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OSCILLATION FOR HIGHER ORDER NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH IMPULSES

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Abstract. In this paper, we study the oscillation of solutions to higher order nonlinear ordinary differential equations with impulses. Several criteria for the oscillations of solutions are given. We find some suitable impulse functions such that all solutions are oscillatory under the impulse control.

1. Introduction

There are many publication on the oscillation of solutions to classical second order nonlinear ordinary differential equations; see for example [1, 2, 3, 8, 9, 10, 13, 14, 15, 16]. There are also some publications on the oscillation of second order ODEs with impulses [4, 7, 12], and some on higher order [5, 6]. In this paper, we study higher order nonlinear ODEs with impulses. Under conditions (A) (B) (C) stated below, we can always find some suitable impulse functions such that all the solutions of the equation become oscillatory under the impulse control. We believe that this oscillation result, under the impulse control, is significant both for the theory and the applications.

2. Main results

We consider the system

$$
\begin{gathered}
x^{(2n)}(t) + f(t, x(t)) = 0, \quad t \geq t_0,\ t \neq t_k, \\
x^{(i)}(t_k^+) = g_{k(i)}(x^{(i)}(t_k)), \quad i = 0, 1, \dots, 2n-1,\ k = 1, 2 \dots, \\
x^{(i)}(t_0^+) = x_0^{(i)},
\end{gathered} \tag{2.1}
$$

where

$$
\begin{gathered}
x^{(i)}(t_k) = \lim_{h \to 0^-} \frac{x^{(i-1)}(t_k + h) - x^{(i-1)}(t_k)}{h}, \\
x^{(i)}(t_k^+) = \lim_{h \to 0^+} \frac{x^{(i-1)}(t_k + h) - x^{(i-1)}(t_k^+)}{h},
\end{gathered}
$$

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$0 < t_0 < t_1 < t_2 < \dots < t_k < \dots$, $k = 1, 2, \dots$, $\lim_{k\to\infty} t_k = +\infty$, $x^{(0)}(t) = x(t)$, and n is a natural number. In this article, we assume that the following conditions:

(A) $f(t, x)$ is continuous on $[t_0, +\infty) \times (-\infty, +\infty)$; $xf(t, x) > 0$ for $x \neq 0$; $\frac{f(t,x)}{\varphi(x)} \geq p(t)$ for $x \neq 0$, where $p(t)$ is positive and continuous on $[t_0, +\infty)$; $x\varphi(x) > 0$ for $x \neq 0$; $\varphi'(x) \geq 0$.

(B) $g_{k(i)}(x)$ is continuous on $(-\infty, +\infty)$, and there exist positive numbers $a_k^{(i)}, b_k^{(i)}$ such that
$$a_k^{(i)} \leq \frac{g_{k(i)}(x)}{x} \leq b_k^{(i)}, i = 0, 1, \dots, 2n-1.$$

(C)
$$\begin{aligned}&(t_1 - t_0) + \frac{a_1^{(i)}}{b_1^{(i-1)}}(t_2 - t_1) + \frac{a_1^{(i)} a_2^{(i)}}{b_1^{(i-1)} b_2^{(i-1)}}(t_3 - t_2) \\ &+ \dots + \frac{a_1^{(i)} a_2^{(i)} \dots a_m^{(i)}}{b_1^{(i-1)} b_2^{(i-1)} \dots b_m^{(i-1)}}(t_{m+1} - t_m) + \dots = +\infty,\end{aligned} \tag{2.2}$$

Definition 2.1. A function $x : [t_0, t_0 + \alpha) \to \mathbb{R}$, $t_0 > 0$, $\alpha > 0$ is said to be a solution of (2.1), if

(i) $x^{(i)}(t_0^+) = x_0^{(i)}$, $i = 0, 1, \dots 2n-1$
(ii) for $t \in [t_0, t_0 + \alpha)$ and $t \neq t_k$, $x(t)$ satisfies $x^{(2n)}(t) + f(t, x(t)) = 0$
(iii) $x^{(i)}(t)$ is left continuous on $t_k \in [t_0, t_0 + \alpha)$, and $x^{(i)}(t_k^+) = g_{k(i)} x^{(i)}(t_k)$, $i = 0, 1, \dots 2n-1$.

Definition 2.2. A solution of (2.1) is said to be non-oscillatory if it is eventually positive or eventually negative. Otherwise,this solution is said to be oscillatory.

Since (2.1) can be transformed into a first-order impulsive differential system, theorems on the existence of solutions, the uniqueness of solutions and the existence of global solutions can be seen in [11]. In the following, we always assume the solutions of (2.1) exists on $[t_0, +\infty)$.

Lemma 2.3. *Let $x(t)$ be a solution of (2.1), and conditions (A), (B), (C) be satisfied. Suppose that there exists an $i \in \{1, 2, \dots, 2n-1\}$ and some $T \geq t_0$, such that $x^{(i)}(t) > 0$ (< 0), $x^{(i+1)}(t) \geq 0$ (≤ 0) for $t \geq T$. Then there exists some $T_1 \geq T$, such that $x^{(i-1)}(t) > 0$ (< 0), for $t \geq T_1$.*

Proof. Without loss of generality, let $T = t_0$, $x^{(i)}(t) > 0$, $x^{(i+1)}(t) \geq 0$ for $t \geq T$. Assume that for any $t_k > T$, $x^{(i-1)}(t_k) < 0$. By $x^{(i+1)}(t) \geq 0$, $x^{(i)}(t) > 0$, $t \in (t_k, t_{k+1}]$, we have that $x^{(i)}(t)$ is monotonically nondecreasing on $(t_k, t_{k+1}]$. For $t \in (t_1, t_2]$, we have
$$x^{(i)}(t) \geq x^{(i)}(t_1^+)$$
Integrating the above inequality, we have
$$x^{(i-1)}(t_2) \geq x^{(i-1)}(t_1^+) + x^{(i)}(t_1^+)(t_2 - t_1) \tag{2.3}$$
Similarly,
$$x^{(i-1)}(t_3) \geq x^{(i-1)}(t_2^+) + x^{(i)}(t_2^+)(t_3 - t_2) \tag{2.4}$$
From $x^{(i)}(t_2) \geq x^{(i)}(t_1^+)$ and (2.3), (2.4), we have
$$x^{(i-1)}(t_3) \geq x^{(i-1)}(t_2^+) + x^{(i)}(t_2^+)(t_3 - t_2)$$

$$
\begin{aligned}
&\geq b_2^{(i-1)} x^{(i-1)}(t_2) + a_2^{(i)} x^{(i)}(t_2)(t_3 - t_2)\\
&\geq b_2^{(i-1)}[x^{(i-1)}(t_1^+) + x^{(i)}(t_1^+)(t_2 - t_1)] + a_2^{(i)} x^{(i)}(t_2)(t_3 - t_2)\\
&\geq b_2^{(i-1)}[x^{(i-1)}(t_1^+) + x^{(i)}(t_1^+)(t_2 - t_1) + \frac{a_2^{(i)}}{b_2^{(i-1)}} x^{(i)}(t_1^+)(t_3 - t_2)]
\end{aligned}
$$

Applying induction, we have that for any natural number m,

$$
\begin{aligned}
x^{(i-1)}(t_m) \geq b_{m-1}^{(i-1)} \dots b_3^{(i-1)} b_2^{(i-1)} \big\{ & x^{(i-1)}(t_1^+) + x^{(i)}(t_1^+)[(t_2 - t_1)\\
&+ \frac{a_2^{(i)}}{b_2^{(i-1)}}(t_3 - t_2) + \dots + \frac{a_2^{(i)} a_3^{(i)} \dots a_{m-1}^{(i)}}{b_2^{(i-1)} b_3^{(i-1)} \dots b_{m-1}^{(i-1)}}(t_m - t_{m-1})] \big\}
\end{aligned}
\tag{2.5}
$$

By condition (C) and $a_k^{(i)} > 0$, $b_k^{(i-1)} > 0$, for all sufficiently large m, we have $x^{(i-1)}(t_m) > 0$. Which is contrary to the assumption. Hence, there exists some j such that $t_j > T$ and $x^{(i-1)}(t_j) \geq 0$. Then

$$
x^{(i-1)}(t_j^+) \geq a_j^{(i-1)} x^{(i-1)}(t_j) \geq 0.
$$

Note that $x^{(i)}(t) > 0$ yields $x^{(i-1)}(t)$ being monotonically increasing on $(t_j, t_{j+1}]$. For $t \in (t_j, t_{j+1}]$, we have

$$
x^{(i-1)}(t) > x^{(i-1)}(t_j^+) \geq 0.
$$

Especially,

$$
x^{(i-1)}(t_{j+1}) > x^{(i-1)}(t_j^+) > 0.
$$

Similarly, for $t \in (t_{j+1}, t_{j+2}]$, we have

$$
x^{(i-1)}(t) > x^{(i-1)}(t_{j+1}^+) \geq a_{j+1}^{(i-1)} x^{(i-1)}(t_{j+1}) > 0.
$$

By induction,for $t \in (t_{j+m-1}, t_{j+m}]$, we have $x^{(i-1)}(t) > 0$. So for $t \geq t_{j+1}$, we have

$$
x^{(i-1)}(t) > 0.
$$

Summing up the above discussion, there exists some $T_1 \geq T$ such that $x^{(i-1)}(t) > 0$, $t \geq T_1$. The proof of the other case in this theorem is similar; so we omit it. The proof of Lemma 2.3 is complete. $\square$

Lemma 2.4. *Let $x(t)$ be a solution of* (2.1) *and conditions (A), (B), (C) be satisfied. Suppose that there exist an $i \in \{1, 2, \dots, 2n\}$ and some $T \geq t_0$ such that $x(t) > 0$, $x^{(i)}(t) \leq 0$, for $t \geq T$, and $x^{(i)}(t)$ is not always equal to 0 in $[t, +\infty)$. Then $x^{(i-1)}(t) > 0$ for all sufficiently large t.*

Proof. Without loss of generality, let $T = t_0$. We claim that $x^{(i-1)}(t_k) > 0$ for any $t_k \geq T$. If it is not true, then there exists some $t_j \geq T$, such that $x^{(i-1)}(t_j) \leq 0$. Since $x^{(i)}(t) \leq 0$, $x^{(i-1)}(t)$ is monotonically non-increasing in $(t_k, t_{k+1}]$ for $k \geq j$. Also because $x^{(i)}(t)$ is not always equal to 0 in $[t, +\infty)$, there exists some $t_l \geq t_j$ such that $x^{(i)}(t)$ is not always equal to 0 in $(t_l, t_{l+1}]$. Without loss of generality, we can assume $l = j$, that is, $x^{(i)}(t)$ is not always equal to 0 in $(t_j, t_{j+1}]$. So we have

$$
x^{(i-1)}(t_{j+1}) < x^{(i-1)}(t_j^+) \leq a_j^{(i-1)} x^{(i-1)}(t_j) \leq 0.
$$

For $t \in (t_{j+1}, t_{j+2}]$, we have

$$
x^{(i-1)}(t_{j+2}) < x^{(i-1)}(t_{j+1}^+) \leq a_{j+1}^{(i-1)} x^{(i-1)}(t_{j+1}) < 0.
$$

By induction, for $t \in (t_{j+m}, t_{j+m+1}]$, we have $x^{(i-1)}(t) < 0$. So we have $x^{(i-1)}(t) < 0, x^{(i)}(t) \leq 0$, $t \in (t_{j+1}, +\infty)$. By Lemma 2.3,for all sufficiently large t,we have $x^{(i-2)}(t) < 0$. Similarly, we can conclude, using Lemma 2.3 repeatedly, that for all sufficiently large t, we have $x(t) < 0$. This is a contradiction to $x(t) > 0$ $(t \geq T)$. Hence, we have $x^{(i-1)}(t_k) > 0$ for any $t_k \geq T$. So we have $x^{(i-1)}(t) > 0$ for all sufficiently large t. The proof of Lemma 2.4 is complete. □

Lemma 2.5. *Let $x(t)$ be a solution of (2.1) and conditions (A), (B), (C) be satisfied. Suppose $T \geq t_0$, $x(t) > 0$ for $t \geq T$. Then there exist some $T' \geq T$ and $l \in \{1, 3, \dots, 2n-1\}$ such that for $t \geq T'$,*

$$\begin{gathered} x^{(i)}(t) > 0, \quad i = 0, 1, \dots, l; \\ (-1)^{i-1} x^{(i)}(t) > 0, \quad i = l+1, \dots, 2n-1; \\ x^{(2n)}(t) \leq 0. \end{gathered} \tag{2.6}$$

Proof. Let $T = t_0$. Since $x(t) > 0 (t \geq t_0)$, by (2.1) and that $p(t)$ is nonnegative and is not always equal to 0 in any $(t, +\infty)$, we have

$$x^{(2n)}(t) = -f(t, x(t)) \leq -p(t)\varphi(x(t)) \leq 0$$

and $x^{(2n)}(t)$ is not always equal to 0 in $(t, +\infty)$. By Lemma 2.4, we have $x^{(2n-1)}(t) > 0$. Without loss of generality, let $x^{(2n-1)}(t) > 0$ for $t \geq t_0$. So $x^{(2n-2)}(t) > 0$ is monotonically nondecreasing on $(t_k, t_{k+1}]$. If for any t_k, $x^{(2n-2)}(t_k) < 0$, then $x^{(2n-2)}(t) < 0 (t \geq t_0)$. If there exists some t_j such that $x^{(2n-2)}(t_j) \geq 0$, by that $x^{(2n-2)}(t)$ is monotonically increasing and $a_k^{(2n-2)} > 0$,we get $x^{(2n-2)}(t) > 0$ for $t > t_j$. So there exists some $T_1 \geq T$, such that one of the following statements hold

$$x^{(2n-1)}(t) > 0, \quad x^{(2n-2)}(t) > 0, \quad \text{for } t \geq T_1 \tag{2.7}$$

$$x^{(2n-1)}(t) > 0, \quad x^{(2n-2)}(t) < 0, \quad \text{for } t \geq T_1 \tag{2.8}$$

When (2.7) holds, Lemma 2.3 yields that $x^{(2n-3)}(t) > 0$ for all sufficiently large t. Using Lemma 2.3 repeatedly, for all sufficiently large t,we can conclude that

$$x^{(2n-1)}(t) > 0, \quad x^{(2n-2)}(t) > 0, \dots, x'(t) > 0, \quad x(t) > 0.$$

When (2.8) holds, by Lemma 2.4, we have $x^{(2n-3)}(t) > 0$, for all sufficiently large t. Hence,there exists some $T_2 \geq T_1$ such that

$$x^{(2n-3)}(t) > 0, \quad x^{(2n-4)}(t) > 0, \quad \text{for } t \geq T_2 \tag{2.9}$$

$$x^{(2n-3)}(t) > 0, \quad x^{(2n-4)}(t) < 0, \quad \text{for } t \geq T_2 \tag{2.10}$$

Repeating the discussion above, we can get, eventually, that there exist some $T' \geq T$ and $l \in \{1, 3, \dots, 2n-1\}$, such that for $t \geq T'$,

$$\begin{gathered} x^{(i)}(t) > 0, \quad i = 0, 1, \dots, l; \\ (-1)^{i-1} x^{(i)}(t) > 0, \quad i = l+1, l+2, \dots, 2n-1; \\ x^{(2n)}(t) \leq 0. \end{gathered}$$

The proof of Lemma 2.5 is complete. □

We remark that if $x(t)$ is an eventually negative solution of (2.1), then there are conclusions similar to Lemma 2.4 and Lemma 2.5.

Theorem 2.6. *If conditions (A),(B),(C) hold, $a_k^{(0)} \geq 1$ and*

$$\begin{aligned}
&\int_{t_0}^{t_1} p(t)dt + \frac{1}{b_1^{(2n-1)}} \int_{t_1}^{t_2} p(t)dt + \frac{1}{b_1^{(2n-1)} b_2^{(2n-1)}} \int_{t_2}^{t_3} p(t)dt + \dots \\
&+ \frac{1}{b_1^{(2n-1)} b_2^{(2n-1)} \dots b_m^{(2n-1)}} \int_{t_m}^{t_{m+1}} p(t)dt + \dots = +\infty
\end{aligned} \tag{2.11}$$

then every solution of (2.1) *is oscillatory.*

Proof. Let $x(t)$ be a non-oscillatory solution of (2.1). Without loss of generality, let $x(t) > 0 (t \geq t_0)$, By Lemma 2.5 and (2.1), there exists $T' \geq t_0$ such that, for $t \geq T'$, we have

$$x^{(2n)}(t) \leq 0, \quad x^{(2n-1)}(t) > 0, \quad x'(t) > 0, \quad x(t) > 0.$$

So $x^{(2n-1)}(t)$ is monotonically non-increasing on $(t_k, t_{k+1}]$ and $x(t)$ is monotonically increasing on $(t_k, t_{k+1}]$. Let

$$u(t) = \frac{x^{(2n-1)}(t)}{\varphi(x(t))}.$$

Then $u(t_k^+) \geq 0$ $(k = 1, 2, \dots)$, $u(t) \geq 0$ $(t \geq t_0)$. Since $\varphi'(x) \geq 0$, for $t \neq t_k$,

$$u'(t) = -\frac{f(t, x(t))}{\varphi(x(t))} - \Big[\frac{x^{(2n-1)}(t)x'(t)}{\varphi^2(x(t))}\Big]\varphi'(x(t)) \leq -p(t) \tag{2.12}$$

$$u(t_k^+) = \frac{x^{(2n-1)}(t_k^+)}{\varphi(x(t_k^+))} \leq \frac{b_k^{(2n-1)} x^{(2n-1)}(t_k)}{\varphi(a_k^{(0)} x(t_k))} \leq \frac{b_k^{(2n-1)} x^{(2n-1)}(t_k)}{\varphi(x(t_k))} \leq b_k^{(2n-1)} u(t_k) \tag{2.13}$$

Integrating (2.12) from t_0 to t_1 we have

$$u(t_1) \leq u(t_0^+) - \int_{t_0}^{t_1} p(t)dt\,, \tag{2.14}$$

$$u(t_1^+) \leq b_1^{(2n-1)} u(t_1) \leq b_1^{(2n-1)} [u(t_0^+) - \int_{t_0}^{t_1} p(t)dt]\,. \tag{2.15}$$

Similar to the above inequality, we have

$$\begin{aligned}
u(t_2^+) &\leq b_2^{(2n-1)} u(t_2) \\
&\leq b_2^{(2n-1)} [u(t_1^+) - \int_{t_1}^{t_2} p(t)dt] \\
&\leq b_2^{(2n-1)} [b_1^{(2n-1)} u(t_0^+) - b_1^{(2n-1)} \int_{t_0}^{t_1} p(t)dt - \int_{t_1}^{t_2} p(t)dt] \\
&\leq b_1^{(2n-1)} b_2^{(2n-1)} [u(t_0^+) - \int_{t_0}^{t_1} p(t)dt - \frac{1}{b_1^{(2n-1)}} \int_{t_1}^{t_2} p(t)dt]
\end{aligned} \tag{2.16}$$

By induction, for any natural number m, we have

$$\begin{aligned} u(t_m^+) \le b_1^{(2n-1)} b_2^{(2n-1)} \dots b_m^{(2n-1)} [u(t_0^+) - \int_{t_0}^{t_1} p(t)dt - \frac{1}{b_1^{(2n-1)}} \int_{t_1}^{t_2} p(t)dt \\ - \dots - \frac{1}{b_1^{(2n-1)} b_2^{(2n-1)} \dots b_{m-2}^{(2n-1)}} \int_{t_{m-2}}^{t_{m-1}} p(t)dt \\ - \frac{1}{b_1^{(2n-1)} b_2^{(2n-1)} \dots b_{m-2}^{(2n-1)} b_{m-1}^{(2n-1)}} \int_{t_{m-1}}^{t_m} p(t)dt] \end{aligned} \tag{2.17}$$

By (2.11) and (2.17), for all sufficiently large m, $u(t_m^+) < 0$. This contradicts $u(t_m^+) \ge 0$. So every solution of (2.1) is oscillatory. The proof of Theorem 2.6 is complete. □

Theorem 2.7. *If conditions (A), (B), (C) hold, $b_k^{(i)} \le 1$, $a_k^{(0)} \ge 1$, $b_k^{(0)} \ge 1$ ($i = 1, 2, \dots, 2n-1$, $k = 1, 2, \dots$) and $\int^{+\infty} t^{2n-1} p(t)dt = +\infty$, then every bounded solution of* (2.1) *is oscillatory.*

Proof. Let $x(t)$ be a non-oscillatory solution of (2.1). Without loss of generality, let $x(t) > 0$ for $t \ge t_0$. By Lemma 2.5, we can divided (2.6) into two cases:
Case (i): If $l = 1$, then $x(t) > 0$, $x'(t) > 0$, $x''(t) < 0$, $x'''(t) > 0$, $x^{(4)}(t) < 0$, $\dots$, $x^{(2n-1)}(t) > 0$, $x^{(2n)}(t) \le 0$.
Case (ii): If $l \ge 3$, then $x(t) > 0$, $x'(t) > 0$, $x''(t) > 0$, $x'''(t) > 0$, $\dots$, $x^{(l)}(t) > 0$, $x^{(l+1)}(t) < 0$, $\dots$, $x^{(2n-1)}(t) > 0$, $x^{(2n)}(t) \le 0$.
Both cases tells us that $x'(t) > 0$, $t \in (t_k, t_{k+1}]$, $k = 1, 2, \dots$. So $x(t)$ is monotonically increasing on $(t_k, t_{k+1}]$. Since $a_k^{(0)} \ge 1$, $x(t)$ is monotonically increasing on $[t_0, +\infty)$, that is, $x(t) \ge x(t_0)$ for $t \ge t_0$. By (2.1), we have

$$x^{(2n)}(s) = -f(s, x(s)) \le -p(s)\varphi(x(t_0)) = -cp(s), \quad s \in (t_k, t_{k+1}] \tag{2.18}$$

where $c = \varphi(x(t_0)) > 0$. Multiplying (2.18) by s^{2n-1} and then integrating it from t_k to t, we have

$$\int_{t_k}^{t} s^{2n-1} x^{(2n)}(s)ds < -c \int_{t_k}^{t} s^{2n-1} p(s)ds, \quad t \in (t_k, t_{k+1}] \,. \tag{2.19}$$

We will consider the following two cases:
(a) if the case (i) holds, then for $t \in (t_k, t_{k+1}]$ we have,

$$\begin{aligned} &\int_{t_k}^{t} s^{2n-1} x^{(2n)}(s)ds \\ &= \int_{t_k}^{t} s^{2n-1} dx^{(2n-1)}(s) \\ &= t^{2n-1} x^{(2n-1)}(t) - t_k^{2n-1} x^{(2n-1)}(t_k^+) - (2n-1) \int_{t_k}^{t} s^{2n-2} x^{(2n-1)}(s)ds \\ &= \dots \\ &= \sum_{i=0}^{2n-1} (-1)^{i+1} \frac{(2n-1)!}{i!} t^i x^{(i)}(t) + \sum_{i=0}^{2n-1} (-1)^i \frac{(2n-1)!}{i!} t_k^i x^{(i)}(t_k^+). \end{aligned}$$

Especially, for any natural number k,

$$\begin{aligned}&\int_{t_k}^{t_{k+1}} s^{2n-1}x^{(2n)}(s)ds\\&=\sum_{i=0}^{2n-1}(-1)^{i+1}\frac{(2n-1)!}{i!}t_{k+1}^i x^{(i)}(t_{k+1})+\sum_{i=0}^{2n-1}(-1)^i\frac{(2n-1)!}{i!}t_k^i x^{(i)}(t_k^+).\end{aligned}$$

No matter if i is odd or even, for $i=1,2,\dots 2n-1$,

$$(-1)^i(x^{(i)}(t_k^+)-x^{(i)}(t_k))\geq(-1)^i(b_k^{(i)}-1)x^{(i)}(t_k)\geq 0.$$

For any natural number m and $t\in(t_m,t_{m+1}]$, we have

$$\begin{aligned}&\int_{t_1}^{t} s^{2n-1}x^{(2n)}(s)ds\\&=\int_{t_1}^{t_2} s^{2n-1}x^{(2n)}(s)ds+\int_{t_2}^{t_3} s^{2n-1}x^{(2n)}(s)ds\\&\quad+\dots+\int_{t_{m-1}}^{t_m} s^{2n-1}x^{(2n)}(s)ds+\int_{t_m}^{t} s^{2n-1}x^{(2n)}(s)ds\\&=\sum_{i=0}^{2n-1}(-1)^{i+1}\frac{(2n-1)!}{i!}t^i x^{(i)}(t)+\sum_{i=0}^{2n-1}(-1)^i\frac{(2n-1)!}{i!}t_1^i x^{(i)}(t_1^+)\\&\quad+\sum_{k=2}^{m}\sum_{i=0}^{2n-1}(-1)^i\frac{(2n-1)!}{i!}t_k^i(x^{(i)}(t_k^+)-x^{(i)}(t_k))\\&\geq-(2n-1)!x(t)+\sum_{i=0}^{2n-1}(-1)^i\frac{(2n-1)!}{i!}t_1^i x^{(i)}(t_1^+)\\&\quad+\sum_{k=2}^{m}\sum_{i=0}^{2n-1}(-1)^i\frac{(2n-1)!}{i!}t_k^i(b_k^{(i)}-1)x^{(i)}(t_k)\\&\geq-(2n-1)!x(t)+\sum_{i=0}^{2n-1}(-1)^i\frac{(2n-1)!}{i!}t_1^i x^{(i)}(t_1^+).\end{aligned}$$

Combining the inequality above and (2.19), we have

$$-(2n-1)!x(t)+\sum_{i=0}^{2n-1}(-1)^i\frac{(2n-1)!}{i!}t_1^i x^{(i)}(t_1^+)\leq -c\int_{t_1}^{t} s^{2n-1}p(s)ds.$$

So $x(t)\to+\infty$, as $t\to+\infty$. This contradicts that $x(t)$ is bounded.
(b) If the case (ii) holds, then $x(t)$ is non-negative and strictly increasing on $t\in[t_1,+\infty)$. Hence, for any natural number m, we have

$$\begin{gathered}x(t)=x(t_m^+)+\int_{t_m}^{t}x'(s)ds,\quad t\in(t_m,t_{m+1}],\\x(t_m)=x(t_{m-1}^+)+\int_{t_{m-1}}^{t_m}x'(s)ds,\\\dots\\x(t_2)=x(t_1^+)+\int_{t_1}^{t_2}x'(s)ds\end{gathered}$$

and

$$x(t)=\sum_{k=2}^{m}(x(t_k^+)-x(t_k))+x(t_1^+)+\sum_{k=1}^{m-1}\int_{t_k}^{t_{k+1}}x'(s)ds+\int_{t_m}^{t}x'(s)ds \tag{2.20}$$

Since $x''(t)>0, t\in(t_k,t_{k+1}], k\geq 1$, we can get

$$x'(t)>x'(t_1^+)\geq a_1^{(1)}x'(t_1),\quad t\in(t_1,t_2]$$
$$x'(t)>x'(t_2^+)\geq a_2^{(1)}x'(t_2)>a_2^{(1)}a_1^{(1)}x'(t_1),\quad t\in(t_2,t_3]\,.$$

Applying induction, for any natural number k,

$$x'(t)>x'(t_k^+)\geq a_k^{(1)}a_{k-1}^{(1)}\dots a_1^{(1)}x'(t_1),\quad t\in(t_k,t_{k+1}]\,.$$

Combining (2.20) and $a_k^{(0)}\geq 1$, we have

$$x(t)>x'(t_1)\sum_{k=1}^{m-1}a_k^{(1)}a_{k-1}^{(1)}\dots a_1^{(1)}(t_{k+1}-t_k),\quad t\in(t_m,t_{m+1}]$$

From the condition (C) and $b_k^{(0)}\geq 1$, we have

$$\sum_{k=1}^{+\infty}a_k^{(1)}a_{k-1}^{(1)}\dots a_1^{(1)}(t_{k+1}-t_k)=+\infty$$

Then $x(t)\to+\infty$ $(t\to+\infty)$, which contradicts that $x(t)$ is bounded. Therefore, every solution of (2.1) is oscillatory. The proof of Theorem 2.7 is complete. □

Theorem 2.8. *If conditions (A), (B), (C) hold,* $\prod_{k=1}^{m}a_k^{(0)}>b>0$ *(*$m=1,2,\dots$*),* $b_k^{(2n-1)}\leq 1$*, and for any* $\delta>0$*,*

$$\Big|\int^{+\infty}\inf_{\delta\leq|x|<+\infty}f(t,x)\,dt\Big|=+\infty \tag{2.21}$$

then every solution of (2.1) *is oscillatory.*

Proof. Let $x(t)$ be a non-oscillatory solution of (2.1). Without loss of generality, let $x(t)>0$, $t\geq t_0$. By Lemma 2.5, $x'(t)\geq 0$, $t\geq t_0$. So $x(t)$ is monotonically nondecreasing on $(t_0,+\infty)$.

$$x(t_1)\geq x(t_0^+), x(t_2)\geq x(t_1^+)\geq a_1^{(0)}x(t_1)\geq a_1^{(0)}x(t_0^+),$$
$$x(t_3)\geq x(t_2^+)\geq a_2^{(0)}x(t_2)\geq a_2^{(0)}a_1^{(0)}x(t_0^+)$$

By induction, we have

$$x(t_{m+1})\geq x(t_m^+)\geq a_m^{(0)}x(t_m)\geq\dots\geq a_1^{(0)}a_2^{(0)}\dots a_m^{(0)}x(t_0^+)>bx(t_0^+).$$

We can assume that $x(t)\geq bx(t_0^+)$, $t\in(t_0,+\infty)$. By (2.21), as $t\to+\infty$, we have

$$\int_{t_0}^{t}f(s,x(s))ds\geq\int_{t_0}^{t}\inf_{bx(t_0^+)\leq|x|<+\infty}f(s,x)ds\to+\infty\,;$$

that is, $\int_{t_0}^{t}f(s,x(s))ds\to+\infty$. Integrating (2.1) from t_0 to t_1, we have

$$x^{(2n-1)}(t_1)+\int_{t_0}^{t_1}f(s,x(s))ds=x^{(2n-1)}(t_0^+)$$

Similar to the above formula, for any natural number integrating (2.1) from t_{k-1} to t_k, we have

$$x^{(2n-1)}(t_k) + \int_{t_{k-1}}^{t_k} f(s, x(s))ds = x^{(2n-1)}(t_{k-1}^+)$$

So, we have

$$\begin{gathered}
x^{(2n-1)}(t_1) + \int_{t_0}^{t_1} f(s, x(s))ds = x^{(2n-1)}(t_0^+),\\
x^{(2n-1)}(t_2) + \int_{t_1}^{t_2} f(s, x(s))ds = x^{(2n-1)}(t_1^+),\\
\dots\\
x^{(2n-1)}(t_m) + \int_{t_{m-1}}^{t_m} f(s, x(s))ds = x^{(2n-1)}(t_{m-1}^+),\\
x^{(2n-1)}(t) + \int_{t_m}^{t} f(s, x(s))ds = x^{(2n-1)}(t_m^+)\,.
\end{gathered}$$

For $t \in (t_m, t_{m+1}]$, we have

$$x^{(2n-1)}(t) + \sum_{i=1}^{m} x^{(2n-1)}(t_i) + \int_{t_0}^{t} f(s, x(s))ds = \sum_{i=0}^{m} x^{(2n-1)}(t_i^+).$$

Then

$$x^{(2n-1)}(t) + \sum_{i=1}^{m} \left(x^{(2n-1)}(t_i) - x^{(2n-1)}(t_i^+)\right) + \int_{t_0}^{t} f(s, x(s))ds = x^{(2n-1)}(t_0^+)\,.$$

Lemma 2.5 shows that $x^{(2n-1)}(t) > 0$ for sufficiently large t. Hence,

$$x^{(2n-1)}(t) \le -\sum_{i=1}^{m} \Big((1-b_k^{(2n-1)})x^{(2n-1)}(t_i)\Big) - \int_{t_0}^{t} f(s, x(s))ds + x^{(2n-1)}(t_0^+) \quad (2.22)$$

By condition $b_k^{(2n-1)} \le 1$ and (2.22), we have $x^{(2n-1)}(t) \le -\int_{t_0}^{t} f(s, x(s))ds + x^{(2n-1)}(t_0^+) \to -\infty$ as $t \to +\infty$. So, for all sufficiently large t, $x^{(2n-1)}(t) < 0$. This contradicts that $x^{(2n-1)}(t) > 0$. So every solution of (2.1) is oscillatory. The proof of Theorem 2.8 is complete. □

Corollary 2.9. *Assume the conditions (A), (B), (C) hold, and $a_k^{(0)} \ge 1$, $b_k^{(2n-1)} \le 1$. If $\int^{+\infty} p(t)dt = +\infty$, then every solution of (2.1) is oscillatory.*

Proof. By $b_k^{(2n-1)} \le 1$, we have

$$\begin{aligned}
&\int_{t_0}^{t_1} p(t)dt + \frac{1}{b_1^{(2n-1)}} \int_{t_1}^{t_2} p(t)dt + \frac{1}{b_1^{(2n-1)} b_2^{(2n-1)}} \int_{t_2}^{t_3} p(t)dt + \dots\\
&+ \frac{1}{b_1^{(2n-1)} b_2^{(2n-1)} \dots b_m^{(2n-1)}} \int_{t_m}^{t_{m+1}} p(t)dt\\
&\ge \int_{t_0}^{t_1} p(t)dt + \int_{t_1}^{t_2} p(t)dt + \int_{t_2}^{t_3} p(t)dt + \dots + \int_{t_m}^{t_{m+1}} p(t)dt\\
&= \int_{t_0}^{t_{m+1}} p(t)dt
\end{aligned}$$

and $\int_{t_0}^{t_{m+1}} p(t)dt \to +\infty$ as $m \to +\infty$. Then (2.11) holds. By Theorem 2.6, every solution of (2.1) is oscillatory. $\square$

Corollary 2.10. *Assume conditions (A), (B), (C) hold, and that there exists a positive number $\alpha > 0$, such that $a_k^{(0)} \geq 1$, $\frac{1}{b_k^{(2n-1)}} \geq (\frac{t_{k+1}}{t_k})^\alpha$. If $\int^{+\infty} t^\alpha p(t)dt = +\infty$, then every solution of (2.1) is oscillatory.*

Proof. By $\frac{1}{b_k^{(2n-1)}} \geq (\frac{t_{k+1}}{t_k})^\alpha$, we have

$$
\begin{aligned}
&\int_{t_0}^{t_1} p(t)dt + \frac{1}{b_1^{(2n-1)}}\int_{t_1}^{t_2} p(t)dt + \frac{1}{b_1^{(2n-1)}b_2^{(2n-1)}}\int_{t_2}^{t_3} p(t)dt + \dots \\
&+ \frac{1}{b_1^{(2n-1)}b_2^{(2n-1)}\dots b_m^{(2n-1)}}\int_{t_m}^{t_{m+1}} p(t)dt \\
&\geq \frac{1}{b_1^{(2n-1)}}\int_{t_1}^{t_2} p(t)dt + \frac{1}{b_1^{(2n-1)}b_2^{(2n-1)}}\int_{t_2}^{t_3} p(t)dt + \dots \\
&\quad + \frac{1}{b_1^{(2n-1)}b_2^{(2n-1)}\dots b_m^{(2n-1)}}\int_{t_m}^{t_{m+1}} p(t)dt \\
&\geq \frac{1}{t_1^\alpha}[\int_{t_1}^{t_2} t_2^\alpha p(t)dt + \int_{t_2}^{t_3} t_3^\alpha p(t)dt + \dots + \int_{t_m}^{t_{m+1}} t_{m+1}^\alpha p(t)dt] \\
&\geq \frac{1}{t_1^\alpha}[\int_{t_1}^{t_2} t^\alpha p(t)dt + \int_{t_2}^{t_3} t^\alpha p(t)dt + \dots + \int_{t_m}^{t_{m+1}} t^\alpha p(t)dt] \\
&= \frac{1}{t_1^\alpha}\int_{t_1}^{t_{m+1}} t^\alpha p(t)dt
\end{aligned}
$$

and $\int_{t_1}^{t_{m+1}} p(t)dt \to +\infty$ as $m \to +\infty$. Then (2.11) holds. By Theorem 2.6, we every solution of (2.1) is oscillatory. $\square$

3. Examples

subsection*Example 3.1 Consider the equation

$$
\begin{gathered}
x^{(2n)}(t) + \frac{1}{4t}x^3 = 0, \quad t \geq \frac{1}{2},\ t \neq k,\ k = 1, 2, \dots \\
x(k^+) = \frac{k+1}{k}x(k), \quad x^{(i)}(k^+) = x^{(i)}(k), \quad i = 1, \dots, 2n-1, \qquad (3.1) \\
x(\frac{1}{2}) = x_0, x^{(i)}(\frac{1}{2}) = x_0^{(i)},
\end{gathered}
$$

where $a_k^{(0)} = b_k^{(0)} = \frac{k+1}{k} > 1$, $a_k^{(i)} = b_k^{(i)} = 1$, $i = 1, 2, \dots, 2n-1$, $p(t) = \frac{1}{4t}$, $\varphi(x) = x^3$, $f(t, x) = \frac{1}{4t}x^3$, $t_k = k$, $t_0 = \frac{1}{2}$. It is obvious that the conditions (A) and (B) are satisfied. For condition (C),we have: For $i > 1$, $a_k^{(i)} = b_k^{(i-1)} = 1$,

$$
\begin{aligned}
&(t_1 - t_0) + (t_2 - t_1) + (t_3 - t_2) + \dots + (t_{m+1} - t_m) + \dots \\
&= \frac{1}{2} + 1 + \dots + 1 + \dots = +\infty.
\end{aligned}
$$

For $i = 1$, $a_k^{(1)} = 1$, $b_k^{(0)} = \frac{k+1}{k}$,

$$\begin{aligned}&(t_1 - t_0) + \frac{1}{2}(t_2 - t_1) + \frac{1}{3}(t_3 - t_2) + \dots + \frac{1}{m+1}(t_{m+1} - t_m) + \dots \\ &= \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{m+1} + \dots = +\infty.\end{aligned}$$

Therefore, condition (C) holds. Since $b_k^{(2n-1)} = 1$, we have

$$\begin{aligned}&\int_{t_0}^{t_1} p(t)dt + \frac{1}{b_1^{(2n-1)}}\int_{t_1}^{t_2} p(t)dt + \frac{1}{b_1^{(2n-1)}b_2^{(2n-1)}}\int_{t_2}^{t_3} p(t)dt + \dots \\ &+ \frac{1}{b_1^{(2n-1)}b_2^{(2n-1)}\dots b_m^{(2n-1)}}\int_{t_m}^{t_{m+1}} p(t)dt \\ &= \int_{t_0}^{t_1} p(t)dt + \int_{t_1}^{t_2} p(t)dt + \int_{t_2}^{t_3} p(t)dt + \dots + \int_{t_m}^{t_{m+1}} p(t)dt \\ &= \int_{t_0}^{t_{m+1}} p(t)dt = \int_{t_0}^{t_{m+1}} \frac{1}{4t}dt \\ &= \frac{1}{4}\ln t|_{t_0}^{t_{m+1}} = \frac{1}{4}(\ln t_{m+1} - \ln t_0)\end{aligned}$$

Since $\ln t_{m+1} \to +\infty$ as $m \to +\infty$, we get that the condition of Theorem 2.6 hold. So every solution of (3.1) is oscillatory.

Example 3.2. Consider the sub-linear system

$$\begin{gathered}x^{(2n)}(t) + \frac{1}{t^2}x^{\frac{1}{3}} = 0, \quad t \geq \frac{1}{2},\ t \neq k, k = 1, 2, \dots, \\ x(k^+) = x(k), x^{(i)}(k^+) = \frac{k}{k+1}x^{(i)}(k), \quad i = 1, \dots, 2n-1, \\ x(\frac{1}{2}) = x_0, \quad x^{(i)}(\frac{1}{2}) = x_0^{(i)},\end{gathered} \tag{3.2}$$

where $a_k^{(0)} = b_k^{(0)} = 1$, $a_k^{(i)} = b_k^{(i)} = \frac{k}{k+1}$, $i = 1, 2, \dots, 2n-1$, $p(t) = \frac{1}{t^2}$, $t_k = k$, $\varphi(x) = x^{\frac{1}{3}}$, $f(t, x(t)) = \frac{1}{t^2}x^{\frac{1}{3}}(t)$, $t_0 = \frac{1}{2}$. It is obvious that the condition (A) and (B) hold. For condition (C), we have: For $i > 1$ and $a_k^{(i)} = b_k^{(i-1)} = \frac{k}{k+1}$,

$$(t_1 - t_0) + (t_2 - t_1) + (t_3 - t_2) + \dots + (t_{m+1} - t_m) + \dots = \frac{1}{2} + 1 + \dots + 1 + \dots = +\infty.$$

For $i = 1$ and $a_k^{(1)} = \frac{k}{k+1}, b_k^{(0)} = 1$,

$$\begin{aligned}&(t_1 - t_0) + \frac{1}{2}(t_2 - t_1) + \frac{1}{3}(t_3 - t_2) + \dots + \frac{1}{m+1}(t_{m+1} - t_m) + \dots \\ &= \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{m+1} + \dots = +\infty.\end{aligned}$$

So, condition (C) holds. Let $\alpha = 1$. Then

$$\frac{1}{b_k^{(2n-1)}} = \frac{k+1}{k} \geq \frac{t_{k+1}}{t_k} = \frac{k+1}{k}\int^{+\infty} tp(t)dt = \int^{+\infty} t\frac{1}{t^2}dt = \int^{+\infty} \frac{1}{t}dt = +\infty\,.$$

Therefore, the conditions of Corollary 2.10 are satisfied. Then every solution of (3.2) is oscillatory.

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