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Non-collision solutions for a class of planar singular Lagrangian systems *

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Abstract

In this paper, we show the existence of non-collision periodic solutions of minimal period for a class of singular second order Hamiltonian systems in $\mathbb{R}^2$ with weak forcing terms. We consider the fixed period problem and the fixed energy problem in the autonomous case.

1 Introduction and statement of results

This paper deals with the existence of non-collision periodic solutions of minimal period for the problem

$$\ddot{q} + V_q(t,q) = 0$$

where $q \in \mathbb{R}^N \setminus \{0\}$ with $N = 2$, the potential V is of the form $V(t,q) = -\frac{1}{|q|^\alpha} + W(q)$ in a neighborhood of $q = 0$ with $1 < \alpha < 2$ and W is such that $|q|^\alpha W(q),\ |q|^{\alpha+1}W'(q) \to 0$ as $|q| \to 0$.

We will consider to cases: *the fixed period problem*

$$\begin{gathered}\ddot{q} + V_q(t,q) = 0\\ q(t+T) = q(t),\end{gathered} \tag{P_T}$$

and *the fixed energy problem* (autonomous case)

$$\begin{gathered}\ddot{q} + V'(q) = 0\\ \tfrac{1}{2}|\dot{q}|^2 + V(q) = h\\ q \text{ periodic.}\end{gathered} \tag{P_h}$$

The case $\alpha \geq 2$ "Strong force" and $N \geq 2$ has been studied by many authors. The existence of classical (non-collision) solutions of (P_T) and (P_h) has been proved via variational methods(See [1, 5, 11, 13, 14]). The case $0 < \alpha < 2$ "weak force" is more complicated because the lose of control of the functional, whose critical points correspond to periodic solutions on the functions passing

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through the origin. Recently, there has been several works which deal with these two problems for $N \geq 3$(See also [1, 2, 4, 8, 17, 18]).

In our situation ($N = 2$), we refer for the study of (P_T) to Degiovanni-Giannoni [10], Ambrosetti-Coti Zelati [3], Serra-Terracini [16] where they treated also case of $N \geq 3$, and to Coti Zelati [7]. In [10], they obtained the existence of classical solutions under a global conditions like

$$\frac{a}{|q|^\alpha} \leq -V(q) \leq \frac{b}{|q|^\alpha}, \ \ \forall\, q \neq 0. \tag{1.1}$$

In [3], they found solutions of large period T. In [16]-[7], they used a radially symmetric assumption on V in a neighborhood of the singularity in order to get a non-collision solution of (P_T). For the study of (P_h), we know the result of Benci-Giannoni [6] where the existence of classical solution strongly depend on the pertubation W. The other result has been obtained by Coti Zelati-Serra [9]. There arguments are based on the fact that the topology of $\{V \leq h\}$ is non trivial; We remark that the case $V(q) = -\frac{1}{|q|^\alpha}$ is excluded in this work.

In the present paper, we are able to find estimates in minima of suitable minimisation perturbed problems using a re-scaling argument. Such estimates give actually non-collision solutions with minimal period to our problems without assuming a radially symmetric condition on V. More precisely, in section 2, we study the fixed period problem; We deal with non-autonomous potentials V satisfying the hypotheses:

(V0) $V \in C^1(\mathbb{R} \times \mathbb{R}^N \setminus \{0\}; \mathbb{R})$ and T-periodic in t;

(V1) $V(t,q) < 0,\ \forall\,(t,q) \in [0,T] \times \mathbb{R}^N \setminus \{0\}$;

(V2) $|\frac{\partial V}{\partial t}(t,q)| \leq -V(t,q),\ \forall\,(t,q) \in [0,T] \times \mathbb{R}^N \setminus \{0\}$;

(V3) There exist $r > 0, 1 < \alpha < 2$ and $W \in C^1(\mathbb{R}^N \setminus \{0\}, \mathbb{R})$ satisfying $|q|^\alpha W(q),\ |q|^{\alpha+1} W'(q) \to 0$ as $|q| \to 0$ such that:

$$V(t,q) = -\frac{1}{|q|^\alpha} + W(q),\ \forall\, 0 < |q| < r.$$

Theorem 1.1 *Assume (V0)-(V3) with $N = 2$. Then for any $T > 0$, (P_T) possesses at least one non-collision solution having T as minimal period.*

Remark 1.1 *For $N \geq 3$, Theorem 1.1 was proved in [17] under condition (V3) by Morse theoretical arguments.*

In section 3, we study the fixed energy problem. Here, we assume:

(V'0) $V \in C^2(\mathbb{R}^N \setminus \{0\}, \mathbb{R})$;

(V'1) $3V'(q)q + V''(q)qq > 0,\ \forall\, q \neq 0$;

(V'2) There exists an constant $\alpha_1 \in]0, 2[$ such that:

$$V'(q)q \geq -\alpha_1 V(q) > 0,\ \forall\, q \neq 0;$$

(V'3) $\liminf[V(q) + \frac{1}{2}V'(q)q] \geq 0$ as $|q| \to \infty$;

(V'4) The same as (V3) with $V(t,q) = V(q)$.

Theorem 1.2 *Assume (V'0)-(V'4) with $N = 2$. Then for any $h < 0$, (P_h) possesses at least one classical solution with a minimal period.*

Remark 1.2 *i) For $N \geq 3$, (V'1) is used in [2] to prove existence of a generalized solution (that may enter the singularity) and in [18] to avoid collision solutions in the case $N = 3$ and $1 < \alpha < \frac{4}{3}$.*
ii) Assumptions (V'1)-(V'2) can be made only in $\{V \leq h\}$ (See [2]).

Notation. For any $u \in H^1([0,T];\mathbb{R}^2)$, we note $u(t) = (|u(t)|, \theta(u)(t))$ in polar coordinates. We consider the following function space:

$$E_0^T = \{u \in H^1([0,T];\mathbb{R}^2);\ u(0) = u(T);\ \int_0^T \dot{\theta}(u)(t)dt = 2\pi\}.$$

i.e., E_0^T is the set of T-periodic functions $u \in H^1([0,T];\mathbb{R}^2)$ such that $\theta : [0,T]/\{0,T\} \sim S^1 \to S^1$ has degre 1.

We shall work in the function set:

$$\Lambda_0^T = \{u \in E_0^T;\ u(t) \neq 0\ \forall\, t\}.$$

2 The fixed period problem

In this section we proof Theorem 1.1. Let us define

$$f(q) = \frac{1}{2}\int_0^T |\dot{q}|^2 dt - \int_0^T V(t,q)dt.$$

It is well known that $f \in C^1(\Lambda_0^T;\mathbb{R})$ and any critical point $u \in \Lambda_0^T$ is a solution of (P_T).

Since we deal with "weak force" potentials, we know the existence of situation where the minimum of f is assumed on functions going through the origin(See [12]). For any $\varepsilon \in]0,1]$, we introduce the perturbed potential:

$$V_\varepsilon(t,q) = V(t,q) - \frac{\varepsilon}{|q|^2}.$$

The corresponding Lagrangian systems are

$$\begin{array}{c} \ddot{q} + (V_\varepsilon)_q(t,q) = 0 \\ q(t+T) = q(t) \end{array} \qquad (P_T)_\varepsilon$$

and the associated functionals are

$$f_\varepsilon(q) = \frac{1}{2}\int_0^T |\dot{q}|^2 dt - \int_0^T V_\varepsilon(t,q)dt.$$

One has that $f_\varepsilon(q_n) \to +\infty$ as $q_n \to \partial\lambda_0^T$ weakly in $H^1([0,T];\mathbb{R}^2)$. We recall that in Λ_0^T,

$$||\dot{u}||_2 = (\int_0^T |\dot{u}|^2 dt)^{\frac{1}{2}}$$

is a norm. Set

$$m_\varepsilon = \inf_{q\in\Lambda_0^T} f_\varepsilon(q).$$

The following result is closely related to this of [11] (See [1]).

Lemma 2.1 *For any* $\varepsilon \in]0,1]$, m_ε *is a critical value for* f_ε*; i.e. there exists* $q_\varepsilon \in \Lambda_0^T$ *such that* $f_\varepsilon(q_\varepsilon) = m_\varepsilon$ *and* $f'_\varepsilon(q_\varepsilon) = 0$.

The fact that $f_\varepsilon(q_\varepsilon) = m_\varepsilon \le m_1$ implies

$$\frac{1}{2}\int_0^T |\dot{q}_\varepsilon|^2 dt \le m_1 \tag{2.1}$$

and

$$\int_0^T V(t, q_\varepsilon) dt \le m_1. \tag{2.2}$$

It follows from 2.1 the existence of $\varepsilon_n \to 0$ such that

$$q_n = q_{\varepsilon_n} \to q \text{ weakly in } H^1([0,T];\mathbb{R}^2) \text{ and uniformly in } [0,T].$$

We say that q is a weak solution of (P_T) in the sense of [1].

Setting $C(q) = \{t \in [0,T],\ q(t) = 0\}$, one can see from 2.2 and (V3), that $\mathrm{mes}C(q) = 0$ (Lebesgue measure). Moreover, we have

$$q_n \to q \text{ in } C^2(K;\mathbb{R}^2),\ \forall\, K \text{ compact } \subset [0,T]\setminus C(q). \tag{2.3}$$

Hence, we have that

$$\ddot{q} + V_q(t,q) = 0,\ \forall\, t \in [0,T]\setminus C(q).$$

Therefore q is a generalized solution of (P_T) in the sense of [5].

Now, we state these properties of approximated solutions q_n:

Lemma 2.2 *(i) There exists an constant* $C_1 > 0$ *independent of* n*, such that*

$$|\frac{1}{2}|\dot{q}_n|^2 + V(t,q_n) - \frac{\varepsilon_n}{|q_n|^2}| \le C_1;$$

(ii)There exist constants $0 < \mu < r$ *and* $C_2 > 0$ *independent of* n*, such that:*

$$\frac{1}{2}\frac{d^2}{dt^2}|q_n(t)|^2 \ge C_2,\ \forall\, t :\ |q_n(t)| < \mu.$$

Proof. (i) follows from (V2) and 2.2, while for (ii), it is a consequence of (i) and (V3). For more details, we refer to [8, 1].

Remark 2.1 *(ii) of Lemma 2.2 does not hold in general when q is merely a generalized solution of (P_T) as in [17].*

Proof of theorem 1.1. We will prove how the function q is actually a non-collision solution of (P_T). We suppose that q has a collision in $\bar{t}$. The contradiction will be showed in two steps.

Step 1. The solution q_n have a self-intersection. We study the angle that the approximated solution q_n describes close to the singularity. By (ii) of Lemma 2.2 and 2.3, we get

$$\frac{1}{2}\frac{d^2}{dt^2}|q(t)|^2 \geq C_2 > 0, \ \forall\, t:\ 0 < |q(t)| < \mu.$$

Take $\mu_0 < \min(\mu, r)$ and $t_1 < \bar{t} < t_2$ such that

$$|q(t_1)| = |q(t_2)| = \frac{\mu_0}{2}.$$

This implies that, for sufficiently large n,

$$\frac{\mu_0}{4} < |q_n(t_1)|,\ |q_n(t_2)| < \mu_0,$$

$$|q_n(t)| < \mu_0,\ \forall\, t \in [t_1, t_2].$$

Let $t_n \in [t_1, t_2]$ be such that

$$|q_n(t_n)| = \min_{t\in[t_1,t_2]} |q_n(t)|.$$

Then, we have

$$\frac{d}{dt}|q_n(t)| < 0,\ \forall\, t \in [t_1, t_n[$$

$$\frac{d}{dt}|q_n(t)| > 0,\ \forall t \in]t_n, t_2].$$

Now, we will use a re-scaling argument as in ([17]-[18]). We set for any $L > 0$,

$$x_n(s) = \delta_n^{-1} q_n(\delta_n^{\frac{\alpha+2}{2}} s + t_n),\ s \in [-L, L]$$

when $\delta_n = |q_n(t_n)| \to 0$. Let us remark that for sufficiently large n, $\delta_n^{\frac{\alpha+2}{2}} s + t_n \in [t_1, t_2]$ for $s \in [-L, L]$ and then $\delta_n |x_n(s)| < \mu$. Hence, $x_n(s)$ satisfies

(i) $|x_n(0)| = 1;\ x_n(0).\dot{x}_n(0) = 0;\ \frac{d}{ds}|x_n(s)| < 0,\ \forall\, s \in [-L, 0[;$
$\frac{d}{ds}|x_n(s)| > 0,\ \forall\, s \in]0, L];$

(ii) $\ddot{x}_n + \frac{\alpha x_n}{|x_n|^{\alpha+2}} + \delta_n^{\alpha+1} W'(\delta_n x_n) + \frac{2\varepsilon_n}{\delta_n^{2-\alpha}} \frac{x_n}{|x_n|^4} = 0$;

(iii) $|\frac{1}{2}|\dot{x}_n|^2 - \frac{1}{|x_n|^\alpha} + \delta_n^\alpha W(\delta_n x_n) - \frac{\varepsilon_n}{\delta_n^{2-\alpha}|x_n|^2}| \le C_1 \delta_n^\alpha$.

We may assume the existence -up a subsequence- of

$$d = \lim_{n\to\infty} \frac{\varepsilon_n}{\delta_n^{2-\alpha}} \in [0, \infty].$$

We consider the following two cases:

Case1: $d < \infty$ From (i) and (iii), we may assume

$$\begin{aligned} x_n(0) &\to e_1 \\ \dot{x}_n(0) &\to \sqrt{2(1+d)}e_2 \end{aligned}$$

where (e_1, e_2) is an orthogonal basis of $\mathbb{R}^2$. By the continuous dependence of solutions in initial data and equations, one can see from (V3) that, $x_n(s)$ converge to a function $y_{\alpha,d}$ in $C^2(-L, L; \mathbb{R}^2)$ where $y_{\alpha,d}$ is the solution of

$$\begin{gathered} \ddot{y} + \frac{\alpha y}{|y|^{\alpha+2}} + \frac{dy}{|y|^4} = 0 \\ y(0) = e_1, \quad \dot{y}(0) = \sqrt{2(1+d)}e_2\,. \end{gathered}$$

Here we state some properties of $y_{\alpha,d}$ (c.f. [17]-[18]).

$$|y_{\alpha,d}(s)| = |y_{\alpha,0}(s)| \ge 1, \quad \forall s \in \mathbb{R}; \tag{2.4}$$
$$|y_{\alpha,d}(s)|^2 \dot{\theta}(y_{\alpha,d})(s) = \sqrt{2(1+d)}, \quad \forall s \in \mathbb{R}; \tag{2.5}$$
$$\lim_{s\to-\infty} \theta(y_{\alpha,0})(s) = -\tfrac{\pi}{2-\alpha}; \tag{2.6}$$
$$\lim_{s\to+\infty} \theta(y_{\alpha,0})(s) = +\tfrac{\pi}{2-\alpha}. \tag{2.7}$$

Since $1 < \alpha < 2$, we get from 2.4-2.7, the existence of $\bar{L} > 0$ such that

$$\begin{aligned} \lim_{n\to\infty} [\theta(x_n)(\bar{L}) - \theta(x_n)(-\bar{L})] &= \theta(y_{\alpha,d})(\bar{L}) - \theta(y_{\alpha,d})(-\bar{L}) \\ &\ge \theta(y_{\alpha,0})(\bar{L}) - \theta(y_{\alpha,0})(-\bar{L}) \\ &> 2\pi. \end{aligned}$$

Thus, for sufficiently large n, there exist $-\bar{L} < s_0 < 0 < s_1 < \bar{L}$ such that

$$x_n(s_0) = x_n(s_1); \;\; \dot{\theta}(x_n)(s) > 0 \text{ for } s = s_0, s_1.$$

Case 2: $d = +\infty$ In this case, we set for $L > 0$

$$z_n(s) = \delta_n^{-1} q_n(\varepsilon_n^{-\frac{1}{2}} \delta_n^2 s + t_n), \; s \in [-L, L].$$

Since $\varepsilon_n^{-\frac{1}{2}}\delta_n^2 \to 0$, we see that $\delta_n|z_n(s)| < \mu$ for sufficiently large n for any $L > 0$. As in case 1, we find:

$$\begin{gathered} |z_n(0)| = 1,\ z_n(0).\dot{z}_n(0) = 0; \\ \tfrac{d}{ds}|z_n(s)| < 0,\ \forall\, s \in [-L, 0[;\ \ \tfrac{d}{ds}|z_n(s)| > 0,\ \forall\, s \in]0, L]; \\ z_n(s) \to y_\infty(s) \text{ in } C^2([-L, L]; \mathbb{R}^2) \end{gathered}$$

where y_∞ is the solution of the system

$$\begin{gathered} \ddot{y} + \frac{2y}{|y|^4} = 0 \\ y(0) = e_1 \quad \dot{y}(0) = \sqrt{2}e_2 \end{gathered}$$

for a suitable orthogonal basis (e_1, e_2) of $\mathbb{R}^2$. Then,

$$y_\infty(s) = e_1 \cos\sqrt{2}s + e_2 \sin\sqrt{2}s.$$

We remark that $\dot{\theta}(z_n) \to \sqrt{2}$ uniformly in $[-L, L]$. So z_n has at least a self intersection for $L > \frac{\sqrt{2}\pi}{2}$.

From the two cases, it follows the existence of $t_{1,n}, t_{2,n} \in]t_1, t_2[$ such that

$$\begin{gathered} q_n(t_{1,n}) = q_n(t_{2,n}); \\ \tfrac{d}{dt}|q_n(t)| \neq 0 \text{ and } \dot{\theta}(q_n)(t) > 0 \text{ for } t = t_{1,n}, t_{2,n}. \end{gathered}$$

Step 2. The solution q_n cannot have a self intersection. Let

$${q_n}^*(t) = \begin{cases} q_n(t) & \text{if } t \notin [t_{1,n}, t_{2,n}] \\ q_n(t_{1,n} + t_{2,n} - t) & \text{if } t \in [t_{1,n}, t_{2,n}]. \end{cases}$$

We have

$$\int_0^T \dot{\theta}({q_n}^*)(t)dt = \int_0^T \dot{\theta}(q_n)(t)dt = 2\pi.$$

Hence ${q_n}^* \in \Lambda_0^T$. Since $f_{\varepsilon_n}({q_n}^*) = f_{\varepsilon_n}(q_n) = m_{\varepsilon_n}$, ${q_n}^*$ must be a solution of $(P_T)_{\varepsilon_n}$ and then of class C^1. This is a contradiction with the fact

$$\lim_{t \to {t_{1,n}}^-} \dot{q_n}^*(t) = \dot{q}_n(t_{1,n}) \neq -\dot{q}_n(t_{2,n}) = \lim_{t \to {t_{1,n}}^+} \dot{q_n}^*(t).$$

Therefore, we proved that q is a non-collision solution of (P_T). The minimality of the period T follows from the fact that $q_n \to q \in \Lambda_0^T$.

3 The fixed energy problem

We give an outline of the proof of Theorem 1.2. According to the variational principle given by [2], we define

$$I(u) = \frac{1}{2}\int_0^1 |\dot{u}|^2 dt \int_0^1 [h - V(u)]dt$$

on the set $M_h = \{u \in \Lambda_0^1;\ g(u) = h\}$ where

$$g(u) = \int_0^1 [V(u) + \frac{1}{2}V'(u)u]dt.$$

We know, if $u \in \Lambda_0^1$ is any possible solution of (P_h), then $g(u) = h$. Moreover, under assumptions (V'0)-(V'4), $M_h \neq \emptyset$ is a C^1 manifold of codimension 1 and if $u \in M_h$ is a critical point of I constrained on M_h such that $I(u) > 0$, set

$$w^2 = \frac{\int_0^1 V'(u)udt}{\int_0^1 |\dot{u}|^2 dt},$$

then $q(t) = u(wt)$ is a non-constant classical solution of (P_h).

We modify V, as in section 2, setting

$$V_\varepsilon(u) = V(u) - \frac{\varepsilon}{|u|^2}, \quad \varepsilon \in]0, 1].$$

Let

$$I_\varepsilon(u) = \frac{1}{2}\int_0^1 |\dot{u}|^2 dt \int_0^1 [h - V_\varepsilon(u)]dt.$$

We remark that

$$g(u) = \int_0^1 [V_\varepsilon(u) + \frac{1}{2}V'_\varepsilon(u)u]dt.$$

It follows from (V'2) that

$$I_\varepsilon(u) \geq \frac{h}{\frac{1}{2} - \frac{1}{\alpha_1}} \int_0^1 |\dot{u}|^2 dt, \quad \forall u \in M_h.$$

Therefore, I_ε is bounded below and coercive on M_h. Since V_ε is a "strong force" potential, one can see that I_ε is lower semi continuous on M_h and has a minimum u_ε on M_h. Set

$${w_\varepsilon}^2 = \frac{\int_0^1 V'_\varepsilon(u_\varepsilon)u_\varepsilon dt}{\int_0^1 |\dot{u}_\varepsilon|^2 dt},$$

the function $q_\varepsilon(t) = u_\varepsilon(w_\varepsilon t)$ is a solution of the modified system $(P_h)_\varepsilon$. Uniform estimates with respect to ε allow to show that u_ε converges uniformly on $[0, 1]$ to u, ${w_\varepsilon}^2 \to w^2 > 0$ and that $q(t) = u(wt)$ satisfies the equations of the system (P_h) for any $t \in \{t \in [0, \frac{1}{w}],\ u(t) \neq 0\}$.

Repeating the argument of section 2, one prove that q is in fact a non-collision solution of (P_h) with minimal period. If not, a new minimizer ${u_n}^* \in M_h$ for large n can be constructed; But ${u_n}^*$ being a minimum of I_{ε_n} on M_h correspond to a solution of $(P_h)_{\varepsilon_n}$, on the other hand it does not have the required regularity.

Remark 3.1 (i)The existence of solutions q_ε of $(P_h)_\varepsilon$ can be found without assuming condition (V'1). The proof relies on an application of the mountain-pass theorem to I_ε. However, $q(t) = \lim q_\varepsilon(t)$ is a generalized solution of (P_h) and collisions are possible.
(ii) Theorem 1.2 can be related to the work of Rabinowitz [15] (see also [9]). He prove under a less restrictive setting than (V'0)-(V'4) that there exists a collision orbit of (P_h). Combining this result with Theorem 1.2 shows the existence of a collision and a non-collision periodic solution of (P_h) for a suitable class of planar singular potentials.

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References

[1] A. Ambrosetti and V. Coti Zelati, "Periodic solutions of singular Lagrangian systems", Birkhauser, Boston, Basel, Berlin, 1993.

[2] A. Ambrosetti and V. Coti Zelati, *Closed orbits of fixed energy for singular Hamiltonian systems*, Arch. Rat. Mech. Anal. 112(1990), 339-362.

[3] A. Ambrosetti and V. Coti Zelati, *Non-collision orbits for a class of Keplerian- like potentials,* Ann. Inst. H. Poincarre. Anal. Non lineaire, 5 (1988), 287-295.

[4] A.Ambrosetti and M. Struwe, *Periodic motions for concervative systems with singular potentials*, No.D.E.A. 1(1994), 179-202.

[5] A.Bahri and P. H. Rabinowitz, *A minmax method for a class of Hamiltonian systems with singular potentials,* J. Funct. Anal. 82, 412-428 (1989).

[6] V.Benci and F. Giannoni, *Periodic solutions of prescribed energy for a class of Hamiltonian systems with singular potentials,* J. Diff. Eq. 82, 60-70 (1989).

[7] V. Coti Zelati, *Periodic solutions for a class of planar singular dynamical systems,* J. Math. pures et appl. 68, 1989, p. 109 -119.

[8] V. Coti Zelati and E. Serra, *Collisions and non-collisions solutions for a class of Keplerian like dynamicals systems,* Tech. report, SISSA, Trieste, Italy , 1991.

[9] V. Coti Zelati and E. Serra, *Multiple brake orbits for some classes of singular Hamiltonian systems,* Nonlinear Analysis, T.M.A., Vol. 20, No. 8, pp. 1001-1012, (1993).

[10] M.Degiovanni and F. Giannoni, *Dynamical systems with Newtonian type potentials,* Schol N. Sup. Pisa P.467-493 (1987).

[11] W. Gordon, *Concervative dynamical systems involving strong forces,* Trans. Amer. Math. Soc. 204 (1975), 113-135.

[12] W. Gordon, *A minimizing property of Keplerian orbits,* Amer. J. Math. 99(1975), 961- 971.

[13] C. Greco, *Periodic solutions of a class of singular Hamiltonian systems,* Nonlinear Analysis, T.M.A. 12 (1988), 259-269.

[14] L. Pisani, *Periodic solutions with prescribed energy for singular conservative systems involving stong forces,* Nonlinear Analysis, T.M.A., Vol 21, No. 3, pp. 167-179, 1993.

[15] P. H. Rabinowitz, *A note on periodic solutions of prescribed energy for singular Hamiltonian systems,* J. Comp. Applied Math. 52 (1994), 147-154.

[16] E. Serra and S. Terracini, *Noncollision solutions to some singular minimisation problems with Keplerian-like potentials,* Nonlinear Analysis, T.M.A., Vol. 22, No. 1, pp. 45-62 , (1994).

[17] K. Tanaka, *Non-collision solutions for a second order singular Hamiltonian system with weak force,* Ann. Inst. H. Poincarre. Anal. Non lineaire, 10(1993), 215-238.

[18] K. Tanaka, *A prescribed energy problem for a singular Hamiltonian system with weak force,*J. Funct. Anal. 113, (1993), pp. 351-390.

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