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EXISTENCE, UNIQUENESS AND MULTIPLICITY OF POSITIVE SOLUTIONS FOR SOME NONLOCAL SINGULAR ELLIPTIC PROBLEMS

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Abstract. In this article, using the sub-supersolution method and Rabinowitz-type global bifurcation theory, we prove some results on existence, uniqueness and multiplicity of positive solutions for some singular nonlocal elliptic problems.

1. Introduction

In this article, we consider the nonlocal elliptic problems

$$\begin{gathered} -a\Big(\int_\Omega |u(x)|^\gamma dx\Big)\Delta u = K(x)u^{-\mu}, \quad x \text{ in } \Omega,\\ u(x) > 0, \quad x \text{ in } \Omega,\\ u(x) = 0, \quad x \text{ on } \partial\Omega \end{gathered} \tag{1.1}$$

and

$$\begin{gathered} -a\Big(\int_\Omega |u(x)|^\gamma dx\Big)\Delta u = \lambda(u^q + K(x)u^{-\mu}), \quad x \text{ in } \Omega,\\ u(x) > 0, \quad x \text{ in } \Omega,\\ u(x) = 0, \quad x \text{ on } \partial\Omega, \end{gathered} \tag{1.2}$$

where $\Omega \subseteq \mathbb{R}^N$ $(N \geq 1)$ is a sufficiently regularity domain, $q > 0$, $\lambda \geq 0$, $\mu > 0$ and $\gamma \in (0, +\infty)$.

Obviously, if $a(t) \equiv 1$ for $t \in [0, +\infty)$, (1.1) and (1.2) are singular elliptic boundary value problems and there are many results on existence, uniqueness and multiplicity of positive solutions, see [12, 13, 14, 15, 18, 20, 21, 22, 23] and their references. Chipot and Lovat [6] considered the model problem

$$\begin{gathered} u_t - a\Big(\int_\Omega u(z,t)dz\Big)\Delta u = f, \quad \text{in } \Omega \times (0,T),\\ u(x,t) = 0, \quad \text{on } \Gamma \times (0,T),\\ u(x,0) = u_0(x), \quad \text{on } \Omega. \end{gathered} \tag{1.3}$$

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Here Ω is a bounded open subset in $\mathbb{R}^N$, $N \geq 1$ with smooth boundary Γ, T is some arbitrary time. Notice that if $u(x,t)$ is independent from t, (1.3) is a nonlocal elliptic problems such as

$$\begin{gathered} -a\Big(\int_\Omega |u(x)|^\gamma dx\Big)\Delta u = f(x,u), \quad x \text{ in } \Omega, \\ u(x) = 0, \quad x \text{ on } \partial\Omega. \end{gathered} \tag{1.4}$$

And a more generalized problem of (1.4) is

$$\begin{gathered} -A(x,u)\Delta u = f(x,u), \quad x \text{ in } \Omega, \\ u(x) > 0, \quad x \text{ in } \Omega, \\ u(x) = 0, \quad x \text{ on } \partial\Omega, \end{gathered} \tag{1.5}$$

where $A : \Omega \times L^p(\Omega) \to R^+$ is a measurable function.

By establishing comparison principles, using the results on fixed point index theory, sub-supersolution method, some authors obtained the existence of at least one positive solutions for (1.4) or (1.5), see [5, 7, 8, 9, 10, 19] and their references. We notice that the nonlocal term $A(x,u)$ or $a(\int_\Omega |u(x)|^\gamma dx)$ causes that the monotonic nondecreasing of f being necessary for using the sub-supersolution method. Up to now, there are fewer results on the existence and multiplicity of positive solutions for (1.4) or (1.5) when $f(x,u)$ is singular at $u = 0$. Very recently, an interesting result on the following problems is obtained

$$\begin{gathered} \begin{aligned} -a\Big(\int_\Omega |u(x)|^\gamma dx\Big)\Delta u &= h_1(x,u) f\Big(\int_\Omega |u(x)|^p dx\Big) \\ &\quad + h_2(x,u) g\Big(\int_\Omega |u(x)|^r dx\Big), \quad x \text{ in } \Omega, \end{aligned} \\ u = 0, \quad x \text{ on } \partial\Omega, \end{gathered} \tag{1.6}$$

where $\gamma, r, p \geq 1$ and in which Alves and Covei showed that the existence of solution for some classes of nonlocal problems without of the monotonic nondecreasing of h_1 (see [4]) as $h_1(x,u) = \frac{1}{u^\alpha}$, $\alpha \in (0,1)$. In [16], applying the change of variable and the theory of fixed point index on a cone, do Ó obtained the multiplicity of radial positive solutions for some nonlocal and nonvariational elliptic systems when the nonlinearities f_i is nondecreasing in u without singularity at $u = 0$, $i = 1, 2, \dots, n$ and $\Omega = \{x \in \mathbb{R}^N | 0 < r_1 < |x| < r_2\}$.

In this article, we consider the existence, uniqueness and multiplicity of positive solutions to (1.1) and (1.2) when $\mu > 0$ is arbitrary.

This paper is organized as follows. In Section 2, according to the idea in [4, 11], we prove a new result on the existence of classical solutions by using sub-supersolution method with maximum principle. In section 3, using Theorem 2.4, the existence and uniqueness of positive solution to (1.1) are presented. In section 4, by Rabinowitz-type global bifurcation theory, we discuss the global results and obtain the multiplicity of positive solutions for (1.2).

2. Sub-supersolution method

Now we consider a general problem

$$\begin{gathered} -a\Big(\int_\Omega |u(x)|^\gamma dx\Big)\Delta u = F(x,u), \quad x \text{ in } \Omega, \\ u = 0, \quad x \text{ on } \partial\Omega, \end{gathered} \tag{2.1}$$

where $\Omega \subseteq \mathbb{R}^N$ is a smooth bounded domain, $\gamma \in (0,+\infty)$ and $a : [0,+\infty) \to (0,+\infty)$ is continuous function with

$$\inf_{t\in[0,+\infty)} a(t) \geq a(0) =: a_0 > 0. \tag{2.2}$$

Let $C(\overline{\Omega}) = \{u : \overline{\Omega} \to R | u$ be a continuous function on $\overline{\Omega}\}$ with norm $\|u\| = \max_{x\in\overline{\Omega}} |u(x)|$.

Definition 2.1. The pair functions α and β with α, $\beta \in C(\overline{\Omega}) \cap C^2(\Omega)$ are subsolution and supersolution of (2.1) if $\alpha(x) \leq u \leq \beta(x)$ for $x \in \Omega$ and

$$\begin{gathered} -\Delta\alpha(x) \leq \frac{1}{b_0} F(x,\alpha(x)), \quad x \text{ in } \Omega, \\ \alpha\big|_{\partial\Omega} \leq 0 \end{gathered}$$

and

$$\begin{gathered} -\Delta\beta(x) \geq \frac{1}{a_0} F(x,\beta(x)), \quad x \text{ in } \Omega, \\ \beta\big|_{\partial\Omega} \geq 0, \end{gathered}$$

where $a_0 = a(0)$ and

$$b_0 = \sup_{t\in[0,\int_\Omega \max\{|\alpha(x)|,|\beta(x)|\}^\gamma dx]} a(t).$$

For a fixed $\lambda > 0$, we state the problem

$$\begin{gathered} -\Delta u + \lambda u(x) = h(x), \quad x \text{ in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{2.3}$$

where $\Omega \subseteq \mathbb{R}^N$ is a smooth bounded domain and give the deformation of Agmon-Douglas-Nirenberg theorem for (2.3).

Theorem 2.2 (Agmon-Douglas-Nirenberg [1])**.** *If $h \in C^\alpha(\overline{\Omega})$, then (2.3) has a unique solution $u \in C^{2+\alpha}(\overline{\Omega})$ such that*

$$\|u\|_{2+\alpha} \leq C_1 \|h|_\infty;$$

if $h \in L^p(\Omega)(p > 1)$, then (2.3) has a unique solution $u \in W_p^2(\Omega)$ such that

$$\|u\|_{2,p} \leq C_2 \|h\|_p,$$

where C_1, C_2 ere independent from u, h.

We define the unique solution $u = (-\Delta + \lambda)^{-1} h$ of (2.3). Obviously $(-\Delta + \lambda)^{-1}$ is a linear operator. To prove our theorem, we need the following Embedding theorem.

Lemma 2.3 ([3]). *Suppose $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with smooth boundary and $p > N$. Then there exists a $C(N,p,\Omega) > 0$ such that*

$$|u|_{k+\alpha} \le C(N,p,\Omega)\|u\|_{k+1,p}, \quad \forall u \in W_p^{k+1}(\Omega),$$

where $\alpha = 1 - \frac{N}{p}$.

Next we give our main theorem.

Theorem 2.4. *Let $\Omega \subseteq \mathbb{R}^N (N \ge 1)$ be a smooth bounded domain and $\gamma \in (0,+\infty)$. Suppose that $F : \Omega \times R \to R$ is a continuous nonnegative function. Assume α and β are the subsolution and supersolution of (2.1) respectively. Then problem (2.1) has at least one solution u such that, for all $x \in \overline{\Omega}$,*

$$\alpha(x) \le u(x) \le \beta(x).$$

Proof. Let

$$\bar{F}(x,u) = \begin{cases} F(x,\alpha(x)), & \text{if } u < \alpha(x); \\ F(x,u), & \text{if } \alpha(x) \le u \le \beta(x); \\ F(x,\beta(x)), & \text{if } u > \beta(x). \end{cases}$$

We will study the modified problem (for $\lambda > 0$)

$$\begin{gathered} -\Delta u + \lambda u = \frac{\bar{F}(x,u)}{a(\int_\Omega |\chi(x,u(x))|^\gamma dx)} + \lambda\chi(x,u), \quad x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{gathered} \tag{2.4}$$

here $\chi(x,u) = \alpha(x) + (u - \alpha(x))^+ - (u - \beta(x))^+$.

Step 1. Every solution u of (2.4) is such that: $\alpha(x) \le u(x) \le \beta(x)$, $x \in \overline{\Omega}$. We prove that $\alpha(x) \le u(x)$ on $\overline{\Omega}$. Obviously, $|\chi(x,u(x))| \le \max\{|\alpha(x)|, |\beta(x)|\}$, which implies that

$$a_0 \le a(\int_\Omega |\chi(x,u(x))|^\gamma dx) \le b_0.$$

By contradiction, assume that $\max_{x\in\bar{\Omega}}(\alpha(x) - u(x)) = M > 0$. Note that $\alpha(x) - u(x) \not\equiv M$ on $\bar{\Omega}$ ($\alpha(x) - u(x) \le 0$, $x \in \partial\Omega$). If $x_0 \in \Omega$ is such that $\alpha(x_0) - u(x_0) = M$, then

$$\begin{aligned} 0 &\le -\Delta(\alpha(x_0) - u(x_0)) \\ &\le \frac{1}{b_0}F(x_0,\alpha(x_0)) - \frac{1}{a(\int_\Omega |\chi(x,u(x))|^\gamma dx)}\bar{F}(x_0,u(x_0)) - \lambda\chi(x_0,u(x_0)) + \lambda u(x_0) \\ &\le -\lambda(\alpha(x_0) - u(x_0)) < 0. \end{aligned}$$

This is a contradiction.

Now we prove that $\beta(x) \ge u(x)$ on $\overline{\Omega}$. By contradiction, assume $\min_{x\in\bar{\Omega}}(\beta(x) - u(x)) = -m < 0$. Note that $\beta(x) - u(x) \not\equiv -m$ on $\bar{\Omega}$ ($\beta(x) - u(x) \ge 0$, $x \in \partial\Omega$). If $x_0 \in \Omega$ is such that $\beta(x_0) - u(x_0) = -m$, then

$$\begin{aligned} 0 &\ge -\Delta(\beta(x_0) - u(x_0)) \\ &\ge \frac{1}{a_0}F(x_0,\beta(x_0)) - \frac{1}{a(\int_\Omega |\chi(x,u(x))|^\gamma dx)}\bar{F}(x_0,u(x_0)) - \lambda\chi(x_0,u(x_0)) + \lambda u(x_0) \\ &\ge \lambda(u(x_0) - \beta(x_0)) > 0. \end{aligned}$$

This is a contradiction. Consequently,

$$\alpha(x) \le u(x) \le \beta(x), \quad x \in \overline{\Omega}.$$

Step 2. Every solution of (2.4) is a solution of (2.1). Every solution of (2.4) is such that :$\alpha(x) \le u(x) \le \beta(x)$. By the definition of $\bar{F}$ and χ, we have

$$\bar{F}(x, u(x)) = F(x, u(x)), \quad \chi(x, u(x)) = u(x), \quad x \in \Omega$$

and u is a solution of (2.1).

Step 3. Problem (2.4) has at least one solution. Choose $p > N$, $\alpha = 1 - \frac{N}{p}$ and define an operator

$$\overline{N} : C(\overline{\Omega}) \to C(\overline{\Omega}) \subseteq L^p(\Omega); u \to \overline{F}(\cdot, u(\cdot)).$$

Since F is continuous, the definition of $\overline{F}$ implies that $\overline{F}$ is continuous also, which guarantees $\overline{N} : C(\overline{\Omega}) \to C(\overline{\Omega})$ is well defined, continuous and maps bounded sets to bounded sets. Since (2.2) is true, a is continuous and

$$\frac{1}{a(\int_\Omega |\chi(x, u(x))|^\gamma dx)} \le \frac{1}{a_0},$$

the operator $\overline{N}_1 u = \frac{1}{a(\int_\Omega |\chi(x,u(x))|^\gamma dx)} \overline{N} u$ is continuous, and maps bounded sets to bounded sets.

For given $\lambda > 0$, we define an operator $\overline{A} : C(\overline{\Omega}) \to C(\overline{\Omega})$ by

$$\overline{A}(u) = (-\Delta + \lambda)^{-1}(\overline{N}_1 u + \lambda \chi(\cdot, u)).$$

Now we show that $\overline{A} : C(\overline{\Omega}) \to C(\overline{\Omega})$ is completely continuous.

(1) By the construction of $\overline{F}$ and χ, we have, for every $u \in C(\overline{\Omega})$,

$$\begin{aligned} &\Big| \frac{\overline{F}(x, u(x))}{a(\int_\Omega |\chi(x, u(x))|^\gamma dx)} + \lambda \chi(x, u(x)) \Big| \\ &\le \frac{1}{a_0} \max_{x \in \overline{\Omega}, \alpha(x) \le u \le \beta(x)} F(x, u) + \lambda \max\{\|\alpha\|, \|\beta\|\}, \end{aligned}$$

for all $x \in \overline{\Omega}$, which guarantees that there exists a $K > 0$ big enough such that $N_1 u + \lambda \chi(\cdot, u) \in B_{L^p}(0, K)$ for all $u \in C(\overline{\Omega})$, where

$$B_{L^p}(0, R) = \{u \in L_p(\Omega) | \|u\|_p \le K\}.$$

By Theorem 2.2, we have

$$\|\overline{A}(u)\|_{2,p} = \|(-\Delta + \lambda)^{-1}(\overline{N}_1 u + \lambda \chi(\cdot, u))\|_{2,p} \le C_2 K, \quad \forall u \in C(\overline{\Omega}). \tag{2.5}$$

Lemma 2.3 implies that $\overline{A}(C(\overline{\Omega}))$ is bounded in $C^\alpha(\overline{\Omega})$. Therefore, $\overline{A}(C(\overline{\Omega}))$ is relatively compact in $C(\overline{\Omega})$.

(2) For u_1, $u_2 \in C(\overline{\Omega})$, by Theorem 2.2, one has

$$\|\overline{A}(u_1) - \overline{A}(u_2)\|_{2,p} \le C_2 \|\overline{N}_1 u_1 + \lambda \chi(\cdot, u_1) - (\overline{N}_1 u_2 + \lambda \chi(\cdot, u_2))\|_p.$$

Lemma 2.3 and the continuity of the operator $N_1 + \lambda \chi$ guarantee that $A : C(\overline{\Omega}) \to C(\overline{\Omega})$ is continuous. Consequently, $A : C(\overline{\Omega}) \to C(\overline{\Omega})$ is completely continuous.

By (2.5) and Lemma 2.3, there exists a $K_1 > 0$ big enough such that

$$\overline{A}(C(\overline{\Omega})) \subseteq B_C(0, K_1),$$

where $B_C(0, K_1) = \{u \in C(\overline{\Omega}) | \|u\| \le K_1\}$, which implies

$$\overline{A}(B_C(0, K_1)) \subseteq B_C(0, K_1).$$

The Schauder fixed point theorem guarantees that there exists a $u \in B_C(0, K_1)$ such that
$$u = \overline{A}u,$$
i.e., u is a solution of (2.4).

Consequently, steps 1 and 2 guarantee that u in the step 3 is a solution of (2.1). The proof is complete. □

We remark that the difference between Theorem 2.4 and [4, Theorem 1] is that the solution u is a classical solution and we use $\gamma > 0$ instead of $\gamma \geq 1$. In the following sections, we assume that $a(t) : [0, +\infty)$ is continuous and increasing on $[0, +\infty)$ for convenience.

3. The existence and uniqueness of positive solution for (1.1)

In this section, we consider the singular elliptic problems (1.1), where $K \in C^\alpha(\overline{\Omega})$ with $K(x) > 0$ for $x \in \overline{\Omega}$, and $\mu > 0$. Let Φ_1 is the eigenfunction corresponding to the principle eigenvalue λ_1 of
$$\begin{gathered} -\Delta u = \lambda u, \quad x \in \Omega \\ u|_{\partial\Omega} = 0. \end{gathered} \tag{3.1}$$
It is found that $\lambda_1 > 0$, and
$$\Phi_1(x) > 0, \quad |\nabla \Phi_1(x)| > 0, \quad \forall x \in \partial\Omega. \tag{3.2}$$

Theorem 3.1. *Let $\Omega \subseteq \mathbb{R}^N$, $N \geq 1$, be a bounded domain with smooth boundary $\partial\Omega$ (of class $C^{2+\alpha}$, $0 < \alpha < 1$). If $K \in C^\alpha(\overline{\Omega})$, $K(x) > 0$ for all $x \in \overline{\Omega}$ and $\mu > 0$, then there exists a unique function $u \in C^{2+\alpha}(\Omega) \cap C(\overline{\Omega})$ such that $u(x) > 0$ for all $x \in \Omega$ and u is a solution of (1.1). If $\mu > 1$, then there exist positive constants b_1 and b_2 such that $b_1\Phi_1(x)^{\frac{2}{1+\mu}} \leq u(x) \leq b_2\Phi_1(x)^{\frac{2}{1+\mu}}$, $x \in \overline{\Omega}$.*

Proof. The proof is based on Theorem 2.4 and the construction of pairs of sub-supersolutions. The construction of supersolutions to (1.1) when $\mu > 1$ is different from that when $0 < \mu \leq 1$.

(1) Assume first that $\mu > 1$. In this case, let $t = 2/(1+\mu)$ and let $\Psi(x) = b\Phi_1(x)^t$ where $b > 0$ is a constant. By (3.1), we deduce that
$$\Delta\Psi(x) + q(x, b)\Psi^{-\mu}(x) = 0, \quad x \in \Omega, \tag{3.3}$$
where $q(x, b) = b^{1+\mu}[t(1-t)|\nabla\Phi_1(x)|^2 + t\lambda_1\Phi_1(x)^2]$. Inequality (3.2) guarantees that $\min_{x\in\overline{\Omega}}[t(1-t)|\nabla\Phi_1(x)|^2 + t\lambda_1\Phi_1(x)^2] > 0$, which implies that there exists a positive constant b such that
$$\frac{1}{a_0}K(x) < q(x, b), \quad \forall x \in \Omega.$$
Let $u(x) = b\Phi_1(x)^t$. Hence,
$$\Delta u(x) + \frac{1}{a_0}K(x)u(x)^{-\mu} = \big[\frac{1}{a_0}K(x) - q(x, b)\big]u^{-\mu}(x) < 0, \quad x \in \Omega. \tag{3.4}$$

(2) Assume that $0 < \mu \leq 1$. Let s be chosen to satisfy the two inequalities
$$0 < s < 1, s(1 + \mu) < 2 \tag{3.5}$$

and $u(x) = c\Phi_1(x)^s$, where c is a large positive constant to be chosen. For $x \in \Omega$, we have

$$\begin{aligned}
&\Delta u(x) + \frac{1}{a_0}K(x)u(x)^{-\mu} \\
&= -\Phi_1(x)^{s-2}|\nabla\Phi_1(x)|^2 cs(1-s) + \frac{1}{a_0}K(x)c^{-\mu}\Phi_1(x)^{-\mu s} - c\lambda_1 s\Phi_1(x)^s \\
&= -\Phi_1(x)^{s-2}\big[|\nabla\Phi_1(x)|^2 cs(1-s) - \frac{1}{a_0}K(x)c^{-\mu}\Phi_1(x)^{2-(1+\mu)s}\big] - c\lambda_1 s\Phi_1(x)^s.
\end{aligned}$$

From (3.2), there exists a open subset $\Omega' \subset\subset \Omega$ and a $\delta > 0$ such that

$$|\nabla\Phi_1(x)| > \delta, \quad \forall x \in \overline{\Omega} - \Omega',$$

which together with $2 - (1+\mu)s > 0$ implies that there exists a $c_1 > 0$ big enough such that for all $c > c_1$,

$$|\nabla\Phi_1(x)|^2 cs(1-s) - \frac{1}{a_0}K(x)c^{-\mu}\Phi_1(x)^{2-(1+\mu)s} > 0, \quad \forall x \in \overline{\Omega} - \Omega',$$

i.e. for all $c > c_1$, $x \in \overline{\Omega} - \Omega'$

$$\begin{aligned}
&-\Phi_1(x)^{s-2}\big[|\nabla\Phi_1(x)|^2 cs(1-s) - \frac{1}{a_0}K(x)c^{-\mu}\Phi_1(x)^{2-(1+\mu)s}\big] - c\lambda_1 s\Phi_1(x)^s \\
&< 0.
\end{aligned} \tag{3.6}$$

Moreover, from $\min_{x\in\overline{\Omega'}}\Phi_1(x) > 0$, there exists a $c_2 > 0$ big enough such that for all $c > c_2$, one has

$$\frac{1}{a_0}K(x)c^{-\mu}\Phi_1(x)^{-\mu s} - c\lambda_1 s\Phi_1(x)^s < 0, \quad \forall x \in \overline{\Omega}',$$

i.e. for all $c > c_2$, $x \in \overline{\Omega}'$,

$$-\Phi_1(x)^{s-2}|\nabla\Phi_1(x)|^2 cs(1-s) + \frac{1}{a_0}K(x)c^{-\mu}\Phi_1(x)^{-\mu s} - c\lambda_1 s\Phi_1(x)^s < 0. \tag{3.7}$$

Now choose a $c > \max\{c_1, c_2\}$. Combining (3.6) and (3.7), we have

$$\begin{aligned}
&\Delta u(x) + \frac{1}{a_0}K(x)u(x)^{-\mu} \\
&= -\Phi_1(x)^{s-2}\big[|\nabla\Phi_1(x)|^2 cs(1-s) - \frac{1}{a_0}K(x)c^{-\mu}\Phi_1(x)^{2-(1+\mu)s}\big] - c\lambda_1 s\Phi_1(x)^s \\
&< 0, \quad x \in \Omega.
\end{aligned} \tag{3.8}$$

Choose $d = \max\{b, c\}$ and define

$$u^*(x) = \begin{cases} d\Phi_1^t(x), & x \in \overline{\Omega} \quad \text{if } \mu > 1; \\ d\Phi_1^s(x), & x \in \overline{\Omega} \quad \text{if } 0 < \mu \le 1. \end{cases}$$

From (3.4) and (3.8), we have

$$\Delta u^*(x) + \frac{1}{a_0}K(x)u^*(x)^{-\mu} < 0, \quad \forall x \in \Omega.$$

It follows that for each $n \in \mathbb{N}$,

$$\Delta u^*(x) + \frac{1}{a_0}K(x)\left(u^*(x) + \frac{1}{n}\right)^{-\mu} < \Delta u^*(x) + \frac{1}{a_0}K(x)u^*(x)^{-\mu} < 0, \tag{3.9}$$

for $x \in \Omega$.

Let $b_0 = a(\int_\Omega |u^*(x)|^\gamma dx)$. Choose $\varepsilon > 0$ small enough such that

$$\frac{1}{b_0}K(x)2^{-\mu} - \varepsilon\lambda_1\Phi_1(x) > 0, \quad \forall x \in \Omega, \tag{3.10}$$

and

$$\varepsilon\Phi_1(x) < \min\{1, u^*(x)\}, \quad \forall x \in \Omega. \tag{3.11}$$

From (3.1), (3.10) and (3.11), one has that for each $n \in \mathbb{N}$,

$$\Delta\varepsilon\Phi_1(x) + \frac{1}{b_0}K(x)\Big(\varepsilon\Phi_1(x) + \frac{1}{n}\Big)^{-\mu} > \frac{1}{b_0}K(x)2^{-\mu} - \varepsilon\lambda_1\Phi_1(x) > 0, \tag{3.12}$$

for $x \in \Omega$.

Let $u_*(x) = \varepsilon\Phi_1(x)$, $x \in \overline{\Omega}$. By the definitions of u_* and u^*, we have

$$\max\{|u_*(x)|, |u^*(x)|\}^\gamma = u^*(x)^\gamma$$

and so

$$\sup_{t\in[0,\int_\Omega \max\{|u_*(x)|,|u^*(x)|\}^\gamma dx]} a(t) = a\Big(\int_\Omega u^*(x)^\gamma dx\Big) = b_0.$$

Then for $n \in \mathbb{N}$, from (3.9) and (3.12), we have for each $n \in \mathbb{N}$,

$$\begin{gathered}\Delta u^*(x) + \frac{1}{a_0}K(x)(u^*(x) + \frac{1}{n})^{-\mu} < 0, \quad x \in \Omega,\\ u^*|_{\partial\Omega} = 0\end{gathered}$$

and

$$\begin{gathered}\Delta u_*(x) + \frac{1}{b_0}K(x)(u_*(x) + \frac{1}{n})^{-\mu} > 0, \quad x \in \Omega,\\ u^*|_{\partial\Omega} = 0.\end{gathered}$$

Now Theorem 2.4 guarantees that for $n \in \mathbb{N}$, there exist $\{u_n\}$ with $u_*(x) \le u_n(x) \le u^*(x)$ for all $x \in \overline{\Omega}$ such that

$$\begin{gathered}a\Big(\int_\Omega |u_n(x)|^\gamma dx\Big)\Delta u_n(x) + K(x)(u_n(x) + \frac{1}{n})^{-\mu} = 0, \quad x \in \Omega,\\ u_n|_{\partial\Omega} = 0.\end{gathered} \tag{3.13}$$

Let $\Omega_k = \{x \in \Omega | u_*(x) > \frac{1}{k}\}$, $k \in \mathbb{N}$. From (3.13), we have

$$|\Delta u_n(x)| \le \frac{1}{a_0}K(x)u_*(x)^{-\mu}\ leq \frac{1}{a_0}\max_{x\in\overline{\Omega}} K(x)(\min_{x\in\overline{\Omega}_k} u_*(x))^{-\mu}, \quad x \in \overline{\Omega}_k,$$

which implies that $\{u_n(x)\}$ is equicontinous and uniformly bounded on $\overline{\Omega}_k$, $k \in \mathbb{N}$. Therefore, $\{u_n(x)\}$ has a uniformly convergent subsequence on every $\overline{\Omega}_k$. By Diagonal method, we can choose a subsequence of $\{u_n(x)\}$ which converges a u_0 on every $\overline{\Omega}_k$ uniformly. Without loss of generality, assume that

$$\lim_{n\to+\infty} u_n(x) = u_0(x), \quad \text{uniformly on } \overline{\Omega}_k,\ k \in \mathbb{N}.$$

Obviously,

$$u_*(x) \le u_0(x) \le u^*(x), \quad x \in \Omega,$$

which implies that

$$\lim_{x\to y\in\partial\Omega} u_0(x) = 0, \quad \forall y \in \partial\Omega.$$

Hence, we define $u_0(x) = 0$, for $x \in \partial\Omega$. And the Dominated Convergence Theorem implies that

$$\lim_{n\to+\infty} \int_\Omega |u_n(x)|^\gamma dx = \int_\Omega |u_0(x)|^\gamma dx,$$

which together with the continuity of $a(t)$ yields

$$\lim_{n\to+\infty} a\Big(\int_\Omega |u_n(x)|^\gamma dx\Big) = a\Big(\int_\Omega |u_0(x)|^\gamma dx\Big).$$

Now we claim that $u_0 \in C^{2+\alpha}(\Omega)$ and that

$$a\Big(\int_\Omega |u_0(x)|^\gamma dx\Big)\Delta u_0(x) + K(x)u_0(x)^{-\mu} = 0, \quad \forall x \in \Omega. \tag{3.14}$$

Although the proof is similar as the standard arguments for the the theory of the Elliptic problems (see [15]), we still give it in details.

Let $x_0 \in \Omega$ and let $r > 0$ be chosen so that $\overline{B(x_0, r)} \subseteq \Omega$, where $B(x_0, r)$ denotes the open ball of radius r centered at x_0. Let Ψ be a C^∞ function which is equal to 1 on $\overline{B(x_0, r/2)}$ and equal to 0 off $\overline{B(x_0, r)}$. We have

$$\Delta(\Psi(x)u_n(x)) = \begin{cases} 2\nabla\Psi(x)\cdot\nabla u_n(x) + u_n(x)\Delta\Psi(x) & \\ +\Psi(x)\frac{1}{a(\int_\Omega |u_n(x)|^\gamma dx)}K(x)u_n^{-\mu}(x), & \forall x \in \overline{B(x_0, r)}, \\ 0, & \forall x \in \Omega - \overline{B(x_0, r)}. \end{cases}$$

Let

$$p_n(x) = \begin{cases} \Psi(x)\frac{1}{a(\int_\Omega |u_n(x)|^\gamma dx)}K(x)u_n^{-\mu}(x), & \forall x \in \overline{B(x_0, r)}, \\ 0, & \forall x \in \Omega - \overline{B(x_0, r)}. \end{cases}$$

It is easy to see that p_n is a term whose L^∞ norm is bounded independently of n (note $\inf_{t\in[0,+\infty)} a(t) \ge a(0) = a_0 > 0$). Therefore, for $n > 1$, we have

$$\Psi(x)u_n(x)\Delta(\Psi(x)u_n(x)) = \sum_{j=1}^N b_{n,j}\frac{\partial(\Psi(x)u_n(x))}{\partial x_j} + q_n,$$

where $b_{n,j}$, $j = 1, 2, \dots, N$, q_n are terms whose L^∞ norm is bounded independently of n. Integrating the above equation, we have that there exist constants $c_3 > 0$, $c_4 > 0$, independent of n, such that

$$\int_{B(x_0,r)} |\nabla(\Psi u_n)|^2 dx \le c_3\Big(\int_{B(x_0,r)} |\nabla(\Psi u_n)|^2 dx\Big)^{\frac{1}{2}} + c_4.$$

From this, it follows that the $L^2(B(x_0, r))$-norm of $|\nabla(\Psi u_n)|$ is bounded independently of n. Hence, $L^2(B(x_0, \frac{r}{2}))$-norm of $|\nabla u_n|$ is bounded independently of n. Let Ψ_1 be a C^∞ function which is equal to 1 on $\overline{B(x_0, r/4)}$ and equal to 0 off $\overline{B(x_0, \frac{r}{2})}$. We have $\Delta(\Psi_1(x)u_n(x)) = 2\nabla\Psi_1(x)\cdot\nabla u_n(x) + p_{n,1}$, $p_{n,1}$ is a term whose $L^\infty(B(x_0, \frac{r}{2}))$ norm is bounded independently of n. From standard elliptic theory, the $W^{2,2}(B(x_0, \frac{r}{2}))$-norm of $\Psi_1 u_n$ is bounded independently of n and hence, the $W^{2,2}(B(x_0, \frac{r}{4}))$-norm of u_n is bounded independently of n. Since the $W^{1,2}(B(x_0, \frac{r}{4}))$-norms of the components of ∇u_n are bounded independently of n, it follows from the Sobolev imbedding theorem that, if $q = 2N/(N-2) > 2$ if $N > 2$ and $q > 2$ is arbitrary if $N \le 2$, then the $L^q(B(x_0, \frac{r}{4}))$-norm of $|\nabla u_n|$ is bounded independently of n. If Ψ_2 is a C^∞ function which is equal to 1 on $\overline{B(x_0, \frac{r}{8})}$ and equal to 0 off $\overline{B(x_0, \frac{r}{4})}$, then $\Delta(\Psi_2(x)u_n(x)) = 2\nabla\Psi_2(x)\cdot\nabla u_n(x) + p_{n,2}$, $p_{n,2}$ is a term whose $L^\infty(B(x_0,\frac{r}{4}))$ norm is bounded independently of n. Since the right-hand side of the above equation is bounded in $L^q(B(x_0,\frac{r}{4}))$, independently of n, the $W^{2,q}(B(x_0,\frac{r}{4}))$-norm of $\Psi_2 u_n$ is also bounded independently of n. Hence, the $W^{2,q}(B(x_0,\frac{r}{8}))$-norm of u_n is bounded independently of n. Continuing the line of reasoning, after a finite number of steps, we find a number $r_1>0$ and $q_1>N/(1-\alpha)$ such that the $W^{2,q_1}(B(x_0,r_1))$-norm of u_n is bounded independently of n. Hence, there is a subsequence of $\{u_n\}$, which we may assume is the sequence itself, which converges in $C^{1+\alpha}(B(x_0,r_1))$. If θ is a C^∞ function which is equal to 1 on $\overline{B(x_0,\frac{r_1}{2})}$ and equal to 0 off $B(x_0,r_1)$, then

$$\Delta(\theta u_n)=\nabla\Psi\nabla u_n+\tilde p_n,$$

where $\tilde p_n=\theta\Delta u_n+u_n\Delta\theta$. The right-hand side of the above equation converges in $C^\alpha(B(x_0,r_1))$. So, by Schauder theory, $\{\theta u_n\}$converges in $C^{2+\alpha}(B(x_0,r_1))$ and hence $\{u_n\}$ converges in $C^{2+\alpha}(B(x_0,\frac{r_1}{2}))$. Since $x_0\in\Omega$ is arbitrary, this shows that $u_0\in C^{2+\alpha}(\Omega)$. Clearly, (3.14) holds.

Consequently, we have

$$\begin{gathered}a\Big(\int_\Omega |u_0(x)|^\gamma dx\Big)\Delta u_0(x)+K(x)u_0(x)^{-\mu}=0,\quad x\in\Omega,\\ u_0|_{\partial\Omega}=0.\end{gathered}$$

By [15, Theorem 1], we have if $\mu>1$, there exist a $b_1>0$ and $b_2>0$ such that

$$b_1\Phi_1(x)^{\frac{2}{1+\mu}}\le u_0(x)\le b_2\Phi_1(x)^{\frac{2}{1+\mu}},\quad \forall x\in\overline{\Omega}.$$

Next we consider the uniqueness of positive solutions of (3.1). Assume that u_1 and u_2 are two positive solutions. Let $c_i=(a(\int_\Omega u_i(x)^\gamma dx))^{1/(\mu+1)}$ and $v_i=c_iu_i$, $i=1,2$. Then v_i satisfies

$$\begin{gathered}-\Delta v_i=K(x)v_i^{-\mu},\\ v_i|_{\partial\Omega}=0.\end{gathered}$$

Now [15] guarantees that

$$\begin{gathered}-\Delta v=K(x)v^{-\mu},\\ v|_{\partial\Omega}=0\end{gathered}$$

has a unique positive solution, which implies $v_1=v_2$, i.e.,

$$\Big(a\Big(\int_\Omega u_1(x)^\gamma dx\Big)\Big)^{1/(\mu+1)}u_1(x)=\Big(a\Big(\int_\Omega u_2(x)^\gamma dx\Big)\Big)^{1/(\mu+1)}u_2(x),\tag{3.15}$$

for $x\in\overline{\Omega}$, and so

$$\Big(a\Big(\int_\Omega u_1(x)^\gamma dx\Big)\Big)^{\gamma/(\mu+1)}u_1^\gamma(x)=\Big(a\Big(\int_\Omega u_2(x)^\gamma dx\Big)\Big)^{\gamma/(\mu+1)}u_2^\gamma(x),\quad \forall x\in\overline{\Omega}.$$

Integration on Ω yields

$$\Big(a\Big(\int_\Omega u_1(x)^\gamma dx\Big)\Big)^{\gamma/(\mu+1)}\int_\Omega u_1^\gamma(x)dx=\Big(a\Big(\int_\Omega u_2(x)^\gamma dx\Big)\Big)^{\gamma/(\mu+1)}\int_\Omega u_2^\gamma(x)dx.$$

The monotonicity of a implies that $(a(t))^{\gamma/(\mu+1)}t$ is increasing on $[0,+\infty)$, which guarantees that

$$\int_\Omega u_1(x)^\gamma dx=\int_\Omega u_2(x)^\gamma dx,$$

and so

$$\Big(a\Big(\int_\Omega u_1(x)^\gamma dx\Big)\Big)^{1/(\mu+1)} = \Big(a\Big(\int_\Omega u_2(x)^\gamma dx\Big)\Big)^{1/(\mu+1)},$$

which together with (3.15) yields $u_1(x) = u_2(x)$. The proof is complete. □

Theorem 3.2. *The solution u of Theorem 3.1 is in $W^{1,2}$ if and only if $\mu < 3$. If $\mu > 1$, then u is not in $C^1(\overline{\Omega})$.*

Proof. Suppose u is a positive solution in Theorem 3.1. Let

$$p(x) = \frac{K(x)}{a\big(\int_\Omega |u(x)|^\gamma dx\big)}.$$

Then $p \in C(\overline{\Omega})$, $p(x) > 0$ for all $x \in \overline{\Omega}$ and $u(x)$ satisfies that

$$\begin{gathered} -\Delta u = p(x)u^{-\mu}, \\ u|_{\partial\Omega} = 0. \end{gathered} \tag{3.16}$$

By [15, Theorem 2], u is in $W^{1,2}$ if and only if $\mu < 3$. If $\mu > 1$, then u is not in $C^1(\overline{\Omega})$.The proof is complete. □

The monotonicity of $a(t)$ on $[0, +\infty)$ is very important for the uniqueness of positive solution to (1.1). For example, assume that $c = \int_\Omega |u_1(x)|dx$, where u_1 is the unique positive solution of the following problem (see [15, Theorem 1]

$$\begin{gathered} -\Delta u = u^{-\mu}, \\ u|_{\partial\Omega} = 0. \end{gathered} \tag{3.17}$$

Let

$$a(t) = \begin{cases} 3, & t = 0; \\ 2 + ((\frac{t}{c})^{-(1+\mu)} - 2)|\sin \frac{t}{c}|^{1+\mu}, & t > 0. \end{cases}$$

It is easy to see that $a(t)$ is not monotone on $[0, +\infty)$. Let $\lambda_k = 2k\pi + \frac{\pi}{2}$. Then

$$a(\lambda_k c) = 2 + ((\lambda_k)^{-(1+\mu)} - 2)|\sin \lambda_k|^{1+\mu} = (\lambda_k)^{-(1+\mu)}, \quad k \in \mathbb{N}. \tag{3.18}$$

Let $u_k(x) = \lambda_k u_1(x)$, $x \in \overline{\Omega}$. Then, from (3.17) and (3.18), we have

$$\Delta u_k(x) = \lambda_k \Delta u_1(x) = -\lambda_k u_1^{-\mu}(x), \quad x \in \Omega,$$

and

$$\begin{aligned} \frac{1}{a(\int_\Omega |u_k(x)|dx)} u_k(x)^{-\mu} &= \frac{1}{a(\int_\Omega \lambda_k |u_1(x)|dx)} u_k(x)^{-\mu} \\ &= \frac{1}{a(\lambda_k c)} (\lambda_k u_1(x))^{-\mu} \\ &= \lambda_k^{1+\mu} \lambda_k^{-\mu} u_1(x)^{-\mu} = \lambda_k u_1(x)^{-\mu} \end{aligned}$$

Hence,

$$\begin{gathered} \Delta u_k(x) + \frac{1}{a(\int_\Omega |u_k(x)|dx)} u_k(x)^{-\mu} = 0, \quad x \in \Omega, \\ u_k|_{\partial\Omega} = 0, \end{gathered}$$

i.e.,

$$a\Big(\int_\Omega |u(x)|dx\Big)\Delta u(x) + u(x)^{-\mu} = 0, \quad x \in \Omega,$$

$$u|_{\partial\Omega} = 0$$

has at infinitely many positive solutions.

4. Global structure of positive solutions for (1.2)

In this section, we consider the singular nonlocal elliptic problems (1.2), where $q \in (0, +\infty)$, $\mu > 0$, $K \in C^{\alpha}(\overline{\Omega})$ with $K(x) > 0$ for all $x \in \overline{\Omega}$.

To sutudy equation (1.2), for each $n \in \mathbb{N}$, we study the equations

$$\begin{gathered} a\Big(\int_\Omega |u(x)|^\gamma dx\Big)\Delta u(x) + \lambda\big[u^q + K(x)\big(u(x) + \frac{1}{n}\big)^{-\mu}\big] = 0, \quad x \in \Omega, \\ u|_{\partial\Omega} = 0. \end{gathered} \tag{4.1}$$

Let u denote the inward normal derivative of u on $\partial\Omega$ and define

$$P = \{u \in C^{1,\alpha}(\overline{\Omega}) : u(x) > 0 \ \forall x \in \Omega,\ u(x) = 0 \text{ on } \partial\Omega \text{ and } \frac{\partial u}{\partial v} > 0 \text{ on } \partial\Omega\},$$

where $\alpha \in (0, 1)$. It follows from [17, Theorem 3.7] that for $n \in \mathbb{N}$ there is a set C_n of solutions of (4.1) which is a connected and unbounded subset of $\mathbb{R}^+ \times (P \cup \{(0, 0)\})$ (in the topology of $\mathbb{R} \times C^{1,\alpha}(\overline{\Omega})$) and contains $(0, 0)$. Obviously,

$$\|u\| \le \|u\|_{1+\alpha}, \quad \forall u \in C_n,$$

which guarantees that

$$\begin{gathered} \|u\| \to +\infty \text{ implies that } \|u\|_{1+\alpha} \to +\infty, \forall u \in C_n, \\ \|u - u_0\|_{1+\alpha} \to 0 \text{ implies that } \|u - u_0\| \to 0. \end{gathered} \tag{4.2}$$

On the other hand, by Lemma 2.3 and Theorem 2.2, for $u \in C_n$, one has

$$\begin{aligned} \|u\|_{1+\alpha} &\le C(n,p,\Omega)\|u\|_{2,p} \\ &\le C(n,p,\Omega)\lambda\frac{1}{a(\int_\Omega |u(x)|^\gamma dx)}\Big(\int_\Omega \big[u^q + K(x)\big(u(x) + \frac{1}{n}\big)^{-\mu}\big]^p dx\Big)^{1/p} \\ &\le C(n,p,\Omega)\lambda\frac{1}{a_0}\Big(\int_\Omega \big[u^q + K(x)\big(u(x) + \frac{1}{n}\big)^{-\mu}\big]^p dx\Big)^{1/p} \\ &\le C(n,p,\Omega)\lambda\frac{1}{a_0}|\Omega|^{1/p}[\|u\|^q + n\|K\|], \quad \forall u \in C_n \end{aligned}$$

and

$$\begin{aligned} \|u - u_0\|_{1+\alpha} &\le C(n,p,\Omega)\|u - u_0\|_{2,p} \\ &\le C(n,p,\Omega)\lambda\Big(\int_\Omega |\Psi_n(u)(x) - \Psi_n(u_0)(x)|^p dx\Big)^{1/p}, \quad \forall u, u_0 \in C_n, \end{aligned}$$

where

$$\Psi_n(u)(x) = \frac{1}{a(\int_\Omega |u(x)|^\gamma dx)}[u^q(x) + \frac{1}{(u(x) + \frac{1}{n})^\mu}],$$

which guarantees that

$$\begin{gathered} \|u\|_{1+\alpha} \to +\infty \text{ implies that } \|u\| \to +\infty, \forall u \in C_n, \\ \|u - u_0\| \to 0 \text{ implies that } \|u - u_0\|_{1+\alpha} \to 0. \end{gathered} \tag{4.3}$$

Combining (4.2) and (4.3), we know that C_n is connected and unbounded in $\mathbb{R} \times C(\overline{\Omega})$.

Let $\phi \in C^{2,\alpha}(\overline{\Omega})$ defined by

$$-\Delta\phi = 1, \quad x \in \Omega; \phi(x) = 0, \quad x \in \partial\Omega. \tag{4.4}$$

Lemma 4.1. *Let $M > 0$ and $(\lambda_n, u_n) \in (0, +\infty) \times P$ be a solution of (4.1) satisfying $\lambda_n \le M$ and $\|u_n\| \le M$. There is a number $\overline{\varepsilon} > 0$ and a pair of functions $\overline{\Gamma}(M) > 0$, $\overline{K}(\beta, M) > 0$ such that if ϕ is given by (4.3) and $0 < \frac{1}{n} < \overline{\varepsilon}$, then*

$$\lambda_n\overline{\Gamma}(M)\phi(x) \le u_n(x) \le \beta + \lambda_n\overline{K}(\beta, M)\phi(x), \quad x \in \Omega \tag{4.5}$$

for $\beta \in (0, M]$.

Proof. Set

$$\overline{K}(\beta, M) = \max\{\frac{1}{a_0}(r^q + K(x)r^{-\mu}) : (x, r) \in \overline{\Omega} \times [\beta, 1 + M]\}. \tag{4.6}$$

Let (λ_n, u_n) be as in the Lemma 4.1, $0 < \frac{1}{n} < 1$ and $\beta \in (0, M]$. Set $A_\beta = \{x \in \Omega | u_n(x) > \beta\}$. By (4.4) and (4.6), one has

$$\begin{aligned}
&-\Delta(\beta + \lambda_n\overline{K}(\beta, M)\phi - u_n) \\
&= \lambda_n\overline{K}(\beta, M) - \lambda_n\frac{1}{a\big(\int_\Omega u_n(x)^\gamma dx\big)}[u_n^q + K(x)(u_n + \frac{1}{n})^{-\mu}] \\
&\ge \lambda_n\overline{K}(\beta, M) - \lambda_n\frac{1}{a_0}[u_n^q + K(x)(u_n)^{-\mu}] \ge 0, \quad x \in A_\beta,
\end{aligned}$$

and

$$u_n(x) = \beta, \quad x \in \partial A_\beta.$$

Thus $\beta + \lambda_n\overline{K}(\beta, M)\phi(x) \ge u_n(x)$ on $\overline{A}_\beta$ by the maximum principle and the right-hand inequality of (4.5) is established.

To obtain the left-hand inequality, choose $R > 0$ so that

$$\frac{1}{a(\int_\Omega(\beta + M\overline{K}(\beta, M)\phi(x))^\gamma dx)}K(x)r^{-\mu} > 1$$

if $0 < r < R$. Define $\overline{\Gamma}(M) = \min\{1, R/(2M\|\phi\|)\}$. Then, for $\frac{1}{n} \in (0, R/2]$, $\eta \in (0, \overline{\Gamma}(M)]$ and $\lambda_n \in (0, M]$, from the right-hand inequality of (4.5) and the monotonicity of $a(t)$, one has

$$\begin{aligned}
-\Delta(\lambda_n\eta\phi(x)) &= \lambda_n\eta \\
&< \lambda_n\frac{1}{a(\int_\Omega(\beta + M\overline{K}(\beta, M)\phi(x))^\gamma dx)}K(x)(\lambda_n\eta\phi + \frac{1}{n})^{-\mu} \\
&\le \lambda_n\frac{1}{a(\int_\Omega u_n(x)^\gamma dx)}[(\lambda_n\eta\phi)^q + K(x)(\lambda_n\eta\phi + \frac{1}{n})^{-\mu}].
\end{aligned} \tag{4.7}$$

From this we will deduce that $\lambda_n\overline{\Gamma}(M)\phi(x) < u_n(x)$, $x \in \Omega$. Since $\frac{\partial u_n}{\partial v}|_{\partial\Omega} > 0$, $u_n(x) > 0$ for $x \in \Omega$, there exists a $\Omega' \subset\subset \Omega$ and $m > 0$ such that $\frac{\partial u_n}{\partial v}|_{\partial\Omega} \ge m > 0$ for all $x \in \overline{\Omega} - \Omega'$ and $u_n(x) \ge m > 0$ for all $x \in \overline{\Omega}'$, which implies that there exists a $s > 0$ such that

$$u_n - \tau\lambda_n\phi \in P, \quad \forall\tau \in [0, s].$$

Since $\lim_{s\to+\infty}\|s\lambda_n\phi\| = +\infty$, there exists a $s' > 0$ such that $u_n - s'\lambda_n\phi \notin P$. Define

$$\eta^* = \sup\{s > 0 | u_n - \tau\lambda_n\phi \in P, \quad \forall\tau \in [0, s]\}.$$

It is easy to see that $0 < \eta^* \leq s'$ and $u_n - \eta\lambda_n\phi \in P$ for $0 < \eta < \eta^*$ and $u_n - \eta^*\lambda_n\phi \notin P$. It suffices to show $\eta^* > \overline{\Gamma}(M)$. If $\eta^* \leq \overline{\Gamma}(M)$, let $w = u_n - \lambda_n\eta^*\phi \geq 0$ in $\overline{\Omega}$ and, by (4.7) for $C > 0$, we have

$$\begin{aligned}
-\Delta w + Cw &= Cw + \lambda_n \frac{1}{a(\int_\Omega u_n(x)^\gamma dx)}[u_n(x)^q + K(x)(u_n(x) + \frac{1}{n})^{-\mu}] - \lambda_n\eta^* \\
&> Cw + \lambda_n \frac{1}{a(\int_\Omega u_n(x)^\gamma dx)}\Big([u_n(x)^q + K(x)(u_n(x) + \frac{1}{n})^{-\mu}] \\
&\quad - [(\lambda_n\eta^*\phi)^q + K(x)(\lambda_n\eta^*\phi + \frac{1}{n})^{-\mu}]\Big).
\end{aligned}$$

By the Mean Value Theorem we have

$$[u_n(x)^q + K(x)(u_n(x) + \frac{1}{n})^{-\mu}] - [(\lambda_n\eta^*\phi(x))^q + K(x)(\lambda_n\eta^*\phi(x) + \frac{1}{n})^{-\mu}] \geq C_0 w,$$

where

$$C_0 = \min_{x\in\overline{\Omega}} \inf_{r\in[\frac{1}{n}, \frac{1}{n} + \|u_n\| + \lambda_n\eta^*\|\phi\|]} K(x)(-\mu)r^{-(1+\mu)}.$$

Choose

$$C + \lambda_n \frac{1}{a(\int_\Omega u_n(x)^\gamma dx)} C_0 > 0.$$

Then

$$-\Delta w + Cw > 0,$$

which means that $w \in P$. This is a contradiction. Consequently, $\eta^* > \overline{\Gamma}(M)$ and so $\lambda_n\overline{\Gamma}(M)\phi(x) < u_n(x)$, $x \in \Omega$. The proof is complete. □

Theorem 4.2. *There is a set C of solutions of (1.2) satisfying the following:*

- (i) *C is connected in $\mathbb{R} \times C(\overline{\Omega})$;*
- (ii) *C is unbounded in $\mathbb{R} \times C(\overline{\Omega})$;*
- (iii) *$(0, 0)$ lies in the closure of C in $\mathbb{R} \times C(\overline{\Omega})$.*

Proof. For $M > 0$, define

$$B((0, 0), M) = \{(\lambda, u) \in \mathbb{R} \times C(\overline{\Omega}) | \lambda^2 + \|u\|^2 < M^2\}.$$

Let $(\lambda_n, u_n) \in \partial B((0, 0), M) \cap (0, +\infty) \times P$ be solutions of (4.1) as above, $n \to +\infty$ and $\lambda_n \to \lambda$. If $\lambda = 0$, we deduce from (4.5) that

$$0 < \limsup_{n\to+\infty} \sup_{x\in\overline{\Omega}} u_n(x) \leq \beta, \quad \forall\beta \in (0, M]$$

and hence that $u_n \to 0$ in $C(\overline{\Omega})$. Then $(\lambda_n, u_n) \to (0, 0)$ as $n \to +\infty$ in $\mathbb{R} \times C(\overline{\Omega})$. Since $(\lambda_n, u_n) \in \partial B((0, 0), M)$, this is impossible. Then $\lambda > 0$.

From (4.5) and $\lambda > 0$, we see that u_n is bounded from below by a function which is positive in Ω and from above by a constant. Arguing as in the proof of Theorem 3.1, without loss of generality, passing to the limit in (4.5), there is a $u_0 \in C(\overline{\Omega})$ such that

$$\lim_{n\to+\infty} u_n(x) = u_0(x), \quad \text{uniformly } x \in \overline{\Omega}_0 \subset \Omega, \tag{4.8}$$

where Ω_0 is arbitrary sub-domain in Ω and

$$\lambda\overline{\Gamma}(M)\phi(x) \leq u(x) \leq \beta + \lambda\overline{K}(\beta, M)\phi(x), \quad x \in \Omega \tag{4.9}$$

for $\beta \in (0, M]$. From (4.5) and (4.9) we have

$$\lim_{x\to\partial\Omega} u_0(x) = 0$$

and

$$\lim_{x\to\partial\Omega} u_n(x) = 0, \quad \text{uniformly for } n \in \mathbb{N}. \tag{4.10}$$

Now (4.8) and (4.10) imply that $u_n \to u_0$ as $n \to +\infty$. It follows that $(\lambda_n, u_n) \to (\lambda, u_0)$ in $\mathbb{R} \times C(\overline{\Omega})$ and hence $(\lambda, u_0) \in \partial B((0,0), M)$.

A standard argument as the proof of Theorem 3.1 shows that u_0 satisfies

$$\begin{gathered} a\Big(\int_\Omega |u_0(x)|^\gamma dx\Big)\Delta u_0(x) + \lambda(u_0(x)^q + K(x)u_0(x)^{-\mu}) = 0, \quad x \in \Omega, \\ u_0|_{\partial\Omega} = 0. \end{gathered}$$

Wee omit the proof.

At this point we have shown that if $B((0,0), M)$ is a bounded neighborhood of $(0,0)$ in $\mathbb{R} \times C(\overline{\Omega})$, then there is a solution $(\lambda, u_0) \in \partial B((0,0), M))$ of (1.2). Since M is arbitrary, $C = \{(\lambda, u_\lambda) \in B((0,0), M) | u_\lambda$ is a positive solution for (1.2). The proof is complete. $\square$

Corollary 4.3. *If $q < 1$, then $\lambda \in (0, +\infty)$. In particular,* (1.2) *with $\lambda = 1$ has a solution.*

Proof. Suppose C is the connected and unbounded set of positive solutions for (1.2) in Theorem 4.2. Now we show that $\lambda \in (0, +\infty)$.

In fact, suppose set $\{\lambda | (\lambda, u) \in C\}$ is finite and let $\Lambda_0 = \{\lambda > 0 | (\lambda, u) \in C\}$. The unboundedness of C means that there exist $\{(\lambda_n, u_n)\}$ such that

$$\lim_{n\to+\infty} \|u_n\| = +\infty.$$

Set $A_1 = \{x \in \Omega | u_n(x) > 1\}$ and

$$\overline{K}_n = \frac{1}{a_0}(\|u_n\|^q + \max_{x\in\overline{\Omega}} K(x)). \tag{4.11}$$

It follows from (4.4) and (4.11) that

$$\begin{aligned} -\Delta(1 + \lambda_n \overline{K}_n \phi - u_n) &= \lambda_n \overline{K}_n - \lambda_n \frac{1}{a(\int_\Omega u_n(x)^\gamma dx)}[u_n^q + K(x)(u_n)^{-\mu}] \\ &\geq \lambda_n \overline{K}_n - \lambda_n \frac{1}{a_0}[\|u_n\|^q + \max_{x\in\overline{\Omega}} K(x)] \geq 0, \quad x \in A_1, \end{aligned}$$

and

$$u_n(x) = 1, \quad x \in \partial A_1.$$

Thus $1 + \lambda_n \overline{K}_n \phi(x) \geq u_n(x)$ on $\overline{A}_1$ by the maximum principle and so

$$u_n(x) \leq 1 + \lambda_n \overline{K}_n \phi(x), \quad \forall x \in \overline{\Omega},$$

which implies

$$\|u_n\| \leq 1 + \Lambda_0(\|u_n\|^q + \max_{x\in\overline{\Omega}} K(x)) \max_{x\in\overline{\Omega}} \phi(x).$$

By $q < 1$, one has

$$1 \leq \lim_{n\to+\infty} \Big[\frac{1}{\|u_n\|} + \Lambda_0(\|u_n\|^{q-1} + \max_{x\in\overline{\Omega}} K(x)/\|u_n\|) \max_{x\in\overline{\Omega}} \phi(x)\Big] = 0.$$

This is a contradiction. Therefore, $\Lambda_0 = +\infty$. The proof is complete. $\square$

Now we consider the case $q > 1$. Let $K(x) = K(|x|)$ and we consider the problem (1.2) when $\Omega = \{x \in \mathbb{R}^N | 0 < r_1 < |x| < r_2\}$ and $N \geq 3$ and discuss the radial positive solutions for (1.2), i.e., (1.2) is equivalent to the problem

$$\begin{aligned} &- a\left(N\omega_N \int_{r_1}^{r_2} r^{N-1}|u(r)|^\gamma dr\right)(u''_{rr} + \frac{N-1}{r}u_r) \\ &= \lambda[u(r)^q + K(|r|)u^{-\mu}(r))], \quad r\text{in } (r_1, r_2), \\ &\qquad u(r) > 0, \quad t \in (r_1, r_2), \\ &\qquad u(r_1) = 0, \quad u(r_2) = 0, \end{aligned} \tag{4.12}$$

where ω_N denotes the area of unit sphere in $\mathbb{R}^N$.

By [16], applying the change of variable $t = l(r)$ and $u(r) = z(t)$ with

$$t = l(r) = -\frac{A}{r^{N-2}} + B \Longleftrightarrow r = (\frac{A}{B-t})^{\frac{1}{N-2}},$$

where

$$A = \frac{(r_1 r_2)^{N-2}}{r_2^{N-2} - r_1^{N-2}}, \quad B = \frac{r_2^{N-2}}{r_2^{N-2} - r_1^{N-2}},$$

we obtain

$$\begin{aligned} &N\omega_N \int_{r_1}^{r_2} r^{N-1}|u(r)|^\gamma dr \\ &= N\omega_N \int_0^1 (\frac{A}{B-s})^{\frac{N-1}{N-2}} A^{\frac{1}{N-2}} \frac{1}{N-2}(B-s)^{-\frac{N-1}{N-2}}|z(s)|^\gamma ds \\ &= A_N \int_0^1 B_N(s)|z(s)|^\gamma ds \end{aligned}$$

where

$$A_N = N\frac{\omega_N}{N-2}A^{\frac{N}{N-2}}, \quad B_N(s) = (B-s)^{\frac{2(N-1)}{2-N}},$$

and

$$\begin{gathered} u'_r = z'_t t'_r = z'_t(-A)(2-N)r^{1-N}, \\ u''_{rr} = z''_{tt}((-A)(2-N)r^{1-N})^2 + z'_t(-A)(2-N)(1-N)r^{-N}, \end{gathered}$$

which implies

$$u''_{rr} + \frac{N-1}{r}u_r = ((-A)(2-N)r^{1-N})^2 z''_{tt}.$$

And then (4.12) is equivalent to the problem

$$\begin{aligned} &- a\Big(A_N \int_0^1 B_N(s)|z(s)|^\gamma ds\Big)z''(t) \\ &= \lambda d(t)[z(t)^q + K((\frac{A}{B-t})^{1/(N-2)})z^{-\mu}(t))], \quad t \text{ in } (0,1), \\ &\qquad z(t) > 0, \quad t \in (0,1), \\ &\qquad z(0) = 0, \quad z(1) = 0, \end{aligned} \tag{4.13}$$

where

$$d(t) = \frac{A^{2/(2-N)}}{(N-2)^2(B-t)^{2(N-1)/(N-2)}}, \quad t \in [0,1]$$

and the related integral equation is

$$
\begin{aligned}
z(t) = \lambda \frac{1}{a\big(A_N \int_0^1 B_N(s)|z(s)|^\gamma ds\big)} \int_0^1 G(t,s)d(s) \\
\times \big[z(s)^q + K((\frac{A}{B-s})^{1/(N-2)}) z^{-\mu}(s)\big] ds,
\end{aligned}
\tag{4.14}
$$

for $t \in (0,1)$, where

$$
G(t,s) = \begin{cases} s(1-t), & 0 \le s \le t \le 1; \\ t(1-s), & 0 \le t \le s \le 1. \end{cases}
$$

Lemma 4.4 (see [2, page 18]). *Suppose $z \in C[0,1]$ is concave on $[0,1]$ with $z(t) \ge 0$ for all $t \in [0,1]$. Then $z(t) \ge \|z\| t(1-t)$ for $t \in [0,1]$*

Corollary 4.5. *If $\lim_{t\to+\infty} \frac{t^{q-1}}{a(t^\gamma)} = +\infty$, then C in Theorem 4.2 satisfies:*

- (i) *there exists $\Lambda_0 >$ satisfying $C \cap ((\Lambda_0, +\infty) \times C_0[0,1]) = \emptyset$;*
- (ii) *for every $\lambda \in (0, \Lambda_0]$, $C \cap ([0,\lambda] \times C_0[0,1])$ is unbounded;*
- (iii) *there exists $\lambda_0 \le \Lambda_0$ such that for every $\lambda \in (0, \lambda_0)$, (4.10) has at least two positive solutions $z_{1,\lambda}$ and $z_{2,\lambda}$ with*

$$
\lim_{\lambda\to 0, (\lambda, z_{1,\lambda}) \in C} \|z_{1,\lambda}\| = 0, \quad \lim_{\lambda\to 0, (\lambda, z_{2,\lambda}) \in C} \|z_{2,\lambda}\| = +\infty.
$$

Proof. (i) Suppose that $(\lambda, z_\lambda) \in C$. Since $z_\lambda''(t) \le 0$ and $z_\lambda(0) = z_\lambda(1) = 0$, we have z is concave on $[0,1]$ with $z(t) \ge 0$ for all $t \in [0,1]$. Now Lemma 4.4 implies

$$
z_\lambda(t) \ge t(1-t)\|z_\lambda\|, \quad \forall t \in [0,1].
$$

If $\|z_\lambda\| \le 1$, it follows from (4.14)

$$
\begin{aligned}
1 &\ge \|z_\lambda\| \\
&= \lambda \frac{1}{a(A_N \int_0^1 B_N(s)|z_\lambda(s)|^\gamma ds)} \max_{t\in[0,1]} \int_0^1 G(t,s)d(s) \\
&\quad \times \big[z_\lambda(s)^q + K((\frac{A}{B-s})^{1/(N-2)}) z_\lambda^{-\mu}(s)\big] ds \\
&> \lambda \frac{1}{a(A_N \int_0^1 B_N(s) ds)} \max_{t\in[0,1]} \int_0^1 G(t,s)d(s) K((\frac{A}{B-s})^{1/(N-2)}) ds,
\end{aligned}
$$

and so

$$
\lambda \le \frac{a(A_N \int_0^1 B_N(s)ds)}{\max_{t\in[0,1]} \int_0^1 G(t,s)d(s)K((\frac{A}{B-s})^{1/(N-2)})ds}. \tag{4.15}
$$

Since

$$
\lim_{t\to+\infty} \frac{t^{q-1}}{a(t^\gamma)} = +\infty,
$$

one has

$$
\lim_{t\to+\infty} \frac{t^{q-1}}{a(t^\gamma A_N \int_0^1 B_N(s)ds)} = \lim_{s\to+\infty} \frac{s^{q-1}(A_N \int_0^1 B_N(s)ds)^{-(q-1)/\gamma}}{a(s^\gamma)} = +\infty,
\tag{4.16}
$$

which implies that there is an $M_0 > 0$ such that

$$
\frac{a(t^\gamma A_N \int_0^1 B_N(s)ds)}{t^{q-1}} \le M_0, \quad \forall t \in [1, +\infty). \tag{4.17}
$$

If $\|z_\lambda\| \geq 1$, from (4.14) and (4.17), one has

$$\begin{aligned}\|z_\lambda\| &\geq \lambda \frac{1}{a\big(\|z\|^\gamma A_N \int_0^1 B_N(s)ds\big)} \max_{t\in[0,1]} \int_0^1 G(t,s)d(s)[z_\lambda(s)^q]ds \\ &\geq \lambda \frac{\|z_\lambda\|^q}{a\big(\|z_\lambda\|^\gamma A_N \int_0^1 B_N(s)ds\big)} \max_{t\in[0,1]} \int_0^1 G(t,s)d(s)[s(1-s)]^q ds,\end{aligned}$$

and so

$$\begin{aligned}\lambda &\leq \frac{a\big(\|z_\lambda\|^\gamma A_N \int_0^1 B_N(s)ds\big)}{\|z\|^{q-1}} \frac{1}{\max_{t\in[0,1]} \int_0^1 G(t,s)d(s)[s(1-s)]^q ds} \\ &\leq M_0 \frac{1}{\max_{t\in[0,1]} \int_0^1 G(t,s)d(s)[s(1-s)]^q ds}.\end{aligned} \tag{4.18}$$

It follows from (4.15) and (4.18) that

$$\begin{gathered}\Lambda_0 = \sup\{\lambda | (\lambda, z_\lambda) \in C\} < +\infty, \\ C \cap ((\Lambda_0, +\infty) \times C_0[0,1]) = \emptyset.\end{gathered}$$

(ii) For every $\lambda \in (0, \Lambda_0]$, we show that $C \cap ([\lambda, \Lambda_0] \times C_0[0,1])$ is bounded. In fact, if $C \cap ([\lambda, \Lambda_0] \times C_0[0,1])$ is unbounded, there is $\{(\lambda_n, z_n)\} \subseteq C \cap ([\lambda, \Lambda_0] \times C_0[0,1])$ such that

$$\lambda_n^2 + \|z_n\|^2 \to +\infty, \quad \text{as } n \to +\infty.$$

Since $\{\lambda_n\} \subseteq [\lambda, \Lambda_0]$ is bounded, without loss of generality, we assume that $\lambda_n \to \lambda' > 0$ as $n \to +\infty$. It implies that

$$\|z_n\|^2 \to +\infty, \quad \text{as } n \to +\infty.$$

From (4.14), one has

$$\begin{aligned}\|z_n\| &\geq \lambda_n \frac{1}{a(\|z_n\|^\gamma A_N \int_0^1 B_N(s)ds)} \max_{t\in[0,1]} \int_0^1 G(t,s)d(s)[z_n(s)^q]ds \\ &\geq \lambda_n \frac{\|z_n\|^q}{a(\|z_n\|^\gamma A_N \int_0^1 B_N(s)ds)} \max_{t\in[0,1]} \int_0^1 G(t,s)d(s)[s(1-s)]^q ds,\end{aligned}$$

and so

$$1 \geq \lambda \frac{\|z_n\|^{q-1}}{a(\|z_n\|^\gamma A_N \int_0^1 B_N(s)ds)} \max_{t\in[0,1]} \int_0^1 G(t,s)d(s)[s(1-s)]^q ds.$$

From (4.16), letting $n \to +\infty$, one has $1 \geq +\infty$. This is a contradiction. Hence, $C \cap ([\lambda, \Lambda_0] \times C_0[0,1])$ is bounded for any $\lambda \in (0, \Lambda_0]$.

(iii) Choose $R > 1 > r > 0$. Suppose $(\lambda, z_\lambda) \in C$ with $r \leq \|z_\lambda\| \leq R$. By

$$z^q + K(x)z^{-\mu} \geq z^q + \min_{x\in\overline{\Omega}} K(|x|)z^{-\mu},$$

there is a $c_0 > 0$ such that

$$z^q + K(x)z^{-\mu} \geq c_0, \quad \forall z \in (0, +\infty), x \in \overline{\Omega}. \tag{4.19}$$

From (4.14) and (4.19) it follows that

$$\begin{aligned}
z_\lambda(t) &= \lambda \frac{1}{a(A_N \int_0^1 B_N(s)|z_\lambda(s)|^\gamma ds)} \int_0^1 G(t,s)d(s) \\
&\quad \times \big[z_\lambda(s)^q + K((\frac{A}{B-s})^{1/(N-2)}) z_\lambda^{-\mu}(s)\big] ds \\
&\ge \lambda \frac{1}{a(R^\gamma A_N \int_0^1 B_N(s)ds)} \int_0^1 G(t,s)d(s)c_0 ds,
\end{aligned}$$

and so

$$\|z_\lambda\| \ge \lambda \frac{1}{a(R^\gamma A_N \int_0^1 B_N(s)ds)} \max_{t\in[0,1]} \int_0^1 G(t,s)d(s)c_0 ds,$$

which guarantees that

$$\lambda \le \frac{Ra(R^\gamma A_N \int_0^1 B_N(s)ds)}{\max_{t\in[0,1]} \int_0^1 G(t,s)d(s)dsc_0} =: \lambda_R. \tag{4.20}$$

One the other hand, since

$$\begin{gathered}
z_\lambda'' + \lambda \frac{1}{a(A_N \int_0^1 B_N(s)|z_\lambda(s)|^\gamma ds)} d(t)[z_\lambda^q(t) + K((\frac{A}{B-t})^{1/(N-2)}) z_\lambda^{-\mu}(t)] = 0, \\
0 < t < 1, \\
z_\lambda(0) = z_\lambda(1) = 0,
\end{gathered}$$

there exists $t_\lambda \in (0,1)$ with $z_\lambda'(t) \ge 0$ on $(0, t_\lambda)$ and $z_\lambda'(t) \le 0$ on $(t_\lambda, 1)$. For $t \in (0, t_\lambda)$ we have

$$\begin{aligned}
-z_\lambda''(t) &\le \lambda \frac{1}{a_0} z_\lambda^{-\mu}(t) d(t) \Big\{ \max_{t\in[0,1]} K((\frac{A}{B-t})^{1/(N-2)}) + z_\lambda^{\mu+q}(t) \Big\} \\
&\le \lambda \frac{1}{a_0} z_\lambda^{-\mu}(t) \max_{t\in[0,1]} d(t) \Big\{ \max_{t\in[0,1]} K((\frac{A}{B-t})^{1/(N-2)}) + R^{\mu+q} \Big\} \\
&= \lambda \frac{1}{a_0} z_\lambda^{-\mu}(t) d_1, \\
d_1 &:= \max_{t\in[0,1]} d(t) \Big\{ \max_{t\in[0,1]} K((\frac{A}{B-t})^{1/(N-2)}) + R^{\mu+q} \Big\}.
\end{aligned}$$

Integrate from t ($t \le t_\lambda$) to t_λ (note $z_\lambda(s)$ is increasing on $[t, t_\lambda]$) to obtain

$$z_\lambda'(t) \le \lambda \frac{1}{a_0} \int_t^{t_\lambda} z_\lambda^{-\mu}(s) ds d_1 \le \lambda \frac{1}{a_0} \int_t^{t_\lambda} z_\lambda^{-\mu}(t) ds d_1 \le \lambda \frac{1}{a_0} d_1 z_\lambda^{-\mu}(t),$$

i.e.

$$z_\lambda^\mu(t) z_\lambda'(t) \le \lambda \frac{1}{a_0} d_1, \tag{4.21}$$

and then integrate (4.21) from 0 to t_λ to obtain

$$\frac{1}{\mu+1} r^{\mu+1} \le \int_0^{t_\lambda} z_\lambda^\mu(t) dz_\lambda(t) \le \lambda \frac{1}{a_0} d_1.$$

Consequently

$$\lambda \ge \frac{r^{\mu+1} a_0}{(\mu+1) d_1} =: \lambda_r. \tag{4.22}$$

It follows from (4.20) and (4.22) that $(\lambda, u_\lambda) \in [\lambda_r, \lambda_R] \times (\{z | r \leq \|z\| \leq R\} \cap P)$ for all $(\lambda, z_\lambda) \in C$ with $r \leq \|z_\lambda\| \leq R$. Since C comes from $(0,0)$, C is connected and $C \cap ((0, \lambda_r) \times C_0[0,1])$ is unbounded, if $\lambda \in (0, \lambda_r)$, there exist at least two $x_{1,\lambda}$ and $x_{2,\lambda}$ with $\|x_{1,\lambda}\| < r$ and $\|x_{2,\lambda}\| > R$.

Let

$$\lambda_0 = \sup\{\lambda_r : (1.2) \text{ has at least two positive solutions for all } \lambda \in (0, \lambda_r)\}.$$

Obviously, $\lambda_0 \leq \Lambda_0$ and (1.2) has at least two positive solutions for all $\lambda \in (0, \lambda_r)$ and has at least one positive solution for all $\lambda \in [\lambda_0, \Lambda_0]$. Since R and r are arbitrary, it follows that (iii) is true. The proof is complete. $\square$

If $N = 1$, we can consider the problem

$$\begin{gathered}
-a\Big(\int_0^1 |z(s)|^\gamma ds\Big) z''(t) = \lambda[z(t)^p + K(t) z^{-\mu}(t))], \quad t \text{ in } (0,1), \\
z(t) > 0, \quad t \in (0,1), \\
z(0) = 0, \quad z(1) = 0,
\end{gathered}$$

and obtain the similar results as Corollary 4.5 for the above problem.

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References

[1] R. A. Adams; *Sobolev spaces*, Academic Press, 1973.

[2] R. P. Agarwal, D. O'Regan; *A survey of recent results for initial and boundary value problems singular in the dependent variable*, Original Research Article Handbook of Differential Equations: Ordinary Differential Equations 1 (2000) 1-68.

[3] S. Agmon, A. Douglis, L. Nirenberg; *Estimates near the boundary for the solutions of elliptic differential equations satisfying general boundary values*, J. Comm. Pure Appl. Math., 12 (1959), 624-727.

[4] C. O. Alves, D. P. Covei; *Existence of solution for a class of nonlocal elliptic problem via sub-supersolution method*, Nonlinear Analysis: Real World Applications, 23 (2015), 1-8.

[5] M. Chipot, F. J. S. A. Corrêa; *Boundary layer solutions to functional elliptic equations*, Bull Braz Math Soc., 40(3) (2009), 381-393.

[6] M. Chipot, B. Lovat; *Some remarks on nonlocal elliptic and parabolic problems*, Nonlinear Anal., 30 (1997), 4619-4627.

[7] M. Chipot, P. Roy; *Existence results for some functional elliptic equations*, Differential and Integral Equations, 27 (2014), 289-300.

[8] F. J. S. A. Corrêa; *On positive solutions of nonlocal and nonvariational elliptic problems*, Nonlinear Analysis, 59 (2004), 1147-1155.

[9] F. J. S. A. Corrêa, M. Delgado, A. Suárez; *Some non-local problems with nonlinear diffusion*, Mathematical and Computer Modelling, 54 (2011), 2293-2305.

[10] F. J. S. A. Corrêa, S. D. B. Menezes, J. Ferreira; *On a class of problems involving a nonlocal operator*, Appl. Math. Comput., 147 (2004), 475-489.

[11] C. De Coster; *Existence and localization of solution for second order elliptic BVP in presence of lower and upper solutions without Any Order*, Journal of differential equations, 145 (1998), 420-452.

[12] M. G. Crandall, P. H. Rabinowitz, L. Tartar; *On a Dirichlet problem with a singular nonlinearity*, Communications in Partial Differential Equations, 2(2) (1997), 193-222.

[13] S. M. Gomes; *On a singular nonlinear elliptic problems*, Siam J. Math. Anal., 17(6) (1986), 1359-1369.

[14] J. Hernández, F. J. Mancebo, J. M. Vega; *Positive solutions for singular nonlinear elliptic equations*, Proceedings of the Royal Society of Edinburgh: Section A Mathematics, 137(1) (2007), 41-62.
[15] A. C. Lazer, P. J. Mckenna; *On a singular nonlinear elliptic boundary-value problems*, Proceeding of the American Mathematical Society, 111(3) (1991), 721-730.
[16] J. M. do Ó, S. Lorca, J. Sánchez, P. Ubilla; *Positive solutions for some nonlocal and nonvariational elliptic systems*, Complex Variables and Elliptic Equations (2015), http://dx.doi.org/10.1080/17476933.2015.1064404.
[17] P. H. Rabinowitz; *Some global results for nonlinear eigenvalue problems*, Journal of Functional Analysis, 7(3) (1971), 487-513.
[18] V. D. Rădulescu; *Singular phenomena in nonlinear elliptic problemsHandbook of Differential Equations Stationary Partial Differential Equations*, 4 (2007), 485-593.
[19] P. Roy; *Existence results for some nonlocal problems*, Differential Equations & Applications, 6(3) (2014), 361-381.
[20] J. Shi, M. Yao; *On a singular nonlinear semilinear elliptic problem*, Proceedings of the Royal Society of Edinburgh, 128A (1995), 1389-1401.
[21] J. Shi, M. Yao; *On a singular nonlinear semilinear elliptic problem*, Proceedings of the Royal Society of Edinburgh: Section A Mathematics, 128(6) (1998), 1389-1401.
[22] Y. Sun, S. Wu, Y. Long; *Combined effects of singular and superlinear nonlinearities in some singular boundary value problems*, Journal of Differential Equations, 176(2) (2001), 511-531.
[23] Z. Zhang, J. Yu; *On a singular nonlinear Dirichlet problem with a convection term*, SIAM J. Math. Anal., 32(4) (2000), 916-927.

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