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BOUNDEDNESS AND EXPONENTIAL STABILITY OF SOLUTIONS TO DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. Making use of the generalized time scales exponential function, we give a new definition for the exponential stability of solutions for dynamic equations on time scales. Employing Lyapunov-type functions on time scales, we investigate the boundedness and the exponential stability of solutions to first-order dynamic equations on time scales, and some sufficient conditions are obtained. Some examples are given at the end of this paper.

1. Introduction

In this paper, we consider the boundedness and exponential stability of solutions to the first-order dynamic equations

$$x^{\Delta} = f(t, x) \quad t \geq t_0,\ t \in \mathbb{T}, \tag{1.1}$$

subject to the initial condition

$$x(t_0) = x_0 \quad t_0 \in \mathbb{T},\ x_0 \in \mathbb{R}^n, \tag{1.2}$$

where $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ is a so called rd-continuous function and t is from a so called "time scale" $\mathbb{T}$.

If $\mathbb{T} = \mathbb{R}$, then $x^{\Delta} = x'$ and (1.1), (1.2) is the following initial value-problem for ordinary differential equations,

$$x' = f(t, x), \tag{1.3}$$

$$x(t_0) = x_0. \tag{1.4}$$

If $\mathbb{T} = \mathbb{Z}$, then $x^{\Delta} = \Delta x$ (the forward difference calculus), and (1.1), (1.2) corresponds to the initial value-problem for the OΔE

$$x(n + 1) - x(n) = f(n, x(n)), \tag{1.5}$$

$$x(n_0) = x_0. \tag{1.6}$$

Recently Raffoul [6] used Lyapunov-type function to formulate some sufficient conditions that ensure all solutions of (1.3), (1.4) are uniformly bounded. In [7], by

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a kind of suitable and easy-to-calculate Lyapunov type I function on time scales, Peterson and Tisdell formulated some appropriate inequalities on these functions that guarantee solutions to (1.1), (1.2) are uniformly bounded. Other results on boundedness can be found, for example, in [3] and [4].

The investigation of stability analysis of nonlinear systems has produced a vast body of important results. In [10], Muhammad made use of non-negative definite Lyapunov functions to study the exponential stability of the zero solution of nonlinear discrete system (1.5), (1.6), and in [11], [12] they gave sufficient conditions for the exponential stability of a class of nonlinear time-varying differential equations.

In this paper, we first define the boundedness and the exponential stability of solutions on time scales, then making use of the Lyapunov-type function on time scales, we get sufficient conditions that guarantee the boundedness and exponential stability of zero solution to (1.1), (1.2), which generalize the results in [10, 11, 12]. Some examples are also presented at the end of this paper.

Throughout this paper, the following notation will be used: $\mathbb{R}^n$ is the n-dimensional Euclidean vector space; $\mathbb{R}^+$ is the set of all non-negative real numbers; $\|x\|$ is the Euclidean norm of a vector $x \in \mathbb{R}^n$.

2. Preliminaries

The theory of dynamic equations on time scales (or more generally, measure chains) was introduced in Stefan Hilger's PhD thesis in 1988. The theory presents a structure where, once a result is established for general time scales, special cases include a result for differential equations (obtained by taking the time scale to be the real numbers) and a result for difference equations (obtained by taking the time scales to be the integers). A great deal of work has been done since 1988, unifying the theory of differential equations and the theory of difference equations by establishing the corresponding results in time scale setting.

Definition 2.1. A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the set $\mathbb{R}$, with the topology and ordering inherited from $\mathbb{R}$. We assume throughout the paper that $\mathbb{T}$ is unbounded above.

Definition 2.2. For $t \in \mathbb{T}$ *the forward jump operator* $\sigma(t) : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}.$$

The jump operator σ gives the classifications of points on time scales. A point $t \in \mathbb{T}$ is called *right dense* if $\sigma(t) = t$, and *right scattered* if $\sigma(t) > t$. *The graininess function* $\mu(t) : \mathbb{T} \to \mathbb{R}$ is defined by $\mu(t) = \sigma(t) - t$.

Definition 2.3. Fix $t \in \mathbb{T}$ and let $x : \mathbb{T} \to \mathbb{R}^n$, define the *delta-derivative* $x^\Delta(t)$ of x at $t \in \mathbb{T}$ to be the vector (if it exists) with the property that given any $\epsilon > 0$, there is a neighborhood $U \subset \mathbb{T}$ of t such that

$$|[x_i(\sigma(t)) - x_i(s)] - x_i^\Delta(t)[\sigma(t) - s]| \le \epsilon|\sigma(t) - s|$$

for all $s \in U$ and $i = 1, 2, \dots, n$. At this time we say $x(t)$ is (delta) differentiable. In the case of $\mathbb{T} = \mathbb{R}$, $x^\Delta(t) = x'(t)$. When $\mathbb{T} = \mathbb{Z}$, $x^\Delta(t)$ is the standard forward difference operator $x(n+1) - x(n)$.

Definition 2.4. For $f : \mathbb{T} \to \mathbb{R}^n$ and $F : \mathbb{T} \to \mathbb{R}^n$, if $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}$, then F is said to be an *antiderivative* of f. And define the *cauchy integral* by the formula

$$\int_a^b f(\tau)\Delta\tau = F(b) - F(a) \quad \text{for } a, b \in \mathbb{T}.$$

Definition 2.5. A function $f : \mathbb{T} \to \mathbb{R}$ is called *right-dense continuous* provided it is continuous at right dense points of $\mathbb{T}$ and its left sided limit exists (finite) at left dense points of $\mathbb{T}$. The set of all right-dense continuous function on $\mathbb{T}$ is denoted by

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}).$$

Consequences of these definitions and properties can be found in [1, 2, 5].

Definition 2.6. A function $p : \mathbb{T} \to \mathbb{R}$ is *regressive* provided that

$$1 + \mu(t)p(t) \neq 0 \quad \text{for all } t \in \mathbb{T}.$$

The set of all regressive and right-dense continuous function is denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$. The set $\mathcal{R}^+$ of all *positively regressive* function is

$$\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}\}.$$

Definition 2.7. For $p(t) \in \mathcal{R}$, we define the *generalized exponential function* as

$$e_p(t, t_0) = \exp\Big(\int_{t_0}^t \frac{Log(1 + \mu(\tau)p(\tau))}{\mu(\tau)} \Delta\tau\Big) \quad \text{for } t_0, t \in \mathbb{T}.$$

Remark 2.8. Consider the dynamic initial-value problem

$$x^\Delta = p(t)x, \quad x(t_0) = x_0, \tag{2.1}$$

where $t_0 \in \mathbb{T}$ and $p(t) \in \mathcal{R}$. The exponential function $x(t) = x_0 e_p(t, t_0)$ is the unique solution to system (2.1).

Theorem 2.9. (i) *If $p \in \mathcal{R}^+$, then $e_p(t, t_0) > 0$;*
(ii) *$e_p(\sigma(t), t_0) = (1 + \mu(t)p(t))e_p(t, t_0)$;*
(iii) *$e_{\ominus p}(t, t_0) = \frac{1}{e_p(t,t_0)}$, where*

$$\ominus p = \frac{-p}{1 + p\mu(t)};$$

(iv) *If $p, q \in \mathcal{R}$, then $e_p(t, t_0)e_q(t, t_0) = e_{p\oplus q}(t, t_0)$;*
(v) *If p is a positive constant, then* $\lim_{t\to\infty} e_p(t, t_0) = \infty, \lim_{t\to\infty} e_{\ominus p}(t, t_0) = 0$.

Other relevant theorems can be found in [1],[2]. In the following discussions, we assume conditions are imposed on system (1.1),(1.2) such that the existence of solutions is guaranteed when $t \in \mathbb{T}_{t_0}^+ = \{t \in \mathbb{T} : t \geq t_0\}$.

3. Boundedness of Solutions

Definition 3.1. We say a solution $x(t)$ of system (1.1),(1.2) is *bounded* if there exists a constant $C(t_0, x_0)$ (that may depend on t_0 and x_0) such that

$$\|x(t)\| \leq C(t_0, x_0) \qquad \text{for} \quad t \in \mathbb{T}_{t_0}^+.$$

We say that solutions of (1.1),(1.2) are *uniformly bounded* if C is independent of t_0.

Assume $V : \mathbb{T}_{t_0}^+ \times \mathbb{R}^n \to \mathbb{R}^+$ is delta differentiable in variable t and continuously differentiable in variable x, and $x(t)$ is any solution of dynamic system (1.1), (1.2), then from [13, 14] we know the delta derivative along $x(t)$ for $V(t,x)$ is the following

$$\begin{aligned}V^\Delta(t,x) = V^\Delta(t,x(t)) &= V_t^\Delta(t,x(\sigma(t))) + \int_0^1 V_x'(t,x(t)+h\mu(t)x^\Delta(t))dh\,x^\Delta(t)\\ &= V_t^\Delta(t,x(\sigma(t))) + \int_0^1 V_x'(t,x(t)+h\mu(t)f(t,x))dh\,f(t,x),\end{aligned}$$

where V_t^Δ is considered as the delta derivative in the first variable t and V_x' is taken as the normal derivative in variable x. Then we call $V(t,x)$ a *Lyapunov-type function* on time scales.

To calculate the derivative is not an easy work generally, but if $V(t,x)$ is explicitly independent of t, i.e. $V : \mathbb{R}^n \to \mathbb{R}^+$ and $V(x) = V_1(x_1) + \dots + V_n(x_n)$, this is the type I Lyapunov function introduced in [7] and $(V \circ x)^\Delta(t)$ is easy to handle at this time.

In this section we want to point out that the results in [7] are held to be true theoretically for general Lyapunov-type function on time scales.

Theorem 3.2. *Assume D is an open and convex set in $\mathbb{R}^n$. Suppose there exists a Lyapunov-type function $V : \mathbb{T}_{t_0}^+ \times D \to \mathbb{R}^+$ that satisfies*

$$\lambda_1\|x\|^p \le V(t,x) \le \lambda_2\|x\|^q, \tag{3.1}$$

$$V^\Delta(t,x) \le \frac{-\lambda_3\|x\|^r + L}{1+M\mu(t)}, \tag{3.2}$$

$$V(t,x) - V^{r/q}(t,x) \le \gamma, \tag{3.3}$$

where $\lambda_1, \lambda_2, \lambda_3, p, q, r$ are positive constants, L and γ are nonnegative constants, and $M = \lambda_3/\lambda_2^{r/q}$. Then all solutions of (1.1), (1.2) *that stay in D are uniformly bounded.*

Proof. Note that $M = \lambda_3/\lambda_2^{r/q}$, so $M \in \mathcal{R}^+$ and $e_M(t,t_0)$ is well defined and is positive. From the derivative formula of products and condition (3.2),

$$\begin{aligned}(V(t,x)e_M(t,t_0))^\Delta &= V^\Delta(t,x)e_M(\sigma(t),t_0) + V(t,x)e_M^\Delta(t,t_0)\\ &\le \frac{-\lambda_3\|x\|^r + L}{1+M\mu(t)}(1+M\mu(t))e_M(t,t_0) + MV(t,x)e_M(t,t_0)\\ &= (-\lambda_3\|x\|^r + L + MV(t,x))e_M(t,t_0).\end{aligned}$$

From (3.1), we have $\|x\|^q \ge V(t,x)/\lambda_2$, consequently $-\|x\|^r \le (V(t,x)/\lambda_2)^{r/q}$. So by (3.3),

$$\begin{aligned}(V(t,x)e_M(t,t_0))^\Delta &\le [-(\lambda_3/\lambda_2^{r/q})V^{r/q}(t,x) + MV(t,x) + L]e_M(t,t_0)\\ &= [M(V(t,x) - V^{r/q}(t,x)) + L]e_M(t,t_0)\\ &\le (M\gamma + L)e_M(t,t_0).\end{aligned}$$

Integrating the above inequality from t_0 to t ($t \in \mathbb{T}^+_{t_0}$), we obtain

$$\begin{aligned} V(t,x)e_M(t,t_0) &\le V(t_0,x_0) + \frac{M\gamma + L}{M}(e_M(t,t_0) - e_M(t_0,t_0)) \\ &\le V(t_0,x_0) + \frac{M\gamma + L}{M} e_M(t,t_0) \\ &\le \lambda_2 \|x_0\|^q + \frac{M\gamma + L}{M} e_M(t,t_0). \end{aligned}$$

Hence

$$\begin{aligned} V(t,x) &\le \lambda_2 \|x_0\|^q e_{\ominus M}(t,t_0) + \frac{M\gamma + L}{M} \\ &\le \lambda_2 \|x_0\|^q + \frac{M\gamma + L}{M}. \end{aligned}$$

From (3.1), we have $\lambda_1 \|x\|^p \le V(t,x)$, which implies

$$\|x(t)\| \le \big(\frac{1}{\lambda_1}\big)^{1/p} \big(\lambda_2 \|x_0\|^q + \frac{M\gamma + L}{M}\big)^{1/p} \quad \text{for all } t \in \mathbb{T}^+_{t_0}.$$

This completes the proof. $\square$

Theorem 3.3. *Assume $D \subset \mathbb{R}^n$ is open and convex, and there exists a Lyapunov-type function $V : \mathbb{T}^+_{t_0} \times D \to \mathbb{R}^+$ that satisfies*

$$\lambda_1(t)\|x\|^p \le V(t,x) \le \lambda_2(t)\|x\|^q, \tag{3.4}$$

$$V^\Delta(t,x) \le \frac{-\lambda_3(t)\|x\|^r + L}{1 + M\mu(t)}, \tag{3.5}$$

$$V(t,x) - V^{r/q}(t,x) \le \gamma, \tag{3.6}$$

for some positive constants p, q, r and positive functions $\lambda_1(t), \lambda_2(t), \lambda_3(t)$, where $\lambda_1(t)$ is nondecreasing, L and γ are nonnegative constants, and

$$M = \inf_{t \in \mathbb{T}^+_{t_0}} \lambda_3(t)/\lambda_2^{r/q}(t) > 0.$$

Then all solutions of (1.1), (1.2) *that stay in D are bounded.*

Proof. For $M = \inf_{t \in \mathbb{T}^+_{t_0}} \lambda_3(t)/\lambda_2^{r/q}(t) > 0$, by calculating $(V(t,x)e_M(t,t_0))^\Delta$ and then by the similar argument as in Theorem 3.2, we obtain

$$\begin{aligned} V(t,x) &\le \lambda_2(t_0)\|x_0\|^q e_{\ominus M}(t,t_0) + \frac{M\gamma + L}{M} \\ &\le \lambda_2(t_0)\|x_0\|^q + \frac{M\gamma + L}{M}. \end{aligned}$$

Using condition (3.4), we arrive at

$$\|x\| \le \big(\frac{V(t,x)}{\lambda_1(t)}\big)^{1/p} \le \big(\frac{V(t,x)}{\lambda_1(t_0)}\big)^{1/p}.$$

Combining the above two inequalities, we get

$$\|x(t)\| \le \big(\frac{1}{\lambda_1(t_0)}\big)^{1/p} \big(\lambda_2(t_0)\|x_0\|^q + \frac{M\gamma + L}{M}\big)^{1/p} \quad \text{for all } t \in \mathbb{T}^+_{t_0},$$

which concludes the proof. $\square$

4. Exponential Stability of Solutions

For the dynamic equation (1.1), we assume $f(t,0)=0$ for all $t\in\mathbb{T}$, so $x(t)=0$ is the trivial solution of (1.1).

There are different definitions for the exponential stability of the zero solution according to different authors ([8, 9]). Pötzsche [8] gave the definition by the regular exponential function $e^{-p(t-t_0)}$(constant $p>0$); and Dacunha [9] defined the exponential stability in terms of $e_{-p}(t,t_0)$ (constant $p>0, -p\in\mathcal{R}^+$). For constant $p>0, -p\in\mathcal{R}^+$, taking into consideration the following relationship

$$e_{-p}(t,t_0)\le e^{-p(t-t_0)}\le e_{\ominus p}(t,t_0)\quad t,t_0\in\mathbb{T}, t\ge t_0,$$

here we introduce a new definition, which is more general than the present, by the use of the generalized time scales exponential function $e_{\ominus p}(t,t_0)$.

Definition 4.1. The zero solution to system (1.1) is *exponentially stable* if any solution $x(t)$ of (1.1), (1.2) satisfies

$$\|x(t)\|\le\beta(\|x_0\|,t_0)e^{\alpha}_{\ominus p}(t,t_0)\quad t\in\mathbb{T}^+_{t_0},$$

where $\beta:\mathbb{R}^+\times\mathbb{T}\to\mathbb{R}^+$ is a nonnegative function, and α,p are positive constants. If $\beta(\|x_0\|,t_0)$ does not depend on t_0, the zero solution is called *uniformly exponentially stable.*

For the rest of this article, to shorten expressions, instead of saying the zero solution is stable, we say that the system (1.1),(1.2) is stable.

Lemma 4.2. *For any time scale $\mathbb{T}$, assume the graininess function $\mu(t)$ is bounded above, i.e. there exists a constant $B_{\mathbb{T}}$ (that may depend on $\mathbb{T}$) such that $\mu(t)\le B_{\mathbb{T}}$, then for any positive constants M,δ satisfying $M<\delta$, we have*

$$t\to\int_{t_0}^t e_{M\ominus\delta}(\tau,t_0)\Delta\tau\quad\textit{is bounded above.}$$

Proof. From the properties of generalized exponential function (Theorem 2.9),

$$\int_{t_0}^t e_{M\ominus\delta}(\tau,t_0)\Delta\tau=\int_{t_0}^t\frac{e_M(\tau,t_0)}{e_\delta(\tau,t_0)}\Delta\tau=\frac{1}{M}\int_{t_0}^t\frac{e^{\Delta}_M(\tau,t_0)}{e_\delta(\tau,t_0)}\Delta\tau.$$

By the integration by parts formula [1, Theorem 1.77], we have

$$\begin{aligned}\int_{t_0}^t e_{M\ominus\delta}(\tau,t_0)\Delta\tau&=\frac{1}{M}\Big(e_{M\ominus\delta}(t,t_0)-1-\int_{t_0}^t e_M(\sigma(\tau),t_0)\frac{-e^{\Delta}_\delta(\tau,t_0)}{e_\delta(\tau,t_0)e_\delta(\sigma(\tau),t_0)}\Delta\tau\Big)\\&=\frac{1}{M}\Big(e_{M\ominus\delta}(t,t_0)-1+\delta\int_{t_0}^t\frac{e_M(\sigma(\tau),t_0)}{e_\delta(\sigma(\tau),t_0)}\Delta\tau\Big)\\&=\frac{1}{M}\Big(e_{M\ominus\delta}(t,t_0)-1+\delta\int_{t_0}^t\frac{(1+\mu(\tau)M)e_M(\tau,t_0)}{e_\delta(\sigma(\tau),t_0)}\Delta\tau\Big).\end{aligned}$$

Due to the assumption, we have

$$\int_{t_0}^t e_{M\ominus\delta}(\tau,t_0)\Delta\tau\le\frac{1}{M}\Big(e_{M\ominus\delta}(t,t_0)-1+\delta(1+B_{\mathbb{T}}M)\int_{t_0}^t\frac{e_M(\tau,t_0)}{e_\delta(\sigma(\tau),t_0)}\Delta\tau\Big).$$

From the formula in [1, Theorem 2.38],

$$\int_{t_0}^t\frac{e_M(\tau,t_0)}{e_\delta(\sigma(\tau),t_0)}\Delta\tau=\frac{1}{M-\delta}\int_{t_0}^t e^{\Delta}_{M\ominus\delta}(\tau,t_0)\Delta\tau.$$

Hence

$$
\begin{aligned}
\int_{t_0}^{t} e_{M\ominus\delta}(\tau,t_0)\Delta\tau &\leq \frac{1}{M}\Big(1-1+\frac{\delta(1+B_{\mathbb{T}}M)}{M-\delta}(e_{M\ominus\delta}(t,t_0)-1)\Big)\\
&= \frac{1}{M}\Big(\frac{\delta(1+B_{\mathbb{T}}M)}{\delta-M}-\frac{\delta(1+B_{\mathbb{T}}M)}{\delta-M}e_{M\ominus\delta}(t,t_0)\Big)\\
&\leq \frac{\delta(1+B_{\mathbb{T}}M)}{M(\delta-M)} = \text{Constant}.
\end{aligned}
$$

That completes the proof. □

The results obtained in this section are under the assumption that Lemma 4.2 holds to be true, i.e., throughout this section we assume $\sup_{t\in\mathbb{T}^+_{t_0}}\mu(t)<\infty$ with its bound dependent on the time scale $\mathbb{T}$.

Theorem 4.3. *Assume $D\subset\mathbb{R}^n$ is an open and convex set containing the origin and there exists a Lyapunov-type function $V:\mathbb{T}^+_{t_0}\times D\to\mathbb{R}^+$ which satisfies*

$$\lambda_1\|x\|^p \leq V(t,x)\leq \lambda_2\|x\|^q, \tag{4.1}$$

$$V^{\Delta}(t,x)\leq \frac{-\lambda_3\|x\|^r+Ke_{\ominus\delta}(t,t_0)}{1+\mu(t)M}, \tag{4.2}$$

where $\lambda_1,\lambda_2,\lambda_3,K,p,q,r,\delta$ are positive numbers and the following two conditions hold for all $(t,x)\in\mathbb{T}^+_{t_0}\times D$

$$\delta>\lambda_3/(\lambda_2)^{r/q}=M, \tag{4.3}$$

and there exists a $\gamma\geq 0$ such that

$$V(t,x)-V^{r/q}(t,x)\leq \gamma e_{\ominus\delta}(t,t_0). \tag{4.4}$$

Then system (1.1), (1.2) is uniformly exponentially stable.

Proof. Let $x(t)$ be a solution of (1.1), (1.2) and let

$$Q(t,x)=V(t,x)e_M(t,t_0).$$

Then

$$Q^{\Delta}(t,x)=V^{\Delta}(t,x)e_M(\sigma(t),t_0)+V(t,x)e^{\Delta}_M(t,t_0).$$

Taking (4.2) into account, for all $t\in\mathbb{T}^+_{t_0}, x\in D$, we have

$$Q^{\Delta}(t,x)\leq \frac{-\lambda_3\|x\|^r+Ke_{\ominus\delta}(t,t_0)}{1+\mu(t)M}(1+\mu(t)M)e_M(t,t_0)+V(t,x)Me_M(t,t_0).$$

By (4.1), we have $\|x\|^q\geq V(t,x)/\lambda_2$, or equivalently $-\|x\|^r\leq -(V(t,x)/\lambda_2)^{r/q}$. Therefore,

$$Q^{\Delta}(t,x)\leq \Big(-V^{r/q}(t,x)\lambda_3/\lambda_2^{r/q}+Ke_{\ominus\delta}(t,t_0)+MV(t,x)\Big)e_M(t,t_0).$$

Since $\lambda_3/\lambda_2^{r/q}=M$, we have

$$Q^{\Delta}(t,x)\leq M(V(t,x)-V^{r/q}(t,x))e_M(t,t_0)+Ke_{M\ominus\delta}(t,t_0).$$

Using (4.4), we obtain

$$Q^{\Delta}(t,x)\leq (M\gamma+K)e_{M\ominus\delta}(t,t_0).$$

Integrating both sides of the above inequality from t_0 to t, we obtain

$$Q(t,x) - Q(t_0,x_0) \le \int_{t_0}^{t} (M\gamma + K) e_{M\ominus\delta}(\tau,t_0)\Delta\tau = (M\gamma+K)\int_{t_0}^{t} e_{M\ominus\delta}(\tau,t_0)\Delta\tau.$$

From Lemma 4.2, we have

$$Q(t,x) - Q(t_0,x_0) \le \frac{(M\gamma+K)}{M}\frac{\delta(1+B_{\mathbb{T}}M)}{\delta - M}.$$

Since $Q(t_0,x_0) = V(t_0,x_0) \le \lambda_2\|x_0\|^q$, we have

$$Q(t,x) \le \lambda_2\|x_0\|^q + \frac{(M\gamma+K)}{M}\frac{\delta(1+B_{\mathbb{T}}M)}{\delta - M} =: \beta(\|x_0\|).$$

We have $Q(t,x) \le \beta(\|x_0\|)$. On the other hand, from (4.1) it follows that

$$\|x\| \le \Big(\frac{V(t,x)}{\lambda_1}\Big)^{1/p}.$$

Substituting $V(t,x) = Q(t,x)e_{\ominus M}(t,t_0)$ in the last inequality, we obtain

$$\|x(t)\| \le \Big(\frac{Q(t,x)e_{\ominus M}(t,t_0)}{\lambda_1}\Big)^{1/p} \le \Big(\frac{\beta(\|x_0\|)}{\lambda_1}\Big)^{1/p} e_{\ominus M}^{1/p}(t,t_0).$$

This inequality shows that system (1.1), (1.2) is uniformly exponentially stable. Therefore the proof is complete. $\square$

Theorem 4.4. *Assume $D \subset \mathbb{R}^n$ is an open and convex set containing the origin and there exists a Lyapunov-type function $V : \mathbb{T}_{t_0}^{+} \times D \to \mathbb{R}^+$ which satisfies*

$$\lambda_1(t)\|x\|^p \le V(t,x) \le \lambda_2(t)\|x\|^q, \tag{4.5}$$

$$V^{\Delta}(t,x) \le \frac{-\lambda_3(t)\|x\|^r + Ke_{\ominus\delta}(t,t_0)}{1+\mu(t)M}, \tag{4.6}$$

$$\delta > \inf_{t\in\mathbb{T}_{t_0}^{+}} \lambda_3(t)/(\lambda_2(t))^{r/q} = M > 0, \tag{4.7}$$

$$\exists\gamma \ge 0, \ \text{such that } V(t,x) - V^{r/q}(t,x) \le \gamma e_{\ominus\delta}(t,t_0), \tag{4.8}$$

where $\lambda_1(t), \lambda_2(t), \lambda_3(t)$ are positive functions, $\lambda_1(t)$ is nondecreasing for all $t \in \mathbb{T}_{t_0}^{+}$, and K, p, q, r, δ are positive constants. Then system (1.1), (1.2) *is exponentially stable.*

Proof. We consider the function

$$Q(t,x) = V(t,x)e_M(t,t_0).$$

By a similar argument used in the proof of Theorem 4.3, we arrive at

$$Q^{\Delta}(t,x) \le \Big(-\lambda_3(t)\|x\|^r + Ke_{\ominus\delta}(t,t_0)\Big)e_M(t,t_0) + V(t,x)Me_M(t,t_0).$$

Taking condition (4.5) into account and by the assumption $\lambda_2(t) > 0$ for all $t \in \mathbb{T}_{t_0}^{+}$, we have $\|x\|^q \ge V(t,x)/\lambda_2(t)$, so equivalently $-\|x\|^r \le -(V(t,x)/\lambda_2(t))^{r/q}$. Therefore,

$$Q^{\Delta}(t,x) \le \Big(-V^{r/q}(t,x)\lambda_3(t)/\lambda_2(t)^{r/q} + Ke_{\ominus\delta}(t,t_0) + MV(t,x)\Big)e_M(t,t_0).$$

Since $\lambda_3(t)/\lambda_2(t)^{r/q} \ge M$, by condition (4.8), we obtain

$$\begin{aligned} Q^{\Delta}(t,x) &\le M(V(t,x) - V^{r/q}(t,x))e_M(t,t_0) + Ke_{M\ominus\delta}(t,t_0) \\ &\le (M\gamma + K)e_{M\ominus\delta}(t,t_0). \end{aligned}$$

Thus integrating both sides of the above inequality from t_0 to t and applying Lemma 4.2,

$$\begin{aligned} Q(t,x) - Q(t_0,x_0) &\le \int_{t_0}^{t} (M\gamma + K) e_{M\ominus\delta}(\tau,t_0)\Delta\tau \\ &= \frac{(M\gamma+K)}{M}\frac{\delta(1+B_{\mathbb{T}}M)}{\delta - M}. \end{aligned}$$

Since $Q(t_0,x_0) = V(t_0,x_0) \le \lambda_2(t_0)\|x_0\|^q$, we have

$$Q(t,x) \le \lambda_2(t_0)\|x_0\|^q + \frac{(M\gamma+K)}{M}\frac{\delta(1+B_{\mathbb{T}}M)}{\delta - M} =: \beta(\|x_0\|,t_0).$$

Furthermore, from (4.5) it follows that

$$\|x\| \le \Big(\frac{V(t,x)}{\lambda_1(t)}\Big)^{1/p}.$$

Since $\lambda_1(t)$ is non-decreasing, hence $\lambda_1(t) \ge \lambda_1(t_0)$. So

$$\|x\| \le \Big(\frac{V(t,x)}{\lambda_1(t_0)}\Big)^{1/p}.$$

Substituting $V(t,x) = Q(t,x)e_{\ominus M}(t,t_0)$ into the last inequality, we obtain

$$\|x(t)\| \le \Big(\frac{Q(t,x)e_{\ominus M}(t,t_0)}{\lambda_1(t_0)}\Big)^{1/p} \le \Big(\frac{\beta(\|x_0\|,t_0)}{\lambda_1(t_0)}\Big)^{1/p} e^{1/p}_{\ominus M}(t,t_0).$$

This inequality shows that system (1.1), (1.2) is exponential stable. □

Theorem 4.5. *Assume $D \subset \mathbb{R}^n$ is an open and convex set containing the origin, and there exists a Lyapunov-type function $V : \mathbb{T}^+_{t_0} \times D \to \mathbb{R}^+$ such that*

$$\begin{gathered} \lambda_1(t)\|x\|^p \le V(t,x) \le \Phi(\|x\|), \\ V^{\Delta}(t,x) \le \frac{\Psi(\|x\|) + Le_{\ominus\delta}(t,t_0)}{1+\mu(t)}, \\ \Psi(\Phi^{-1}(V(t,x))) + V(t,x) \le \gamma e_{\ominus\delta}(t,t_0), \end{gathered}$$

where $\Phi : [0,\infty) \to [0,\infty)$, $\Psi : [0,\infty) \to (-\infty,0]$, $\lambda_1(t) : \mathbb{T}^+_{t_0} \to (0,+\infty)$, Ψ is nonincreasing, $\lambda_1(t), \Phi$ is nondecreasing, and Φ^{-1} exists, L and γ are nonnegative constants, $\delta > 1$. Then system (1.1), (1.2) is uniformly exponentially stable.

Proof. Let $x(t)$ be a solution of system (1.1), (1.2), then

$$\begin{aligned} [V(t,x)e_1(t,t_0)]^{\Delta} &= V^{\Delta}(t,x)e_1(\sigma(t),t_0) + V(t,x)e_1^{\Delta}(t,t_0) \\ &\le \frac{\Psi(\|x\|) + Le_{\ominus\delta}(t,t_0)}{1+\mu(t)}(1+\mu(t))e_1(t,t_0) + V(t,x)e_1(t,t_0) \\ &\le (\Psi(\Phi^{-1}(V(t,x))) + Le_{\ominus\delta}(t,t_0))e_1(t,t_0) + V(t,x)e_1(t,t_0) \\ &\le (\gamma+L)e_{1\ominus\delta}(t,t_0)). \end{aligned}$$

Integrating both sides from t_0 to t, we obtain

$$\begin{aligned} V(t,x(t))e_1(t,t_0) &\le V(t_0,x_0) + (\gamma+L)\int_{t_0}^{t} e_{1\ominus\delta}(\tau,t_0)\Delta\tau \\ &\le V(t_0,x_0) + (\gamma+L)\frac{\delta(1+B_{\mathbb{T}})}{\delta-1} =: \beta(\|x_0\|,t_0), \end{aligned}$$

so that $V(t,x(t)) \le \beta(\|x_0\|,t_0)e_{\ominus 1}(t,t_0)$. From the assumption, we have

$$\|x\| \le \Big(\frac{1}{\lambda_1(t)}\beta(\|x_0\|,t_0)e_{\ominus 1}(t,t_0)\Big)^{1/p} \le \Big(\frac{1}{\lambda_1(t_0)}\beta(\|x_0\|,t_0)\Big)^{1/p} e_{\ominus 1}^{1/p}(t,t_0),$$

which completes the proof. □

Remark 4.6. In Theorems 4.4 and 4.5, we can replace the nondecreasing assumption of $\lambda_1(t)$ by the following assumption: There exists $a > 0$ such that $a < M$ and

$$\lambda_1(t) \ge e_{\ominus a}(t,t_0), \quad \text{for all } t \in \mathbb{T}^+_{t_0},$$

where $M = \inf_{t\in\mathbb{T}^+_{t_0}} \lambda_3(t)/(\lambda_2(t))^{r/q}$. Take $M = 1$ in Theorem 4.5.

The following theorem does not require an upper bound on the Lyapunov function $V(t,x)$.

Theorem 4.7. *Assume $D \subset \mathbb{R}^n$ is an open and convex set containing the origin. Let $V : \mathbb{T}^+_{t_0} \times D \to \mathbb{R}^+$ be a given Lyapunov-type function satisfying*

$$\lambda_1\|x\|^p \le V(t,x), \tag{4.9}$$

$$V^\Delta(t,x) \le \frac{-\lambda_2 V(t,x) + K e_{\ominus\delta}(t,t_0)}{1+\mu(t)\varepsilon}, \tag{4.10}$$

for some positive constants $\lambda_1, \lambda_2, p, K, \delta, \varepsilon$ with $\varepsilon \le \lambda_2, \varepsilon < \delta$. Then system (1.1), (1.2) *is exponentially stable.*

Proof. Let

$$Q(t,x) = V(t,x)e_\varepsilon(t,t_0).$$

By an argument similar to the one used in the proof of the two theorems above, and taking into consideration conditions (4.9), (4.10), we arrive at

$$\begin{aligned}
Q^\Delta(t,x) &= V^\Delta(t,x)e_\varepsilon(\sigma(t),t_0) + V(t,x)e^\Delta_\varepsilon(t,t_0)\\
&\le \frac{-\lambda_2 V(t,x) + K e_{\ominus\delta}(t,t_0)}{1+\mu(t)\varepsilon}(1+\mu(t)\varepsilon)e_\varepsilon(t,t_0) + \varepsilon V(t,x)e_\varepsilon(t,t_0)\\
&= \left(-\lambda_2 V(t,x) + K e_{\ominus\delta}(t,t_0)\right)e_\varepsilon(t,t_0) + \varepsilon V(t,x)e_\varepsilon(t,t_0)\\
&= \left(-\lambda_2 + \varepsilon\right)V(t,x)e_\varepsilon(t,t_0) + K e_{\varepsilon\ominus\delta}(t,t_0)\\
&\le K e_{\varepsilon\ominus\delta}(t,t_0).
\end{aligned}$$

So by Lemma 4.2,

$$\begin{aligned}
V(t,x)e_\varepsilon(t,t_0) &\le V(t_0,x_0) + K\int_{t_0}^t e_{\varepsilon\ominus\delta}(\tau,t_0)\Delta\tau\\
&\le V(t_0,x_0) + K\frac{\delta(1+B_{\mathbb{T}}\varepsilon)}{\varepsilon(\delta-\varepsilon)} =: \beta(\|x_0\|,t_0).
\end{aligned}$$

Hence $V(t,x) \le \beta(\|x_0\|,t_0)e_{\ominus\varepsilon}(t,t_0)$. From (4.9), we get

$$\|x\| \le \Big(\frac{V(t,x)}{\lambda_1}\Big)^{1/p} \le \Big(\frac{\beta(\|x_0\|,t_0)}{\lambda_1}\Big)^{1/p} e^{1/p}_{\ominus\varepsilon}(t,t_0).$$

This completes the proof. □

Now we will present some examples to illustrate the theory developed above.

Example 4.8. Consider the dynamic equation

$$x^{\Delta}(t) = a\,x + R\,x^{1/3}\,e_{\ominus\delta}^{1/3}\,(t,t_0), \tag{4.11}$$

where $a < 0, a \in \mathcal{R}^+$, $R > 0$ and $\delta > 0$. If there exist positive constants λ_3, K such that

$$\begin{gathered}
\delta > \lambda_3,\\
(1+\mu(t)\lambda_3)(2a+\mu(t)a^2+\frac{4}{3}R+\frac{4}{3}\mu(t)aR+\frac{1}{3}\mu(t)R^2) \le -\lambda_3,\\
(1+\mu(t)\lambda_3)(\frac{2}{3}R+\frac{2}{3}\mu(t)aR+\frac{2}{3}\mu(t)R^2) \le K,
\end{gathered} \tag{4.12}$$

then system (4.11) is uniformly exponentially stable.

To see this, let $V(t,x) = x^2$. By calculating $V^{\Delta}(t,x)$ along the solutions of (4.11), we obtain

$$\begin{aligned}
V^{\Delta}(t,x) &= 2xf(t,x) + \mu(t)(f(t,x))^2\\
&= 2x\left(ax + Rx^{1/3}e_{\ominus\delta}^{1/3}(t,t_0)\right) + \mu(t)\left(ax + Rx^{1/3}e_{\ominus\delta}^{1/3}(t,t_0)\right)^2\\
&= \left(2a+\mu(t)a^2\right)x^2 + \left(2Re_{\ominus\delta}^{1/3}(t,t_0) + 2\mu(t)aRe_{\ominus\delta}^{1/3}(t,t_0)\right)x^{4/3}\\
&\quad + \mu(t)R^2e_{\ominus\delta}^{2/3}(t,t_0)x^{2/3}.
\end{aligned}$$

Using the Young's inequality ($wz < \frac{w^e}{e} + \frac{z^f}{f}$ with $\frac{1}{e}+\frac{1}{f} = 1$), we have

$$\begin{aligned}
x^{4/3}e_{\ominus\delta}^{1/3}(t,t_0) &\le \Big(\frac{(x^{4/3})^{3/2}}{3/2} + \frac{(e_{\ominus\delta}^{1/3}(t,t_0))^3}{3}\Big) = \frac{2}{3}x^2 + \frac{1}{3}e_{\ominus\delta}(t,t_0),\\
x^{2/3}e_{\ominus\delta}^{2/3}(t,t_0) &\le \Big(\frac{(x^{2/3})^3}{3} + \frac{(e_{\ominus\delta}^{2/3}(t,t_0))^{3/2}}{3/2}\Big) = \frac{1}{3}x^2 + \frac{2}{3}e_{\ominus\delta}(t,t_0).
\end{aligned}$$

Thus

$$\begin{aligned}
V^{\Delta}(t,x) &\le \left(2a+\mu(t)a^2\right)x^2 + (2R+2\mu(t)aR)\left(\frac{2}{3}x^2 + \frac{1}{3}e_{\ominus\delta}(t,t_0)\right)\\
&\quad + \mu(t)R^2\Big(\frac{1}{3}x^2 + \frac{2}{3}e_{\ominus\delta}(t,t_0)\Big)\\
&\le \Big(2a+\mu(t)a^2+\frac{4}{3}R+\frac{4}{3}\mu(t)aR+\frac{1}{3}\mu(t)R^2\Big)x^2\\
&\quad + \Big(\frac{2}{3}R+\frac{2}{3}\mu(t)aR+\frac{2}{3}\mu(t)R^2\Big)e_{\ominus\delta}(t,t_0)\\
&= \frac{1}{1+\mu(t)\lambda_3}\Big\{(1+\mu(t)\lambda_3)\left(2a+\mu(t)a^2+\frac{4}{3}R+\frac{4}{3}\mu(t)aR+\frac{1}{3}\mu(t)R^2\right)x^2\\
&\quad + (1+\mu(t)\lambda_3)\left(\frac{2}{3}R+\frac{2}{3}\mu(t)aR+\frac{2}{3}\mu(t)R^2\Big)e_{\ominus\delta}(t,t_0)\right\}.
\end{aligned}$$

Under the above assumptions, one can check that conditions (4.1)–(4.4) of Theorem 4.3 are satisfied. Hence system (4.11) is uniformly exponentially stable.

In fact, if there exist constants $\lambda_3 > 0,\ K > 0$, such that

$$\begin{gathered} \delta > \lambda_3, \\ (1 + \lfloor\mu\rfloor\lambda_3)(2a + \lceil\mu\rceil a^2 + \frac{4}{3}R + \frac{4}{3}\lfloor\mu\rfloor aR + \frac{1}{3}\lceil\mu\rceil R^2) \le -\lambda_3, \\ (1 + \lceil\mu\rceil\lambda_3)(\frac{2}{3}R + \frac{2}{3}\lfloor\mu\rfloor aR + \frac{2}{3}\lceil\mu\rceil R^2) \le K, \end{gathered} \tag{4.13}$$

here $\lceil\mu\rceil = \sup_{t\in\mathbb{T}} \mu(t)$, $\lfloor\mu\rfloor = \inf_{t\in\mathbb{T}} \mu(t)$, then (4.12) will hold evidently.
Case 1: If $\mathbb{T} = \mathbb{R}$, then $\mu(t) = \lceil\mu\rceil = \lfloor\mu\rfloor = 0$ and the conditions in (4.13) reduce to that positive constants λ_3, K need to exist such that

$$\delta > \lambda_3 \le -(2a + \frac{4}{3}R), \quad \frac{2}{3}R \le K,$$

then system (4.11) is uniformly exponentially stable.
Case 2: If $\mathbb{T} = h\mathbb{Z}$, then $\mu(t) = \lceil\mu\rceil = \lfloor\mu\rfloor = h$. The conditions in (4.13) reduce to that there exist $\lambda_3 > 0, K > 0$ such that

$$\begin{gathered} \delta > \lambda_3, \\ 2a + ha^2 + \frac{4}{3}R + \frac{4}{3}haR + \frac{1}{3}hR^2 \le -\lambda_3/(1 + h\lambda_3), \\ \frac{2}{3}R + \frac{2}{3}haR + \frac{2}{3}hR^2 \le K/(1 + h\lambda_3), \end{gathered}$$

then system (4.11) is uniformly exponentially stable.
Case 3: When $\mathbb{T} = \bigcup_{k=0}^{\infty}[k(l+h), k(l+h)+l)]$, here l, h are positive constants, this kind of time scales could exactly describe many phenomena which are common in nature, such as the life span of certain species and the change of electric circuit with time progressing etc. [1, 15]. At this time, $\lfloor\mu\rfloor = 0$, $\lceil\mu\rceil = h$. If there exist constants $\lambda_3 > 0$, $K > 0$, such that

$$\begin{gathered} \lambda_3 < \delta, \\ 2a + ha^2 + \frac{4}{3}R + \frac{1}{3}R^2 h \le -\lambda_3, \\ (1 + h\lambda_3)(\frac{2}{3}R + \frac{2}{3}R^2 h) \le K, \end{gathered}$$

then (4.13) will hold. So system (4.11) is uniformly exponentially stable.

Example 4.9. Consider the system

$$x_1^\Delta = -ax_1 + ax_2, \tag{4.14}$$
$$x_2^\Delta = -ax_1 - ax_2, \tag{4.15}$$
$$(x_1(t_0), x_2(t_0)) = (c, d), \tag{4.16}$$

for certain constants $a > 0$, $-a \in \mathcal{R}^+$, and c, d are any constants. If there is a constant $\lambda_2 > 0$ such that for all $t \in \mathbb{T}_{t_0}^+$

$$\lambda_2/(1 + \lambda_2\mu(t)) \le 2a(1 - a\mu(t)), \tag{4.17}$$

then system (4.14)–(4.16) is exponentially stable.

Proof. We will show that, under the above assumptions, the conditions of Theorem 4.7 are satisfied. Choose $D = \mathbb{R}^2$, $V(t,x) = \|x\|^2 = x_1^2 + x_2^2$, so (4.9) holds. From [7], we have

$$\begin{aligned} V^{\Delta}(t,x) &= 2xf(t,x) + \mu(t)\|f(t,x)\|^2 \\ &= -2a(1 - a\mu(t))\|x\|^2 \\ &\le \frac{-\lambda_2 V(t,x)}{1+\lambda_2\mu(t)} \\ &\le \frac{-\lambda_2 V(t,x) + Ke_{\ominus\delta}(t,t_0)}{1+\lambda_2\mu(t)} \\ &= \frac{-\lambda_2 V(t,x) + Ke_{\ominus\delta}(t,t_0)}{1+\varepsilon\mu(t)}. \end{aligned}$$

Here we let $\varepsilon = \lambda_2$, $\delta > \varepsilon > 0$, and K be arbitrary positive constant, so (4.10) holds. Therefore, system (4.14)–(4.16) is exponentially stable.

In fact, if there exists a constant $\lambda_2 > 0$, such that

$$2a(1 - a\lceil\mu\rceil)(1 + \lambda_2\lfloor\mu\rfloor) \ge \lambda_2, \tag{4.18}$$

then condition (4.17) will hold.

Case 1: If $\mathbb{T} = \mathbb{R}$, then $\mu(t) = \lceil\mu\rceil = \lfloor\mu\rfloor = 0$ and (4.18) will hold to be true if there exists a constant λ_2 satisfying

$$0 < \lambda_2 \le 2a.$$

So system (4.14)–(4.16) is exponentially stable.

Case 2: If $\mathbb{T} = h\mathbb{Z}$, then $\mu(t) = \lceil\mu\rceil = \lfloor\mu\rfloor = h$. If we can find a constant $\lambda_2 > 0$ such that

$$\frac{\lambda_2}{1 + h\lambda_2} \le 2a(1 - ah),$$

then condition (4.18) would hold. So system (4.14)–(4.16) is exponentially stable.

Case 3: If $\mathbb{T} = \bigcup_{k=0}^{\infty}[k(l+h), k(l+h)+l]$ (as in the above example), then $\lfloor\mu\rfloor = 0$, $\lceil\mu\rceil = h$. The condition (4.18) reduces to that a constant λ_2 exists such that

$$0 < \lambda_2 \le 2a(1 - ah),$$

so that system (4.14)–(4.16) is exponentially stable. $\square$

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