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PERIODICITY OF MILD SOLUTIONS TO HIGHER ORDER DIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract. We give necessary and sufficient conditions for the periodicity of mild solutions to the the higher order differential equation $u^{(n)}(t) = Au(t) + f(t)$, $0 \le t \le T$, in a Banach space E. Applications are made to the cases, when A generates a C_0-semigroup or a cosine family, and when E is a Hilbert space.

1. Introduction

This paper concerns the periodicity of solutions to the higher order Cauchy problem

$$\begin{aligned} u^{(n)}(t) &= Au(t) + f(t), \quad 0 \le t \le T \\ u^{(i)}(0) &= x_i, \quad i = 0, 1, \dots, n-1, \end{aligned} \tag{1.1}$$

where A is a linear and closed operator on a Banach space E, and f is a function from $[0, T]$ to E. The asymptotic behavior and, in particular, the periodicity of solutions of (1.1) has been subject to intensive study in recent decades. It is well-known [6] that, if A is an $n \times n$ matrix on $\mathbb{C}^n$, then the first order Cauchy problem

$$\begin{aligned} u'(t) &= Au(t) + f(t), \quad 0 \le t \le T, \\ u(0) &= x \end{aligned} \tag{1.2}$$

in $E = \mathbb{C}^n$ admits a unique T-periodic solution for each continuous T-periodic forcing term f if and only if $\lambda_k = 2k\pi t/T$, $k \in \mathbb{Z}$, are not eigenvalues of A. This result was extended by Krein and Dalecki [2, 9] to the Cauchy problem in an abstract Banach space. In [2, Theorem II 4.3] it was claimed that, if A is a linear bounded operator on E, then (1.2) admits a unique T-periodic solution for each $f \in C[0, T]$ if and only if $2k\pi i/T \in \varrho(A)$, $k \in \mathbb{Z}$. Here $\varrho(A)$ denotes the resolvent set of A. Unfortunately, the above result does not hold any more when A is an unbounded operator (see [5]). For the case, when A generates a strongly continuous semigroup, periodicity of solutions of (1.2) was studied in [8, 15]. Corresponding results on the periodic solutions of the second order Cauchy problem were obtained in [12, 16], when A is generator of a cosine family. Related results can also be found in [3, 7, 10, 11, 13, 17] and the references therein.

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In this paper we investigate the periodicity of mild solutions of the higher order Cauchy problem (1.1) when A is a linear, unbounded operator. The main tool we use here is the Fourier series method. For an integrable function $f(t)$ from $[0,T]$ to E, the Fourier coefficient of $f(t)$ is defined as

$$f_k = \frac{1}{T}\int_0^T f(s)e^{-2k\pi is/T}ds, \quad k \in \mathbb{Z}.$$

Then $f(t)$ can be represented by Fourier series

$$f(t) \approx \sum_{k=-\infty}^{\infty} e^{2k\pi it/T} f_k.$$

First, we establish the relationship between the Fourier coefficients of the periodic solutions of (1.1) and those of the inhomogeneity f. We then give different equivalent conditions so that (1.1) admits a unique periodic solution for each inhomogeneity f in a certain function space. As applications, in Section 3 we show a short proof of the Gearhart's Theorem: If A is generator of a strongly continuous semigroup $T(t)$, then $1 \in \varrho(T(1))$ if and only if $2k\pi i \in \varrho(A)$ and $\sup_{k\in\mathbb{Z}} \|R(2k\pi i, A)\| < \infty$. Corresponding result for the spectrum of a cosine family is also presented.

Let us fix some notation. A continuous function on $[0,T]$ is said to be T-periodic if $u(0) = u(T)$. For the sake of simplicity (and without loss of generality) we assume $T = 1$ and put $J := [0,1]$. For $p \geq 1$, $L_p(J)$ denotes the space of E-valued functions on J with $\int_0^1 \|f(t)\|^p dt < \infty$ and $C(J)$ the space of functions on J with and $\|f\| = \sup_J \|f(t)\| < \infty$. Moreover, for $m > 0$ we define the following function spaces

(1) $W_p^m(J) := \{f \in L_p(J) : f', f'', \dots, f^{(m)} \in L_p(J)\}$. $W_p^m(J)$ is then a Banach space with the norm

$$\|f\|_{W_p^m} := \sum_{k=0}^{m} \|f^{(k)}\|_{L_p(J)}.$$

(2) $P^m(J) := \{f \in C(J) : f, f', \dots, f^{(m)}$ are in $P(J)\}$. That means $P^m(J)$ is the space of all functions on J, which can be extended to 1-periodic, m-times continuously differentiable functions on $\mathbb{R}$. $P^m(J)$ is a Banach space with the norm

$$\|f\|_{P^m(J)} := \sum_{k=0}^{m} \|f^{(k)}\|_{C(J)}.$$

(3) $WP_p^m(J) := P^{m-1}(J) \cap W_p^m(J)$. It is easy to see that $WP_p^m(J)$ is a Banach space with $W_p^m(J)$-norm.

We will use the following simple lemma.

Lemma 1.1. *If F is a continuous function on J such that $f = F' \in L_p(J)$, then for $k \neq 0$ we have*

$$F_k = \frac{1}{2k\pi i} f_k + \frac{F(0) - F(1)}{2k\pi i},$$

where f_k and F_k are the Fourier series of f and F, respectively.

2. Periodic Mild Solutions of Higher Order Differential Equations

Let J be the interval $[0,1]$ and $p \geq 1$. For each function $f \in L_p(J)$ we define the function If by $If(t) := \int_0^t f(s)ds$ and, for $n \geq 2$, the function $I^n f$ by $I^n f(t) := I(I^{n-1}f)(t)$.

Definition 2.1. (1) A continuous function u is called a mild solution of (1.1) on J, if $I^n u(t) \in D(A)$ and, for all $t \in J$,

$$u(t) = \sum_{i=0}^{n-1} \frac{t^i}{i!} x_i + AI^n u(t) + I^n f(t)\,. \tag{2.1}$$

(2) A function u is a classical solution of (1.1) on J, if $u(t) \in D(A)$, u is n-times continuously differentiable, and (1.1) holds for $t \in J$.

Remarks.

(i) If $n = 1$ and A is the generator of a C_0 semigroup $T(t)$, then a continuous function $u : J \to E$ is a mild solution of (1.1) if and only if it has the form

$$u(t) = T(t)x_0 + \int_0^t T(t-r)f(r)dr, \quad t \in J.$$

(See [1]).

(ii) Similarly, if $n = 2$ and A generates a cosine family $(C(t))$ on E, then any continuously differentiable function u on E of the form

$$u(t) = C(t)x_0 + S(t)x_1 + \int_0^t S(t-\tau)f(\tau)d\tau, \quad t \in J,$$

where $(S(t))$ is the associated sine family, is a mild solution of (1.1) (see Section 3 for more details).

The mild solution to (1.1) defined by (2.1) is really an extension of a classical solution in the sense that every classical solution is a mild solution and conversely, if a mild solution is n-times continuously differentiable, then it is a classical solution. That statement is actually contained in the following lemma.

Lemma 2.2. *Suppose $0 \le m \le n$ and u is a mild solution of* (1.1)*, which is m-times continuously differentiable. Then we have $(I^{n-m}u)(t) \in D(A)$ and*

$$u^{(m)}(t) = \sum_{j=m}^{n-1} \frac{t^{j-m}}{(j-m)!} x_j + AI^{n-m}u(t) + I^{n-m}f(t). \tag{2.2}$$

Proof. If $m = 0$, then (2.2) coincides with (2.1). We prove for $m = 1$: Let $v(t) := AI^n u(t)$. Then, by (2.1), v is continuously differentiable and

$$v'(t) = u'(t) - \sum_{j=1}^{n-1} \frac{t^{j-1}}{(j-1)!} x_j - I^{n-1}f(t).$$

Let $h > 0$ and put

$$v_h := \frac{1}{h} \int_t^{t+h} I^{n-1}u(s)ds.$$

Then $v_h \to (I^{n-1}u)(t)$ for $h \to 0$ and

$$\begin{aligned}\lim_{h\to 0} Av_h = & \lim_{h\to 0} \frac{1}{h}\Big(A\int_0^{t+h} I^{n-1}u(s)ds - A\int_0^t I^{n-1}u(s)ds\Big) \\ &= \frac{1}{h}(v(t+h) - v(t)) \\ &= v'(t).\end{aligned}$$

Since A is a closed operator, we obtain that $I^{n-1}u(t) \in D(A)$ and

$$AI^{n-1}u(t) = u'(t) - \sum_{j=1}^{n-1} \frac{t^{j-1}}{(j-1)!} x_j - I^{n-1}f(t),$$

from which (2.2) with $m = 1$ follows. If $m > 1$, we obtain (2.2) by repeating the above process $(m-1)$ times. □

In particular, if the mild solution u is n-times continuously differentiable, then (2.2) becomes $u^{(n)}(t) = Au(t) + f(t)$, i.e. u is a classical solution of (1.1).

We now consider the mild solutions of (1.1), which are $(n-1)$ times continuously differentiable. The following proposition describes the connection between the Fourier coefficients of such solutions and those of $f(t)$.

Proposition 2.3. *Suppose $f \in L_p(J)$ and u is a mild solution of* (1.1)*, which is $(n-1)$ times continuously differentiable. Then*

$$\frac{((2k\pi i)^n - A)u_k - f_k}{(2k\pi i)^n} = \sum_{j=0}^{n-1} \frac{u^{(j)}(0) - u^{(j)}(1)}{(2k\pi i)^{j+1}} \tag{2.3}$$

for $k \neq 0$.

Proof. Let $u_k^{(j)}$ be the k^{th} Fourier coefficient of $u^{(j)}$. Using the identity

$$u_k^{(j)} = \frac{u^{(j)}(0) - u^{(j)}(1)}{2k\pi i} + \frac{1}{2k\pi i} u_k^{(j+1)} \tag{2.4}$$

for $j = 0, 1, 2, \dots, n-2$ (by Lemma 1.1), we obtain

$$u_k = \sum_{j=0}^{n-2} \frac{u^{(j)}(0) - u^{(j)}(1)}{(2k\pi i)^{j+1}} + \frac{1}{(2k\pi i)^{n-1}} u_k^{(n-1)}. \tag{2.5}$$

Since u is $(n-1)$ times continuously differentiable, by Lemma 2.2,

$$u^{(n-1)}(t) = u^{(n-1)}(0) + AIu(t) + If(t). \tag{2.6}$$

Taking the k^{th} Fourier coefficient on both sides of (2.6) and using (2.4), we have

$$\begin{aligned} u_k^{(n-1)} &= A(Iu)_k + (If)_k \\ &= A\Big(\frac{Iu(0) - Iu(1)}{2k\pi i} + \frac{1}{2k\pi i}(Iu)'_k\Big) + \Big(\frac{If(0) - If(1)}{2k\pi i} + \frac{1}{2k\pi i}(If)'_k\Big) \\ &= \frac{-(AIu(1) + If(1))}{2k\pi i} + \frac{Au_k + f_k}{2k\pi i} \\ &= \frac{u^{(n-1)}(0) - u^{(n-1)}(1)}{2k\pi i} + \frac{Au_k + f_k}{2k\pi i}. \end{aligned} \tag{2.7}$$

Here we have also used $Iu(0) = If(0) = 0$, $(Iu)'_k = u_k$ and $(If)'_k = f_k$. Combining (2.5) and (2.7), we obtain

$$u_k = \sum_{j=0}^{n-1} \frac{u^{(j)}(0) - u^{(j)}(1)}{(2k\pi i)^{j+1}} + \frac{Au_k + f_k}{(2k\pi i)^n},$$

from which (2.3) follows. □

The interesting point of Proposition 2.3 is that the Fourier coefficients of the mild solution u depend not only on u but also on its derivatives. If u is a mild solution in $P^{(n-1)}(J)$, then we have a nice relationship between Fourier coefficients of u and those of f, as the following proposition shows.

Proposition 2.4. *Suppose $f \in L_p(J)$ and u is a mild solution of (1.1), which is $(n-1)$ times continuously differentiable. Then $u \in P^{(n-1)}(J)$ if and only if*

$$((2k\pi i)^n - A)u_k = f_k \tag{2.8}$$

for every $k \in \mathbb{Z}$.

Proof. Suppose u is a mild 1-periodic solution of (1.1) in $P^{n-1}(J)$. If $k \neq 0$, then (2.8) follows directly from (2.3). If $k = 0$, using (2.2) with $m = n-1$ and $t = 1$ we obtain

$$\begin{aligned} u^{(n-1)}(1) &= u^{(n-1)}(0) + A\int_0^1 u(s)ds + \int_0^1 f(s)ds \\ &= u^{(n-1)}(0) + Au_0 + f_0. \end{aligned}$$

Due to the 1-periodicity of $u^{(n-1)}$ we obtain $Au_0 + f_0 = 0$, from which (2.8) holds for $k = 0$. Conversely, suppose (2.8) holds for all $k \in \mathbb{Z}$. Then, by (2.3),

$$\sum_{j=0}^{n-1} \frac{u^{(j)}(0) - u^{(j)}(1)}{(2k\pi i)^j} = 0 \tag{2.9}$$

all $k \neq 0$. That means that for any positive integer K, the vector

$$X = \Big(u(0) - u(1), u'(0) - u'(1), \dots, u^{(n-1)}(0) - u^{(n-1)}(1)\Big)^T$$

is a solution of the system of linear equations

$$\begin{pmatrix} 1 & \frac{1}{2\pi i} & \cdots & \frac{1}{(2\pi i)^{n-1}} \\ 1 & \frac{1}{2\cdot 2\pi i} & \cdots & \frac{1}{(2\cdot 2\pi i)^{n-1}} \\ \vdots & & \ddots & \vdots \\ 1 & \frac{1}{2K\pi i} & \cdots & \frac{1}{(2K\pi i)^{n-1}} \end{pmatrix}_{n\times K} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = 0.$$

This can only happen if $X = 0$, i.e. $u^{(j)}(0) - u^{(j)}(1) = 0$ for $j = 0, 1, 2, \dots, (n-1)$. Hence, $u \in P^{(n-1)}(J)$, and the proposition is proved. $\square$

From Proposition 2.4 we obtain

Corollary 2.5. *Suppose $f \in L_p(J)$. Then*

- (i) *If $((2k\pi i)^n - A)$ is injective for $k \in \mathbb{Z}$, then Equation (1.1) has at most one 1-periodic mild solution, which belongs tp $P^{n-1}(J)$.*
- (ii) *If there exists a number $k \in \mathbb{Z}$ such that $f_k \notin Range((2k\pi i)^n - A)$, then Equation (1.1) has no periodic mild solution which belongs to $P^{n-1}(J)$.*
- (iii) *Let u be a mild solution of $u^{(n)} = Au$, which is $(n-1)$ times continuously differentiable. Then u belongs to P^{n-1} if and only if*

$$(2k\pi i)^n u_k = Au_k,$$

i.e., u_k is an eigen-vector of A corresponding to $(2k\pi i)^n$, $k \in \mathbb{Z}$.

We are now in a position to state the main results.

Theorem 2.6. *Let A be a closed operator on E and $0 \le m \le n$. The following statements are equivalent.*

(i) *For each function $f \in WP_p^m(J)$, Equation (1.1) admits a unique mild solution in $WP_p^n(J)$*

(ii) *For each $k \in \mathbb{Z}$, $2k\pi i \in \varrho(A)$ and there exists a constant $C > 0$ such that*

$$\| \sum_k ((2k\pi i)^n - A)^{-1} e^{2k\pi i \cdot} x_k)\|_{W_p^n(J)} \le C \cdot \| \sum_k e^{2k\pi i \cdot} x_k\|_{W_p^m(J)} \tag{2.10}$$

for any finite sequence $\{x_k\} \subset E$

If E is a Hilbert space, and $p = 2$, then (i) and (ii) are equivalent to

(iii) *For every $k \in \mathbb{Z}$, $(2k\pi i)^n \in \varrho(A)$ and*

$$\sup_{k\in\mathbb{Z}} \|k^{n-m}((2k\pi i)^n - A)^{-1}\| < \infty \tag{2.11}$$

We will need the following lemma.

Lemma 2.7. *Let $F_1 := WP_p^m(J)$ and $F_2 := WP_p^n(J)$. Then the following are equivalent:*

(1) For each function $f \in F_1$, (1.1) admits a unique mild solution u in F_2.

(2) There exists a dense subset D in F_1 such that:

(i) *For each function $f \in D$, (1.1) admits a unique mild solution u in F_2;*

(ii) *There exists a constant $C > 0$ such that for all $f \in D$,*

$$\|u\|_{F_2} \le C\|f\|_{F_1} . \tag{2.12}$$

Proof. (1)$\Rightarrow$ (2): We will prove (2) with $D = F_1$. It is easy to see that (i) is automatically satisfied. To show (ii), we define the operator $G : F_1 \mapsto F_2$ by $Gf := u$, where u is the unique mild solution of (1.1) in F_2. Then G is a linear, everywhere defined operator. We will prove the boundedness of G by showing that G is a closed operator. To this end, let $\{f_j\} \subset F_1$ a sequence such that $f_j \to f$ in F_1 and $Gf_j \to u$ in F_2 for $j \to \infty$. For each $t \in J$, let $v_j := I^n(Gf_j)(t)$, then

$$\lim_{j\to\infty} v_j = I^n u(t).$$

Moreover, from the identity

$$(Gf_j)(t) = \sum_{i=0}^{n-1} \frac{t^j}{j!}(Gf_j)(0) + AI^n(Gf_j)(t) + I^n f_j(t)$$

we have

$$\begin{aligned} Av_j &= AI^n(Gf_j)(t) \\ &= (Gf_j)(t) - \sum_{i=0}^{n-1} \frac{t^i}{i!}(Gf_j)(0) - I^n f_j(t) \to u(t) - \sum_{i=0}^{n-1} \frac{t^i}{i!}u(0) - I^n f(t) \end{aligned}$$

as $j \to \infty$. Since A is a closed operator, $I^n u(t) \in D(A)$ and

$$AI^n u(t) = u(t) - \sum_{i=0}^{n-1} \frac{t^i}{i!}u(0) - I^n f(t),$$

i.e., u is a mild solution of (1.1) and consequently, $Gf = u$. So, G is a bounded operator from F_1 to F_2, from which (2.12) follows with $C = \|G\|$.

(2)⇒ (1). For any $f \in F_1$ there exists a sequence $\{f_j\} \subset D$ such that $f_j \to f$ for $j \to \infty$. Let u_j be the mild solution in F_2 corresponding to f_j, then, by (2.12), $u_j \to u$ for some $u \in F_2$. With the same manner as in the previous part, we can prove that u is a mild solution of (1.1) corresponding to f. The uniqueness of this solution comes directly from (2.12). □

Proof of Theorem 2.6. (i) → (ii): We first show that $(2k\pi i)^n \in \varrho(A)$ for $k \in \mathbb{Z}$. To this end, let $f(t) = e^{2k\pi i t}x$, $x \in E$ and $u(t)$ be the unique mild solution to (1.2) corresponding to f. By Lemma 2.4 we have $((2k\pi i)^n - A)u_k = x$. Hence $((2k\pi i)^n - A)$ is surjective. On the other side, if $((2k\pi i)^n - A)$ is not injective, i.e. there is a non-zero vector $x_0 \in E$ such that $((2k\pi i)^n - A)x_0 = 0$, then it is not hard to check that $u_1 :\equiv 0$ and $u_2(t) := e^{2k\pi i t}x_0$ are two distinct 1-periodic mild (classical) solution f $u^{(n)}(t) = Au(t)$. It is contradicting to the uniqueness of u. So $((2k\pi i)^n - A)$ is injective and hence bijective, i.e. $(2k\pi i)^n \in \varrho(A)$. Let now $f(t) := \sum_k e^{2k\pi i t}x_k$, where $\{x_k\}$ is any finite sequence in E. Then, by Lemma 2.4, $u(t) = \sum_k ((2k\pi i)^n - A)^{-1} e^{2k\pi i t}x_k$ is the unique 1-periodic mild solution to (1.1) corresponding to f. Thus, (2.10) is obtained by inequality (2.12).
(ii) → (i): Put

$$\mathcal{M} := \{f(t) = \sum_k e^{2k\pi i t}x_k : \{x_k\} \text{ is a finite sequence in } E\}.$$

Observe that $\mathcal{M}$ is dense in $WP_p^m(J)$. Moreover, if f is a function in $\mathcal{M}$, i.e., if $f(t) = \sum_k e^{2k\pi t}x_k$, then it is easy to check that $u(t) = \sum_k ((2k\pi i)^n - A)^{-1} e^{2k\pi i t}x_k$ is a unique 1-periodic mild solution of (1.1) corresponding to f and, by Corollary 2.5*(i)*, it is the unique one. From (2.12) it follows that $\|u\|_{W_p^n(J)} \le C\|f\|_{W_p^m(J)}$ for all $f \in \mathcal{M}$. By Lemma 2.7, that implies *(i)*.

Finally, if E is a Hilbert space, then $WP_2^m(J)$ is a Hilbert space for any $0 \le m \le n$. Moreover, for $f(t) = \sum_k e^{2k\pi i t}x_k$ and $u(t) = \sum_k ((2k\pi i)^n - A)^{-1} e^{2k\pi i t}x_k$ we have

$$\|f\|_{W_2^m(J)} = \sum_{j=0}^{m} \Big(\sum_k (2k\pi)^{2j} \|x_k\|^2 \Big)^{1/2} \tag{2.13}$$

and

$$\|u\|_{W_2^n(J,E)} = \sum_{j=0}^{n} \Big(\sum_k (2k\pi)^{2j} \|((2k\pi i)^n - A)^{-1} x_k\|^2 \Big)^{1/2}. \tag{2.14}$$

Suppose *(ii)* holds, i.e., $\|u\|_{W_2^n(J)} \le C\|f\|_{W_2^m(J)}$ for $f \in \mathcal{M}$. For any $k \in \mathbb{Z}$, take $f(t) := e^{2k\pi i t}x$. From (2.13) and (2.14), we have

$$\|f\|_{W_2^m(J)} = \sum_{j=0}^{m} \|(2k\pi)^j x\| \le (2\pi)^m (m+1)\|k^m x\| \tag{2.15}$$

and

$$\|u\|_{W_2^n(J)} = \sum_{j=0}^{n} \|(2k\pi)^j ((2k\pi i)^n - A)^{-1} x\| \ge (2\pi)^n \|k^n ((2k\pi i)^n - A)^{-1} x\|. \tag{2.16}$$

Combining (2.10), (2.15) and (2.16) we obtain

$$(2\pi)^n \|k^n ((2k\pi i)^n - A)^{-1} x\| \le C \cdot (2\pi)^m (m+1)\|k^m x\|,$$

from which (2.11) follows.

Conversely, suppose (iii) holds, i.e., there is a positive constant C such that $\|(2k\pi i)^n - A)^{-1}\| \le C|k|^{m-n}$ for $k \in \mathbb{Z}$. Using that inequality for the right hand side of (2.14) we obtain

$$\begin{aligned}
\|\sum_k ((2k\pi i)^n - A)^{-1} e^{2k\pi i\cdot} x_k\|_{W_2^n(J)} &\le C \sum_{j=0}^{n} \Big(\sum_k (2k\pi)^{2j} k^{2m-2n} \|x_k\|^2\Big)^{1/2} \\
&\le C_1 \sum_{j=0}^{n} \Big(\sum_k (2k\pi)^{2j+2m-2n} \|x_k\|^2\Big)^{1/2} \\
&\le C_1(n+1)\Big(\sum_k (2k\pi)^{2m} \|x_k\|^2\Big)^{1/2} \\
&\le C_1(n+1) \sum_{j=0}^{m} \Big(\sum_k (2k\pi)^{2j} \|x_k\|^2\Big)^{1/2} \\
&= C_1(n+1) \|\sum_k e^{2k\pi i\cdot} x_k\|_{W_2^m(J)},
\end{aligned}$$

where $C_1 = C(2\pi)^{n-m}$. Thus, (2.10) holds and the theorem is proved. $\square$

The next theorem shows the relationship between the regularity of the inhomogeneity and that of the corresponding mild solution.

Theorem 2.8. *If A is a closed operator on E, then the following statements are equivalent.*

- (i) *For each $f \in L_p(J)$ Eq.* (1.1) *admits a unique mild solution in $P^{n-1}(J)$.*
- (ii) *$0 \in \varrho(A)$ and for each $f \in L_p(J)$ with $\int_0^1 f(s)ds = 0$, Equation* (1.1) *admits a unique mild solution in $P^{n-1}(J)$.*
- (iii) *For each $f \in WP_p^1(J)$, Equation* (1.1) *admits a unique 1-periodic classical solution.*

Proof. If (i) or (iii) holds, then, by the same reasoning as in the proof of Theorem 2.6, we can prove that $2k\pi i \in \varrho(A)$ for $k \in \mathbb{Z}$.
(i) $\to$ (iii): Let F be any function in $WP_p^1(J)$. Then F can be written as by $F(t) = \int_0^t f(s)ds + x_0$, where $f \in L_p(J)$ and x_0 is a vector in E. Since F is 1-periodic we have $\int_0^1 f(s)ds = 0$. Let u be the mild solution to (1.1) corresponding to f, which is in $P^{n-1}(J)$, and put

$$U(t) = \int_0^t u(s)ds + A^{-1}u^{n-1}(0) - A^{-1}x_0.$$

From identity (2.2) with $m = n-1$ we have

$$u^{(n-1)}(1) = u^{n-1}(0) + A\int_0^1 u(s)ds + \int_0^1 f(s)ds. \tag{2.17}$$

Note that $u^{(n-1)}(1) = u^{(n-1)}(0)$ and $\int_0^1 f(s)ds = 0$. Thus, from (2.17) we obtain $A\int_0^1 u(s)ds = 0$, which implies, due to $0 \in \varrho(A)$, $\int_0^1 u(s)ds = 0$. Hence, U is a

1-periodic function. Moreover,

$$\begin{aligned} U^{(n)}(t) &= u^{(n-1)}(t) \\ &= u^{n-1}(0) + A\int_0^t u(s)ds + \int_0^t f(s)ds \\ &= u^{n-1}(0) + A[U(t) - A^{-1}u_{n-1} + A^{-1}x_0] + (F(t) - x_0) \\ &= AU(t) + F(t). \end{aligned}$$

So, U is an 1-periodic classical solution. The uniqueness of this solution follows from the fact that $u \equiv 0$ is the unique 1-periodic mild solution to the homogeneous equation $u^{(n)}(t) = Au(t)$, which, in turn, follows from (i).

(iii) $\to$ (ii): Let f be a function in $L_p(J)$ with $\int_0^1 f(s)ds = 0$. Define $F(t) := \int_0^t f(s)ds$, then it is easy to see that $F \in WP_p^1(J)$. Let U be the unique 1-periodic classical solution of (1.2) corresponding to F and put $u := U'$. Then $u \in P^{n-1}(J)$ and $U(t) = \int_0^t u(s)ds + U(0)$. By the definition of U and F, the equation $U^{(n)}(t) = AU(t) + F(t)$ means

$$u^{(n-1)}(t) = AU(0) + A\int_0^t u(s)ds + \int_0^t f(s)ds.$$

Hence, by Lemma 2.2, u is a mild solution to (1.1) corresponding to f. The uniqueness of u follows from Corollary 2.5.

(ii) $\to$ (i): Let f be a function in $L_p(J)$. Define $\tilde{f}(t) := f(t) - f_0$, where $f_0 = \int_0^1 f(s)ds$, then $\int_0^1 \tilde{f}(s)ds = 0$. Let $\tilde{u}$ be the 1-periodic mild solution to (1.1) corresponding to $\tilde{f}$ and put $u(t) := \tilde{u}(t) - A^{-1}f_0$. Then u, as $\tilde{u}$, is in $P^{n-1}(J)$. Moreover,

$$\begin{aligned} u(t) &= \tilde{u}(t) - A^{-1}f_0 \\ &= \Big(\sum_{k=0}^{n-1} \frac{t^k}{k!}\tilde{u}^{(k)}(0) + AI^n\tilde{u}(t) + I^n\tilde{f}(t)\Big) - A^{-1}f_0 \\ &= \Big(u(0) + A^{-1}f_0 + \sum_{k=1}^{n-1}\frac{t^k}{k!}u^{(k)}(0)\Big) + AI^n\Big(u(t) + A^{-1}f_0\Big) \\ &\quad + I^n\Big(f(t) - f_0\Big) - A^{-1}f_0 \\ &= \sum_{k=0}^{n-1}\frac{t^k}{k!}u^{(k)}(0)) + AI^n u(t) + I^n f(t). \end{aligned}$$

Hence, u is a mild solution to (1.1) corresponding to f. The uniqueness of u follows from Corollary 2.5. □

3. Applications

A semigroup case. Here, we consider the first order Cauchy problem

$$\begin{aligned} u'(t) &= Au(t) + f(t) \quad 0 \le t \le T \\ u(0) &= x, \end{aligned} \tag{3.1}$$

where A generates a C_0-semigroup $(T(t))_{t\geq 0}$. Recall that in this case the mild solution is of the form

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)ds. \tag{3.2}$$

We have the following result, in which the equivalence between (i) and (v) is the Gearhart's Theorem [4].

Theorem 3.1. *Let A generate a C_0-semigroup $(T(t))_{t\geq 0}$. Then the following statements are equivalent:*

(i) $1 \in \varrho(T(1))$;

(ii) *For every function $f \in L_p(J)$, Equation* (3.1) *admits a unique 1-periodic mild solution;*

(iii) *For every function $f \in WP_p^1(J)$, Equation* (3.1) *admits a unique mild solution in $WP_p^1(J)$;*

(iv) *For every function $f \in WP_p^1(J)$, Equation* (3.1) *admits a unique 1-periodic classical solution*

If E is a Hilbert space, all the above statements are equivalent to

(v) $\{2k\pi i : k \in \mathbb{Z}\} \subset \varrho(A)$ *and*

$$\sup_{k\in\mathbb{Z}} \|(2k\pi i - A)^{-1}\| < \infty.$$

Proof. The equivalence (i) $\Leftrightarrow$ (ii) was proved in [15]. The equivalence (ii) $\Leftrightarrow$ (iv) follows from Theorem 2.8 and, if E is a Hilbert space, (iii) $\Leftrightarrow$ (v) follows from Theorem 2.6. The inclusion (iv) $\Rightarrow$ (iii) is obvious. So, it remains to show (iii) $\rightarrow$ (iv).

To this end, let u be the unique mild solution of (3.1), which belong to $WP_p^1(J)$. Since $\int_0^t T(t-s)f(s)ds \in D(A)$ and $t \rightarrow \int_0^t T(t-s)f(s)ds$ is continuously differentiable for any $f \in W_p^1(J)$ (see e.g. [14]), we obtain that $T(\cdot)u(0) \in W_p^1(J)$. It follows that $T(t)u(0) \in D(A)$ for $t > 0$ (since $t \mapsto T(t)x$ is differentiable at t_0 if and only if $T(t_0)x \in D(A)$). Hence, $u(1)$, and thus, $x = u(1)$ belongs to $D(A)$. So u is a classical solution. The uniqueness of the 1-periodic classical solution is obvious. □

A cosine family case. We now consider the second order Cauchy problem

$$\begin{gathered} u''(t) = Au(t) + f(t) \quad 0 \leq t \leq T \\ u(0) = x, u'(0) = y, \end{gathered} \tag{3.3}$$

where A is generator of a cosine family $(C(t))_{t\in\mathbb{R}}$ on E. Recall (see u.g. [1]) that in this case there exists a Banach space F such that $D(A) \hookrightarrow F \hookrightarrow E$ and such that the operator

$$\mathcal{A} := \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$$

with $D(\mathcal{A}) = D(A) \times F$ generates the C_0-semigroup

$$\mathcal{T}(t) := \begin{pmatrix} C(t) & S(t) \\ C'(t) & C(t) \end{pmatrix}$$

on $F \times E$, where $S(t)$ is the associated sine family. Moreover, it is not difficult to check that u is a mild solution of (3.3), which is continuously differentiable (a mild solution, which is in $WP_p^2(J)$, or a classical solution of (3.3), respectively), if and only if $\mathcal{U} = (u, u')^T$ is a mild solution (a mild solution, which is in $WP_p^1(J)$, or a classical solution, respectively) of the first order differential equation

$$\begin{gathered} \mathcal{U}'(t) = \mathcal{A}\mathcal{U}(t) + (0, f(t))^T, \quad 0 \le t \le T, \\ \mathcal{U}(0) = (x, y)^T \end{gathered} \tag{3.4}$$

in the space $F \times E$. Using (3.2), we have the explicit form of u by

$$u(t) = C(t)x + S(t)y + \int_0^t S(s-\tau)f(\tau)d\tau.$$

Theorem 3.2. *Let A generate a cosine family $(C(t))_{t\in\mathbb{R}}$ in E. Then the following statements are equivalent:*

(i) $1 \in \varrho(C(1))$;

(ii) *For each function $f \in L_p(J)$, Equation* (3.3) *has a unique 1-periodic mild solution, which is continuously differentiable;*

(iii) *For each function $f \in WP_p^1(J)$, Equation* (3.3) *admits a unique mild solution in $WP_p^2(J)$;*

(iv) *For each function $f \in WP_p^1(J)$, Equation* (3.3) *admits a unique 1-periodic classical solution;*

If E is a Hilbert space, all the above statements are equivalent to

(v) $\{-4k^2\pi^2 : k \in \mathbb{Z}\} \subset \varrho(A)$ *and* $\sup_{k\in\mathbb{Z}} \|k(4k^2\pi^2 + A)^{-1}\| < \infty$.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is virtually proved in [16]. The equivalence (ii) $\Leftrightarrow$ (iv) from Theorem 2.8 and, if E is a Hilbert space, (iii) $\Leftrightarrow$ (v) follows from Theorem 2.6. The inclusion (iv) $\Rightarrow$ (iii) is obvious. So, it remains to show (iii) $\rightarrow$ (iv). To this end, let u be the 1-periodic mild solution of (3.3), which is in $WP_p^2(J)$, then $\mathcal{U} = (u, u')^T$ is the 1-periodic mild solution of (3.4), which is in $WP_p^1(J, F \times E)$. Since $\mathcal{A}$ is the generator of a C_0-semigroup, we can show (with the same manner as in the proof of Theorem 3.1) that $\mathcal{U}$ is a 1-periodic classical solution of (3.4). It follows that u is a 1-periodic classical solution of (3.3). $\square$

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