Phi(\vec{x};\vec{h})\ \vec{h} + \vec{g}(\vec{x}) + D\ \vec{g}(\vec{x})\ \vec{h} \\ &\qquad + \Psi(\vec{x};\vec{h})\ \vec{h} \\ &= (\vec{f} + \vec{g})(\vec{x}) + \{D\ \vec{f}(\vec{x}) + D\ \vec{g}(\vec{x})\}\ \vec{h} \\ &\qquad + \{\Phi(\vec{x};\vec{h}) + \Psi(\vec{x};\vec{h})\}\ \vec{h}\ .\end{aligned}$$

Since $\lim_{\vec{h}\to\vec{0}} \{\Phi(\vec{x};\vec{h}) + \Psi(\vec{x};\vec{h})\} = \theta$, then $\vec{f} + \vec{g}$ is differentiable at $\vec{x}$,

and

$D\ (\vec{f} + \vec{g})(\vec{x}) = D\ \vec{f}(\vec{x}) + D\ \vec{g}(\vec{x})$, and

$d\ (\vec{f} + \vec{g})(\vec{x}) = d\ \vec{f}(\vec{x};\vec{h}) + d\ \vec{g}(\vec{x};\vec{h})$.

4.1.9 Theorem.

1. $\vec{f}$ is a function from E^n to E^m.

2. $\vec{g}$ is a function from E^n to E^m.

3. $\vec{f}$ is differentiable at $\vec{x}$.

4. $\vec{g}$ is differentiable at $\vec{x}$.

$\Longrightarrow$ $\vec{f} \cdot \vec{g}$ is differentiable at $\vec{x}$, and

$\vec{D}\ (\vec{f} \cdot \vec{g})(\vec{x}) = \vec{f}(\vec{x}) \cdot D\ \vec{g}(\vec{x}) + \vec{g}(\vec{x}) \cdot D\ \vec{f}(\vec{x})$, and

$d\ (\vec{f} \cdot \vec{g})(\vec{x}) = \vec{f}(\vec{x}) \cdot d\ \vec{g}(\vec{x};\vec{h}) + \vec{g}(\vec{x}) \cdot d\ \vec{f}(\vec{x};\vec{h})$.

Proof:

Let $\vec{f} = (f_1(\vec{x}), f_2(\vec{x}), \ldots, f_m(\vec{x}))$, and

$\vec{g} = (g_1(\vec{x}), g_2(\vec{x}), \ldots, g_m(\vec{x}))$.

Then $\vec{f} \cdot \vec{g} = f_1\ g_1 + f_2\ g_2 + \ldots + f_m\ g_m$.

$$\begin{aligned}
\vec{D}\ (\vec{f} \cdot \vec{g})(\vec{x}) &= \vec{D}\ (f_1\ g_1 + f_2\ g_2 + \ldots + f_m\ g_m) \\
&= f_1\ \vec{D}\ g_1 + g_1\ \vec{D}\ f_1 + f_2\ \vec{D}\ g_2 + g_2\ \vec{D}\ f_2 + \ldots \\
&\qquad + f_m\ \vec{D}\ g_m + g_m\ \vec{D}\ f_m \\
&= (f_1\ \vec{D}\ g_1 + f_2\ \vec{D}\ g_2 + \ldots + f_m\ \vec{D}\ g_m) \\
&\qquad + (g_1\ \vec{D}\ f_1 + g_2\ \vec{D}\ f_2 + \ldots + g_m\ \vec{D}\ f_m) \\
&= \vec{f} \cdot D\ \vec{g} + \vec{g} \cdot D\ \vec{f}\ \text{, and}
\end{aligned}$$

$d\ (\vec{f} \cdot \vec{g})(\vec{x}) = \vec{f}(\vec{x}) \cdot d\ \vec{g}(\vec{x};\vec{h}) + \vec{g}(\vec{x}) \cdot d\ \vec{f}(\vec{x};\vec{h})$.

4.1.10 Theorem.

1. $\vec{f}$ is a function from E^n to E^3.
2. $\vec{g}$ is a function from E^n to E^3.
3. $\vec{f}$ is differentiable at $\vec{x}$.
4. $\vec{g}$ is differentiable at $\vec{x}$.

$\Longrightarrow$ $\vec{f} \times \vec{g}$ is differentiable at $\vec{x}$, and

$D\ (\vec{f} \times \vec{g})(\vec{x}) = \vec{f}(\vec{x}) \times D\ \vec{g}(\vec{x}) + [D\ \vec{f}(\vec{x})] \times \vec{g}(\vec{x})$, and

$d\ (\vec{f} + \vec{g})(\vec{x}) = \vec{f}(\vec{x}) \times d\ \vec{g}(\vec{x};\vec{h}) + [d\ \vec{f}(\vec{x};\vec{h})] \times \vec{g}(\vec{x})$.

Proof:

Let $\vec{f}(\vec{x}) = (f_1(\vec{x}), f_2(\vec{x}), f_3(\vec{x}))$, and

$\vec{g}(\vec{x}) = (g_1(\vec{x}), g_2(\vec{x}), g_3(\vec{x}))$.

Then $\vec{f} \times \vec{g} = \begin{vmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \end{vmatrix}$, and

$$D\ (\vec{f} \times \vec{g})(\vec{x}) = D \begin{vmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 \\ f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \end{vmatrix} + \begin{vmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ \vec{D}\ f_1 & \vec{D}\ f_2 & \vec{D}\ f_3 \\ g_1 & g_2 & g_3 \end{vmatrix}$$

$$+ \begin{vmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ f_1 & f_2 & f_3 \\ \vec{D}\ g_1 & \vec{D}\ g_2 & \vec{D}\ g_3 \end{vmatrix}$$

$$= \begin{vmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ \vec{D}\ f_1 & \vec{D}\ f_2 & \vec{D}\ f_3 \\ g_1 & g_2 & g_3 \end{vmatrix} + \begin{vmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ f_1 & f_2 & f_3 \\ \vec{D}\ g_1 & \vec{D}\ g_2 & \vec{D}\ g_3 \end{vmatrix}$$

$$= D\ \vec{f} \times \vec{g} + \vec{f} \times D\ \vec{g}$$

$$= \vec{f}(\vec{x}) \times D\ \vec{g}(\vec{x}) + [D\ \vec{f}(\vec{x})] \times \vec{g}(\vec{x}).$$

Then, $d\ (\vec{f} \times \vec{g})(\vec{x}) = \vec{f}(\vec{x}) \times d\ \vec{g}(\vec{x};\vec{h}) + [d\ \vec{f}(\vec{x};\vec{h})] \times \vec{g}(\vec{x})$.

4.1.11 **Theorem.**

1. $\vec{f}$ is a function from E^n to E^m.
2. u is a function from E^n to E^1.
3. $\vec{f}$ is differentiable at $\vec{x}$.
4. u is differentiable at $\vec{x}$.

$\Longrightarrow$ $u\ \vec{f}$ is differentiable at $\vec{x}$, and

$D\ (u\ \vec{f})(\vec{x}) = u(\vec{x})\ D\ \vec{f}(\vec{x}) + [\vec{D}\ u(\vec{x})]\ \vec{f}(\vec{x})$, and

$d\ (u\ \vec{f})(\vec{x}) = u(\vec{x})\ d\ \vec{f}(\vec{x};\vec{h}) + [d\ u(\vec{x};\vec{h})]\ \vec{f}(\vec{x})$.

Proof:

Let $\vec{f} = (f_1, f_2, \ldots, f_m)$, then

$u\,\vec{f} = (uf_1, uf_2, \ldots, uf_m)$, and

$$
\begin{aligned}
D\,(u\,\vec{f})(\vec{x}) &= (\vec{D}\,u\,f_1,\ \vec{D}\,u\,f_2,\ \ldots,\ \vec{D}\,u\,f_m) \\
&= (u\,\vec{D}\,f_1 + f_1\,\vec{D}\,u,\ u\,\vec{D}\,f_2 + f_2\,\vec{D}\,u, \ldots, u\,\vec{D}\,f_m + f_m\,\vec{D}\,u) \\
&= (u\,\vec{D}\,f_1,\ u\,\vec{D}\,f_2,\ \ldots,\ u\,\vec{D}\,f_m) \\
&\qquad + (f_1\,\vec{D}\,u,\ f_2\,\vec{D}\,u,\ \ldots,\ f_m\,\vec{D}\,u) \\
&= u\,(\vec{D}\,f_1,\ \vec{D}\,f_2,\ \ldots,\ \vec{D}\,f_m) + (\vec{D}\,u)\,(f_1,\ f_2,\ \ldots,\ f_m) \\
&= u(\vec{x})\ D\,\vec{f}(\vec{x}) + [\vec{D}\,u(\vec{x})]\ \vec{f}(\vec{x}).
\end{aligned}
$$

4.1.12 Theorem.

1. $\vec{f}(\vec{x}) = \vec{c}$ is a constant function from E^n to E^m.

$\Longrightarrow \quad D\,\vec{c} = \theta_{mn}$.

Proof:

Let $\vec{c} = (c_1,\ c_2,\ \ldots,\ c_m)$, where the c_k, $k = 1,2,3,\ldots,m$, are constants.

$$
\text{Now, } D\,\vec{f}(\vec{x}) = D\,\vec{c} = \begin{pmatrix} D_1\,c_1 & D_2\,c_1 & \cdots & D_n\,c_1 \\ D_1\,c_2 & D_2\,c_2 & \cdots & D_n\,c_2 \\ \vdots & \vdots & \vdots\,\vdots\,\vdots & \vdots \\ D_1\,c_m & D_2\,c_m & \cdots & D_n\,c_m \end{pmatrix}
$$

$$
= \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots\,\vdots\,\vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}
$$

$$
= \theta_{mn}\ .
$$

4.2 The Chain Rule.

We will consider the composition of vector-valued functions of a vector.

4.2.1 Definition. The statement that $\vec{f} \circ \vec{g}$ is the composition of $\vec{f}$ with $\vec{g}$, where $\vec{g}$ is a function from E^n to E^m and $\vec{f}$ is a function from E^m to E^p means $\vec{f} \circ \vec{g}$ is the function from E^n to E^p with rule of correspondence $(\vec{f} \circ \vec{g})(\vec{x}) = \vec{f}(\vec{g}(\vec{x}))$ and with domain $\text{Dom } (\vec{f} \circ \vec{g}) = \{\vec{x} \mid \vec{x} \in \text{Dom } \vec{g}, \vec{g}(\vec{x}) \in \text{Dom } \vec{f}\}$. If $\vec{f} = (f_1, f_2, \ldots, f_p)$, then $\vec{f} \circ \vec{g} = (f_1 \circ \vec{g}, f_2 \circ \vec{g}, \ . \ . \ . \ , f_p \circ \vec{g})$

4.2.2 Theorem.

1. $\vec{g}$ is a function from E^n to E^m.
2. $\lim\limits_{\vec{a}} \vec{g} = \vec{b}$.
3. $\vec{f}$ is a function from E^m to E^p.
4. $\vec{f} \in C^0$ at $\vec{b}$.
5. $\vec{a}$ is an accumulation point of Dom $(\vec{f} \circ \vec{g})$.

$\Longrightarrow \quad \lim\limits_{\vec{a}} \vec{f} \circ \vec{g} = \vec{f}(\vec{b})$.

Proof:

Let $\varepsilon > 0$. Since $\vec{f} \in C^0$ at $\vec{b}$, there exists a number $r > 0$ such that $|\vec{f}(\vec{y}) - \vec{f}(\vec{b})| < \varepsilon$ whenever $\vec{y} \in \text{Dom } \vec{f}$ and $|\vec{y} - \vec{b}| < r$.

Since $\lim\limits_{\vec{a}} \vec{g} = \vec{b}$, there exists a number $\delta > 0$ such that $|\vec{g}(\vec{x}) - \vec{b}| < r$ whenever $\vec{x} \in \text{Dom } \vec{f}$ and $0 < |\vec{x} - \vec{a}| < \delta$.

If $\vec{x} \in \text{Dom } (\vec{f} \circ \vec{g})$ and $0 < |\vec{x} - \vec{a}| < \delta$, then $|\vec{g}(\vec{x}) - \vec{b}| < r$, and $|\vec{f}(\vec{g}(\vec{x})) - \vec{f}(\vec{b})| < \varepsilon$.

Thus, $\lim\limits_{\vec{a}} \vec{f} \circ \vec{g} = \vec{f}(\vec{b})$.

4.2.3 Theorem.

1. $\vec{g}$ is a function from E^n to E^m.

2. $\vec{g} \in C^0$ at $\vec{a}$.

3. $\vec{f}$ is a function from E^m to E^p.

4. $\vec{f} \in C^0$ at $\vec{g}(\vec{a})$.

$\Longrightarrow$ $\vec{f} \circ \vec{g} \in C^0$ at $\vec{a}$.

Proof:

If $\vec{a}$ is not an accumulation point of Dom $\vec{f} \circ \vec{g}$, then $\vec{f} \circ \vec{g}$ is continuous at $\vec{a}$.

If $\vec{a}$ is an accumulation point of Dom $\vec{f} \circ \vec{g}$, then, since Dom $\vec{f} \circ \vec{g} \subset$ Dom $\vec{g}$, $\vec{a}$ must be an accumulation point of Dom $\vec{g}$ and $\lim_{\vec{a}} \vec{g} = \vec{g}(\vec{a})$.

Then, by 4.2.2, $\lim_{\vec{a}} \vec{f} \circ \vec{g} = \vec{f}(\vec{g}(\vec{a})) = (\vec{f} \circ \vec{g})(\vec{a})$, and, hence, $\vec{f} \circ \vec{g} \in C^0$ at $\vec{a}$.

Now, we will state the Chain Rule for differentiating the composition of functions.

4.2.4 Theorem.

1. $\vec{g}$ is a function from E^n to E^m.

2. $\vec{g}$ is differentiable on an open set E.

3. $\vec{f}$ is a function from E^m to E^p.

4. $\vec{f}$ is differentiable on an open set containing g(E).

$\Longrightarrow$ $\vec{f} \circ \vec{g}$ is differentiable on E, and for each $\vec{x} \in E$, the following formulae are true:

$D\,(\vec{f} \circ \vec{g})(\vec{x}) = D\,\vec{f}(\vec{g}(\vec{x}))\;D\,\vec{g}(\vec{x})$, and

$d\,(\vec{f} \circ \vec{g})(\vec{x}) = D\,\vec{f}(\vec{g}(\vec{x}))\;d\,\vec{g}(\vec{x};\vec{h}) = d\,\vec{f}(\vec{g}(\vec{x});d\vec{g}(\vec{x};\vec{h}))$.

Proof:

Take $\vec{x} \in E$. Since $\vec{f}$ is differentiable at $\vec{g}(\vec{x})$, there exists a $p \times m$ matrix A such that for all points $\vec{g}(\vec{x}) + \vec{k}$ in some deleted neighborhood $V^*(\vec{g}(\vec{x});s)$ of $\vec{g}(\vec{x})$,

(1) $\quad \vec{f}(\vec{g}(\vec{x}) + \vec{k}) = \vec{f}(\vec{g}(\vec{x})) + [A + \Phi(\vec{k})]\ \vec{k}$, where $\lim\limits_{\vec{k}\to\vec{0}} \Phi(\vec{k}) = \theta$.

We will define $\Phi(\vec{0})$ to be the $p \times m$ zero matrix θ and observe that Φ is continuous at $\vec{0}$. Also, (1) will hold for all $\vec{g}(\vec{x}) + \vec{k} \ \in V(\vec{g}(\vec{x});s)$. Since $\vec{g}$ is differentiable, then $\vec{g}$ is continuous at $\vec{x}$, and there exists a neighborhood $V(\vec{x};r)$ of $\vec{x}$ such that $\vec{g}(V(\vec{x};r)) \quad V(\vec{g}(\vec{x});s)$, and there exists an $m \times n$ matrix B such that, for all $\vec{x} + \vec{h} \in V^*(\vec{x};r)$,

(2) $\quad \vec{g}(\vec{x} + \vec{h}) = \vec{g}(\vec{x}) + [B + \Psi(\vec{h})]\ \vec{h}$, where $\lim\limits_{\vec{h}\to\vec{0}} \Psi(\vec{h}) = \theta$.

Now, take $\vec{x} + \vec{h}$ in $V^*(\vec{x};r)$, and let $\vec{k}(\vec{h}) = \vec{g}(\vec{x} + \vec{h}) - \vec{g}(\vec{x})$.

Then, $\lim\limits_{\vec{h}\to\vec{0}} \vec{k}(\vec{h}) = \vec{0}$.

From (1) and (2), we obtain

$$
\begin{aligned}
(\vec{f} \circ \vec{g})(\vec{x} + \vec{h}) &= \vec{f}(\vec{g}(\vec{x}) + \vec{k}(\vec{h})) \\
&= \vec{f}(\vec{g}(\vec{x})) + [A + \Phi(\vec{k}(\vec{h}))]\ \vec{k}(\vec{h}) \\
&= \vec{f}(\vec{g}(\vec{x})) + [A + \Phi(\vec{k}(\vec{h}))]\ [B + \Psi(\vec{h})]\ \vec{h} \\
&= \vec{f}(\vec{g}(\vec{x})) + A\,B\,\vec{h} + \theta(\vec{h})\ \vec{h}.
\end{aligned}
$$

Since $\lim\limits_{\vec{h}\to\vec{0}} \theta(\vec{h}) = \lim\limits_{\vec{h}\to\vec{0}} [\Phi(\vec{k}(\vec{h}))\ B + A\ \Psi(\vec{h}) + \Phi(\vec{k}(\vec{h}))\ \Psi(\vec{h})] = \theta$, $\vec{f} \circ \vec{g}$ is differentiable at $\vec{x}$.

If we use the fact that $A = D\,\vec{f}(\vec{g}(\vec{x}))$ and $B = D\,\vec{g}(\vec{x})$, we have

$D\,(\vec{f} \circ \vec{g})(\vec{x}) = D\,\vec{f}(\vec{g}(\vec{x}))\ D\,\vec{g}(\vec{x})$, and

$d\,(\vec{f} \circ \vec{g})(\vec{x}) = D\,\vec{f}(\vec{g}(\vec{x}))\ d\,(\vec{g}(\vec{x});\vec{h}) = d\,\vec{f}(\vec{g}(\vec{x});d\vec{g}(\vec{x};\vec{h}))$, and the proof is complete.

Remark: From $D(\vec{f} \circ \vec{g})(\vec{x}) = D\vec{f}(\vec{g}(\vec{x}))\, D\vec{g}(\vec{x})$, we see that the entry $\vec{D}_j(f_i \circ \vec{g})(\vec{x})$ in the i^{th} row and the j^{th} column of $D(\vec{f} \circ \vec{g})(\vec{x})$ is the i^{th} row of $D\vec{f}(\vec{g}(\vec{x}))$ times the j^{th} column of $D\vec{g}(\vec{x})$; i.e.,

(3) $$\vec{D}_j\,(f_i \circ \vec{g})(\vec{x}) = \vec{D}\,f_i(\vec{g}(\vec{x})) \cdot D_j\,\vec{g}(\vec{x}),$$

where $D_j\,\vec{g} = (\vec{D}_j\,g_1, \vec{D}_j\,g_2, \ldots, \vec{D}_j\,g_m)$.

We call (3) the Chain Rule, also.

CHAPTER V

LINE INTEGRALS

5.1 Introduction.

The line integral is an important type of integral which appears in many physical applications. This type of integral is an integral of a vector-valued function of a vector along some curve in the domain of the function. In our development, we will restrict ourselves to the consideration of functions and curves which are of the type which occur commonly in physical applications. Line integrals are called, sometimes, curvilinear integrals. The integral is a generalization of the ordinary Riemann integral, in which the interval $[a,b]$ is replaced by a curve in E^n described by a vector-valued function $\vec{x} = (x_1,x_2,x_3,\ldots,x_n)$. In this generalization, the integrand is a vector-valued function $\vec{f} = (f_1,f_2,f_3,\ldots,f_n)$, and is a function from E^n to E^n which is continuous on an open set containing the curve C described by the mapping $\vec{x}$ of $[a,b]$, and C is a smooth curve in E^n; i.e., we assume that $\vec{x}'$ is continuous and nonzero on $[a,b]$. We write the integral $\int_C \vec{f} \cdot d\,\vec{x}$, and the dot in this symbol is used purposely to suggest an inner product of two vectors. The fact is, line integrals can be considered as generalizations of Riemann-Stieltjes integrals in which both the integrand $\vec{f}$ and the integrator $\vec{x}$ are vector-valued functions, and in fact, they could be defined and developed in an analogous manner to that in which the RS-integrals are defined and developed. Some of the theorems would be very analogous and could be proved analogously to those which corresponded in the theory of RS-integrals. We will make a different approach, however.

5.2 Definitions and Theorems Concerning Line Integration.

5.2.1 Definition. The statement that $\int_C \vec{f} \cdot d\,\vec{x}$ is the line integral of $\vec{f}$ along the smooth curve C which is described by the mapping $\vec{x}$ of [a,b] means $\int_C \vec{f} \cdot d\,\vec{x} = \int_a^b \vec{f}(\vec{x}(t)) \cdot \dot{\vec{x}}(t)\, d\, t$.

Remark: Since we assumed that $\dot{\vec{x}}$ is continuous on [a,b] and that $\vec{f}$ is continuous on C, the integral on the right in our definition exists.

From the definition of the line integral and the properties of the Riemann integral, it is shown quite easily that

$\int_C c\,\vec{f} \cdot d\,\vec{x} = c\int_C \vec{f} \cdot d\,\vec{x}$, and

$\int_C (\vec{f} + \vec{g}) \cdot d\,\vec{x} = \int_C \vec{f} \cdot d\,\vec{x} + \int_C \vec{g} \cdot d\,\vec{x}$.

If C is the smooth curve described by $\vec{x} = \vec{g}(t)$, $t \in [a,b]$, then we denote $-$ C by the curve traced out in a direction opposite to that of C; i.e., $-$ C is described by $\vec{x} = \vec{g}(-\,t)$, $t \in [-b,-a]$. Hence,

$$\int_{-C} \vec{f} \cdot d\,\vec{x} = -\int_{-b}^{-a} \vec{f}(\vec{g}(-t)) \cdot \dot{\vec{g}}(-t)\, d\, t.$$

If we let $u = -t$, we obtain

$$\int_{-C} \vec{f} \cdot d\,\vec{x} = \int_b^a \vec{f}(\vec{g}(u)) \cdot \dot{\vec{g}}(u)\, d\,u = -\int_a^b \vec{f}(\vec{g}(u)) \cdot \dot{\vec{g}}(u)\, d\,u$$

$$= -\int_C \vec{f} \cdot d\,\vec{x}.$$

Also, if the curve C is composed of the curves C_1 and C_2; i.e., if C is traced out by tracing out C_1 and then C_2, then

$$\int_C \vec{f} \cdot d\,\vec{x} = \int_{C_1} \vec{f} \cdot d\,\vec{x} + \int_{C_2} \vec{f} \cdot d\,\vec{x}\ .$$

Suppose C is described by the mapping $\vec{x}$ of $[a,c]$, and $[c,b]$, respectively, where $c \in (a,b)$, then

$$\int_C \vec{f} \cdot d\,\vec{x} = \int_a^b \vec{f}(\vec{x}(t)) \cdot \dot{\vec{x}}(t)\, d\,t$$

$$= \int_a^c \vec{f}(\vec{x}(t)) \cdot \dot{\vec{x}}(t)\, d\,t + \int_c^b \vec{f}(\vec{x}(t)) \cdot \dot{\vec{x}}(t)\, d\,t$$

$$= \int_{C_1} \vec{f} \cdot d\,\vec{x} + \int_{C_2} \vec{f} \cdot d\,\vec{x}\ .$$

We can extend the definition of the line integral to a path composed of a number of smooth curves which do not necessarily form a smooth curve.

5.2.2 Definition. The statement that a curve C is a piecewise smooth curve means C is a curve consisting of a finite number of smooth curves.

5.2.3 Definition. The statement that $\int_C \vec{f} \cdot d\,\vec{x}$ is the line integral of a function $\vec{f}$ with respect to a curve C composed of the smooth curves C_k, $k = 1,2,3,\ldots,m$, means $\int\limits_C \vec{f} \cdot d\,\vec{x} = \sum\limits_{k=1}^{m} \int\limits_{C_k} \vec{f} \cdot d\,\vec{x}$,

where $\vec{f}$ is continuous on an open set containing C.

5.2.4 Theorem.

1. $\vec{f}$ is continuous on an open set E.
2. $\vec{x}_1$ and $\vec{x}_2 \in E$.
3. $\vec{f} = \vec{D}\, g$ on E.
4. C is any piecewise smooth curve in E from $\vec{x}_1$ to $\vec{x}_2$.

$$\Longrightarrow \quad \int\limits_C \vec{f} \cdot d\,\vec{x} = g(\vec{x}_2) - g(\vec{x}_1).$$

Proof:

Let C be described by the mapping $\vec{x}$ of $[a,b]$ and let $h(t) = g(\vec{x}(t))$.
Then, $h'(t) = \vec{D}\, g(\vec{x}(t)) \cdot \dot{\vec{x}}(t)$, and

$$\begin{aligned}\int\limits_C \vec{f} \cdot d\,\vec{x} = \int\limits_C \vec{D}\, g \cdot d\,\vec{x} &= \int_a^b \vec{D}\, g(\vec{x}(t)) \cdot \dot{\vec{x}}(t)\, d\,t \\ &= \int_a^b h'(t)\, d\,t \\ &= h(b) - h(a) \\ &= g(\vec{x}_2) - g(\vec{x}_1).\end{aligned}$$

Remark: In this theorem we applied the Second Fundamenaal Theorem of Integral Calculus to the function h' which is a piecewise continuous function on [a,b]. A function is piecewise continuous on [a,b] if it is continuous at all but a finite number of points of [a,b] and at each point of discontinuity the right-hand and the left-hand limits of the function exist. Although the Second Fundamental Theorem is stated, usually, for functions with continuous derivatives, the theorem is true in the case where the derivatives are piecewise continuous functions.

Theorem 5.2.4 states in its conclusion that the line integral of a function which is the derivative of some function g depends only on the values of g at the endpoints $\vec{x}_1$ and $\vec{x}_2$ of the curve, and this means that in this case, the line integral of such a function is independent of the piecewise smooth curve in C which joins $\vec{x}_1$ and $\vec{x}_2$, so in this case we say that the line integral in question is independent of the path in E.

5.2.5 Definition. The statement that C is a closed curve means C is a curve which is such that its endpoints coincide.

5.2.6 Corollary.

1. $\vec{f}$ is a continuous function on an open set E.
2. $\vec{f} = \vec{D}\, g$ on E.
3. C is a piecewise smooth closed curve in E.

$$\Longrightarrow \int_C \vec{f} \cdot d\,\vec{x} = 0.$$

Proof:

If C is a closed curve, then the endpoints $\vec{x}_1$ and $\vec{x}_2$ coincide, and from 5.2.4, we have $\int_C \vec{f} \cdot d\,\vec{x} = g(\vec{x}_1) - g(\vec{x}_1) = 0.$

5.2.7 Definition. The statement that $\vec{f} \cdot d\,\vec{x}$ is an exact differential on an open set E in E^n means there is a function g from E^n to E^1 such that $\vec{f} = \vec{D}\, g$ on E, and hence,

$\vec{f}(\vec{x}) \cdot d\,\vec{x} = \vec{D}\, g(\vec{x}) \cdot d\,\vec{x} = d\, g(\vec{x};d\vec{x}).$

Note that Theorem 5.2.4 shows that if $\vec{f} \cdot d\,\vec{x}$ is an exact differential on E, then the line integral $\int_C \vec{f} \cdot d\,\vec{x}$ is independent of the path in E.

If $\vec{f} \in C^1$ on E and there exists a function g such that $\vec{f} = \vec{D}\,g$ on E, then $g \in C^2$ on E, and hence, $D_{ij}\,g = D_{ji}\,g$ on E, $i,j = 1,2,3,\dots,n$, or what is the same, $\vec{D}_i\,f_j = \vec{D}_j\,f_i$ on E, $i = 1,2,3,\dots,n$.

This gives a necessary condition for $\vec{f} \cdot d\,\vec{x}$ to be an exact differential on E. Hence, if $\vec{D}_i\,f_j(\vec{x}) \neq \vec{D}_j\,f_i(\vec{x})$ for some $\vec{x} \in E$ and some i and j, then $\vec{f}$ is not the derivative of a function on E.

However, continuity and equality of the partial derivatives $\vec{D}_i\,f_j$ and $\vec{D}_j\,f_i$ are not sufficient to ensure that $\vec{f} \cdot d\,\vec{x}$ is an exact differential on E. Some restriction must be replaced on the open set E.

It is true, also, that the converse of Theorem 5.2.4 does not hold unless there is some restriction placed on E. The set E is arcwise connected if for any two points $\vec{x}_1$ and $\vec{x}_2$ of E there is a piecewise smooth curve in E with endpoints $\vec{x}_1$ and $\vec{x}_2$. It is possible to show that if a set E is arcwise connected, then it is connected. The converse is not true in general.

If E, however, is open and connected, then E is arcwise connected.

5.2.8 Theorem.

1. $\vec{f}$ is continuous on an open connected set E.

2. $\int_C \vec{f} \cdot d\,\vec{x}$ is independent of the path C is E.

$\Longrightarrow$ $\vec{f} \cdot d\,\vec{x}$ is an exact differential in E.

Proof:

Let $\vec{x}_o \in E$. Then if $\vec{x} \in E$, let $g(\vec{x}) = \int_C \vec{f} \cdot d\,\vec{x}$, where C is a piecewise smooth curve from $\vec{x}_o$ to $\vec{x}$ and lies in E. Since the integral is independent of the path in E, the value $g(\vec{x})$ does not depend on the choice of the curve C. Now consider a particular point $\vec{x}$ in E and let C_1 be a piecewise smooth curve from $\vec{x}_o$ to $\vec{x}$ and lying in E. Since E is open, there is a neighborhood $V(\vec{x};\delta)$ of $\vec{x}$ which is contained in E. Hence, for $|h| < \delta$, the line segment $C_2 = \{\vec{x} + t\,h\,\vec{u}_k \mid t \in (0,1)\}$, where $\vec{u}_k$ is the unit vector in the direction of the X_k-axis, and lies in E. Let C_3 be the path composed of C_1 and C_2, and we have

$$g(\vec{x} + h\,\vec{u}_k) - g(\vec{x}) = \int_{C_3} \vec{f} \cdot d\,\vec{x} - \int_{C_1} \vec{f} \cdot d\,\vec{x} = \int_{C_2} \vec{f} \cdot d\,\vec{x}$$

$$= \int_0^1 \vec{f}(\vec{x} + t\,h\,\vec{u}_k) \cdot h\,\vec{u}_k\,d\,t$$

$$= h \int_0^1 f_k(\vec{x} + t\,h\,\vec{u}_k)\,d\,t$$

$$= h\,f_k(\vec{x} + \theta\,h\,\vec{u}_k), \text{ for some } \theta \in (0,1), \text{ using}$$

in the last step the First Mean Value Theorem for Integrals.

Since $\vec{f} \in C^o$ on E, then

$$\lim_{h \to 0} \frac{g(\vec{x} + h\,\vec{u}_k) - g(\vec{x})}{h} = \lim_{h \to 0} f_k(\vec{x} + \theta\,h\,\vec{u}_k)$$

$$= f_k(\vec{x});$$

i.e., $D_k\,g(\vec{x}) = f_k(\vec{x})$, which shows that $\vec{D}\,g = \vec{f}$ on E.

Hence, $\vec{f} \cdot d\,\vec{x}$ is an exact differential on E.

5.3 Applications to Mechanics.

Let $\vec{F}$ be an E^3 force field; i.e., $\vec{F}$ is a function which assigns to each point $\vec{x}$ in some region E of E^3 the force $\vec{F}(\vec{x})$ which acts on a particle at this point. We will define the work done by the force field in moving a particle along a curve C in E^3. The work done by a force in moving a particle from one position to another is the component of the force in the direction of motion multiplied by the distance moved. Let C be a smooth curve described by the equation $\vec{x} = \vec{x}(t)$, $a \leqq t \leqq b$. At the point $\vec{x}(t)$ the component of the force in the direction of motion is $\vec{F}(\vec{x}(t)) \cdot \dfrac{\vec{x}'(t)}{|\vec{x}'(t)|}$, where $\dfrac{\vec{x}'(t)}{|\vec{x}'(t)|}$ is a unit tangent vector in the direction of the parameter increasing. So, if we take a partition $\{t_{k,n} \mid k = 0,1,2,\ldots,n\}$ of the interval $[a,b]$, the work done by the force field in moving a particle along C is approximately $\sum_{k=1}^{n} \vec{F}(\vec{x}(t_{k,n})) - \vec{x}'(\bar{t}_{k,n})\ (t_{k,n} - t_{k-1,n})$, where $t_{k-1,n} \leqq \bar{t}_{k,n} \leqq t_{k,n}$, $k = 1,2,3,\ldots,n$. If these approximating sums approach a limit which is a number as the norm of the partitions approach zero, then this limit is defined to be the work done by the force field. If we assume $\vec{F}(\vec{x}(t))$ is piecewise continuous, then this limit exists and is $\int_a^b \vec{F}(\vec{x}(t)) \cdot \vec{x}'(t)\, d\, t = \int_C \vec{F} \cdot d\, \vec{x}$.

Hence, the work done by a force field $\vec{F}$ moving a particle along a curve C is defined to be $\int_C \vec{F} \cdot d\, \vec{x}$.

Remark: In order to ensure the existence of the line integral $\int_C \vec{F} \cdot d\,\vec{x}$, we will assume throughout this section that $\vec{F} \in C^0$ defined on an open set E and C denotes a piecewise smooth curve contained in E.

If $\vec{x}_1$ and $\vec{x}_2$ are the endpoints of the curve C, then this line integral along C is denoted by $\int_{C\vec{x}_1}^{\vec{x}_2} \vec{F} \cdot d\,\vec{x}$.

Remark: Newton's Second Law of Motion states that a particle of mass m subject to a force field $\vec{F}$ will move according to the equation $m\,\vec{x}(t) = \vec{F}(\vec{x}(t))$, where $\vec{x}(t)$ is the position of the particle at time t.

Hence,

$$m\,\ddot{\vec{x}}(t) \cdot \dot{\vec{x}}(t) = \tfrac{1}{2}\, m\, D_t\, \{\dot{\vec{x}}(t) \cdot \dot{\vec{x}}(t)\}$$
$$= \vec{F}(\vec{x}(t)) \cdot \dot{\vec{x}}(t)\ .$$

$$(1) \qquad \tfrac{1}{2}\, m\, |\vec{v}(t_2)|^2 - \tfrac{1}{2}\, m\, |\vec{v}(t_1)|^2 = \int_{t_1}^{t_2} \vec{F}(\vec{x}(t)) \cdot \dot{\vec{x}}(t)\ d\ t$$
$$= \int_{C\vec{x}(t_1)}^{\vec{x}(t_2)} \vec{F} \cdot d\,\vec{x}.$$

The quantity $\tfrac{1}{2}\, m\, |\vec{v}(t)|^2$ is called the kinetic energy of the particle at time t. Hence, (1) states: As a particle moves along its trajectory C from $\vec{x}(t_1)$ to $\vec{x}(t_2)$, the change in kinetic energy is equal to the work done by the force field.

5.3.1 Definition. The statement that the force field $\vec{F}$ defined on an open set E is conservative means the work done along each closed curve of E is zero.

This definition states that the force field $\vec{F}$ is conservative if the work done in moving a particle from one position to another is independent of the path along which it moves. If $\vec{F}$ is conservative and the domain of $\vec{F}$ is an open connected set E, then Theorem 5.5.2 implies that there exists a real-valued function U defined on E, called a potential function, such that $\vec{D}\,U = -\vec{F}$ on E. Also from Theorem 5.2.4, if $\vec{F}$ has a potential function U, then $\vec{F}$ is conservative, and

$$\int_{C\,\vec{x}_1}^{\vec{x}_2} \vec{F} \cdot d\,\vec{x} = U(\vec{x}_1) - U(\vec{x}_2) .$$

Hence, when the force field is conservative, we can write

$$\tfrac{1}{2}\, m\, |\vec{v}(t_2)|^2 - \tfrac{1}{2}\, m\, |\vec{v}(t_1)|^2 = \int_{C\,\vec{x}_1}^{\vec{x}_2} \vec{F} \cdot d\,\vec{x} \quad \text{as}$$

$$\tfrac{1}{2}\, m\, |\vec{v}(t_2)|^2 + U(\vec{x}(t_2)) = \tfrac{1}{2}\, m\, |\vec{v}(t_1)|^2 + U(\vec{x}(t_1)).$$

This is the Law of Conservation of Energy: If the force field is conservative, the sum of the kinetic energy and the potential energy is a constant.

If U is a potential function for $\vec{F}$, then $\vec{F} = -\vec{D}\,U$, and this relation implies that, at a point on the surface through $\vec{x}$, U is constant. Such a surface is called an equipotential.

We will clsse this chapter with some problems relating to conservative force fields.

Problem 1. At a point $\vec{x}$ the force acting on a particle of mass m due to the earth's gravitational field is $\vec{F}(\vec{x}) = -m\,(0,0,g)$. Show that this force field is conservative.

Solution:

We must show that there is a potential function U such that $-\vec{D}\,U = \vec{F}$. Such will be the case, iff, $D_1\,U = 0$, $D_2\,U = 0$, and $D_3\,U = mg$. A solution of these equations is $U(x,y,z) = m\,g\,z$. Hence, the force field is conservative. The equipotential surfaces are clearly horizontal planes.

Problem 2. Suppose a particle of mass m with initial velocity (a,0,b) and initial position (0,0,0) moves under the influence of the gravitational force field $\vec{F}(x,y,z) = -m\,(0,0,g)$. Verify the Law of Conservation of Energy.

Solution:

The particle moves according to Newton's Law, $\vec{F} = m\,\vec{a}$.

Hence, if $\vec{x}(t)$ is the position of the particle at time t,

$$\ddot{\vec{x}}(t) = (0,0,-g)$$

$$\dot{\vec{x}}(t) = (a,0,-gt + b)$$

$$\vec{x}(t) = (at,0,-\tfrac{1}{2}gt^2 + bt).$$

At time t, since $U(x,y,z) = m\,g\,z$, then

$$\begin{aligned}\tfrac{1}{2}\,m\,|\vec{v}(t)|^2 + U(\vec{x}(t)) &= \tfrac{1}{2}\,m\,(a^2 + g^2t^2 - 2bgt + b^2) + m\,g\,(-\tfrac{1}{2}gt^2 + bt)\\ &= \tfrac{1}{2}\,m\,(a^2 + b^2)\,.\end{aligned}$$

Remark: In a conservative force field a particle is in stable equilibrium at points where the potential energy has a relative minimum.

Problem 3. In the gravitational force field $\vec{F}(x,y,z) = -m\ (0,0,g)$ determine the points on the surface whose equation is $9\ x^2 + 4\ y^2 - y\ z + 4 = 0$, where a particle of mass m is in stable equiligrium.

Solution:

The potential function is $U(x,y,z) = m\ g\ z$, where $m > 0$ and $g > 0$, and we will determine the point where z has a relative minimum. Now,

$$z = f(x,y) = 4\ y + \frac{9\ x^2 + 4}{y}$$, and we must test $f(x,y)$ for a relative minimum. We have

$$D_1\ f(x,y) = \frac{18\ x}{y} = 0$$

$$D_2\ f(x,y) = 4 - \frac{9\ x^2 + 4}{y^2} = 0,$$

and thus, $x = 0$, and $y = \pm 1$.

Since $D_{11}\ f(x,y) = \frac{18}{y}$, $D_{12}\ f(x,y) = -\frac{18\ x}{y^2}$, and

$D_{22}\ f(x,y) = \frac{2}{y^2}(9\ x^2 + 4)$, then the expression

$(D_{11}\ f)\ (D_{22}\ f) - (D_{12}\ f)^2 > 0$ at the points $(0,1)$ and $(0,-1)$.

Also, f has a relative minimum at $(0,1)$ and a relative maximum at $(0,-1)$. Thus, the only point of stable equilibrium on the given surface is the point $(0,1,8)$.

CHAPTER VI

VECTOR FIELDS

6.1 Introduction.

Often in the applications of mathematics to physics and engineering we deal with the concept of vector fields. In the mathematical sense, a vector field is a vector-valued function defined on some set. As an example, suppose that to each point $\vec{x}$ in the atmosphere there is assigned a vector $\vec{v}(\vec{x})$ which represents the wind velocity, then this defines a vector field. If $\vec{v}(\vec{x})$ is expressed in terms of its components relative to some basis $\{\vec{u}_1,\vec{u}_2,\vec{u}_3\}$, we can write

$$\vec{v}(\vec{x}) = v_1(\vec{x})\ \vec{u}_1 + v_2(\vec{x})\ \vec{u}_2 + v_3(\vec{x})\ \vec{u}_3.$$

The components v_1, v_2, v_3 are three real-valued functions called scalar fields. The temperature, for example, of each point of the atmosphere defines a scalar field.

In physical problems involving vector fields one must know not only the vector $\vec{v}(\vec{x})$ at each point $\vec{x}$, but also how this vector changes as one moves from one point to another. We have at our disposal the machinery of partial derivatives to study this change, and this can be applied to the components of $\vec{v}$. In general, these partial derivatives do not depend on the choice of the basis relative to which the components have been determined. Thus, partial derivatives are not entirely satisfactory for describing certain physical quantities, and in particular when these quantities have meaning independent of the basis. We have recourse, then, to special combinations of the partial derivatives, known as the divergence and curl, to describe the behavior of vector fields. The divergence and the curl are independent of the basis (if the basis is orthonormal), and they have a definite physical significance. We will define and study these concepts.

6.2 The Gradient Field in E^n.

If Φ is a real-valued function (a scalar field) defined on an open set S in E^n, the gradient of Φ, denoted by $\nabla\Phi$, or by grad Φ, is a vector-valued function defined by

(1) $$\operatorname{grad}\Phi(\vec{x}) = \nabla\Phi(\vec{x}) = (D_1\Phi(\vec{x}), D_2\Phi(\vec{x}), \ldots, D_n\Phi(\vec{x})),$$

at each point $\vec{x}$ in S where these partial derivatives exist. The following properties of the gradient are consequences, immediately, of the definition:

6.2.1 Theorem.

1. Φ and Ψ are real-valued functions such that $\nabla\Phi$ and $\nabla\Psi$ both exist on an open set S in E^n.

$\Longrightarrow$ (a) $\nabla(\Phi + \Psi) = \nabla\Phi + \nabla\Psi$

(b) $\nabla(\Phi \cdot \Psi) = \Phi\nabla\Psi + \Psi\nabla\Phi$

(c) $\nabla\left(\frac{\Phi}{\Psi}\right) = \frac{(\Psi\nabla\Phi - \Phi\nabla\Psi)}{\Psi^2}$, at points $\vec{x}$ where $\Psi(\vec{x}) \neq 0$.

In case $n = 3$, the gradient has a useful geometric interpretation. Suppose c is a constant, and consider the set S_c of points $\vec{x}$ in S where $\Phi(\vec{x}) = c$. In some cases S_c is a surface. If S_c has a tangent plane at a point $\vec{a} = (a_1,a_2,a_3)$, then from elementary calculus, the equation of this plane is

$$D_1\Phi(\vec{a})(x_1 - a_1) + D_2\Phi(\vec{a})(x_2 - a_2) + D_3\Phi(\vec{a})(x_3 - a_3) = 0.$$

Then $\nabla\ \Phi(\vec{a})$ is normal to the plane (and thus normal to S_c) at the point $\vec{a}$. The tangent plane exists whenever $\nabla\ \Phi(\vec{a}) = \vec{0}$.
The scalar field Φ whose gradient is $\nabla\ \Phi$ is called the potential function of the vector field $\nabla\ \Phi$. The corresponding surfaces S_c are called equipotential surfaces (or level surfaces). In this case of E^2 fields, each set S_c is a plane curve called an equipotential line (or level line). The equipotential surfaces (lines) are orthogonal to the gradient vector at each point $\vec{a}$ where $\nabla\ \Phi(\vec{a}) = \vec{0}$.

6.3 The Curl of a Vector Field in E^3.

6.3.1 Definition. The statement that curl $\vec{f}$ is the curl of the vector-valued function $\vec{f} = (f_1,f_2,f_3)$ defined on an open set S in E^3 means curl $\vec{f} = (D_2f_3 - D_3f_2,\ D_3f_1 - D_1f_3,\ D_1f_2 - D_2f_1)$, whenever the partial derivatives on the right exist.
Symbolically, we can write

$$\operatorname{curl} \vec{f} = \nabla \times \vec{f} = \begin{vmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ D_1 & D_2 & D_3 \\ f_1 & f_2 & f_3 \end{vmatrix}.$$

6.3.2 Theorem.

1. $\vec{f}$ and $\vec{g}$ are vector fields on an open set S in E^3.

2. curl $\vec{f}$ and curl $\vec{g}$ exist on S.

$\Longrightarrow$ curl $(\vec{f} + \vec{g})$ = curl $\vec{f}$ + curl $\vec{g}$.

Proof:

Let $\vec{f} = (f_1, f_2, f_3)$, and

$\vec{g} = (g_1, g_2, g_3)$.

$$\text{curl } (\vec{f} + \vec{g}) = \begin{vmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ D_{x_1} & D_{x_2} & D_{x_3} \\ f_1 + g_1 & f_2 + g_2 & f_3 + g_3 \end{vmatrix}$$

$$= \{D_{x_2} (f_3 + g_3) - D_{x_3} (f_2 + g_2)\} \vec{u}_1$$
$$- \{D_{x_1} (f_3 + g_3) - D_{x_3} (f_1 + g_1)\} \vec{u}_2$$
$$+ \{D_{x_1} (f_2 + g_2) - D_{x_2} (f_1 + g_1)\} \vec{u}_3$$

$$= (D_{x_2} f_3 + D_{x_2} g_3 - D_{x_3} f_2 - D_{x_3} g_2) \vec{u}_1$$
$$- (D_{x_1} f_3 + D_{x_1} g_3 - D_{x_3} f_1 - D_{x_3} g_1) \vec{u}_2$$
$$+ (D_{x_1} f_2 + D_{x_1} g_2 - D_{x_2} f_1 - D_{x_2} g_1) \vec{u}_3$$

$$= \{(D_{x_2} f_3 - D_{x_3} f_2) \vec{u}_1 - (D_{x_1} f_3 - D_{x_3} f_1) \vec{u}_2$$
$$+ (D_{x_1} f_2 - D_{x_2} f_1) \vec{u}_3\}$$
$$+ \{(D_{x_2} g_3 - D_{x_3} g_2) \vec{u}_1 - (D_{x_3} g_1 - D_{x_1} g_3) \vec{u}_2$$
$$+ (D_{x_1} g_2 - D_{x_2} g_1) \vec{u}_3\}$$

$$= \text{curl } \vec{f} + \text{curl } \vec{g} .$$

6.3.3 Theorem.

1. $\vec{f}$ is a vector field defined on an open set S in E^3.

2. Φ is a scalar field.

3. curl $\vec{f}$ exists on S.

4. $\nabla \Phi$ exists on S.

$\Longrightarrow$ curl $(\Phi \vec{f}) = \Phi$ curl $\vec{f} + \nabla \Phi \times \vec{f}$.

Proof:

From the definition of curl we know, since $\vec{f} = (f_1, f_2, f_3)$, that

$$\text{curl}\ (\Phi \vec{f}) = \begin{vmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ D_{x_1} & D_{x_2} & D_{x_3} \\ \Phi f_1 & \Phi f_2 & \Phi f_3 \end{vmatrix}$$

$$= \{D_{x_2}(\Phi f_3) - D_{x_3}(\Phi f_2)\}\ \vec{u}_1 - \{D_{x_1}(\Phi f_3) - D_{x_3}(\Phi f_1)\}\ \vec{u}_2 + \{D_{x_1}(\Phi f_2) - D_{x_2}(\Phi f_1)\}\ \vec{u}_3$$

$$= \{\Phi\ D_{x_2} f_3 + (D_{x_2}\Phi)\ f_3 - \Phi\ D_{x_3} f_2 - (D_{x_3}\Phi)\ f_2\}\ \vec{u}_1 - \{\Phi\ D_{x_1} f_3 + (D_{x_1}\Phi)\ f_3 - \Phi\ D_{x_3} f_1 - (D_{x_3}\Phi)\ f_1\}\ \vec{u}_2 + \{\Phi\ D_{x_1} f_2 + (D_{x_1}\Phi)\ f_2 - \Phi\ D_{x_2} f_1 - (D_{x_2}\Phi)\ f_1\}\ \vec{u}_3$$

$$= \Phi\ (D_{x_2} f_3)\ \vec{u}_1 + (D_{x_2}\Phi)\ f_3\ \vec{u}_1 - \Phi\ (D_{x_3} f_2)\ \vec{u}_1 - (D_{x_3}\Phi)\ f_2\ \vec{u}_1 - \Phi\ (D_{x_1} f_3)\ \vec{u}_2 - (D_{x_1}\Phi)\ f_3\ \vec{u}_2 + \Phi\ (D_{x_3} f_1)\ \vec{u}_2 + (D_{x_3}\Phi)\ f_1\ \vec{u}_2 + \Phi\ (D_{x_1} f_2)\ \vec{u}_3 + (D_{x_1}\Phi)\ f_2\ \vec{u}_3 - \Phi\ (D_{x_2} f_1)\ \vec{u}_3 - (D_{x_2}\Phi)\ f_1\ \vec{u}_3$$

$$= \{\Phi(D_{x_2}f_3)\vec{u}_1 - \Phi(D_{x_2}f_2)\vec{u}_1\} - \{\Phi(D_{x_1}f_3)\vec{u}_2 - \Phi(D_{x_3}f_1)\vec{u}_2\}$$

$$+ \{\Phi(D_{x_1}f_2)\vec{u}_3 - \Phi(D_{x_2}f_1)\vec{u}_3\} + \{(D_{x_2}\Phi)f_3\vec{u}_1 - (D_{x_3}\Phi)f_2\vec{u}_1\}$$

$$- \{(D_{x_1}\Phi)f_3\vec{u}_2 - (D_{x_3}\Phi)f_1\vec{u}_2\} + \{(D_{x_1}\Phi)f_2\vec{u}_3 - (D_{x_2}\Phi)f_1\vec{u}_3\}$$

$$= \Phi(D_{x_2}f_3 - D_{x_3}f_2)\vec{u}_1 - \Phi(D_{x_1}f_3 - D_{x_3}f_1)\vec{u}_2$$

$$+ \Phi(D_{x_1}f_2 - D_{x_2}f_1)\vec{u}_3 + \{(D_{x_2}\Phi)f_3 - (D_{x_3}\Phi)f_2\}\vec{u}_1$$

$$- \{(D_{x_1}\Phi)f_3 - (D_{x_3}\Phi)f_1\}\vec{u}_2$$

$$+ \{(D_{x_1}\Phi)f_2 - (D_{x_2}\Phi)f_1\}\vec{u}_3$$

$$= \Phi \text{ curl } \vec{f} + (\nabla \Phi) \times \vec{f}.$$

Hence, curl $(\Phi \vec{f}) = \Phi$ curl $\vec{f} + (\nabla \Phi) \times \vec{f}$.

Remark: The curl can be given a physical interpretation; for example, suppose a rigid body to be rotating about a fixed axis with constant angular velocity $\vec{\omega}$. The basis $(\vec{u}_1,\vec{u}_2,\vec{u}_3)$ is chosen so that the velocity vector $\vec{x}'$ of a point P of the body is given by

$\vec{x}' = (\omega\, \vec{u}_3) \times \vec{x} = -\,\omega\, x_2\, \vec{u}_1 + \omega\, x_1\, \vec{u}_2$,

where $\vec{x} = x_1\, \vec{u}_1 + x_2\, \vec{u}_2 + x_3\, \vec{u}_3$ is the position vector $\overrightarrow{OP}$

The vector $\vec{\omega} = \omega\, \vec{u}_3$ is called the angular velocity of the body.

The curl of $\vec{x}'$ is

$$\text{curl } \vec{x}' = \begin{vmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ D_1 & D_2 & D_3 \\ -\omega x_2 & \omega x_1 & 0 \end{vmatrix} = 2\,\omega\, \vec{u}_3 = 2\, \vec{\omega}.$$

This means that the curl of the velocity of a rigid body rotating with angular velocity $\vec{\omega}$ is $2\, \vec{\omega}$.

6.3.4 Theorem.

1. $\phi(\vec{x}) \in C^2$ on an open set S of E^3.

$\Longrightarrow$ curl (grad ϕ) $= \vec{0}$.

Proof:

grad $\phi(\vec{x})$, $n = 3$, is

grad $\phi(\vec{x}) = (D_1\phi(\vec{x}), D_2\phi(\vec{x}), D_3\phi(\vec{x}))$.

Hence,

$$\text{curl (grad } \phi) = \begin{vmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ D_1 & D_2 & D_3 \\ D_1\phi & D_2\phi & D_3\phi \end{vmatrix}$$

$$= \begin{vmatrix} D_1 & D_2 \\ D_1\phi & D_2\phi \end{vmatrix} \vec{u}_1 - \begin{vmatrix} D_1 & D_3 \\ D_1\phi & D_3\phi \end{vmatrix} \vec{u}_2 + \begin{vmatrix} D_1 & D_2 \\ D_1\phi & D_2\phi \end{vmatrix} \vec{u}_3$$

$$= (D_1D_2\phi - D_2D_1\phi)\,\vec{u}_1 - (D_1D_3\phi - D_3D_1\phi)\,\vec{u}_2 + (D_1D_2\phi - D_2D_1\phi)\,\vec{u}_3$$

$$= 0\,\vec{u}_1 + 0\,\vec{u}_2 + 0\,\vec{u}_3$$

$$= \vec{0},$$

since $\phi \in C^2$ on an open set S in E^3.

6.3.5 Definition. The statement that a vector field $\vec{f}$ is irrotational means curl $\vec{f} = \vec{0}$.

6.4 The Divergence of a Vector Field in E^n.

Consider the equation $\nabla \times \vec{g} = \vec{f}$. One might ask: When is a given vector field $\vec{f}$ the curl of another vector field $\vec{g}$? A necessary and sufficient condition for solving such an equation can be stated simply in terms of a scalar field known as the divergence, whose properties we will develop.

6.4.1 Definition. The statement that div $\vec{f}$ is the divergence of a vector function $\vec{f} = (f_1, f_2, f_3, \ldots, f_n)$ which is a vector field defined on an open set S in E^n means div $\vec{f} = D_1 f_1 + D_2 f_2 + \ldots + D_n f_n$, whenever the partial derivatives on the right exist.

Remark. div $\vec{f}$ is written, also, as $\nabla \cdot \vec{f}$, and

$$\text{div } \vec{f} = \nabla \cdot \vec{f} = \sum_{k=1}^{n} D_{x_k} f_k .$$

6.4.2 Theorem.

1. $\vec{f}$ and $\vec{g}$ are vector fields defined on an open set S in E^n.
2. div $\vec{f}$ and div $\vec{g}$ exist on S.

$$\Longrightarrow \quad \text{div } (\vec{f} + \vec{g}) = \text{div } \vec{f} + \text{div } \vec{g}.$$

Proof:

From the definition of divergence, we know that

$$\text{div } \vec{f} = \sum_{k=1}^{n} D_{x_k} f_k \text{ , where } \vec{f} = (f_1, f_2, \ldots, f_n), \text{ and}$$

$$\text{div } \vec{g} = \sum_{k=1}^{n} D_{x_k} g_k \text{ , where } \vec{g} = (g_1, g_2, \ldots, g_n).$$

Then,

$$\text{div } (\vec{f} + \vec{g}) = \sum_{k=1}^{n} D_{x_k} (f_k + g_k)$$

$$= \sum_{k=1}^{n} (D_{x_k} f_k + D_{x_k} g_k)$$

$$= \sum_{k=1}^{n} D_{x_k} f_k + \sum_{k=1}^{n} D_{x_k} g_k$$

$$= \text{div } \vec{f} + \text{div } \vec{g}.$$

Thus, div $(\vec{f} + \vec{g}) = \text{div } \vec{f} + \text{div } \vec{g}$.

6.4.3 Theorem.

1. $\vec{f}$ is a vector field defined on an open set S of E^n.
2. ϕ is a scalar field defined on S.
3. div $\vec{f}$ exists on S.
4. $\nabla \phi$ exists on S.

$\Longrightarrow$ div $(\phi \vec{f}) = \phi \text{ div } \vec{f} + (\nabla \phi) \cdot \vec{f}$.

Proof:

Let $\vec{f} = (f_1, f_2, \ldots, f_n)$, then

$$\phi \vec{f} = (\phi f_1, \phi f_2, \ldots, \phi f_n).$$

$$\nabla (\phi f) = \sum_{k=1}^{n} D_{x_k} \phi f_k = \sum_{k=1}^{n} \{\phi D_{x_k} f_k + (D_{x_k} \phi) f_k\}$$

$$= \phi \sum_{k=1}^{n} D_{x_k} f_k + \sum_{k=1}^{n} (D_{x_k} \phi) f_k$$

$$= \phi \text{ div } \vec{f} + (\nabla \phi) \cdot \vec{f}.$$

Hence, div $(\phi \vec{f}) = \phi \text{ div } \vec{f} + (\nabla \phi) \cdot \vec{f}$.

6.4.3 Theorem.

1. $\vec{f} = (f_1, f_2, f_3)$ is a vector field defined on an open set S of E^3.
2. $\vec{f}$ has continuous cross-derivatives on S.

$\Longrightarrow$ div (curl $\vec{f}$) $= 0$.

Proof:

We know that

$$\operatorname{curl} \vec{f} = \begin{vmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ D_{x_1} & D_{x_2} & D_{x_3} \\ f_1 & f_2 & f_3 \end{vmatrix} .$$

$$\operatorname{curl} \vec{f} = \nabla \times \vec{f} = (D_{x_2} f_3 - D_{x_3} f_2)\, \vec{u}_1 - (D_{x_1} f_3 - D_{x_3} f_1)\, \vec{u}_2 + (D_{x_1} f_2 - D_{x_2} f_1)\, \vec{u}_3$$

Now, $\operatorname{div} (\operatorname{curl} \vec{f}) = \nabla \cdot (\nabla \times \vec{f})$

$$= D_{x_1} (D_{x_2} f_3 - D_{x_3} f_2) + D_{x_2} (D_{x_3} f_1 - D_{x_1} f_3) + D_{x_3} (D_{x_1} f_2 - D_{x_2} f_1)$$

$$= D_{x_1} D_{x_2} f_3 - D_{x_1} D_{x_3} f_2 + D_{x_2} D_{x_3} f_1 - D_{x_2} D_{x_1} f_3 + D_{x_3} D_{x_1} f_2 - D_{x_3} D_{x_2} f_1$$

$$= (D_{x_1} D_{x_2} f_3 - D_{x_2} D_{x_1} f_3) + (D_{x_3} D_{x_1} f_2 - D_{x_1} D_{x_3} f_2) + (D_{x_2} D_{x_3} f_1 - D_{x_3} D_{x_2} f_1)$$

$$= 0 + 0 + 0$$

$= 0$, since $\vec{f}$ has continuous cross derivatives on S.

Hence, $\operatorname{div} (\operatorname{curl} \vec{f}) = 0$.

6.4.5 Theorem.

1. $\vec{f} = (f_1, f_2, f_3)$ is a vector field defined on an open set S of E^3.

2. f_1, f_2, and $f_3 \in C^1$ on S.

3. div $\vec{f}(\vec{x}) = 0$, for each $\vec{x}$ in S.

$\Longrightarrow$ There exists a vector field $\vec{g} = (g_1, g_2, g_3)$ such that curl $\vec{g} = \vec{f}$.

Proof:

We will construct $\vec{g}$ explicitly as follows:

Let $\vec{y} = (y_1, y_2, y_3)$ be a fixed point in S. For each point $\vec{x} = (x_1, x_2, x_3)$ in S, define

$$g_1(x_1,x_2,x_3) = \int_{y_3}^{x_3} f_2(x_1,y_2,t_3)\, d\, t_3 - \int_{y_2}^{x_2} f_3(x_1,t_2,y_3)\, d\, t_2 \,.$$

Then, taking the derivatives,

(1) $D_3\, g_1(x_1,x_2,x_3) = f_2(x_1,y_2,x_3)$, and

$D_2\, g_1(x_1,x_2,x_3) = - f_3(x_1,x_2,y_3)$.

Now, place

$$g_2(x_1,x_2,x_3) = \int_{y_3}^{x_3} \left[\int_{y_1}^{x_1} D_3 f_3(t_1,x_2,t_3)\, d\, t_1\right] d\, t_3$$

Taking the derivatives, we have

(2) $D_3\, g_2(x_1,x_2,x_3) = \int_{y_1}^{x_1} D_3 f_3(t_1,x_2,x_3)\, d\, t_1$

and

$$(3)\ D_1\ g_2(x_1,x_2,x_3) = \int_{y_3}^{x_3} D_3 f_3(x_1,x_2,t_3)\ d\ t_3$$

$$= f_3(x_1,x_2,x_3) - f_3(x_1,x_2,y_3).$$

Then, define

$$g_3(x_1,x_2,x_3) = - \int_{y_2}^{x_2} \left[\int_{y_1}^{x_1} D_2 f_2(t_1,t_2,x_3)\ d\ t_1\right] d\ t_2$$

$$+ \int_{y_2}^{x_2} f_1(x_1,t_2,x_3)\ d\ t_2$$

$$+ \int_{y_2}^{x_2} \left[\int_{y_1}^{x_1} \{D_2 f_2(t_1,t_2,x_3) + D_3 f_3(t_1,t_2,x_3)\} dt_1\right] dt_2 .$$

Taking the derivatives, we have

$$(4)\ D_2\ g_3(x_1,x_2,x_3) = - \int_{y_1}^{x_1} D_2 f_2(t_1,x_2,x_3)\ d\ t_1 + f_1(\vec{x})$$

$$+ \int_{y_1}^{x_1} \{D_2 f_2(t_1,x_2,x_3) + D_3 f_3(t_1,x_2,x_3)\}\ d\ t_1$$

and

$$(5)\ D_1\ g_3(x_1,x_2,x_3) = - \int_{y_2}^{x_2} D_2 f_2(x_1,t_2,x_3)\ d\ t_2$$

$$+ \int_{x_2}^{y_2} \operatorname{div} f\ (x_1,t_2,x_3)\ d\ t_2$$

$$= - \int_{y_2}^{x_2} D_2 f_2(x_1,t_2,x_3)\ d\ t_2$$

$$= - f_2(\vec{x}) + f_2(x_1,y_2,x_3).$$

From (4) and (2) we have

$D_2\ g_3(\vec{x}) - D_3\ g_2(\vec{x}) = f_1(\vec{x})$.

From (1) and (5) we have

$D_3\ g_1(\vec{x}) - D_1\ g_3(\vec{x}) = f_2(\vec{x})$.

From (3) and (1) we have

$D_1\ g_2(\vec{x}) - D_2\ g_1(\vec{x}) = f_3(\vec{x})$.

Thus, curl $\vec{g}(\vec{x}) = \vec{f}(\vec{x})$, and our proof is complete.

6.4.6 Definition. The statement that a vector field $\vec{f}$ is solenoidal means div $\vec{f} = 0$.

6.5 The Laplacian Operator.

If ϕ is a scalar field defined on an open set S in E^n, then the definition of divergence gives the formula

div $(\nabla\ \phi) = D_{11}\phi + D_{22}\phi + \ .\ .\ .\ + D_{nn}\phi$,

whenever the partial derivatives on the right exist.

The divergence of the gradient is expressed symbolically as $\nabla\cdot\nabla\ \phi$, and is written usually as $\nabla^2\ \phi$. The operator ∇^2 is called the Laplacian operator, and when applied to scalar fields yields the result given above. The partial derivative equation $D^2\ \phi = 0$ is called Laplace's equation.

A function ϕ is harmonic on S if it satisfies Laplace's equation on S.

The operator ∇^2 can be applied, also, to a vector field $\vec{f} = (f_1,f_2,f_3,\ldots,f_n)$ by defining

$\nabla^2\ \vec{f} = (\nabla^2 f_1,\ \nabla^2 f_2,\ \nabla^2 f_3,\ .\ .\ .\ ,\ \nabla^2 f_n)$.

The four operators: gradient, curl, divergence, and Laplacian are related by the following identity:

curl (curl $\vec{f}$) = grad (div $\vec{f}$) - $\nabla^2 \vec{f}$.

6.6 Surfaces.

In order to consider further the study of vector fields in E^3 we need the use of surface integrals. The surface integral is the E^2 analog of a line integral in which the path of integration is a surface instead of a curve.

A surface, speaking generally, is the locus of a point which moves in space with two degrees of freedom of movement. Several ways of describing such a locus by mathematical formulae exist. If we use the usual x y z cartesian coordinate system of analytic geometry, we can obtain a surface by imposing one restriction on a variable point (x,y,z), written in the form $F(x,y,z) = 0$, and an equation of this kind is called an implicit representation of the surface. If we are able to solve this equation explicitly for one of the variables x, y, z in terms of the other two variables, say z in terms of x and y, we obtain an equation of the form $z = f(x,y)$, and we have what is called an explicit representation of the surface. We can, apparently, write such a representation in the form which is an implicit equation as $f(x,y) - z = 0$.

While these two representations are useful and fairly common in use, a different way of describing surfaces is more useful for theoretical purposes. This is the parametric representation or vector representation of a surface. In such a representation, we have three equations in which x, y, and z are expressed as functions of two parameters, say u and v:

$$x = x(u,v), \qquad y = y(u,v), \qquad z = z(u,v).$$

This means the point (u,v) varies over some E^2 region R in the uv-plane, and the corresponding points (x,y,z) trace out a portion of a surface in E^3 space. This procedure is analogous to representing a space curve by three parametric equations which involve only one parameter.

The question arises, naturally, as to what restrictions must be placed on the functions defined by this parametric representation discussed above. Serious complications result in the theory when any attempt is made to obtain a great amount of generality in regard to these surfaces. We will, accordingly, place considerable restriction on the types of surfaces which we are intending to consider in this investigation. However, most of the familiar surfaces of solid analytic geometry will be covered under the scope of the definitions we are going to make.

In order to use the vector notation more effectively, we will write (x_1,x_2,x_3) instead of (x,y,z) and (t_1,t_2) instead of (u,v).

6.6.1 Definition. Let Γ be a rectifiable Jordan curve in E^2 and let R be the union of Γ with its interior. Suppose there exists an open set R' which contains R and vector-valued functions $\vec{x} = (x_1,x_2,x_3)$ such that $\vec{x} \in C^1$ on R'. Then the image of R under $\vec{x}$, say $S = \vec{x}(R)$ is called a parametric surface described by $\vec{x}$. If, also, $\vec{x}$ is one-to-one on R, then S is a simple parametric surface. In such case the image of Γ will be a rectifiable Jordan curve called the edge of S.

Remark: The definition above is too general for our purposes, so we will impose the following further restrictions on the function $\vec{x}$. Define vectors $D_1 \vec{x}$ and $D_2 \vec{x}$ as follows:

(1) $$D_1 \vec{x}(\vec{t}) = D_1 x_1(\vec{t}) \vec{u}_1 + D_1 x_2(\vec{t}) \vec{u}_2 + D_1 x_3(\vec{t}) \vec{u}_3,$$

$$D_2 \vec{x}(\vec{t}) = D_2 x_1(\vec{t}) \vec{u}_1 + D_2 x_2(\vec{t}) \vec{u}_2 + D_2 x_3(\vec{t}) \vec{u}_3,$$

where $\vec{t} = (t_1,t_2) \varepsilon R$. Points $\vec{x}(\vec{t})$ on S, where the cross product $D_1 \vec{x}(\vec{t}) \times D_2 \vec{x}(\vec{t}) \neq \vec{0}$, are called regular points of $\vec{x}$, and points where $D_1 \vec{x}(\vec{t}) \times D_2 \vec{x}(\vec{t}) = \vec{0}$ are called singular points of $\vec{x}$. We will assume in our development that all excepting possibly a finite number of points of S are regular points of $\vec{x}$.

Let us consider a horizontal line segment in R. Its image under $\vec{x}$ is a curve (called a t_1-curve) which lies on the surface S. The vector $D_1 \vec{x}$ represents the vector velocity of this curve. In like manner, $D_2 \vec{x}$ is the velocity vector of a t_2-curve, obtained by setting t_1 as a constant. There is a t_1 curve and a t_2 curve passing through each point of the surface. The restriction $D_1 \vec{x}(\vec{t}) \times D_2 \vec{x}(\vec{t}) \neq \vec{0}$ means that the velocity vectors $D_1 \vec{x}(\vec{t})$ and $D_2 \vec{x}(\vec{t})$ are not collinear at this point. Thus, for each regular point, $D_1 \vec{x}(\vec{t})$ and $D_2 \vec{x}(\vec{t})$ determine a plane called the tangent plane to the surface at the point $\vec{x}(\vec{t})$. The vector $D_1 \vec{x}(\vec{t}) \times D_2 \vec{x}(\vec{t})$ is normal to this plane.

The cross-product $D_1 \vec{x} \times D_2 \vec{x}$ plays an important role in the theory of surfaces. Its components can be expressed as Jacobians by means of the following theorem.

6.6.2 Theorem.

If $D_1 \vec{x}$ and $D_2 \vec{x}$ are defined as in (1) of this section, then

$$(2) \qquad D_1 \vec{x} \times D_2 \vec{x} = \frac{\partial(x_2,x_3)}{\partial(t_1,t_2)} \vec{u}_1 + \frac{\partial(x_3,x_1)}{\partial(t_1,t_2)} \vec{u}_2 + \frac{\partial(x_1,x_2)}{\partial(t_1,t_2)} \vec{u}_3 .$$

Proof:

We have

$$(2) \qquad D_1 \vec{x} \times D_2 \vec{x} = \begin{vmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ D_1x_1 & D_1x_2 & D_1x_3 \\ D_2x_1 & D_2x_2 & D_2x_3 \end{vmatrix}$$

$$= \begin{vmatrix} D_1x_2 & D_1x_3 \\ D_2x_2 & D_2x_3 \end{vmatrix} \vec{u}_1 - \begin{vmatrix} D_1x_1 & D_1x_3 \\ D_2x_1 & D_2x_3 \end{vmatrix} \vec{u}_2$$

$$+ \begin{vmatrix} D_1x_1 & D_1x_2 \\ D_2x_1 & D_2x_2 \end{vmatrix} \vec{u}_3$$

$$= \frac{\partial(x_2,x_3)}{\partial(t_1,t_2)} \vec{u}_1 + \frac{\partial(x_3,x_1)}{\partial(t_1,t_2)} \vec{u}_2 + \frac{\partial(x_1,x_2)}{\partial(t_1,t_2)} \vec{u}_3 .$$

6.7 Explicit Representation of a Parametric Surface.

Suppose we write (2) of the preceding section in the form $D_1 \vec{x}(\vec{t}) \times D_2 \vec{x}(\vec{t}) = J_1(\vec{t})\ \vec{u}_1 + J_2(\vec{t})\ \vec{u}_2 + J_3(\vec{t})\ \vec{u}_3$, where $\vec{t} = (t_1,t_2) \in R$, and where J_1, J_2, J_3 denote the corresponding Jacobians. At a regular point not all three of these Jacobians can be zero. Suppose, to fix the ideas, that $J_3(\vec{t}_o) \neq 0$ at an interior point of R, and write the vector equation of S as three scalar equations, say

$$(3) \qquad x_1 - x_1(t_1,t_2) = 0, \qquad x_2 - x_2(t_1,t_2) = 0,$$
$$x_3 - x_3(t_1,t_2) = 0.$$

Since $J_3(\vec{t}_o) \neq 0$, we can solve the first two equations in (3) for t_1 and t_2 in terms of x_1 and x_2; i.e., if $y_1 = x_1(\vec{t}_o)$ and $y_2 = x_2(\vec{t}_o)$, then there is an E^2 neighborhood $V(y_1,y_2)$ and a vector-valued function $\vec{g} = (g_1,g_2)$ such that the equations

$$(4) \qquad t_1 = g_1(x_1,x_2), \qquad \text{and } t_2 = g_2(x_1,x_2)$$

are valid whenever $(x_1,x_2) \in V(y_1,y_2)$, and when we substitute (4) into (3), the first two equations in (2) are satisfied identically. The third equation in (2) becomes

$$(5) \qquad x_3 = x_3\big(g_1(x_1,x_2),g_2(x_1,x_2)\big) = \phi(x_1,x_2), \text{ say.}$$

This implies that we have, always, an explicit representation of S, at least locally, in a neighborhood of each regular point.

It can happen that equation (5) describes all of S. In such a case, we can identify the t_1t_2-plane and the x_1x_2-plane and the vector equations of S can be written,

$$(6) \qquad \vec{x}(\vec{t}) = t_1\,\vec{u}_1 + t_2\,\vec{u}_2 + \phi(t_1,t_2)\,\vec{u}_3, \text{ where } t \in R.$$

When a parametric surface is described by an equation of the form (6), the set R is called the projection of S on the x_1x_2-plane. When (6) holds, we see that the fundamental vector product becomes

$$D_1\,\vec{x} \times D_2\,\vec{x} = -\,D_1\,\phi\,\vec{u}_1 - D_2\,\phi\,\vec{u}_2 + \vec{u}_3.$$

Hence, the vector $D_1\,\vec{x} \times D_2\,\vec{x}$, has always, a positive component in the $\vec{u}_1$ direction. Similar statements to the above hold if we interchange the roles of x_2 and x_3 or those of x_1 and x_3.

6.8 Area Number of a Parametric Surface.

Let us consider a parametric surface S described by a vector-valued function $\vec{x}$ defined on a region R of E^2. Let us write $\vec{V}_1 = D_1 \vec{x}(\vec{t})$ and $\vec{V}_2 = D_2 \vec{x}(\vec{t})$, where $\vec{t} = (t_1,t_2) \in R$. If we consider t_1 and t_2 ar representing time, then, when t_1 increases by Δt_1, a point originally at $\vec{x}(\vec{t})$ moves along a t_1-curve a distance equal, approximately, to $|\vec{V}_1|\ \Delta t_1$ (since $|\vec{V}_1|$ represents the velocity along the t_1-curve). Similarly, in time Δt_2 a point of a t_2-curve moves a distance equal, approximately, to $|\vec{V}_2|\ \Delta t_2$. Thus, a rectangle in R having area $\Delta t_1\ \Delta t_2$ is traced onto a portion of S that is approximately a parallelogram whose sides are the vectors $\vec{V}_1\ \Delta t_1$ and $\vec{V}_2\ \Delta t_2$. The area number of the parallelogaam determined by the vectors $\vec{V}_1\ \Delta t_1$ and $\vec{V}_2\ \Delta t_2$ is the length of their cross product; namely,

$$\begin{aligned} |(\vec{V}_1\ \Delta t_1) \times (\vec{V}_2\ \Delta t_2)| &= |\vec{V}_1 \times \vec{V}_2|\ \Delta t_1\ \Delta t_2 \\ &= |D_1\ \vec{x}(\vec{t}) \times D_2\ \vec{x}(\vec{t})|\ \Delta t_1\ \Delta t_2\ . \end{aligned}$$

Thus, the number $|D_1\ \vec{x}(\vec{t}) \times D_2\ \vec{x}(\vec{t})|$ represents what is called a local magnification factor for area, and this observation suggests the following definition for surface area.

6.8.1 Definition. Let S be a parametric surface described by a vector-valued function $\vec{x}$ defined on a region R in E^2. The area number of S is defined to be the value of the following double integral:

$$\iint_R |D_1\ \vec{x}(\vec{t}) \times D_2\ \vec{x}(\vec{t})|\ d\ (t_1,t_2)\ .$$

6.9 The Sum of Parametric Surfaces.

Let R_1 and R_2 be two closed regions in E^2, the boundaries of which are Γ_1 and Γ_2, respectively, and Γ_1 and Γ_2 are rectifiable Jordan curves. Let us assume that the inner region of Γ_2 is outside that of Γ_1 and $\Gamma_1 \cap \Gamma_2$ is an arc joining two distinct points. Let S_1 and S_2 be parametric surfaces described by vector-valued functions $\vec{x}$ and $\vec{y}$ defined on R_1 and R_2, respectively. Assume that $\vec{x}$ and $\vec{y}$ map $\Gamma_1 \cap \Gamma_2$ onto the same arc; i.e., assume that $\vec{x}(\Gamma_1 \cap \Gamma_2) = \vec{y}(\Gamma_1 \cap \Gamma_2)$. Let $C_1 = \vec{x}(\Gamma_1)$ and $C_2 = \vec{y}(\Gamma_2)$ be the edges of S_1 and S_2 and assume, further, that $S_1 \cap S_2 = C_1 \cap C_2$. This means that S_1 and S_2 must intersect at least along part of an edge, but at no points other than points of $C_1 \cap C_2$. The union $S_1 \cup S_2$ is called the sum of the surfaces S_1 and S_2 and is denoted by $S_1 + S_2$.

If $C_1 \cap C_2 = C_1 = C_2$, then the sum $S_1 + S_2$ is called a closed surface. Otherwise, the set $(C_1 \cup C_2) - (C_1 \cap C_2)$ is called the edge of $S_1 + S_2$. In our investigation, we will restrict ourselves to the consideration of those surfaces $S_1 + S_2$, whose edges are the union of at most a finite number of simple closed curves.

If the sum $S_1 + S_2$ is not a closed surface, then it has an edge (say C) and we can define $(S_1 + S_2) + S_3$, where S_3 is an appropriate parametric surface. We must assume that the regions R_1 and R_2

associated with S_1 and S_2 have exactly one arc in common. The functions which describe S_2 and S_3 must map $R_2 \cap R_3$ onto the same set, and we must have $(S_1 + S_2) \cap S_3 = C \cap C_3$. When these conditions hold, the union $(S_1 + S_2) \cup S_3$ is called the sum $(S_1 + S_2) + S_3$. The addition can be shown to be associative. We can extend the process to a finite number of summands, provided that the addition is not defined if one of the summands is a closed surface. We will restrict our work to surfaces formed in this way by adding a finite number of parametric surfaces. In addition, we will assume the edge (if any) is the union of a finite number of simple closed areas. The area of a sum of parametric surfaces is defined to be the sum of the areas of the individual parts.

6.10 Surface Integrals.

Suppose S to be a parametric surface described by a vector-valued function $\vec{x} = (x_1, x_2, x_3)$ defined on a region R in E^2. At the regular points we can define two vector-valued functions $\vec{n}_1$ and $\vec{n}_2$:

$$(1) \qquad \vec{n}_1(\vec{t}) = \frac{D_1\,\vec{x}(\vec{t}) \times D_2\,\vec{x}(\vec{t})}{|D_1\,\vec{x}(\vec{t}) \times D_2\,\vec{x}(\vec{t})|},$$

$$\vec{n}_2(\vec{t}) = -\,\vec{n}_1(\vec{t}), \text{ where } \vec{t} \in R.$$

For each $\vec{t}$, both vectors $\vec{n}_1(\vec{t})$ and $\vec{n}_2(\vec{t})$ are unit vectors normal to the surface.

6.10.1 Definition. Let $\vec{f} = (f_1,f_2,f_3)$ be a vector-valued function defined on the parametric surface S described above. Define $\vec{F}(\vec{t}) = \vec{f}(\vec{x}(\vec{t}))$, where $\vec{t} \in R$, and let $\vec{n}$ denote either of the two normals $\vec{n}_1$ or $\vec{n}_2$ described by (1). The surface integral of $\vec{f} \cdot \vec{n}$ over S, denoted by $\iint_S \vec{f} \cdot \vec{n} \, d\,\sigma$, is defined by the following equations

$$(2) \quad \iint_S \vec{f} \cdot \vec{n} \, d\,\sigma = \iint_R \vec{F}(\vec{t}) \cdot \vec{n}(\vec{t}) \, |D_1 \vec{x}(\vec{t}) \times D_2 \vec{x}(\vec{t})| \, d\,(t_1,t_2),$$

whenever the double integral on the right exists.

Due to Theorem 6.6.2, the double integral in (2) can be written as the sum of three double integrals,

$$\pm \left\{ \iint_R F_1 \frac{\partial(x_2,x_3)}{\partial(t_1,t_2)} \, d\,(t_1,t_2) + \iint_R F_2 \frac{\partial(x_3,x_1)}{\partial(t_1,t_2)} \, d\,(t_1,t_2) + \iint_R F_3 \frac{\partial(x_1,x_2)}{\partial(t_1,t_2)} \, d\,(t_1,t_2) \right\},$$

where the plus or minus is used according as $\vec{n} = \vec{n}_1$, or $\vec{n} = \vec{n}_2$, respectively.

If S is described explicitly by an equation of the form $\vec{x}(\vec{t}) = t_1 \vec{u}_1 + t_2 \vec{u}_2 + \phi(t_1,t_2) \vec{u}_3$, we have

$$\iint_S \vec{f} \cdot \vec{n} \, d\,\sigma = \pm \iint_R (- F_1 D_1 \phi - F_2 D_2 \phi + F_3) \, d\,(t_1,t_2) .$$

6.11 Triple Integrals.

Let V be a region in E^3 enclosed by a surface S, and suppose that $f(x_1,x_2,x_3)$ is a function which is single-valued and continuous in V. Let us make a partition Δ_n of V into n subregions $V_{k,n}$, $k = 1,2,3,\ldots,n$, with respective volumes $\Delta V_{k,n}$, where the norm of the partition, $\|\Delta_n\|$, is the diameter of the $V_{k,n}$ of maximum diameter, $k = 1,2,3,\ldots,n$. Let $\vec{x}_{k,n} = (x'_{k,n},x''_{k,n},x'''_{k,n})$ be a point of $V_{k,n}$, $k = 1,2,3,\ldots,n$. Then, we will define the triple integral of f over V to be

$$(1) \qquad \iiint_V f \; d\,V = \lim_{\|\Delta_n\|\to 0} \sum_{k=1}^{n} f(x'_{k,n},x''_{k,n},x'''_{k,n}) \; \Delta V_{k,n},$$

when this limit exists.

This limit is independent of the manner in which V is divided to subregions, since f is single-valued and continuous in V.

We are interested in triple integrals which have integrands which are vector functions. Thus, if $\vec{F} = (f_1,f_2,f_3)$ is a vector field which is single-valued and continuous in V, we will have, following the definition of integration of vectors,

$$(2) \qquad \iiint_V \vec{F} \; d\,V = \vec{u}_1 \int_V F_1 \; d\,V + \vec{u}_2 \int_V F_2 \; d\,V + \vec{u}_3 \int_V F_3 \; d\,V \; .$$

The triple integrals are evaluated using the methods developed in elementary calculus.

6.12 Green's Theorems.

We will consider Green's Theorems in both E^2 and E^3 in this section.

6.12.1 Green's Theorem in E^2.

1. S is a closed region in E^2 bounded by a curve C.
2. $\vec{F}$ is a vector field which is continuous and has continuous first derivatives in S.
3. $\vec{t}$ is a unit tangent vector to C in the positive direction.
4. $\vec{u}_3$ is a unit vector which forms with unit vectors $\vec{u}_1$ and $\vec{u}_2$ a right handed triad.

$$\Longrightarrow \quad \iint_S \vec{u}_3 \cdot (\nabla \times \vec{F})\, d\,\sigma = \int_C \vec{F} \cdot \vec{t}\, d\, S\ .$$

Proof:

We will note that the conclusion of the theorem can be written

$$(1) \qquad \iint_S \left(\frac{\partial F_1}{\partial x_2} - \frac{\partial F_2}{\partial x_2}\right) d\,\sigma = -\int_C (F_1\, d\, x_1 + F_2\, d\, x_2)\ .$$

We will prove, first, that

$$(2) \qquad \iint_S \frac{\partial F_1}{\partial x_2}\, d\,\sigma = -\int_C F_1\, d\, x_1\ ,$$

in the case where C can be intersected by a straight line parallel to the x_2 axis in two points at most. Suppose there are two points D and E where the tangent to C is parallel to the x_2-axis. Let d and e be the abscissae of D and E, respectively. These points divide C into two parts C' and C". At a general point $\vec{x} = (x_1,x_2)$ in S we will introduce an element of area lying in a strip parallel to the x_2-axis, the left edge of the strip intersecting C' and C" at the points $\vec{x}' = (x_1,x_2')$ and $\vec{x}'' = (x_1,x_2'')$, respectively.

Then,

$$\iint_S \frac{F_1}{x_2}\, d\,\sigma = \int_d^e \left[\int_{x_2''}^{x_2'} \frac{F_1}{x_2}\, d\,x_2\right] d\,x_1$$

$$= \int_d^e \Big[F_1(x_1, x_2)\Big]_{x_2''}^{x_2'} d\,x_1$$

$$= \int_d^e F_1(x_1, x_2')\; d\,x_1 - \int_d^e F_1(x_1, x_2'')\; d\,x_1$$

$$= -\int_e^d F_1(x_1, x_2')\; d\,x_1 - \int_d^e F_1(x_1, x_2'')\; d\,x_1$$

$$= -\int_C F_1\; d\,x_1\;.$$

Let us consider, now, the case where C can be intersected by a straight line parallel to the x_2-axis in more than two points. Here we need only to join the points F and G where there are tangents parallel to the x_2-axis by a curve K which is contained in S and which cannot be intersected by a straight line parallel to the x_2-axis in more than one point. The curve K divides S into two parts for both of which (2) holds. Hence, if we apply (2) to both portions, the two line integrals over K cancel, and we establish, thus, (2) for the entire region S. In a similar manner, we can establish (2) for the case where several curves such as K are required. We can proceed likewise when S is multiply connected; i.e., when S has holes and C consists of several isolated parts. In a manner analogous to the above, we can prove

(3) $$\iint_S \frac{F_2}{x_1}\, d\,S = -\int_C F_2\; d\,x_2\;.$$

Then, subtracting (3) from (2), we have (1), which completes our proof.

6.12.2 Green's Theorem in E^3.

1. V is a closed E^3 region bounded by a surface S.
2. $\vec{F}$ is a vector field which is continuous and has continuous first derivatives in V.
3. $\vec{n}$ is the unit outer normal vector to S.

$$\Longrightarrow \quad \iiint_V \nabla \cdot \vec{F} \, d\, V = \iint_S \vec{b} \cdot \vec{n} \, d\, \sigma \, .$$

Proof:

This theorem is called, also, the divergence theorem, and can be written in the form

$$(1) \qquad \iiint_V \left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}\right) d\, V = \iint_S (b_1 n_1 + b_2 n_2 + b_3 n_3)\, d\, \sigma \, .$$

We will prove, first, that

$$(2) \qquad \iiint_V \frac{\partial F_3}{\partial x_3}\, d\, V = \iint_S b_3 n_3 \, d\, \sigma,$$

in the case where S can be intersected by a straight line parallel to the x_3-axis in two points at most. Let T be the projection of S on the x_1x_2 plane. On S there is a curve C consisting of points where the tangent plane to S is parallel to the x_3-axis. The curve C divides S into two parts S' and S". At a point $\vec{x} = (x_1, x_2, x_3)$ in V we will introduce an element of volume lying in a prism parallel to the x_3-axis, the vertical line through $\vec{x}$ meeting S' and S" at the points $\vec{x}' = (x_1, x_2, x_3')$ and $\vec{x}'' = (x_1, x_2, x_3'')$.

Thus,

$$(3) \qquad \iiint_V \frac{\partial F_3}{\partial x_3} \, d\,V = \iint_T \left[\int_{x_3''}^{x_3'} \frac{\partial F_3}{\partial x_3} \, d\,x_3 \right] d\,(x_2,x_1)$$

$$= \iint_T \{F_3(x_1,x_2,x_3') - F_3(x_1,x_2,x_3'')\} \, d\,(x_2,x_1) \ .$$

Let $\vec{n}'$ be the unit outer normal vector at $\vec{x}'$ and let dS' be the area number of the element cut from S' by the vertical prism. Let us define $\vec{n}''$ and dS" at $\vec{x}''$ in a similar manner. Then,

$$d(x_2,x_1) = n_3' \, d\,S' = - n_3'' \, d\,S'',$$

and we can write (3) as

$$\iiint_V \frac{\partial F_3}{\partial x_3} \, d\,V = \iint_{S'} F_3(x_1,x_2,x_3') \, n_3' \, d\,S' + \iint_{S'} F_3(x_1,x_2,x_3'') \, n_3'' \, d\,S''$$

$$= \iint_S F_3 \, n_3 \, d\,\sigma \ .$$

Let us consider, now, the case where S can be intersected by a vertical line in more than two points. In such cases, we can divide V into a number of regions V_1, V_2, V_3, . . . , V_m by intersecting V by a number of surfaces k_1, k_2, k_3, . . . , k_m so chosen that the boundary of each of the regions V_i, $i = 1,2,3,\ldots,m$, can be intersected by a vertical line in at most two points. The proof above of (2) applies, then, to the regions V_i, $i = 1,2,3,\ldots,m$. If we proceed in this manner, the surface integrals over the k_i, $i = 1,2,3,\ldots,m$, cancel, and we establish, thus, (2) for the region V.

In a similar manner to the above, we can prove that

$$(4) \qquad \iiint_V \frac{\partial F_2}{\partial x_2} \, d\,V = \iint_S b_2 \, n_2 \, d\,\sigma \ , \text{ and } \iiint_V \frac{\partial F_1}{\partial x_1} \, d\,V = \iint_S b_1 \, n_1 \, d\,\sigma.$$

When we add equations (2) and (4), we obtain (1), and our proof is complete.

Remark. If f is a scalar field with continuous second-order derivatives, then we can set $\vec{F} = \nabla f$ and substituting in

$$\iiint_V \nabla \cdot \vec{F} \, d\, V = \iint_S \vec{F} \cdot \vec{n} \, d\, \sigma,$$

we obtain

$$\iiint_V \nabla \cdot (\nabla f) \, d\, V = \iint_S \nabla f \cdot \vec{n} \, d\, \sigma,$$

or,

$$(5) \qquad \iiint_V (\nabla \cdot \nabla) f \, d\, V = \iint_S D_n f \, d\, \sigma,$$

where $\nabla \cdot \nabla$ is the Laplacian operator ∇^2 and $D_n f$ is the directional derivative of f in the direction of the outer normal to the surface S.

6.12.3 The Symmetric Form of Green's Theorem.

Let f and g be scalar fields with continuous second derivatives in a closed region V bounded by a surface S. Then, we can apply Green's Theorem as stated, but with the vector $\vec{F}$ replaced by $f \nabla g$. We have

$$(1) \qquad \iiint_V \nabla \cdot (f \nabla g) \, d\, V = \iint_S f \nabla g \cdot \vec{n} \, d\, \sigma .$$

However, $\nabla \cdot \nabla g = f (\nabla \cdot \nabla) g + \nabla f \cdot \nabla g$

$$= f \nabla^2 g + \nabla f \cdot \nabla g.$$

Also, $\nabla g \cdot \vec{n}$ is equal to the directional derivative $D_n g$ of g in the direction of the outer normal $\vec{n}$ to S. Thus, (1) becomes

$$(2) \qquad \iiint_V (f \nabla^2 g + \nabla f \cdot \nabla g)\, d V = \iint_S f D_n g \, d \sigma .$$

In a similar manner, by making an interchange of f and g in the above relation, we have

$$(3) \qquad \iiint_V (g \nabla^2 f + \nabla g \cdot \nabla f)\, d V = \iint_S g D_n f \, d \sigma.$$

Subtraction of (3) from (2) gives

$$(4) \qquad \iiint_V (f \nabla^2 g - g \nabla^2 f)\, d V = \iint_S (f D_n g - g D_n f)\, d \sigma .$$

This equation is known as the symmetric form of Green's Theorem.

6.13 Stokes' Theorem.

1. S is a closed region on a surface.
2. C is the boundary of S.
3. $\vec{n} = (n_1, n_2, n_3)$ is the unit vector normal to S on the positive side.
4. The positive direction on C is that in which an observer would travel to have the interior of S on his left.
5. $\vec{t}$ is the unit vector tangent to C in the positive direction.
6. $\vec{F}$ is a vector field with continuous first derivatives in the closed region S.

$$\Longrightarrow \qquad \iint_S \vec{n} \cdot (\nabla \times \vec{F})\, d \sigma = \int_C \vec{F} \cdot \vec{t} \, d S,$$

where the integration around C is carried out in the positive direction.

Proof:

The theorem can be written, also, in the form

$$(1) \qquad \iint_S \{n_1 (\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}) + n_2 (\frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}) + n_3 (\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2})\} \, d\sigma$$

$$= \int_C (b_1 \, d x_1 + b_2 \, d x_2 + b_3 \, d x_3) \ .$$

First, we will prove that

$$(2) \qquad \iint_S \vec{n} \cdot (\nabla \times F_1 \, \vec{u}_1) \, d\sigma = \int_C F_1 \, d x_1 \ ,$$

in the case when S is a regular surface element and the positive side of S is the side on which the unit normal vector $\vec{n}$ points in the direction of increasing x_3. $\vec{t}$ is the unit tangent vector of C and the region S' of the $x_1 x_2$ plane is the region into which S projects.

Now,

$$(3) \qquad \iint_S \vec{n} \cdot (\nabla \times F_1 \, \vec{u}_1) \, d\sigma = \iint_S \vec{n} \cdot (\vec{u}_2 \frac{\partial F_1}{\partial x_2} - \vec{u}_3 \frac{\partial F_1}{\partial x_2}) \, d\sigma \ .$$

Suppose the equation of the surface is $x_3 = g(x_1,x_2)$. Then, on S, we have $F_1(x_1,x_2,x_3(x_1,x_2)) = c_1(x_1,x_2)$,

$$(4) \qquad \frac{\partial c_1}{\partial x_2} = \frac{\partial F_1}{\partial x_2} + \frac{\partial F_1}{\partial x_3} \frac{\partial x_3}{\partial x_2} \ .$$

Then, we substitute from this equation for $\frac{\partial F_1}{\partial x_2}$ in equation (3), and obtain

$$(5) \qquad \iint_S \vec{n} \cdot (\nabla \times F_1 \, \vec{u}_1) \, d S = - \iint_S \vec{n} \cdot \vec{u}_3 \frac{\partial c_1}{\partial x_2} \, d S$$

$$+ \iint_S \vec{n} \cdot (\vec{u}_2 + \vec{u}_3 \frac{\partial x_3}{\partial x_2}) \frac{\partial F_1}{\partial x_3} \, d S$$

$$= - I_1 + I_2 \ ,$$

where I_1 and I_2 are the two integrals on the right-hand side of equation (5).

We will consider I_1. We have $\vec{n} \cdot \vec{u}_3 \, d\,S = n_3 \, d\,S = d\,S'$, where $d\,S'$ is the projection of $d\,S$ on the x_1x_2 plane. Since c_1 is a function of x_1 and x_2 only, we have

$$I_1 = -\iint_{S'} \frac{\partial c_1}{\partial x_2} \, d\,\sigma \, .$$

By Green's Theorem in the plane, we have

$$(6) \qquad I_1 = \int_{C'} c_1(x_1,x_2) \, d\,x_1 = \int_{C'} b_1(x_1,x_2,x_3(x_1,x_2)) \, d\,x_1$$

$$= \int_C b_1 \, d\,x_1 \, .$$

Now, consider I_2. The position vector of a point $\vec{x}$ on S is $\vec{x} = x_1 \, \vec{u}_1 + x_2 \, \vec{u}_2 + x_3(x_1,x_2) \, \vec{u}_3$.

Hence,

$$\frac{\partial \vec{x}}{\partial x_1} = \vec{u}_2 + \frac{\partial x_3}{\partial x_2} \vec{u}_3, \text{ and}$$

$$(7) \qquad I_2 = \iint_S \vec{n} \cdot \frac{\partial \vec{x}}{\partial x_2} \frac{\partial F_1}{\partial x_3} \, d\,\sigma \, .$$

However, the vector $\frac{\partial \vec{x}}{\partial x_2}$ is tangent at $\vec{x}$ to the curve of intersection of S and a plane parallel to the x_2x_3 plane. Thus, this vector is tangent to S and is perpendicular to the unit normal vector $\vec{n}$, and we have $\vec{n} \cdot \frac{\partial \vec{x}}{\partial x_2} = 0$.

Thus, equation (7) means $I_2 = 0$, and from equations (5) and (6), we can conclude that the conclusion (2) is true.

If the positive side of S is so chosen that the unit normal vector $\vec{n}$ points in the direction of decreasing x_3, the proof of

$\iint_S \vec{n} \cdot (\nabla \times F_1 \vec{u}_1)\, d\sigma = \int_C F_1\, d x_1$ is similar to that above, the only differences in the proof being that in the present case n_1 is negative and the direction of integration around the curve C is opposite to that in the proof above.

If the surface S is not a regular surface element, we divide it into a number of regular surface elements S_k, $k = 1,2,3,\ldots,m$, by a number of curves L_k, $k = 1,2,3,\ldots,m$. The proof above for (2) applies to the regions S_k, $k = 1,2,3,\ldots,m$. If we apply (2) to these regions, and add, the line integrals over L_k, $k = 1,2,3,\ldots,m$, cancel, and we have equation (2) is true for the region S.

In a manner similar to that employed above, we can prove

(8) $$\iint_S \vec{n} \cdot (\nabla \times F_2 \vec{u}_2)\, d\sigma = \int_C F_2\, d x_2$$

and

$$\iint_S \vec{n} \cdot (\nabla \times F_3 \vec{u}_3)\, d\sigma = \int_C F_3\, d x_3 \ .$$

On the addition of equations (2) and (8), we have equation (1), and our proof is complete.

6.14 Integration Formulae.

We have established Green's Theorem in E^3 and Stoke's Theorem. These theorems are integration formulae written in the form

(1) $$\iiint_V \nabla \cdot \vec{F}\, d V = \iint_S \vec{n} \cdot \vec{F}\, d\sigma \quad ,$$

and

(2) $$\iint_S (n \times \nabla) \cdot \vec{F} \, d\sigma = \int_C \vec{t} \cdot \vec{F} \, dS .$$

Each of these formulae involves a vector field $\vec{F}$, and (1) represents a transformation from a triple integral to a surface integral, and (2) represents a transformation from a surface integral to a line integral. We will introduce, now, four other integration formulae in the form of two theorems.

6.14.1 Theorem.

1. V is a closed region in E^3 bounded by a surface S with the unit outer normal $\vec{n}$ as in the case of Green's Theorem in E^3.
2. f is a scalar field with continuous first derivatives in V.
3. $\vec{F}$ is a vector field with continuous first derivatives in V.

$\Longrightarrow$ (3) $$\iiint_V \nabla f \, dV = \iint_S \vec{n} f \, d\sigma \text{ , and}$$

(4) $$\iiint_V \nabla \times \vec{F} \, dV = \iint_S \vec{n} \times \vec{b} \, d\sigma .$$

Proof:

Let $\vec{c}$ be a constant vector field. If, in equation (1) of this section, we set $\vec{F} = f \vec{c}$, we have

(5) $$\iiint_V \nabla \cdot (f \vec{c}) \, dV = \iint_S \vec{n} \cdot f \vec{c} \, d\sigma .$$

However, $\nabla \cdot (f \vec{c}) = \nabla f \cdot \vec{c}$, since $\vec{c}$ is a constant vector. Then, equation (5) can be written

$$\vec{c} \cdot \{\iiint_V \nabla f \, dV - \iint_S \vec{n} f \, d\sigma\} = 0 .$$

Since $\vec{c}$ is a constant vector, then

$$\iiint_V \nabla f \, d V - \iint_S \vec{n} f \, d \sigma = 0,$$

and thus we have equation (3).

To prove (4), we introduce, as in the proof of (3), the constant vector field $\vec{c}$, but in equation (1) we replace $\vec{F}$ by $\vec{F} \times \vec{c}$ to obtain

$$(6) \qquad \iiint_V \nabla \cdot (\vec{F} \times \vec{c}) \, d V = \iint_S \vec{n} \cdot (\vec{F} \times \vec{c}) \, d \sigma .$$

Since $\vec{c}$ is a constant vector, by the permutation theorem for scalar triple products, we have

$$\nabla \cdot (\vec{F} \times \vec{c}) = \vec{c} \cdot (\nabla \times \vec{F}), \text{ and}$$

$$\vec{n} \cdot (\vec{F} \times \vec{c}) = \vec{c} \cdot (\vec{n} \times \vec{F}) .$$

Thus, equation (6) can be written

$$\vec{c} \cdot \{\iiint_V \nabla \times \vec{F} \, d V - \iint_S \vec{n} \times \vec{F} \, d \sigma\} = 0.$$

Since $\vec{c}$ is a constant vector, we can conclude that (4) is true.

6.14.2 Theorem.

1. S is a closed region lying on a surface and bounded by a curve C.
2. $\vec{n}$ is the unit positive normal vector to S.
3. $\vec{t}$ is the unit positive tangent vector to C, as in the case of Stoke's Theorem.
4. f is a scalar field with continuous first derivatives in S.
5. $\vec{F}$ is a vector field with continuous first derivatives in S.

$$\Longrightarrow \quad (7) \qquad \iint_S (\vec{n} \times \nabla)\, f \, d\,\sigma = \int_C \vec{t}\, f \, d\,S$$

$$(8) \qquad \iint_S (\vec{n} \times \nabla) \times \vec{F} \, d\,\sigma = \int_C \vec{t} \times \vec{F} \, d\,S.$$

Proof:

To prove (7) and (8), we replace $\vec{F}$ in equation (2) be f $\vec{c}$ and then by $\vec{F} \times \vec{c}$. The procedure follows that of the previous proof:

6.14.3 A Compact Form for the Integration Formulae.

The six integration formulae which we have derived may be written very compactly in the form

$$(9) \qquad \iiint_V \nabla * \, T \, d\,V = \iint_S n * \, T \, d\,\sigma,$$

$$(10) \qquad \iint_S (\vec{n} \times \nabla) * \, T \, d\,\sigma = \int_C \vec{t} * \, T \, d\,S ,$$

where T can denote a scalar field or a vector field, and the asterisk has the following meanings: if T is a scalar field, then * denotes the multiplication of a vector and a scalar; and if T denotes a vector field, then * denotes either scalar or vector multiplication.

6.15 Irrotational Vectors.

A vector field $\vec{F}(x_1,x_2,x_3)$ is irrotational in a region V in E^3, iff, everywhere in V, we have

$$(1) \qquad \nabla \times \vec{F} = \vec{0}.$$

Suppose ϕ is any scalar field with continuous second derivatives; and let us write $\vec{F} = \nabla\, \phi$.

Then,

$$\nabla \times \vec{F} = \nabla \times \nabla \phi = \vec{0};$$

hence, a vector $\vec{F}$ defined as the gradient of a scalar field is irrotational.

We will show that an irrotational vector field has the following properties:

(a) Its integral around every reducible circuit in V is zero;

(b) When V is simply connected $\vec{F}$ is the gradient of a scalar field.

To prove property (a), we will consider a general circuit in V which is reducible; i.e., it can be contracted to a point without leaving V. Suppose S is a surface which lies entirely in V and is bounded by C. Let us assume $\vec{F}$ has continuous first derivatives, then Stokes' Theorem gives

$$\int_C \vec{F} \cdot \vec{t} \; d\, S = \iint_S \vec{n} \cdot (\nabla \times \vec{F}) \; d\, \sigma = 0,$$

by (1).

To prove property (a), let $\vec{x}$ be a general point in V, and let $\vec{x}_o$ be a given point. Also, let C' and C" be any two paths in V from $\vec{x}_o$ to $\vec{x}$. Property (a) informs us that the line integral of $\vec{F}$ from $\vec{x}_o$ to $\vec{x}$ is the same for paths C' and C" and hence has the same value for all paths in V from $\vec{x}_o$ to $\vec{x}$. Thus, if we write

$$(2) \qquad \phi = \int_{\vec{x}_o}^{\vec{x}} \vec{F} \cdot d\, \vec{x} \; ,$$

then ϕ depends only on the coordinates (x_1,x_2,x_3) of $\vec{x}$. If we take the derivative of equation (2) with respect to S, we have

$$(3) \qquad \frac{d\phi}{dS} = \vec{F} \cdot \frac{d\vec{x}}{dS} \; .$$

But $\frac{d\phi}{dS}$ is the directional derivative of ϕ, and is equal to

$\nabla \phi \cdot (D_S \vec{x})$.

Hence, equation (3) can be written as $(\nabla \phi - \vec{F}) \cdot D_S \vec{x} = 0$, and since $D_S \vec{x}$ is an arbitrary vector, then

(4) $\quad \vec{F} = \nabla \phi,$

and the proof is complete.

The function ϕ is called a scalar potential function.

6.16 Solenoidal Vectors.

We say a vector field $\vec{F} = \vec{F}(x_1,x_2,x_3)$ is solenoidal in a region V, iff, everywhere in V we have

(1) $\quad \nabla \cdot \vec{F} = 0.$

Suppose $\vec{\phi}$ is a vector field with continuous second derivatives, and let us write

$$\vec{F} = \nabla \times \vec{\phi}.$$

Then, $\nabla \cdot \vec{F} = \nabla \cdot (\nabla \times \vec{\phi}) = 0.$

We will show that if $\vec{F}$ is any solenoidal field, there exists a vector field $\vec{\phi}$ such that $\vec{F} = \nabla \times \vec{\phi}$.

In order to prove this result, we must solve the scalar equations

(2) $\quad F_1 = D_{x_2} \phi_3 - D_{x_3} \phi_2,$

(3) $\quad F_2 = D_{x_3} \phi_1 - D_{x_1} \phi_3,$

(4) $\quad F_3 = D_{x_1} \phi_2 - D_{x_2} \phi_1,$

for ϕ_1, ϕ_2, and ϕ_3, where F_1, F_2, and F_3 are given functions subject to the condition

$$(5) \qquad D_{x_1} F_1 + D_{x_2} F_2 + D_{x_3} F_3 = 0.$$

Let us choose $\phi_1 = 0$. Then, from equations (3) and (4) by partial integrations with respect to x_1, we have

$$(6) \qquad \phi_2 = \int_{a_1}^{x_1} F_3 \, d\,\bar{x}_1 + \psi_2(x_2,x_3) \ ,$$

$$\phi_2 = -\int_{a_1}^{x_1} F_2 \, d\,\bar{x}_1 + \psi_3(x_2,x_3) \ ,$$

where a_1 is a constant and ψ_2 and ψ_3 are functions of x_2 and x_3 whose choices are arbitrary. To satisfy (2), we must have

$$F_1 = -\int_{a_1}^{x_1} (D_{x_2} F_2 + D_{x_3} F_3) \, d\,\bar{x}_1 + D_{x_2} \psi_3 - D_{x_3} \psi_2 \ .$$

Using equation (5), we can write

$$F_1 = \int_{a_1}^{x_1} D_{\bar{x}_1} F_1 \, d\,\bar{x}_1 + D_{x_2} \psi_3 - D_{x_3} \psi_2$$

$$= F_1(x_1,x_2,x_3) - b_1(a_1,x_2,x_3) + D_{x_2} \psi_3 - D_{x_3} \psi_2 \ .$$

This equation is satisfied if we choose

$$\psi_2 = 0,$$

$$\psi_3 = \int_{a_2}^{x_1} F_1(a_1,\bar{x}_2,x_3) \, d\,\bar{x}_2 \ ,$$

where a_2 is a constant.

Then, we have

$$\phi_1 = 0,$$

$$\phi_2 = \int_{a_1}^{x_1} F_3(\bar{x}_1, x_2, x_3)\, d\,\bar{x}_1 \quad ,$$

$$\phi_3 = -\int_{a_1}^{x_1} F_2(\bar{x}_1, x_2, x_3)\, d\,\bar{x}_1 + \int_{a_2}^{x_2} F_1(a_1, \bar{x}_2, x_3)\, d\,\bar{x}_2 \ ,$$

where all the integrations are partial integrations, and a_1 and a_2 are constants.

The function $\vec{\phi}$ is called a vector potential function.

In the proof above, we made several selections which were arbitrary, and this indicates that the solenoidal vector field $\vec{F}$ does not possess an unique vector potential function. To understand this fact, we let $\vec{\phi}$ be one vector potential function corresponding to the solenoidal vector field $\vec{F}$, and let f be any scalar field.

Then,

$\nabla \times (\vec{\phi} + \nabla f) = \nabla \times \vec{\phi} + \nabla \times \nabla f = \nabla \times \vec{\phi} = \vec{F}.$

Thus, $\vec{\phi} + \nabla f$ is a vector potential function; also, corresponding to the vector field $\vec{F}$.

If $\vec{F}$ is any vector field having continuous second derivaties in a region V, then it is possible to show that $\vec{F}$ can be expressed as the sum of an irrotational vector and a solenoidal vector, although we are not offering a proof of this result in the present investigation.

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