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# POSITIVE SOLUTIONS FOR ELLIPTIC EQUATIONS WITH SINGULAR NONLINEARITY

JUNPING SHI, MIAOXIN YAO

Abstract. We study an elliptic boundary-value problem with singular nonlinearity via the method of monotone iteration scheme:

$$
\begin{gathered}
-\Delta u(x) = f(x, u(x)), \quad x \in \Omega, \\
u(x) = \phi(x), \quad x \in \partial\Omega,
\end{gathered}
$$

where $\Delta$ is the Laplacian operator, $\Omega$ is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, $\phi \geq 0$ may take the value 0 on $\partial\Omega$, and $f(x, s)$ is possibly singular near $s = 0$. We prove the existence and the uniqueness of positive solutions under a set of hypotheses that do not make neither monotonicity nor strict positivity assumption on $f(x, s)$, which improvements of some previous results.

## 1. Introduction

Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^N$, $N \geq 2$. We assume that the boundary $\partial\Omega$ of $\Omega$ is of $C^{2,\theta}$ for some $\theta \in (0, 1)$. Let $\phi(x)$ be a nonnegative function belonging to $C^{2,\theta}(\partial\Omega)$ and $f(x, s)$ be a function defined on $\overline{\Omega} \times (0, +\infty)$ which is locally Hölder continuous with exponent $\theta$. We consider the existence and the uniqueness of positive solutions for the nonlinear boundary-value problem

$$
-\Delta u(x) = f(x, u(x)), \quad x \in \Omega, \tag{1.1}
$$

$$
u(x) = \phi(x), \quad x \in \partial\Omega, \tag{1.2}
$$

where $\Delta$ is the Laplacian operator.

A positive solution of problem (1.1)-(1.2) is a function $u(x) \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ satisfying (1.1)-(1.2) and $u(x) > 0$ for $x \in \Omega$.

Many articles treat the problem of the existence and/or the uniqueness of positive solutions for (1.1)-(1.2) under a variety of hypotheses on function $f(x, s)$. When $f(x, s)$ is locally Lipschiz in $\Omega \times [0, +\infty)$, the existence and uniqueness of positive solutions (for some cases) are well understood. However, if there is a sequence $\{(x_i, s_i)\}$ in $\Omega \times (0, +\infty)$, for which $x_i$ converges to some point in the set $\{x \in \partial\Omega | \phi(x) = 0\}$ and $s_i$ tends to 0 as $i \to +\infty$, such that $f(x_i, s_i) \to \infty$, then problem (1.1)-(1.2) is singular, it does not have a solution in $C^2(\overline{\Omega})$, and the existence or

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uniqueness results do not follow from the results obtained for nonsingular equations in the literature.

It is well-known that such singular elliptic problems arise in the contexts of chemical heterogeneous catalysts, non-Newtonian fluids and also the theory of heat conduction in electrically conducting materials, see [3, 4, 6, 7] for a detailed discussion.

In [7], the existence of a positive solution of such a singular problem is established under a set of assumptions in which $f(x,s)$ is assumed to be non-increasing in $s$. Thus if $f(x,s)$ is defined, say, by

$$f(x,s) = g(x)\ln^2 s, \tag{1.3}$$

or by

$$f(x,s) = g(x)s^{-\alpha} + h(x)s^{\beta} - k(x)s^{\rho}, \tag{1.4}$$

where $\alpha > 0, \beta \in (0,1), \rho \geq 1$ , $g$ and $k$ are nonnegative Hölder continuous functions, then the existence of positive solutions does not follow from the results in [7].

the authors in [12] and [5] treat the singular problem with no monotonicity assumption on $f(x,s)$, and the results there may imply the existence of positive solutions even when $f(x,s)$ is given by (1.3), (1.4), in which $k(x) = 0, g(x), h(x) > 0$ for $x \in \overline{\Omega}$, or, by (see [12])

$$f(x,s) = 1 + \{1 + \cos\frac{1}{s}\}s^{1/2}e^{1/s}. \tag{1.5}$$

Some uniqueness results are also given in [12] and [5]. However, the method of proof in [12] and [5] requires that $f(x,s)$ be strictly positive near $s = 0$, i.e., $f(x,s)$ is bounded away from 0 as $s \to 0^+$, for $x \in \overline{\Omega}$ (See $(H_2), (H_2')$ in [12] and $(g_1)$ in [5]). Therefore, if $f(x,s)$ is given, say, by (1.3), (1.4), with $g(x)$ and $h(x)$ vanishing on some non-empty subset of $\Omega$, or given by

$$f(x,s) = s^{1/2}e^{\frac{1}{s}(1+\cos\frac{1}{s})}, \tag{1.6}$$

then no conclusion regarding the existence of positive solutions can be derived from the results in [12] and [5].

For the special case where $f(x,s) = g(x)s^{-\alpha}$ in which $g$ is a sufficiently regular function and is positive in $\Omega$, and $\alpha > 0$, [9] gives some results when $g(x)$ is vanishing or tending to $\infty$ near $\partial\Omega$ with a suitable rate, and the positivity of $f(x,s)$ for $x \in \Omega$ is still assumed.

Recently the case where $f(x,s) = g(x)s^{-\alpha} + h(x)s^p$ is studied with $p \in (0,1)$ and the restriction that $\alpha \in (0, \frac{1}{N})$, also assumed the positivity hypotheses on functions $g(x)$ and $h(x)$ on whole $\Omega$.

In the present article, neither monotonicity nor positivity on whole $\Omega$ is assumed for $f(x,s)$, and the results are more general, implying the existence of positive solutions for (1.1)-(1.2) even with $f(x,s)$ given by any of (1.3)–(1.6), where $g(x)$ and $h(x)$ may be 0, and even $h(x)$ may be negative, in some subset of $\Omega$ . Also a uniqueness result is obtained. If we assume that for each $x \in \Omega$ either $s^{-1}f(x,s)$ is strictly decreasing in $s$ for $s > 0$, or $f(x,s)$ and $s^{-1}f(x,s)$ are both nonincreasing in $s$, and that function $f$ satisfies some certain conditions in addition to the conditions for existence results, then we can further prove that the solution is unique. When $f(x,s)$ is locally Lipschiz in $\Omega \times [0,+\infty)$ and hence not singular, and $s^{-1}f(x,s)$ is strictly decreasing in $s$ for $s > 0$ at every $x$ in $\Omega$, this kind of uniqueness result is well-known (see for example, [10]), however, our result extends it to include singular nonlinearity cases, which covers the special case where $f$ is given by (1.4), and also applies to the case where $s^{-1}f(x,s)$ need'nt be strictly decreasing in $s$ for all $x$ in $\Omega$.

The precise hypotheses and main results are stated in Section 2, and the proof for the results is given in Section 3. The proof for the existence results is based on a monotone convergence argument with solutions of (1.1) corresponding to the boundary data $\phi(x)+\frac{1}{k}$, which are obtained by using a monotone iteration scheme started with certain supersolutions and subsolutions particularly chosen; the proof for the uniqueness result makes use of a comparison lemma, which stems from some idea of a lemma in [2].

## 2. Hypotheses and Main Results

We assume that the function $f$ that defines the nonlinear term in (1.1) satisfies the following conditions:

(F1) $f\colon \overline{\Omega}\times(0,+\infty)\to\mathbb{R}$ is Hölder continuous with exponent $\theta\in(0,1)$ on each compact subset of $\overline{\Omega}\times(0,+\infty)$.

(F2)
$$\limsup_{s\to+\infty}\Big(s^{-1}\max_{x\in\overline{\Omega}} f(x,s)\Big)<\lambda_1,$$
where $\lambda_1$ is the first eigenvalue of $-\Delta$ on $\Omega$ with Dirichlet boundary value.

(F3) For each $t>0$, there exists a constant $D(t)>0$ such that
$$f(x,r)-f(x,s)\ge -D(t)(r-s)$$
for $x\in\overline{\Omega}$ and $r\ge s\ge t$. (Without loss of generality we assume that $D(s)\le D(t)$ for $s\ge t>0$.)

For the case in which $\phi(x)\not\equiv 0$ on $\partial\Omega$, we have the following result.

**Theorem 2.1.** *Suppose that $f$ satisfies (F1)–(F3) and $\phi\in C^{2,\theta}(\partial\Omega)$. If $\phi(x)\ge 0$ and $\phi(x)\not\equiv 0$ on $\partial\Omega$, and if there exist $\gamma,\delta>0$ such that*
$$f(x,s)\ge-\gamma s,\quad \textit{for}\quad x\in\overline{\Omega}\ s\in(0,\delta), \tag{2.1}$$
*then there exists at least one positive solution $u(x)$ of problem (1.1) (1.2) such that for any compact subset $G$ of $\Omega\cup\{x\in\partial\Omega|\phi(x)>0\}$, $u(x)\in C^{2,\theta}(G)$.*

For the general case where $\phi(x)$ may be 0 for all $x\in\partial\Omega$, we have the following theorems.

**Theorem 2.2.** *Suppose that $f$ satisfies (F1)–(F3) and $\phi\in C^{2,\theta}(\partial\Omega)$. If $\phi(x)\ge 0$ on $\partial\Omega$ and if there exist positive numbers $\delta,\gamma$ and a nonempty open subset $\Omega_0$ of $\Omega$ such that*
$$f(x,s)\ge-\gamma s,\quad \textit{for } x\in\overline{\Omega}\ s\in(0,\delta), \tag{2.2}$$
$$s^{-1}f(x,s)\to+\infty\quad \textit{as } s\to 0^+ \textit{ uniformly for } x\in\Omega_0, \tag{2.3}$$
*then the conclusion of Theorem 2.1 holds.*

**Theorem 2.3.** *Suppose that $f$ satisfies (F1)–(F3) and $\phi\in C^{2,\theta}(\partial\Omega)$. If $\phi(x)\ge 0$ on $\partial\Omega$ and if there exists $\delta>0$ such that*
$$f(x,s)\ge\lambda_1 s\quad \textit{for } x\in\overline{\Omega}\ s\in(0,\delta), \tag{2.4}$$
*then the conclusion of Theorem 2.1 holds.*

The following theorem concerns to the uniqueness of positive solutions for problem (1.1)-(1.2). We use the hypotheses

(F4) Either $f(x, s)$ is nonincreasing in $s$ for each $x$ in $\Omega$, or, $s^{-1}f(x, s)$ is strictly decreasing in $s$ for each $x$ in an open subset $\Omega_0$ of $\Omega$ and both $f(x, s)$ and $s^{-1}f(x, s)$ are nonincreasing in $s$ for all $x$ in the remainder part $\Omega - \Omega_0$,

(F5) The function
$$F(s, t) = \max_{d(x)=s} |f(x, t)|, \quad \text{with } d(x) = \operatorname{dist}(x, \partial\Omega),$$
either is bounded on $(0, \delta) \times (0, \delta)$, or is a sum of such a bounded function and some function that is decreasing in $t$ on $(0, \delta)$ for any $s \in (0, \delta)$, and
$$\int_0^\delta F(s, c_0 s)ds < +\infty, \quad \text{for all } c_0 \in (0, 1).$$

**Theorem 2.4.** *Under the assumption of any of Theorems 2.1–2.3, if in addition the function $f(x, s)$ satisfies (F4) and (F5), then problem* (1.1)-(1.2) *has one and only one positive solution in $C^0(\overline{\Omega}) \cap C^{2,\theta}(\Omega)$.*

**Remarks.**

(1) Examples of $f(x, s)$ , at a point $x$, satisfying the condition in (F4) that both $f(x, s)$ and $s^{-1}f(x, s)$ are non-increasing in $s$, are $f(x, s) = f_1(x)s^{\rho_1}$ for $s > 0$ with $\rho_1 \leq 0$ and $f_1(x) \geq 0$ , $f(x, s) = f_2(x)s^{\rho_2}$ for $s > 0$ with $\rho_2 \geq 1$ and $f_2(x) \leq 0$, and so on.

(2) If $f(x, s)$ is a sum of a function $f_1(x, s)$ that is bounded on $\Omega \times (0, \delta)$ and some function $f_2(x, s)$ that is decreasing in $s$ on $(0, \delta)$ for any $x \in \Omega$, and if for any $c_0 \in (0, 1)$, there exists $\alpha_0 < 1$ such that
$$|f_2\left(x, c_0 d(x)\right)| = O\left((d(x))^{-\alpha_0}\right), \quad \text{as}\, d(x) \to 0,$$
then (F5) is obviously satisfied.

By the above remarks, we can easily derive from Theorems 2.2 and 2.4 the following corollary, in which $h^+$ and $h^-$ stand respectively for the positive part and the negative part of $h$, i.e., $h^+(x) = \max\{h(x), 0\}$, $h^-(x) = \max\{-h(x), 0\}$.

**Corollary 2.5.** *The singular nonlinear elliptic problem*
$$\Delta u + g(x)u^{-\alpha} + h(x)u^\beta - k(x)u^\rho = 0, \quad x \in \Omega,$$
$$u(x) = 0, \quad x \in \partial\Omega,$$
*with $\beta \in (0, 1)$, $\rho \geq 1$, and $\alpha > 0$, possesses a positive solution $u$ in $C^0(\overline{\Omega}) \cap C^{2,\theta}(\Omega)$, provided that functions $g, h$ and $k$ are $\theta-$Hölder continuous on $\overline{\Omega}$ , $g, k$ are nonnegative, $g + h^+$ is not identically zero, and $h^-(x) \leq \sigma_0 g(x), \forall x \in \Omega$, for some constant $\sigma_0 > 0$.*

*If in addition the function $h$ is non-negative or non-positive on whole $\Omega$, and for some $\alpha_0 < 1$,*
$$g(x) = O\left((d(x))^{\alpha-\alpha_0}\right), \quad \text{as } d(x) \to 0, x \in \Omega,$$
*then the solution $u$ is unique.*

This is an example in which the behavior of a coefficient function near the boundary affects the existence and uniqueness of solutions. Moreover, the result here makes improvement to some results in the literature [5] [7] [12] and [13].

## 3. Proof of Results

Let $(P_k)$ denote the boundary-value problem:

$$\begin{gathered}-\Delta u(x) = f(x, u(x)), \quad x \in \Omega,\\ u(x) = \phi(x) + \frac{1}{k}, \quad x \in \partial\Omega,\end{gathered} \tag{3.1}$$

where $k$ is a positive integer. We say that a function $u$ is a supersolution, or a subsolution, of (3.1) if $u$ belongs to $C^2(\Omega) \cap C^0(\overline{\Omega})$ and satisfies (3.1) with sign $=$ replaced by signs $\geq$, or $\leq$, respectively.

In this Section we first prove Theorem 2.1 in detail, then we outline the proofs for Theorems 2.2 and 2.3. After we state and prove a lemma we finally prove Theorem 2.4.

*Proof of Theorem 2.1. Step 1.* Let $m, k$ be positive integers and denote by $\psi_{m,k}(x)$ (resp. $\psi_{m,\infty}(x)$) the unique solution in $C^2(\overline{\Omega})$ of problem

$$\begin{gathered}-\Delta\psi(x) + \gamma\psi(x) = 0, \quad x \in \Omega,\\ \psi(x) = \frac{1}{m}\phi(x) + \frac{1}{k}, \quad x \in \partial\Omega,\\ (\text{resp.} \quad \psi(x) = \frac{1}{m}\phi(x), \quad x \in \partial\Omega.)\end{gathered}$$

Then it follows from the estimates of Schauder type [8] and the maximum principle for $-\Delta + \gamma$ that there exists a positive integer $m_0$ such that

$$\begin{gathered}0 < \psi_{m_0,\infty}(x) < \psi_{m_0,k}(x), \quad x \in \Omega,\ k \geq m_0,\\ 0 < \psi_{m_0,k+1}(x) < \psi_{m_0,k}(x) < \delta, \quad x \in \overline{\Omega},\ k \geq m_0.\end{gathered}$$

Hence, by (2.1), $\psi_{m_0,k}(x)$ is a subsolution of (3.1) for every $k \geq m_0$. Let

$$\delta_k = \min_{x\in\overline{\Omega}} \psi_{m_0,k}(x),$$

we have

$$0 < \delta_{k+1} < \delta_k, \quad k \geq m_0.$$

By (F2) we may take $\lambda_0 > 0$ such that

$$\limsup_{s\to+\infty} \big(s^{-1} \max_{x\in\overline{\Omega}} f(x, s)\big) < \lambda_0 < \lambda_1,$$

and then consider the problem

$$\begin{gathered}-\Delta\xi(x) \geq \lambda_0\xi(x), \quad x \in \Omega,\\ \xi(x) > 0, \quad x \in \overline{\Omega}.\end{gathered}$$

The existence of solutions to this problem is established in [11]. Let $\xi(x)$ be such a function and $k_0$ be a positive integer sufficiently large. Then it's easy to verify that $k_0\xi(x)$ is a supersolution of (3.1) for every $k \geq m_0$, and we may have

$$k_0\xi(x) \geq \psi_{m_0,k}(x) + \max_{x\in\overline{\Omega}} \phi(x), \quad x \in \overline{\Omega},\ k \geq m_0.$$

*Step 2.* We define the iteration scheme below, as in the standard supersolution and subsolution argument,

$$-\Delta w_n(x) + D(\delta_{m_0})w_n(x) = f(x, w_{n-1}(x)) + D(\delta_{m_0})w_{n-1}(x), \quad x \in \Omega,$$

$$w_n(x) = \phi(x) + \frac{1}{m_0}, \quad x \in \partial\Omega,$$

noting that (F3) implies that for each $x \in \Omega, s \mapsto f(x,s) + D(\delta_{m_0})s$ is an increasing function on $[\delta_{m_0}, +\infty)$. Thus, as in the proof of Theorem 1 in [1], by setting $w_0(x) = \psi_{m_0,m_0}(x)$ (or $k_0\xi(x)$) for $x \in \Omega$, we obtain a monotonic sequence that converges to a solution $u_{m_0}(x) \in C^2(\overline{\Omega})$ of $(P_{m_0})$ such that

$$\psi_{m_0,m_0}(x) \le u_{m_0}(x) \le k_0\xi(x), \quad x \in \overline{\Omega}.$$

Using the same iteration scheme with $m_0$ replaced by $m_0+1$, and setting $w_0(x) = \psi_{m_0,m_0+1}(x)$ (or $u_{m_0}(x)$), we can obtain, as in above, a positive solution $u_{m_0+1}(x) \in C^2(\overline{\Omega})$ of $(P_{m_0+1})$. Furthermore, by the maximum principle for $-\Delta + D(\delta_{m_0+1})$, we have

$$\psi_{m_0,m_0+1}(x) \le u_{m_0+1}(x) \le u_{m_0}(x), \quad x \in \overline{\Omega}.$$

Hence, by repeating the above process, we obtain the sequence $\{u_k(x)\}_{k\ge m_0}$ satisfying

$$\psi_{m_0,\infty}(x) \le u_{k+1}(x) \le u_k(x) \le k_0\xi(x), \quad x \in \overline{\Omega},\ k \ge m_0. \tag{3.2}$$

and $u_k(x)$ solves (3.1) for any $k \ge m_0$.

*Step 3.* We can define function $u$ by

$$u(x) = \lim_{k\to+\infty} u_k(x), \quad x \in \overline{\Omega},$$

because $\{u_k(x)\}_{k\ge m_0}$ is a decreasing sequence uniformly bounded from below by $\psi_{m_0,\infty}(x)$ on $\overline{\Omega}$. Now, we have from (3.2) that

$$\psi_{m_0,\infty}(x) \le u(x) \le k_0\xi(x), \quad x \in \overline{\Omega}.$$

Thus, if $G$ is a compact subset of $\Omega \cup \{x \in \partial\Omega | \phi(x) > 0\}$, then there exist two positive constants $E_1(G)$ and $E_2(G)$ such that

$$E_1(G) \le u(x) \le E_2(G), \quad x \in G,\ k \ge m_1.$$

Therefore, using the same reasoning as that in [12] and [9] and the Schauder theory as stated in [8], we conclude that $u(x)$ satisfies (1.1) and belong to $C^{2,\theta}(G)$.

On the other hand, by the hypotheses about function $f$, the number

$$H := \inf_{k\ge m_0}\{\min_{x\in\overline{\Omega}} f(x, u_k(x))\}$$

exists, hence by the maximum principle we have

$$Q(x) \le u_k(x), \quad x \in \overline{\Omega},\ k \ge m_0,$$

and hence

$$Q(x) \le u(x), \quad x \in \overline{\Omega},$$

where $Q(x)$ is the solution of problem

$$-\Delta Q(x) = H, \quad x \in \Omega,$$
$$Q(x) = \phi(x), \quad x \in \partial\Omega.$$

Furthermore, it is easy to see that if $x_0 \in \partial\Omega$, then for any $\varepsilon > 0$ there exist $r_0 > 0$ and an integer $m_1 \ge m_0$ such that

$$Q(x) \le u_k(x) \le \phi(x_0) + \varepsilon,$$

for all $k \ge m_1$ and $x \in \Omega$ for which $|x - x_0| < r_0$. Therefore, $u(x)$ is continuous on $\overline{\Omega}$ satisfying (1.2). This completes the proof. $\square$

Functions $\psi_{m_0,\infty}(x)$ and $\psi_{m_0,k}(x)$ play an important role in the proof above. For the proof of Theorem 2.2 or 2.3, we only show the way for obtaining these two functions, the remainder of the proof is almost the same as that of Theorem 2.1 and is omitted.

*Proof of Theorem 2.2.* Choose an $\eta(x) \in C_0^\infty(\Omega)$ such that $0 \le \eta(x) \le 1$ for $x \in \overline{\Omega}$, $\eta(x) \not\equiv 0$, and $\operatorname{supp}\eta \subset \Omega_0$. By (F1) and (2.3), there exist $c_1, c_2 > 0$ such that $c_1 < \delta$, hence $f_\gamma(x, c_1) \ge 0, x \in \overline{\Omega}$, here $f_\gamma(x,s) \equiv f(x,s) + \gamma s$, and

$$c_1 \le f_\gamma(x, c_1) \le c_2,\ x \in \Omega_0, \tag{3.3}$$

then we denote by $\psi_{m,k}(x)$ ( resp. $\psi_{m,\infty}(x)$) the unique solution in $C^2(\overline{\Omega})$ of the problem

$$\begin{gathered}
-\Delta\psi(x) + \gamma\psi(x) = \frac{1}{m}\eta(x) f_\gamma(x, c_1), \quad x \in \Omega,\\
\psi(x) = \frac{1}{k}, \quad x \in \partial\Omega.\\
(\text{resp.} \quad \psi(x) = 0, \quad x \in \partial\Omega.)
\end{gathered}$$

We have for all $m, k \ge 1$ that

$$\begin{gathered}
\psi_{m,\infty}(x) = \frac{1}{m}\psi_{1,\infty}(x), \quad x \in \overline{\Omega}\\
\psi_{m,k}(x) \ge \psi_{m,\infty}(x) > 0, \quad x \in \Omega\\
\psi_{m,k}(x) \ge \psi_{m^*,k^*}(x) > 0, \quad x \in \overline{\Omega}, \text{if } m^* \ge m \text{ and } k^* \ge k.
\end{gathered}$$

Clearly there exist $d_1, d_2 > 0$ such that

$$d_1 \le \psi_{1,\infty}(x) \le d_2 \quad \text{for } x \in \operatorname{supp}\eta. \tag{3.4}$$

By the Schauder estimates [8], we can make $\psi_{m,k}(x)$, uniformly for $x \in \overline{\Omega}$, as small as we want by taking $m$ and $k$ both large enough. Hence there exists integer $m_0$ such that

$$\frac{f_\gamma(x, \psi_{m,k}(x))}{\psi_{m,k}(x)} \ge \frac{c_2}{d_1}, \quad m, k \ge m_0,\ x \in \operatorname{supp}\eta,$$

by (2.3), now by (2.2),

$$f_\gamma(x, \psi_{m,k}(x)) \ge 0, \quad x \in \Omega.$$

Therefore, if $x \in \operatorname{supp}\eta$ and $m, k \ge m_0$,

$$\begin{aligned}
-\Delta\psi_{m,k}(x) - f(x, \psi_{m,k}(x)) &= \frac{1}{m} f_\gamma(x, c_1)\big[\eta(x) - \frac{f_\gamma(x, \psi_{m,k}(x))}{\psi_{m,k}(x)} \frac{\psi_{m,k}(x)}{\frac{1}{m} f_\gamma(x, c_1)}\big]\\
&\le \frac{1}{m} f_\gamma(x, c_1)\big[\eta(x) - \frac{c_2}{d_1}\frac{\psi_{1,\infty}(x)}{f_\gamma(x, c_1)}\big]\\
&\le 0;
\end{aligned}$$

(by (3.3) and (3.4)). If $x \in \Omega \setminus \operatorname{supp}\eta$,

$$-\Delta\psi_{m,k}(x) - f\big((x, \psi_{m,k}(x)\big) = -f_\gamma\big(x, \psi_{m,k}(x)\big) \le 0.$$

Thus, if $m, k \ge m_0$, then $\psi_{m,k}(x)$ is a subsolution of (3.1). Therefore, the functions $\psi_{m_0,k}(x)$ and $\psi_{m_0,\infty}(x)$ meet the needs. $\square$

*Proof.* Proof of Theorem 2.3 We point our that it suffices to let $\psi_{m,\infty}(x)$ be the function $m^{-1}\Phi_1(x)$ and $\psi_{m,k}(x)$ be the unique solution of the problem

$$-\Delta\psi(x) = \frac{\lambda_1}{m}\Phi_1(x), \quad x \in \Omega,$$
$$\psi(x) = \frac{1}{k}, \quad x \in \partial\Omega$$

where $\Phi_1$ is the first eigenfunction of $-\Delta$ with zero boundary value which satisfies $\max_{x\in\Omega}\Phi_1(x) = 1$. □

To prove Theorem 2.4, we need the following lemma, which is an extension of a lemma in [2].

**Lemma 3.1.** *Let $\Omega$ be a domain with a $C^2$ boundary $\partial\Omega$ or no boundary in $\mathbb{R}^N$, $N \geq 2$. Suppose that $f : \Omega \times (0, +\infty) \to \mathbb{R}$ is a continuous function such that the assumption (F4) is satisfied, and let $w, v \in C^2(\Omega)$ satisfy:*

(a) $\Delta w + f(x, w) \leq 0 \leq \Delta v + f(x, v)$ *in* $\Omega$
(b) $w, v > 0$ *in* $\Omega$*,* $\liminf_{|x|\to+\infty}\big(w(x)-v(x)\big)$*, and* $\liminf_{x\to\partial\Omega}\big(w(x)-v(x)\big) \geq 0$
(c) $\Delta v \in L^1(\Omega)$.

*Then $w(x) \geq v(x)$ for all $x \in \Omega$.*

*Proof.* The proof for the case where $f(x, s)$ is non-increasing in $s$ at each $x$ in $\Omega$ is trivial, so we only prove for the second case in assumption (F4).

Without loss of generality, we assume that $\Omega = \Omega_1 \cup \Omega_2$ in which

$$\Omega_1 = \{x \in \Omega : f(x, s) \text{ and } s^{-1}f(x, s) \text{ are nonincreasing in } s\},$$

$\Omega_1 \neq \Omega$ and

$$\Omega_2 = \Omega_0 - \Omega_1$$

which is an anon-empty and open subset of $\Omega$, since $\Omega_1$ is a relative closed subset of $\Omega$.

To prove the lemma by contradiction, we let $S_\delta$ be the set $\{x \in \Omega \mid w(x) < v(x) - \delta\}$ for $\delta \geq 0$ and suppose that $S_0 \neq \emptyset$. Then by the condition (b), there exists some $\sigma > 0$ such that $S_\sigma \neq \emptyset$ and $\overline{S_\sigma} \subset \Omega$ .

If $S_\sigma \cap \Omega_2 = \emptyset$, then $\overline{S_\sigma} \subset \Omega_1$. Noting that, at the boundary of $S_\sigma, w(x) = v(x) - \sigma$, and that, for $x \in s_\sigma$,

$$\Delta(w(x) - (v(x) - \sigma)) \leq f(x, v(x)) - f(x, w(x) \leq 0$$

by the assumption on $f(x, s)$ for $x \in \Omega_1$ and the condition $(a)$, one could have $w(x) \geq v(x) - \sigma$ for all $x \in s_\sigma$ by the aid of the maximum principle applied on $S_\sigma$. But this is a contradiction to the definition of $S_\sigma$.

If $S_\sigma \cap \Omega_2 \neq \emptyset$, then it is easily seen from the assumption on $f(x, s)$ for $x \in \Omega_0$ that there exist $\varepsilon_0 > 0$ and a closed ball $\overline{B} \subset (S_\sigma \cap \Omega_2)$ such that

$$v(x) - w(x) \geq \varepsilon_0, \; x \in \mathrm{B}, \tag{3.5}$$

and

$$\delta_0 := \int_B vw\Big(\frac{f(x, w)}{w} - \frac{f(x, v)}{v}\Big)dx > 0. \tag{3.6}$$

Let

$$M = \max\{1, \|\Delta v\|_{L^1(\Omega)}\}, \quad \varepsilon = \min\{1, \varepsilon_0, \frac{\delta_0}{4M}\}.$$

Let $\theta$ be a smooth function on $\mathbb{R}$ such that $\theta(t) = 0$ if $t \leq 1/2$, $\theta(t) = 0$ if $t \geq 1, \theta(t) \in (0,1)$ if $t \in (\frac{1}{2}, 1)$, and $\theta'(t) \geq 0$ for $t \in \mathbb{R}$. Then, for $\varepsilon > 0$, define the function $\theta_\varepsilon(t)$ by

$$\theta_\varepsilon(t) = \theta\big(\frac{t}{\varepsilon}\big),\ t \in \mathbb{R}.$$

It then follows from condition (a) and the fact that $\theta_\varepsilon(t) \geq 0$ for $t \in \mathbb{R}$ that

$$(w\Delta v - v\Delta w)\theta_\varepsilon(v-w) \geq vw\big(\frac{f(x,w)}{w} - \frac{f(x,v)}{v}\big)\theta_\varepsilon(v-w), \quad x \in \Omega.$$

On the other hand, by the continuity of $w, v$ and $\theta_\varepsilon$, and condition (b), we can take an open set $D$ with a smooth boundary such that $\overline{\mathrm{B}} \subset D \subset S_\delta$, here $\delta = \min\{\sigma, \frac{\varepsilon}{4}\}$, and $v(x) - w(x) \leq \frac{\varepsilon}{2}$, for all $x \in S_0 - D$. Then we have

$$\int_D (w\Delta v - v\Delta w)\theta_\varepsilon(v-w)dx \geq \int_D vw\big(\frac{f(x,w)}{w} - \frac{f(x,v)}{v}\big)\theta_\varepsilon(v-w)dx.$$

Denote

$$\Theta_\varepsilon(t) = \int_0^t s\theta'(s)ds,\ t \in \mathbb{R},$$

then it is easy to verify that

$$0 \leq \Theta_\varepsilon(t) \leq 2\varepsilon, \quad t \in \mathbb{R}, \quad \text{and} \quad \Theta_\varepsilon(t) = 0, \text{ if } t < \frac{\varepsilon}{2}. \tag{3.7}$$

Therefore,

$$\begin{aligned}
&\int_D (w\Delta v - v\Delta w)\theta_\varepsilon(v-w)dx \\
&= \int_{\partial D} w\theta_\varepsilon(v-w)\frac{\partial v}{\partial n}ds - \int_D (\nabla v \cdot \nabla w)\theta_\varepsilon(v-w)dx \\
&\quad - \int_D w\theta'_\varepsilon(v-w)\nabla v \cdot (\nabla v - \nabla w)dx - \int_{\partial D} v\theta_\varepsilon(v-w)\frac{\partial w}{\partial n}ds \\
&\quad + \int_D (\nabla w \cdot \nabla v)\theta_\varepsilon(v-w)dx + \int_D v\theta'_\varepsilon(v-w)\nabla w \cdot (\nabla v - \nabla w)dx \\
&= \int_D v\theta'_\varepsilon(v-w)(\nabla w - \nabla v) \cdot (\nabla v - \nabla w)dx \\
&\quad + \int_D (v-w)\theta'_\varepsilon(v-w)\nabla v \cdot (\nabla v - \nabla w)dx \\
&\leq \int_D \nabla v \cdot \nabla\left(\Theta_\varepsilon(v-w)\right)dx \\
&= \int_{\partial D} \Theta_\varepsilon(v-w)\frac{\partial v}{\partial n}ds - \int_D \Theta_\varepsilon(v-w)\Delta v dx \\
&\leq 2\varepsilon \int_D |\Delta v| dx \quad (\text{ by } (3.7)) \\
&\leq 2\varepsilon M < \frac{\delta_0}{2}\,.
\end{aligned}$$

However,

$$\int_D vw\big(\frac{f(x,w)}{w} - \frac{f(x,v)}{v}\big)\theta_\varepsilon(v-w)dx \geq \int_{\mathrm{B}} vw\big(\frac{f(x,w)}{w} - \frac{f(x,v)}{v}\big)\theta_\varepsilon(v-w)dx$$

$$
\begin{aligned}
&= \int_{\mathrm{B}} vw\Big(\frac{f(x,w)}{w} - \frac{f(x,v)}{v}\Big)dx \quad \text{(by (3.5))} \\
&\geq \delta_0 \quad \text{by (3.6))},
\end{aligned}
$$

which is a contradiction. Thus $S_0$ must be empty, and the lemma is proved. $\square$

*Proof of Theorem 2.4.* Let $u_1, u_2 \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ be two positive solutions of problem (1.1)-(1.2). We prove that $u_1(x) = u_2(x)$, $x \in \overline{\Omega}$. From the proofs of Theorems 1-3, we can easily see that if $v = \psi_{m_0,\infty}$ then

$$
\begin{gathered}
\Delta v(x) + f(x, v(x)) \geq 0, \quad x \in \Omega, \\
v(x) > 0, \quad x \in \Omega, \\
\phi(x) \geq v(x) \geq 0, \quad x \in \partial\Omega,
\end{gathered}
$$

and $\Delta v \in L^1(\Omega)$. Therefore it follows from Lemma 3.1 that

$$
u_i(x) \geq v(x), \quad x \in \overline{\Omega},\ i = 1, 2.
$$

Moreover, by the Hopf's strong maximum principle, we have $\frac{\partial v}{\partial n} < 0$ on $\partial\Omega$, hence there exists $c_0 > 0$ such that $u_i(x) \geq c_0 d(x)$, $x \in \overline{\Omega}$, $i = 1, 2$, where $d(x) = \operatorname{dist}(x, \partial\Omega)$. Let $\Omega_\varepsilon = \{x \in \Omega : d(x) \leq \varepsilon\}$ for $\varepsilon > 0$ and $U_i(\delta) = \{x \in \Omega : u_i(x) \leq \delta\}$, $i = 1, 2$. Since $\partial\Omega \in C^{2,\theta}$, there exist $\varepsilon \in (0, \delta)$ such that if $x \in \Omega_\varepsilon$, then there is a unique $y_x \in \partial\Omega$ such that $\operatorname{dist}(x, y_x) = d(x), c_0 d(x) < \delta$. Thus, for some $M > 0$ only depending on $\partial\Omega$,

$$
\begin{aligned}
\int_{\Omega_\varepsilon} |f(x, c_0 d(x))|dx &\leq M \int_{\partial\Omega} \int_0^\varepsilon |f(y - sn_y, c_0 s)| ds\, dy \\
&\leq M \int_{\partial\Omega} \int_0^\varepsilon F(s, c_0 s) ds\, dy \\
&\leq M^* < +\infty,
\end{aligned}
$$

where

$$
M^* = M \int_{\partial\Omega} \int_0^\delta F(s, c_0 s) ds\, dy\,.
$$

By the hypothesis (F5), there exists $M_0 > 0$ such that

$$
0 \leq F(r, s) \leq F(r, t) + M_0 \quad \text{for } \delta \geq s \geq t > 0\ r \in (0, \delta).
$$

Therefore,

$$
\begin{aligned}
\int_{\Omega_\varepsilon \cap U_i(\delta)} |f(x, u_i(x))|dx &\leq \int_{\Omega_\varepsilon} |f\big(x, c_0 d(x)\big)|dx + M_0 \operatorname{meas}(\Omega) \\
&\leq M^* + M_0 \operatorname{meas}(\Omega) < +\infty, \quad i = 1, 2.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\int_\Omega |f\big(x, u_i(x)\big)|dx &\leq \int_{\Omega_\varepsilon \cap U_i(\delta)} |f\big(x, u_i(x)\big)|dx + \int_{\Omega \setminus (\Omega_\varepsilon \cap U_i(\delta))} |f\big(x, u_i(x)\big)|dx \\
&\leq M^* + M_0 \operatorname{meas}(\Omega) + M_i^{**} \operatorname{meas}(\Omega) < +\infty,
\end{aligned}
$$

where

$$
M_i^{**} = \max_{x \in \overline{\Omega},\, \delta \leq s \leq \delta_i^*} |f(x, s)|, \quad \delta_i^* = \max_{x \in \overline{\Omega}} u_i(x), \quad i = 1, 2.
$$

Therefore,

$$
\int_\Omega |\Delta u_i| dx = \int_\Omega |f(x, u_i)| dx < +\infty, \quad i = 1, 2.
$$

i.e., $\Delta u_i \in L^1(\Omega)$, $i = 1, 2$. Hence, it follows from Lemma 3.1 that

$$u_1(x) = u_2(x), \quad x \in \overline{\Omega},$$

and the theorem is proved. □

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