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VARIABLE LORENTZ ESTIMATE FOR GENERALIZED STOKES SYSTEMS IN NON-SMOOTH DOMAINS

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Abstract. We prove a global Calderón-Zygmund type estimate in the framework of Lorentz spaces for the variable power of the gradient of weak solution pair (u, P) to the generalized steady Stokes system over a bounded non-smooth domain. It is assumed that the leading coefficients satisfy the small BMO condition, the boundary of domain belongs to Reifenberg flatness, and the variable exponent $p(x)$ is log-Hölder continuous.

1. Introduction

Let $\Omega \subset \mathbb{R}^n (n \geq 2)$ be a given bounded domain with a rough boundary specified later. The aim of this article is to study a global Lorentz estimate for the variable power of the gradient of weak solution to the generalized steady Stokes problem

$$\begin{gathered} \operatorname{div}(\mathbf{A}(x)\nabla u) - \nabla P = \operatorname{div}\mathbf{F}, \quad \text{in } \Omega, \\ \operatorname{div} u = 0, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{1.1}$$

Throughout this article, as usual we assume that the fourth-order tensor $\mathbf{A}(x) = \left(A_{ij}^{\alpha\beta}\right)_{i,j,\alpha,\beta=1}^{n} : \Omega \to \mathbb{R}^{n^2\times n^2}$ satisfies a uniform boundedness and ellipticity for constants $0 < \nu \leq \Lambda < +\infty$:

$$\nu|\xi|^2 \leq \mathbf{A}(x)\xi \cdot \xi \leq \Lambda|\xi|^2, \tag{1.2}$$

where $x \in \Omega$ a.e., $\xi \in \mathbb{R}^{n^2}$, and $\mathbf{F}(x) = (F_i^{\alpha})_{i,\alpha=1}^{n}$. The unknown velocity of vectorial-value functions is denoted by $u = (u^1, u^2, \dots, u^n) : \Omega \to \mathbb{R}^n$ and $P : \Omega \to \mathbb{R}$ is the pressure.

Let us recall some recent progresses of the Calderón-Zygmund theory concerning partial differential equations with discontinuous coefficients. The interior and global $W^{2,p}$ estimates for nondivergence linear elliptic equations with the VMO discontinuous coefficients were presented by Chiarenza-Frasca-Longo [14, 15]. Since then, there has been continuous attention on the Calderón-Zygmund theory for various elliptic and parabolic problems with discontinuous coefficients. Apart from an earlier technique by using singular integral operators and its commutators, there are three kinds of important arguments to deal with the Calderón-Zygmund theory

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concerning elliptic and parabolic problems with the VMO or small BMO discontinuous coefficients. The first one is the so-called geometrical approach originally traced from Byun-Wang's work [11], which is used to attain the global L^p estimate by way of weak compactness based on the boundedness of the Hardy-Littlewood maximal operators and the modified Vitali covering theory of distributional functions regarding the gradients of solutions. Indeed, this can also be regarded as a development from Caffarelli-Peral's work [13] to obtain local $W^{1,p}_{\rm loc}$-estimates for solutions of a class of elliptic problems of p-Laplace type. Secondly, Dong-Kim-Krylov (for examples, see [20, 25]) presented a unified approach of studying L^p solvability for elliptic and parabolic problems on the basis of the Fefferman-Stein theorem on sharp functions and the Hardy-Littlewood maximal function theorem for the spatial derivatives of solutions. The third technique is called the large-M-inequality principle originated from Acerbi-Mingione's work [1, 2], which is directly applied to argue on certain Calderón-Zygmund-type coverings instead of the maximal function operator and other harmonic techniques such as the good-λ-inequality. Here, we would like to mention that recently Byun et al have obtained numerous global Calderón-Zygmund-type results to various nonlinear elliptic and parabolic problems over non-smooth domains by combining the large-M-inequality principle with a geometrical approach [10, 11, 12]. As we have seen, Byun-Ok-Wang [10] attained a global Calderón-Zygmund estimate with the variable exponent of gradients of solution to the zero Dirichlet problem for linear elliptic systems in the divergence form with partial BMO coefficients and log-Hölder continuity $p(x)$, which implies that

$$\mathbf{F} \in L^{p(x)}(\Omega, \mathbb{R}^n) \Rightarrow Du \in L^{p(x)}(\Omega, \mathbb{R}^n).$$

Later, Tian-Zheng [31] further extended it to a global Calderón-Zygmund-type estimate for variable power of the gradient of solution in Lorentz spaces for the same problem with partial BMO coefficients.

We know that solvability and optimal regularity of Stokes system under minimal regular datum have been a classical and important problem in the theory of partial differential equations and fluid dynamics. In the past decades, we have seen a great deal of literature concerning the interior regularity of Stokes system [7, 17, 18] and global regularity of the generalized Stokes system in the Lipschitz domain (cf. [9, 16, 19, 21, 22, 26]), the domain with the Reifenberg flatness boundary [12, 24]. It is quite necessary to mention some recent advances concerning the generalized steady Stokes problems (1.1) with discontinuous coefficients. Daněček-John-Stará [16] investigated the Morrey regularity for the gradient of weak solution (u, P) to a generalized Stokes system with symmetric elliptic coefficients whose entries satisfy the boundedness and VMO discontinuity. Gu-Shen [22] considered a uniform $W^{1,p}$-regularity in the homogenization theory of the generalized Stokes system with setting of rapidly oscillating periodic coefficients, and obtained the global $W^{1,p}$-estimates with $1 < p < \infty$ to a family of generalized Stokes systems with the VMO periodic coefficients in a bounded C^1 domain. Very recently, Bulíček-Burczak-Schwaarzacher [9] dealt with the following steady Stokes system in a bounded domain with $\partial\Omega \in C^1$:

$$\begin{gathered}
\operatorname{div}\big(\frac{1}{2}(\nabla u - \nabla^T u)\big) - \nabla P = \operatorname{div}\mathbf{F}, \quad \text{in } \Omega,\\
\operatorname{div} u = d, \quad \text{in } \Omega,\\
\gamma(u) = g, \quad \text{on } \partial\Omega,
\end{gathered}$$

where γ is the trace operator. They established a global weighted $W^{1,p}$ result, which means that

$$\mathbf{F} \in L^q_\omega(\Omega)^{n^2} \text{ and } d, g \in L^q_\omega(\Omega) \ \Rightarrow\ u \in W^{1,q}_\omega(\Omega)^n \text{ and } P - \langle P \rangle \in L^q_\omega(\Omega)$$

for $1 < q < \infty$. Byun-So [12] obtained the global weighted L^q-estimates for the gradient of weak solution and an associated pressure to the Dirichlet problem of the generalized Stokes system (1.1) under assumptions that the coefficients have small BMO semi-norms and the domain is flatness in the Reifenberg sense, which implies that

$$\mathbf{F} \in L^q_\omega(\Omega)^{n^2} \ \Rightarrow\ \nabla u \in L^q_\omega(\Omega)^{n^2} \text{ and } P \in L^q_\omega(\Omega).$$

For the results on the L^q theory for the generalized Stokes problem of p-Laplacian type and evolutional Stokes equations, one can refer to [17, 24] etc.

On the other hand, Lorentz spaces are a two-parameter scale of Lebesgue spaces by refining Lebesgue spaces in the fashion of second index. In recent years, a number of theoretical issues concerning Lorentz regularity of various PDEs has received considerable attention. For examples, Mengesha-Phuc [27] derived weighted Lorentz estimates for the gradients of solutions to quasilinear p-Laplace type equations based on the geometrical approach. Meanwhile, Baroni [5, 6] obtained Lorentz estimates for the gradients of solutions to evolutionary p-Laplacian systems and parabolic p-Laplacian with the given obstacle function $D\psi \in L(\gamma, q)$ locally in Ω_T respectively, by using the large-M-inequality principle, which means that

$$\mathbf{F}, D\psi \in L(\gamma, q) \text{ locally in } \Omega_T \ \Rightarrow\ Du \in L(\gamma, q) \text{ locally in } \Omega_T$$

with $\gamma > p$ and $q \in (0, \infty]$. Later, Zhang-Zhou [35] extended the result of [27] to the quasilinear elliptic $p(x)$-Laplacian equations by using a geometrical argument. Adimurthi-Phuc [3] showed global Lorentz and Lorentz-Morrey estimates below the natural exponent of quasilinear equations. Zhang-Zheng [33, 34] studied Lorentz estimates of fully nonlinear parabolic and elliptic equations with small BMO nonlinearities, and obtained weighted Lorentz estimates of the Hessian of strong solution for nondivergence linear elliptic equations with partial BMO coefficients.

Motivated by the progresses mentioned above, in this work we focus on a global Calderón-Zygmund type estimate for the variable power of the gradient of weak solution in the framework of Lorentz spaces to the Dirichlet problem of the generalized steady Stokes systems (1.1) in the non-smooth domain. Here, we allow the coefficient tensor $\mathbf{A}(x)$ to be discontinuous, but it suffices to impose a small BMO regular condition, the variable exponent $p(x)$ satisfies log-Hölder continuity, and the boundary of domain belongs to Reifenberg flatness. This study is also inspired by elegant results presented in [1, 2, 5, 6, 12]. That is, we apply the mixed argument of large-M-inequality principle and the geometric approach to prove global Lorentz estimates for the variable power of the gradient of weak solution to the Dirichlet problem of (1.1) over a bounded Reifenberg flatness domain. Indeed, the key ingredient is based on making use of Calderón-Zygmund type covering, approximate estimate and iteration arguments to obtain an estimate of the measure of the super-level set for the variable power of the gradient of weak solution.

The rest of this artivcle is organized as follows. In section 2, we recall the definition of weak solution to problem (1.1) and state our main result. In Section 3 we present some technical lemmas. In Section 4 we prove of our main result.

2. Preliminaries and statement of main results

In this section, we present some related definition and notations and state our main result on the Dirichlet problem of the generalized steady Stokes systems (1.1).

Denote by $c(n, \nu, \Lambda, \dots)$ a universal constant depending only on prescribed quantities and possibly varying from line to line in the following context. Let us recall the Lorentz space $L(t, q)(U)$ with an open subset $U \subset \mathbb{R}^n$ for any parameters $1 \leq t < \infty$ and $0 < q < \infty$. This is defined by requiring that for a measurable function $g : U \to \mathbb{R}$, it holds

$$\|g\|^q_{L(t,q)(U)} := t \int_0^\infty \Big(\mu^t |\{\xi \in U : |g(\xi)| > \mu\}|\Big)^{q/t} \frac{d\mu}{\mu} < \infty;$$

while the Lorentz space $L(t, \infty)$ for $1 \leq t < \infty$ and $q = \infty$ is defined by the Marcinkiewicz space $\mathcal{M}^t(U)$ as usual, which is the space of measurable functions g with

$$\|g\|_{L(t,\infty)} = \|g\|_{\mathcal{M}^t(U)} := \sup_{\mu>0} \Big(\mu^t |\{\xi \in U : |g(\xi)| > \mu\}|\Big)^{1/t} < \infty.$$

The local variance of such spaces is defined in the usual way. We remark that if $t = q$, then the Lorentz space $L(t, t)(U)$ is nothing but a classical Lebesgue space. Indeed, by Fubini's theorem it gives

$$\|g\|^t_{L^t(U)} = t \int_0^\infty \mu^t |\{\xi \in U : |g(\xi)| > \mu\}| \frac{d\mu}{\mu} = \|g\|^t_{L(t,t)(U)},$$

which implies $L^t(U) = L(t, t)(U)$, see [33, 4,5,26,31,[33]].

Remark 2.1. Because of the lack of sub-additivity, the quantity $\| \cdot \|_{L(t,q)(U)}$ is just a quasi-norm. Nevertheless, the mapping $g \mapsto \|g\|_{L(t,q)(U)}$ is still weak lower semi-continuous, for details see [29, Remark 3].

Definition 2.2. Let $\mathbf{F} \in L^2(\Omega)^{n^2}$. If $u \in W_0^{1,2}(\Omega)^n$, $\operatorname{div} u = 0$ and satisfies

$$\int_\Omega \langle \mathbf{A}(x)\nabla u, \nabla v\rangle \, dx = \int_\Omega \langle \mathbf{F}, \nabla v\rangle \, dx \tag{2.1}$$

for all $v \in W_0^{1,2}(\Omega)^n$ and $\operatorname{div} v = 0$, then u is called a weak solution of the zero Dirichlet problem of the generalized Stokes system (1.1). If u is such a weak solution and $P \in L^2(\Omega)$ satisfies

$$\int_\Omega \big(\langle \mathbf{A}(x)\nabla u, \nabla v\rangle - \langle P, \operatorname{div} v\rangle\big) dx = \int_\Omega \langle \mathbf{F}, \nabla v\rangle \, dx \tag{2.2}$$

for all $v \in W_0^{1,2}(\Omega)^n$, then (u, P) is said to be a weak solution pair of the generalized Stokes system (1.1) and P is called an associated pressure of u.

The traditional assumption on the variable exponent $p(\cdot)$ is log-Hölder continuity, which ensures that the Hardy-Littlewood maximal operator is bounded within the framework of generalized Lebesgue space. Briefly, we recall that $p(x)$ is log-Hölder continuous, denoted by $p(x) \in LH(\Omega)$, if there exist constants c_0 and $\delta > 0$ such that for all $x, y \in \Omega$ with $|x - y| < \delta$, it holds

$$|p(x) - p(y)| \leq \frac{c_0}{-\log(|x - y|)}.$$

In this context, we assume that $p(x) : \Omega \to \mathbb{R}$ is a log-Hölder continuous function, and there exist positive constants γ_1 and γ_2 such that

$$2 < \gamma_1 \le p(x) \le \gamma_2 < \infty, \quad \forall x \in \Omega, \tag{2.3}$$

$$|p(x) - p(y)| \le \omega(|x-y|), \quad \forall x, y \in \Omega, \tag{2.4}$$

where $\omega : [0,\infty) \to [0,\infty)$ is a modulus of continuity of $p(x)$. Without loss of generality, we suppose that ω is a non-decreasing continuous function with

$$\omega(0) = 0, \quad \limsup_{r\to 0} \omega(r) \log\big(\frac{1}{r}\big) < \infty.$$

With the above assumptions, it is clear that $p(x) \in LH(\Omega)$ and there exists a positive constant A such that

$$\omega(r)\log(\frac{1}{r}) \le A \;\Leftrightarrow\; r^{-\omega(r)} \le e^A \text{ for any } r \in (0,1). \tag{2.5}$$

On the other hand, the generalization of the classical steady Stokes system consists of general second order elliptic equations in the divergence form instead of the standard Stokes equation. It is rather necessary to impose certain regular assumptions on the leading coefficients $\mathbf{A}(x)$ and the geometric structure on the boundary of domain. To this end, we let

$$B_r(y) = \{x \in \mathbb{R}^n : |x-y| < r\}$$

for $y \in \Omega$ and radius $r > 0$. Denote that

$$\Omega_r(y) = B_r(y) \cap \Omega, \quad \partial_\omega \Omega_r(y) = B_r(y) \cap \partial\Omega,$$
$$B_r^+ = B_r(0) \cap \{x_n > 0\}, \quad T_r = B_r(0) \cap \{x_n = 0\}.$$

For any bounded domain $U \subset \mathbb{R}^n$, we denote

$$f_U = \fint_U f(x)dx = \frac{1}{|U|}\int_U f(x)dx.$$

In what follows, a key assumption is that the coefficient tensor $\mathbf{A}(x)$ is allowed to be sufficiently small BMO discontinuous, and the boundary $\partial\Omega$ of domain is Reifenberg flat.

Definition 2.3. We say that the pair $(\mathbf{A}, \Omega)$ is (δ, R_0)-vanishing, if for any $x \in \Omega$ and for each number $r \in (0, R_0]$ with

$$\operatorname{dist}(x, \partial\Omega) = \min_{z\in\partial\Omega} \operatorname{dist}(x,z) > \sqrt{2}r,$$

there exists a coordinate system depending on x and r such that in the new coordinate system x is the origin and satisfies

$$\fint_{B_r(x)} \big|\mathbf{A}(y) - \mathbf{A}_{B_r(x)}\big|^2 dy \le \delta^2. \tag{2.6}$$

While, for any $x \in \Omega$ and for each number $r \in (0, R_0]$ with

$$\operatorname{dist}(x, \partial\Omega) = \min_{z\in\partial\Omega} \operatorname{dist}(x,z) = \operatorname{dist}(x, z_0) \le \sqrt{2}r$$

for some $z_0 \in \partial\Omega$, there exists a coordinate system depending on x and r such that in the new coordinate system z_0 is the origin and satisfies

$$B_{3r}(z_0) \cap \{x_1 \ge 3\delta r\} \subset B_{3r}(z_0) \cap \Omega \subset B_{3r}(z_0) \cap \{x_1 \ge -3\delta r\}, \tag{2.7}$$

$$
\fint_{B_{3r}(z_0)} |\mathbf{A}(y) - \mathbf{A}_{B_{3r}(z_0)}|^2 \, dx \le \delta^2, \tag{2.8}
$$

where $A(x)$ is a zero-extension from $B_{3r}(z_0) \cap \Omega$ to B_{3r}, and the parameters $\delta > 0$ and R_0 will be specified later.

We assume that δ is a small positive constant, saying $0 < \delta < 1/8$. Notice that Ω is a (δ, R_0)-Reifenberg flat domain. Obviously, it is an A-type domain, which implies the following measure density condition [11]:

$$
\sup_{0<r\le R} \sup_{y\in\Omega} \frac{|B_r(y)|}{|B_r(y)\cap\Omega|} \le \big(\frac{2}{1-\delta}\big)^n \le \big(\frac{16}{7}\big)^n, \tag{2.9}
$$

and for any $y \in \partial\Omega$ and $r \in (0, R_0]$ it holds

$$
\frac{|B_r(y)\cap\Omega^c|}{|B_r(y)|} \ge \big(\frac{1-\delta}{2}\big)^n \ge \big(\frac{7}{16}\big)^n. \tag{2.10}
$$

This ensures that it holds a reverse Hölder's inequality for the gradients of solutions at the neighborhood of boundary point. Now, we are ready to summarize our main result.

Theorem 2.4. *Let the variable exponent $p(x)$ satisfy (2.3), (2.4) and (2.5), and the given pair $(\mathbf{A}, \Omega)$ is (δ, R_0)-vanishing with $R_0 > 0$. Suppose that (u, P) satisfying $u \in W_0^{1,2}(\Omega)^n$, $\operatorname{div} u = 0$ and is a weak solution pair of the generalized Stokes system (1.1)–(1.2). If*

$$
|F|^{p(x)} \in L(t,q)(\Omega), \quad \text{for } t \ge 1 \text{ and } q \in (0, +\infty],
$$

then there exists a small constant $\delta_0 = \delta_0(n, \gamma_1, \gamma_2, \nu, \Lambda) > 0$ such that for every $\delta \in (0, \delta_0]$, we have $(|Du| + |P|)^{p(x)} \in L(t,q)(\Omega)$ with the estimate

$$
\big\|(|Du| + |P|)^{p(x)}\big\|_{L(t,q)(\Omega)} \le c\Big(\||F|^{p(x)}\|_{L(t,q)(\Omega)} + 1\Big)^{\gamma_2/\gamma_1}, \tag{2.11}
$$

where the constant c depends only on $n, \gamma_1, \gamma_2, \nu, \Lambda, t, q, \delta_0, R_0, \Omega, \omega(\cdot)$ and $|\Omega|$ (except in the case $q = \infty$).

3. Technical tools

In this section, we present some useful technical lemmas, which will play an essential role in proving our main result. We start with recalling the existence and energy estimate of weak solution pair to the generalized Stokes system (1.1), see [12, Lemma 2.9].

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^n$ for $n \ge 2$ be an open bounded (δ, R_0)-Reifenberg flat domain with sufficiently small $\delta > 0$, and $\mathbf{F} \in L^2(\Omega)^{n^2}$. Then there exists a unique solution pair $(u, P) \in W_0^{1,2}(\Omega)^n \times L^2(\Omega)$ to the generalized Stokes system (1.1) with $\operatorname{div} u = 0$ and $\int_\Omega P \, dx = 0$ such that the following standard estimate holds*

$$
\|\nabla u\|_{L^2(\Omega)^{n^2}} + \|P\|_{L^2(\Omega)} \le c\|\mathbf{F}\|_{L^2(\Omega)^{n^2}}, \tag{3.1}
$$

where $c = c(n, \nu, \Lambda, \Omega)$.

In addition, if $u \in W_0^{1,q}(\Omega)^n$ with $\operatorname{div} u = 0$ and $\mathbf{F} \in L^q(\Omega)^{n^2}$ for $q \in [2, +\infty)$, then

$$
\|P\|_{L^q(\Omega)} \le c\big(\|\mathbf{F}\|_{L^q(\Omega)^{n^2}} + \|\nabla u\|_{L^q(\Omega)^{n^2}}\big), \tag{3.2}
$$

where $c = c(n, \nu, \Lambda, \Omega)$.

The following lemma is regarding the principle of local reverse Hölder's inequality, which can be obtained form [23, Prop. 1.2 Chapter 5].

Lemma 3.2. *Let $0 < r < d_0 \le \operatorname{dist}(x, \partial\Omega)$ for $x \in \Omega$. Suppose that $g(x), h(x) \in L^p(B_{2R}, \mathbb{R}^N)$ with $1 < p < \infty$ satisfy*

$$\fint_{B_R} |g(x)|^p dx \le \theta \fint_{B_{2R}} |g(x)|^p dx + c \fint_{B_{2R}} |h(x)|^p dx + c\Big(\fint_{B_{2R}} |g(x)|^s dx \Big)^{p/s},$$

where $1 \le s < p$ and $0 \le \theta < \infty$. Then there exists $p' = p'(\theta, p, N, c) > p$ such that there is $g \in L^{p'}_{\rm loc}(\Omega, \mathbb{R}^N)$ with the estimate

$$\Big(\fint_{B_R} |g(x)|^{p'} dx \Big)^{\frac{1}{p'}} \le c\Big(\fint_{B_{2R}} |g(x)|^p dx \Big)^{1/p} + c\Big(\fint_{B_{2R}} |h(x)|^p dx \Big)^{1/p}.$$

Based on Lemmas 3.1 and 3.2, we now prove a higher integrability for the gradients of weak solutions and the pressure P to the generalized Stokes system (1.1) in the admissible set $W^{1,2}_0(\Omega)^n \times L^2(\Omega)$ with $\operatorname{div} u = 0$.

Lemma 3.3. *Let $(u, P) \in W^{1,2}_0(B_{2r})^n \times L^2(B_{2r})$ be a weak solution pair to the generalized Stokes system (1.1) with $\operatorname{div} u = 0$ under the usual assumption (1.2). If $|\mathbf{F}|^{p(x)} \in L^t(B_{2r})$ with $p(x) > \gamma_1 > 2$ and $t > 1$ with $B_{2r} \Subset \Omega$, then there exist positive constants $c = c(n, \gamma_1, \gamma_2, \nu, \Lambda)$ and small $\sigma_0 > 0$ with*

$$\sigma_0 < \frac{t\gamma_1}{2} - 1,$$

such that for any $0 < \sigma \le \sigma_0$, it holds

$$\begin{aligned} &\Big(\fint_{B_r} |\nabla u|^{2(1+\sigma)} dx \Big)^{\frac{1}{1+\sigma}} + \Big(\fint_{B_r} |P|^{2(1+\sigma)} dx \Big)^{\frac{1}{1+\sigma}} \\ &\le c \fint_{B_{2r}} |\nabla u|^2 dx + c \fint_{B_{2r}} |P|^2 dx + c\Big(\fint_{B_{2r}} |\mathbf{F}|^{2(1+\sigma)} dx \Big)^{\frac{1}{1+\sigma}}. \end{aligned} \tag{3.3}$$

Proof. Let $\eta \in C^\infty_0(B_{2r})$ be a cut-off function such that $0 \le \eta \le 1$, $\eta \equiv 1$ on B_r and $|\nabla \eta| \le 2/r$. Taking $v = \eta^2(u - (u)_{2r})$ into (2.1) as a test function, we obtain

$$\int_{B_{2r}} \langle \mathbf{A}(x)\nabla u, \nabla\big(\eta^2(u - (u)_{2r})\big)\rangle \, dx = \int_{B_{2r}} \langle \mathbf{F}, \nabla\big(\eta^2(u - (u)_{2r})\big)\rangle \, dx. \tag{3.4}$$

In view of the ellipticity and boundedness (1.2), for $0 < \varepsilon_1 < 1$, using Young's inequality we have

$$\begin{aligned} &\nu \int_{B_{2r}} \eta^2 |\nabla u|^2 dx \\ &\le \int_{B_{2r}} \langle \mathbf{A}(x)\nabla u, \nabla\big(\eta^2(u - (u)_{2r})\big)\rangle \, dx + 2\Lambda \int_{B_{2r}} |\eta \nabla u| \cdot |\nabla \eta (u - (u)_{2r})| dx \\ &\le \int_{B_{2r}} \langle \mathbf{A}(x)\nabla u, \nabla\big(\eta^2(u - (u)_{2r})\big)\rangle \, dx + c\varepsilon_1 \int_{B_{2r}} \eta^2 |\nabla u|^2 dx \\ &\quad + c(\varepsilon_1) \int_{B_{2r}} |\nabla \eta|^2 |u - (u)_{2r}|^2 dx. \end{aligned}$$

It follows from the Sobolev-Poincáre inequality with $2_* = \frac{2n}{n+2}$ that

$$
\begin{aligned}
&\nu \int_{B_{2r}} \eta^2 |\nabla u|^2 dx \\
&\le \int_{B_{2r}} \langle \mathbf{A}(x)\nabla u, \nabla\big(\eta^2(u-(u)_{2r})\big)\rangle\, dx + c\varepsilon_1 \int_{B_{2r}} \eta^2 |\nabla u|^2 dx \\
&\quad + c(\varepsilon_1) r^{n-\frac{2n}{2_*}} \Big(\int_{B_{2r}} |\nabla u|^{2_*} dx\Big)^{2/2_*}.
\end{aligned}
\tag{3.5}
$$

Similarly, we can find the estimate to the right-hand side of formula (3.4):

$$
\begin{aligned}
&\int_{B_{2r}} \langle \mathbf{F}, \nabla\big(\eta^2(u-(u)_{2r})\big)\rangle\, dx \\
&\le \varepsilon_2 \int_{B_{2r}} \eta^2 |\nabla u|^2 dx \\
&\quad + c(\varepsilon_2,\varepsilon_3) \int_{B_{2r}} |\mathbf{F}|^2 dx + c(\varepsilon_3) r^{n-\frac{2n}{2_*}} \Big(\int_{B_{2r}} |\nabla u|^{2_*} dx\Big)^{2/2_*}
\end{aligned}
\tag{3.6}
$$

for $0 < \varepsilon_2$ and $\varepsilon_3 < 1$. Using (3.4)-(3.6) and choosing $c\varepsilon_1 + \varepsilon_2 = \nu/2$ yields

$$
\fint_{B_r} |\nabla u|^2 dx \le c\Big(\fint_{B_{2r}} |\nabla u|^{2_*} dx\Big)^{2/2_*} + c\fint_{B_{2r}} |\mathbf{F}|^2 dx.
$$

From Lemma 3.2 by taking $g(x) = \nabla u,\ h(x) = \mathbf{F}$, $p = 2$, $\theta = 0$ and $s = 2_* < 2$, it follows that there exists a σ satisfying $2 < 2(1+\sigma) < t\gamma_1$ such that

$$
\begin{aligned}
\Big(\fint_{B_r} |\nabla u|^{2(1+\sigma)} dx\Big)^{\frac{1}{1+\sigma}} &\le c\fint_{B_{2r}} |\nabla u|^2 dx + c\fint_{B_{2r}} |\mathbf{F}|^2 dx \\
&\le c\fint_{B_{2r}} |\nabla u|^2 dx + c\Big(\fint_{B_{2r}} |\mathbf{F}|^{2(1+\sigma)} dx\Big)^{\frac{1}{1+\sigma}}.
\end{aligned}
\tag{3.7}
$$

By (3.7) and (3.2) with $q = 2((1+\sigma)) > 2$, we obtain

$$
\begin{aligned}
&\Big(\fint_{B_r} |P|^{2(1+\sigma)} dx\Big)^{\frac{1}{2(1+\sigma)}} \\
&\le c\Big(\fint_{B_{2r}} |\nabla u|^{2(1+\sigma)} dx\Big)^{\frac{1}{2(1+\sigma)}} + c\Big(\fint_{B_{2r}} |\mathbf{F}|^{2(1+\sigma)} dx\Big)^{\frac{1}{2(1+\sigma)}} \\
&\le c\Big(\fint_{B_{2r}} |\nabla u|^2 dx\Big)^{\frac{1}{2}} + c\Big(\fint_{B_{2r}} |\mathbf{F}|^{2(1+\sigma)} dx\Big)^{\frac{1}{2(1+\sigma)}}.
\end{aligned}
\tag{3.8}
$$

Combining (3.7) and (3.8), we arrive at the desired estimate (3.3). □

The following higher integrability on the boundary version is a self-improving result due to the Reifenberg flatness condition of domain being an A-type domain as shown in (2.9).

Lemma 3.4. *Let $(u,P) \in W_0^{1,2}(\Omega_{2r})^n \times L^2(\Omega_{2r})$ be a weak solution pair to the generalized Stokes system* (1.1) *with* $\operatorname{div} u = 0$ *under condition* (1.2)*. Suppose that $|\mathbf{F}|^{p(x)} \in L^t(\Omega_{2r})$ with $p(x) > \gamma_1 > 2$ and $t > 1$, and the boundary of Ω satisfies local (δ, R_0)-Reifenberg flatness: there exist $R_0 > 0$ and $\delta_0 > 0$ such that for $0 < r \le R_0$ and $0 < \delta < \delta_0$ it holds*

$$
B_{2r}^+ \subset \Omega_{2r} \subset B_{2r} \cap \{x_n > -4\delta r\}.
$$

Then there exists a small positive constant $\sigma_0 > 0$ with $\sigma_0 < \frac{t\gamma_1}{2} - 1$ such that for any $0 < \sigma \le \sigma_0$, we have $\nabla u \in L^{2(1+\sigma)}(\Omega_r)$ with the estimate

$$
\begin{aligned}
&\Big(\fint_{\Omega_r} |\nabla u|^{2(1+\sigma)} dx\Big)^{\frac{1}{1+\sigma}} + \Big(\fint_{\Omega_r} |P|^{2(1+\sigma)} dx\Big)^{\frac{1}{1+\sigma}} \\
&\le c\fint_{\Omega_{2r}} |\nabla u|^2 dx + c\fint_{\Omega_{2r}} |P|^2 dx + c\Big(\fint_{\Omega_{2r}} |\mathbf{F}|^{2(1+\sigma)} dx\Big)^{\frac{1}{1+\sigma}},
\end{aligned}
\tag{3.9}
$$

where $c = c(n, \gamma_1, \gamma_2, \nu, \Lambda, \Omega) > 0$.

Proof. Without loss of generality, for any fixed boundary point $y \in \partial\Omega$ we set $\Omega_{2r} = \Omega_{2r}(y)$. By a zero extension of u in $B_{2r}(y)\backslash\Omega_{2r}$, we can take $v = \eta^2(u-(u)_{2r})$ as a test function in the neighbourhood of boundary point. Using the arguments analogous to the proof of Lemma 3.3 due to the measure density property of Ω (cf. (2.10)), we can obtain the estimate (3.9) immediately. □

Here, let us recall a basic property that the generalized Stokes problem (1.1) is invariant under scaling transformation and normalization, see [12, Lemma 2.6].

Lemma 3.5. *For fixed $K > 1$ and $0 < \rho < 1$, let*

$$
\tilde{\mathbf{A}}(x) = \mathbf{A}(\rho x), \quad \tilde{u}(x) = \frac{u(\rho x)}{K\rho}, \quad \tilde{P}(x) = \frac{P(\rho x)}{K}, \quad \tilde{\mathbf{F}}(x) = \frac{\mathbf{F}(\rho x)}{K}
$$

and let $\tilde{\Omega} = \{\frac{x}{\rho} : x \in \Omega\}$. Then the following three statements are true.

(i) *If (u, P) is a weak solution pair to system (1.1), then $(\tilde{u}, \tilde{P})$ is a weak solution pair to*

$$
\begin{gathered}
\operatorname{div}(\tilde{\mathbf{A}}(x)\nabla\tilde{u}) - \nabla\tilde{P} = \operatorname{div}\tilde{\mathbf{F}}, \quad \text{in } \tilde{\Omega} \\
\operatorname{div}\tilde{u} = 0, \quad \text{in } \tilde{\Omega}, \\
\tilde{u} = 0, \quad \text{on } \partial\tilde{\Omega}.
\end{gathered}
$$

(ii) *If $\mathbf{A}$ satisfies condition (1.2), then so dose $\tilde{\mathbf{A}}$ with the same constants ν and Λ.*

(iii) *If $(\mathbf{A}, \Omega)$ is (δ, R_0)-vanishing, then $(\tilde{\mathbf{A}}, \tilde{\Omega})$ is $\big(\delta, \frac{R_0}{\rho}\big)$-vanishing.*

Let us consider the comparison estimates at the interior point and boundary point. For simplicity, we fix $x_i, z_i \in \Omega$ and let $r_i < \frac{R}{250}$ for each i. Set

$$
R \le \min\Big\{\frac{R_0}{2}, \frac{R_0}{c^*}, 1\Big\}, \tag{3.10}
$$

where $c^* = c^*(n, \gamma_1, \gamma_2, \nu, \Lambda, \omega(\cdot), |\Omega|)) \ge |\Omega| + 1$. Let

$$
\begin{gathered}
B_i^0 = B_{r_i}(x_i), \quad B_i^j = B_{5jr_i}(x_i), \quad j = 1, \dots, 6; \\
\Omega_i^0 = \Omega_{r_i}(z_i), \quad \Omega_i^j = \Omega_{25jr_i}(0), \quad j = 1, \dots, 6.
\end{gathered}
$$

For a boundary point $y_i \in B_{30r_i}(x_i) \cap \partial\Omega$, there exists a new coordinate system in $z_i = (z_i^1, z_i^2, \dots, z_i^n)$-variables such that

$$
\begin{gathered}
z_i = x_i; \quad y_i + 150\delta r_i(0, \dots, 0, 1) \quad \text{is the origin}, \\
B_{150r_i}^+(0) \subset \Omega_{150r_i}(0) \subset B_{150r_i}(0) \cap \{z_i^n > -300\delta r_i\}.
\end{gathered}
\tag{3.11}
$$

Choose $0 < \delta < 1/300$ such that $\Omega_{r_i}(z_i) \subset \Omega_{40r_i}(0)$ in the new z-coordinate system. Then we have

$$B_i^{j+} = B_{25jr}^+(0) \subset \Omega_i^j \subset B_{25jr}(0) \cap \{y_n > -300\delta r\}, \tag{3.12}$$

$$\Omega_i^j \subset \Omega_{150r_i}(0) \subset \Omega_{190r_i}(z_i). \tag{3.13}$$

By a scaling transformation to (3.12), without loss of generality, we set

$$B_6^+ \subset \Omega_6 \subset B_6 \cap \{x_n > -12\delta\}.$$

We now consider a series of localizing problems in the neighbourhood of boundary point as follows:

$$\begin{gathered} \operatorname{div}(\mathbf{A}(x)\nabla u) - \nabla P = \operatorname{div}\mathbf{F}, \quad \text{in } \Omega_6, \\ \operatorname{div} u = 0, \quad \text{in } \Omega_6, \\ u = 0, \quad \text{on } \partial_\omega \Omega_6; \end{gathered} \tag{3.14}$$

$$\begin{gathered} \operatorname{div}(\mathbf{A}(x)\nabla v) - \nabla P_v = 0, \quad \text{in } \Omega_5, \\ \operatorname{div} v = 0, \quad \text{in } \Omega_5, \\ v = u, \quad \text{on } \partial \Omega_5; \end{gathered} \tag{3.15}$$

$$\begin{gathered} \operatorname{div}(\mathbf{A}_{B_4^+}\nabla w) - \nabla P_w = 0, \quad \text{in } \Omega_4, \\ \operatorname{div} w = 0, \quad \text{in } \Omega_4, \\ w = v, \quad \text{on } \partial \Omega_4; \end{gathered} \tag{3.16}$$

and

$$\begin{gathered} \operatorname{div}(\mathbf{A}_{B_4^+}\nabla h) - \nabla P_h = 0, \quad \text{in } B_4^+, \\ \operatorname{div} h = 0, \quad \text{in } B_4^+, \\ h = w, \quad \text{on } T_4. \end{gathered} \tag{3.17}$$

Let us recall an approximating estimate in accordance with Byun-So's work, see [12, Lemma 3.6].

Lemma 3.6. *For any $0 < \epsilon < 1$, if (w, P_w) is a weak solution pair of problem* (3.16) *and satisfies*

$$\fint_{\Omega_4} \big(|\nabla w|^2 + |P_w|^2\big) dx \le c.$$

Then, there exists a weak solution pair (h, P_h) of problem (3.17) *with*

$$\fint_{B_4^+} \big(|\nabla h|^2 + |P_h|^2\big)\, dx \le c,$$

$$\fint_{B_4^+} |w - h|^2 dx \le \epsilon^2.$$

A local Lipschitz regularity in the neighbourhood of any boundary point for the Dirichlet problem (3.17) is well-established, see [12, Lemma 3.5].

Lemma 3.7. *Let (h, P_h) be the weak solution pair of problem* (3.17)*. Then we have*

$$\|\nabla \bar{h}\|_{L^\infty(\Omega_3)^{n^2}} = \|\nabla h\|_{L^\infty(B_3^+)^{n^2}} \le c\|\nabla h\|_{L^2(B_4^+)^{n^2}},$$

$$\|P_{\bar{h}}\|_{L^\infty(\Omega_3)} = \|P_h\|_{L^\infty(B_3^+)} \le c\Big(\|\nabla h\|_{L^2(B_4^+)^{n^2}} + \|P_h\|_{L^2(B_4^+)}\Big),$$

where $\bar{h}$ is the zero extension of h from B_3^+ to Ω_3, and $P_{\bar{h}}$ is an associated pressure of $\bar{h}$ by extending it from B_3^+ to Ω_3.

Lemma 3.8. *Let (u,P) be a weak solution pair of problem (3.14). If for any $0<\epsilon<1$ there exists a constant $\delta=\delta(\epsilon,\gamma_1,\gamma_2)$ such that*
$$\fint_{\Omega_5}\big(|\nabla u|^2+|P|^2\big)dx\le 1,\quad \fint_{\Omega_5}|\mathbf{F}|^2dx\le\delta^{\gamma_1/\gamma_2},\quad \fint_{\Omega_6}|\mathbf{A}-\mathbf{A}_{\Omega_6}|^2\,dx\le\delta^2,\tag{3.18}$$
then there exists a weak solution pair (h,P_h) of problem (3.17) such that
$$\fint_{B_4^+}\big(|\nabla h|^2+|P_h|^2\big)dx\le c,$$
$$\fint_{\Omega_3}\big(|\nabla u-\nabla\bar h|^2+|P-P_{\bar h}|^2\big)dx\le\epsilon^2,$$
where $\bar h$ and $P_{\bar h}$ are the same as given in Lemma 3.7.

Proof. Let (v,P_v) and (w,P_w) be weak solution pairs to problems (3.15) and (3.16), respectively. It follows Lemma 3.1 and (3.18) that
$$\fint_{\Omega_5}\big(|\nabla u-\nabla v|^2+|P-P_v|^2\big)dx\le c\fint_{\Omega_5}|\mathbf{F}|^2dx\le c\delta^{\gamma_1/\gamma_2}.\tag{3.19}$$
Combining (3.16) and (3.15) leads to
$$\begin{gathered}\operatorname{div}(\mathbf{A}_{B_4^+}\nabla(v-w))-\nabla(P_v-P_w)=-\operatorname{div}(\mathbf{A}(x)-\mathbf{A}_{B_4^+})\nabla v),\quad\text{in }\Omega_4,\\ \operatorname{div}v-w=0,\quad\text{in }\Omega_4,\\ v-w=0,\quad\text{on }\partial\Omega_4.\end{gathered}$$

In view of higher integrability on the boundary and normalization conditions (3.18), similar to the proof of [12, Lemma 3.7], using (3.1) we obtain
$$\fint_{\Omega_4}\Big(|\nabla v-\nabla w|^2+|P_v-P_w|^2\Big)dx\le c\big(\delta^2+\delta^3\big)^{\frac{r_1-2}{r_1}},\tag{3.20}$$
where $r_1>2$ is the same as given in Lemma 3.4. Combining (3.19) and (3.20) leads to
$$\fint_{\Omega_4}\Big(|\nabla u-\nabla w|^2+|P-P_w|^2\Big)dx\le c\big(\delta^{\gamma_1/\gamma_2}+\delta^{2-\frac{4}{r_1}}+\delta^{3-\frac{6}{r_1}}\big),\tag{3.21}$$
from which and (3.18) we obtain
$$\fint_{\Omega_4}\Big(|\nabla w|^2+|P_w|^2\Big)dx\le c.$$
It follows Lemma 3.6 that there exists a weak solution pair (h,P_h) of problem (3.17) with
$$\fint_{B_4^+}\Big(|\nabla h|^2+|P_h|^2\Big)dx\le c,\quad \fint_{B_4^+}|w-h|^2dx\le\epsilon_*^2,$$
where ϵ_*^2 is to be determined later.

Notice that $(\bar h,P_{\bar h})$ is a weak solution pair of
$$\begin{gathered}\operatorname{div}(\mathbf{A}_{B_4^+}\nabla\bar h)-\nabla P_{\bar h}=-\frac{\partial}{\partial x_n}\Big(\bar a_{nn}^{\alpha\beta}\frac{\partial h^\alpha}{\partial x_n}(x',0)\chi_{\mathbb{R}^n_-}(x)\Big),\quad\text{in }\Omega_4,\\ \operatorname{div}\bar h=0,\quad\text{in }\Omega_4,\\ \bar h=0,\quad\text{on }\partial_\omega\Omega_4,\end{gathered}$$

where $\mathbf{A}_{B_4^+} = \bar{a}_{ij}^{\alpha\beta}$, $x' = (x_1, x_2, \dots, x_{n-1})$ and χ is a standard characteristic function. Processing in a similar manner to the proof of [12, Lemma 3.7], we deduce that

$$\fint_{\Omega_2} |\nabla w - \nabla \bar{h}|^2 dx \le \epsilon_*^2 + c\delta + c\delta^{\frac{2}{n}}, \tag{3.22}$$

and

$$\begin{aligned}
&\fint_{\Omega_2} |P_w - P_{\bar{h}}|^2 \, dx \\
&\le c\fint_{\Omega_2} |\nabla w - \nabla \bar{h}|^2 \, dx + c\fint_{\Omega_2} \big|\bar{a}_{nn}^{\alpha\beta} \frac{\partial h^\alpha}{\partial x_n}(x', 0)\chi_{\mathbb{R}^n_-}(x)\big|^2 \, dx \\
&\le c\big(\epsilon_*^2 + \delta + \delta^{\frac{2}{n}}\big).
\end{aligned} \tag{3.23}$$

Taking $\epsilon_*, \delta > 0$ small enough and making use of (3.21)-(3.23) such that

$$c\Big(\delta^{\frac{\gamma_1}{\gamma_2}} + \delta^{2-\frac{4}{r_1}} + \delta^{3-\frac{6}{r_1}} + \epsilon_*^2 + \delta + \delta^{\frac{2}{n}}\Big) < \epsilon^2,$$

consequently, by (3.22) and (3.23) we arrive at the desired result. $\square$

For the interior case, one can process in an analogous but simple way as for the boundary case. Similar to Lemmas 3.7 and 3.8, we replace Ω_j by B_{j-1} with $B_6 \Subset \Omega$. The first one is a local Lipschitz regularity of ∇w to the Dirichlet problem for a limiting system of local constant coefficients, and the second one is a comparison between limiting system and the local version of system (1.1) under the normalization. For details, see [12, Lemmas 3.1 and 3.2].

Lemma 3.9. *Suppose that (w, P_w) is a weak solution pair to*

$$\begin{gathered}
\operatorname{div}(\mathbf{A}_{B_3} \nabla w) - \nabla P_w = 0, \quad \text{in } B_3, \\
\operatorname{div} w = 0, \quad \text{in } B_3 \\
w = v, \quad \text{on } \partial B_3,
\end{gathered} \tag{3.24}$$

and u is a local weak solution of system (1.1) with

$$\fint_{B_4} \big(|\nabla u|^2 + |P|^2\big) dx \le 1 \quad \text{and} \fint_{B_4} |\mathbf{F}|^2 dx \le \delta^{\gamma_1/\gamma_2}.$$

Then

$$\|\nabla w\|_{L^\infty(B_2)^{n^2}} + \|P_w\|_{L^\infty(B_2)} \le c_2,$$

where $c_2 > 1$.

Lemma 3.10. *Let u be a weak solution of (1.1). If for any $0 < \epsilon < 1$, there exists a constant $\delta = \delta(\epsilon, \gamma_1, \gamma_2)$ with*

$$\fint_{B_4} |\mathbf{F}|^2 dx \le \delta^{\gamma_1/\gamma_2} \quad \text{and} \quad \fint_{B_4} |\mathbf{A}(x) - \mathbf{A}_{B_4}|^2 \, dx \le \delta^2,$$

then there exists a weak solution pair (w, P_w) to (3.24) such that

$$\fint_{B_3} \big(|\nabla u - \nabla w|^2 + |P - P_w|^2\big) \, dx \le \epsilon^2.$$

Proof. Let (v, P_v) be a weak solution pair to (3.15) in B_4. From (3.15) and (3.14) it follows that

$$\begin{gathered}
\operatorname{div}(\mathbf{A}(x)\nabla(u - v)) - \nabla(P - P_v) = \operatorname{div} \mathbf{F}, \quad \text{in } B_4, \\
\operatorname{div}(u - v) = 0, \quad \text{in } B_4
\end{gathered}$$

$$u - v = 0, \quad \text{on } \partial B_4.$$

By Lemma 3.1 and inequality (1.2), and Hölder's inequality it follows that

$$\fint_{B_4} \big(|\nabla(u-v)|^2 + |P - P_v|^2\big) dx \le \fint_{B_4} |\mathbf{F}|^2 \, dx \le c\delta^{\gamma_1/\gamma_2}. \tag{3.25}$$

Similarly, let (w, P_w) be a weak solution pair to (3.24). Regarding (3.15) in B_4, we have

$$\begin{gathered} \operatorname{div}(\mathbf{A}_{B_3}\nabla(v-w)) - \nabla(P_v - P_w) = -\operatorname{div}\big((\mathbf{A}(x) - \mathbf{A}_{B_3})\nabla v\big), \quad \text{in } B_3, \\ \operatorname{div}(v-w) = 0, \quad \text{in } B_3, \\ v - w = 0, \quad \text{on } \partial B_3. \end{gathered}$$

Processing in a way analogous to the proof of [12, Lemma 3.3], we obtain

$$\fint_{B_3} \big(|\nabla(v-w)|^2 + |P_v - P_w|^2\big) \, dx \le c\delta^{2-\frac{4}{r_2}}. \tag{3.26}$$

Combining (3.25) and (3.26) gives

$$\fint_{B_3} \big(|\nabla u - \nabla w|^2 + |P - P_w|^2\big) \, dx \le c\big(\delta^{\gamma_1/\gamma_2} + \delta^{2-\frac{4}{r_2}}\big).$$

Taking $\delta > 0$ small enough such that $c(\delta^{\gamma_1/\gamma_2} + \delta^{2-\frac{4}{r_2}}) = \epsilon^2$, hence we obtain the desired result. $\square$

Let us recall the embedding relation with respect to the Lorentz spaces, see [27, Proposition 3.9].

Proposition 3.11. *Let U be a bounded measurable subset of $\mathbb{R}^n$. Then the following three statements are true.*

(i) *If $0 < q_1,\ q_2 \le \infty$, and $1 \le t_1 < t_2 < \infty$, then $L(t_2, q_2)(U) \subset L(t_1, q_1)(U)$ with the estimate*

$$\|g\|_{L(t_1,q_1)(U)} \le c(t_1, t_2, , q_1, q_2, U)\|g\|_{L(t_2,q_2)(U)}.$$

(ii) *If $1 \le t < \infty$ and $0 < q_1 < q_2 \le \infty$, then $L(t, q_1)(U) \subset L(t, q_2)(U) \subset L(t, \infty)(U)$ and*

$$\|g\|_{L(t,q_2)(U)} \le c(t, q_1, q_2)\|g\|_{L(t,q_1)(U)}.$$

(iii) *For $0 < \alpha < \infty$, if $|g|^\alpha \in L(t, q)(U)$, then $g \in L(\alpha t, \alpha q)(U)$ with the estimate*

$$\||g|^\alpha\|_{L(t,q)(U)} = \|g\|^\alpha_{L(\alpha t,\alpha q)(U)}. \tag{3.27}$$

The following two lemmas are actually variant versions of classic Hardy's inequality and reverse Hölder's inequality, respectively, see [5, Lemmas 3.4 and 3.5].

Lemma 3.12. *Let $f : [0, +\infty) \to [0, +\infty)$ be a measurable function such that*

$$\int_0^\infty f(\lambda) d\lambda < \infty. \tag{3.28}$$

Then for any $\alpha \ge 1$ and $r > 0$ it holds

$$\int_0^\infty \lambda^r \Big(\int_\lambda^\infty f(\mu) d\mu\Big)^\alpha \frac{d\lambda}{\lambda} \le \big(\frac{\alpha}{r}\big)^\alpha \int_0^\infty \lambda^r \Big(\lambda f(\lambda)\Big)^\alpha \frac{d\lambda}{\lambda}.$$

Lemma 3.13. *Let $h:[0,+\infty)\to[0,+\infty)$ be a non-increasing measurable function. Suppose that $\alpha_1\le\alpha_2$ and $r>0$. If $\alpha_2<\infty$, then*

$$\Big(\int_\lambda^\infty \big(\mu^r h(\mu)\big)^{\alpha_2}\frac{d\mu}{\mu}\Big)^{1/\alpha_2}\le \varepsilon\lambda^r h(\lambda)+\frac{c}{\varepsilon^{\frac{\alpha_2}{\alpha_1}-1}}\Big(\int_\lambda^\infty \big(\mu^r h(\mu)\big)^{\alpha_1}\frac{d\mu}{\mu}\Big)^{1/\alpha_1}$$

for $\varepsilon\in(0,1]$ and $\lambda\ge0$. If $\alpha_2=\infty$, then

$$\sup_{\mu>\lambda}\left(\mu^r h(\mu)\right)\le c\lambda^r h(\lambda)+c\Big(\int_\lambda^\infty \left(\mu^r h(\mu)\right)^{\alpha_1}\frac{d\mu}{\mu}\Big)^{1/\alpha_1},$$

where c depends on α_1, α_2 and r.

The following lemma is regarding an iteration argument, see [23] or [30, Lemma 4.1].

Lemma 3.14. *Let $\varphi:[r_1,2r_1]\to[0,\infty)$ be a function such that*

$$\varphi(\rho_1)\le\frac12\varphi(\rho_2)+B_0(\rho_2-\rho_1)^{-\beta}+L$$

for $r_1<\rho_1<\rho_2<2r_1$, where B_0, $L\ge0$ and $\beta>0$. Then we have

$$\varphi(r_1)\le c(\beta)B_0r_1^{-\beta}+cL.$$

4. Proof of Theorem 2.4

With aid of the lemmas presented in the preceding section, now we are in a position to prove Theorem 2.4 by means of the large-M-inequality principle [2] and a geometric argument [10]. To present our discussion in a straightforward and lucid manner, we separate our proof into six steps.

Proof. We only treat the boundary case. For the interior case, one can process in a similar but much simpler way. For the boundary case, a proper translation and rotation of the original coordinates does not change the corresponding features. Without loss of generality, we may assume that $R_0\le1$. For any fixed $x_0\in\Omega$, we set

$$p^-=\inf_{\Omega_{2R}(x_0)}p(x),\quad p^+=\sup_{\Omega_{2R}(x_0)}p(x),$$
$$p_i^-=\inf_{\Omega_i^5(x_0)}p(x),\quad p_i^+=\sup_{\Omega_i^5(x_0)}p(x).$$

Step 1. In this step, we present a modified Vitali's covering. Let u be the weak solution of system (1.1). For $\Omega_R=\Omega_R(x_0)$, we define

$$\lambda_0:=\fint_{\Omega_{2R}}(|\nabla u|+|P|)^{2p(x)/p^-}\,dx+\frac1\delta\Big(\fint_{\Omega_{2R}}\Big(|\mathbf{F}|^{2p(x)/p^-}+1\Big)^\eta\,dx\Big)^{1/\eta},\tag{4.1}$$

where $\delta>0$ and $\eta>1$ will be specified later. We now introduce the super-level set

$$E\left(\lambda,\Omega_R\right):=\big\{x\in\Omega_R:(|\nabla u|+|P|)^{2p(x)/p^-}>\lambda\big\}$$

for some $\lambda>M\lambda_0\ge1$ with $M=(\frac{8000}{7})^n$.

For $x_i\in E\left(\lambda,\Omega_R\right)$ and radii $0<r\le R$, let

$$CZ(\Omega_r(x_i)):=\fint_{\Omega_r(x_i)}(|\nabla u|+|P|)^{2p(x)/p^-}\,dx+\frac1\delta\Big(\fint_{\Omega_r(x_i)}|\mathbf{F}|^{\frac{2p(x)}{p^-}\eta}\,dx\Big)^{1/\eta}.\tag{4.2}$$

Note that $\frac{R}{250} \le r \le R$. Expanding the domain of integration gives

$$
\begin{aligned}
&CZ(\Omega_r(x_i)) \\
&\le \frac{|\Omega_{2R}|}{|\Omega_r(x_i)|} \fint_{\Omega_{2R}} (|\nabla u| + |P|)^{2p(x)/p^-}\, dx + \Big(\frac{|\Omega_{2R}|}{|\Omega_r(x_i)|}\Big)^{1/\eta} \frac{1}{\delta}\Big(\fint_{\Omega_{2R}} |\mathbf{F}|^{\frac{2p(x)}{p^-}\eta}\, dx\Big)^{1/\eta} \\
&\le \frac{|\Omega_{2R}|}{|\Omega_r(x_i)|}\Big[\fint_{\Omega_{2R}} (|\nabla u| + |P|)^{2p(x)/p^-}\, dx + \frac{1}{\delta}\Big(\fint_{\Omega_{2R}} |\mathbf{F}|^{\frac{2p(x)}{p^-}\eta} dx\Big)^{1/\eta}\Big] \\
&\le \frac{|B_{2R}|}{|B_r(x_i)|}\frac{|B_r(x_i)|}{|\Omega_r(x_i)|}\lambda_0 \\
&\le \big(\frac{2R}{r}\big)^n \big(\frac{16}{7}\big)^n \lambda_0 \\
&\le \big(\frac{8000}{7}\big)^n \lambda_0 < \lambda.
\end{aligned}
$$

This indicates that for $\frac{R}{250} \le r \le R$, we have $CZ(\Omega_r(x_i)) < \lambda$.

On the other hand, by Lebesgue's differentiation theorem we find that for $0 < r << 1$, it holds $CZ(\Omega_r(x_i)) > \lambda$. Thus, in view of absolute continuity of the integral, we can pick a maximal radius $r_i = r_{x_i}$ such that

$$
\begin{aligned}
&CZ(\Omega_{r_i}(x_i)) \\
&= \fint_{\Omega_{r_i}(x_i)} (|\nabla u| + |P|)^{2p(x)/p^-}\, dx + \frac{1}{\delta}\Big(\fint_{\Omega_{r_i}(x_i)} |\mathbf{F}|^{\frac{2p(x)}{p^-}\eta}\, dx\Big)^{1/\eta} = \lambda
\end{aligned}
\tag{4.3}
$$

for any point $x_i \in E(\lambda, \Omega_R)$. Moreover, for $r \in (r_i, R]$ one has

$$
CZ(\Omega_r(x_i)) < \lambda. \tag{4.4}
$$

From (4.3), we obtain the following alternatives:

$$
\frac{\lambda}{2} \le \fint_{\Omega_{r_i}(x_i)} (|\nabla u| + |P|)^{2p(x)/p^-}\, dx \quad \text{or} \quad \Big(\frac{\delta\lambda}{2}\Big)^{\eta} \le \fint_{\Omega_{r_i}(x_i)} |\mathbf{F}|^{\frac{2p(x)}{p^-}\eta}\, dx. \tag{4.5}
$$

Suppose that the first case of (4.5) is valid, and split the integral as

$$
\begin{aligned}
&\fint_{\Omega_{r_i}(x_i)} (|\nabla u| + |P|)^{2p(x)/p^-}\, dx \\
&\le \frac{|\Omega_{r_i}(x_i)\backslash E(\frac{\lambda}{4}, \Omega_{2R})|}{|\Omega_{r_i}(x_i)|} \fint_{\Omega_{r_i}(x_i)\backslash E(\frac{\lambda}{4},\Omega_{2R})} (|\nabla u| + |P|)^{2p(x)/p^-}\, dx \\
&\quad + \frac{1}{|\Omega_{r_i}(x_i)|}\int_{\Omega_{r_i}(x_i)\cap E(\frac{\lambda}{4},\Omega_{2R})} (|\nabla u| + |P|)^{2p(x)/p^-}\, dx \\
&\le \frac{\lambda}{4} + c\frac{|\Omega_{r_i}(x_i)\cap E(\frac{\lambda}{4}, \Omega_{2R})|}{|\Omega_{r_i}(x_i)\cap E(\frac{\lambda}{4}, \Omega_{2R})|^{\frac{1}{1+\sigma_1}}} \frac{1}{|\Omega_{r_i}(x_i)|} \\
&\quad \times \Big(\int_{\Omega_{r_i}(x_i)} (|\nabla u| + |P|)^{\frac{2p(x)}{p^-}(1+\sigma_1)}\, dx\Big)^{\frac{1}{1+\sigma_1}} \\
&\le \frac{\lambda}{4} + c\Big(\frac{|\Omega_{r_i}(x_i)\cap E(\frac{\lambda}{4}, \Omega_{2R})|}{|\Omega_{r_i}(x_i)|}\Big)^{1-\frac{1}{1+\sigma_1}} \Big(\fint_{\Omega_{r_i}(x_i)} (|\nabla u| + |P|)^{\frac{2p(x)}{p^-}(1+\sigma_1)}\, dx\Big)^{\frac{1}{1+\sigma_1}}.
\end{aligned}
$$

Take

$$
0 < \sigma_1 \le \frac{\gamma_1(1+\sigma)}{\gamma_1 + \omega(2R)} - 1,
$$

where σ is the same as in Lemma 3.4. Then

$$\frac{p(x)}{p^-}(1+\sigma_1) = \Big(1+\frac{p(x)-p^-}{p^-}\Big)(1+\sigma_1) \le \Big(1+\frac{\omega(2R)}{p^-}\Big)(1+\sigma_1) \le (1+\sigma).$$

Let $\eta = 1+\sigma_1$. In view of (4.4), it follows by Hölder's inequality (given in Lemma 3.4) that

$$\begin{aligned}
&\Big(\fint_{\Omega_{r_i}(x_i)} (|\nabla u|+|P|)^{\frac{2p(x)}{p^-}(1+\sigma_1)}\,dx\Big)^{\frac{1}{1+\sigma_1}}\\
&\le c\Big[\fint_{\Omega_{2r_i}(x_i)} |\nabla u|^{2p(x)/p^-}\,dx + \fint_{\Omega_{2r_i}(x_i)} |P|^{2p(x)/p^-}\,dx\\
&\quad + \Big(\fint_{\Omega_{2r_i}(x_i)} |\mathbf{F}|^{\frac{2p(x)}{p^-}(1+\sigma_1)}\,dx\Big)^{\frac{1}{1+\sigma_1}} + 1\Big]\\
&\le c\lambda.
\end{aligned}$$

Hence, using (4.5) in combination with

$$\frac{\lambda}{4} \le c\Big(\frac{|\Omega_{r_i}(x_i)\cap E(\frac{\lambda}{4},\Omega_{2R})|}{|\Omega_{r_i}(x_i)|}\Big)^{1-\frac{1}{1+\sigma_1}}\lambda$$

we have

$$|\Omega_{r_i}(x_i)| \le c\big|\Omega_{r_i}(x_i)\cap E\big(\frac{\lambda}{4},\Omega_{2R}\big)\big|, \tag{4.6}$$

where the constant c depends on $n,\gamma_2,\gamma_2,\nu,\Lambda$ and t.

For the second estimate in (4.5), it follows from Fubini's theorem that

$$\begin{aligned}
\Big(\frac{\lambda\delta}{2}\Big)^{\eta} &\le \fint_{\Omega_{r_i}(x_i)} |\mathbf{F}|^{\frac{2p(x)}{p^-}\eta}dx\\
&= \frac{\eta}{|\Omega_{r_i}(x_i)|}\int_0^\infty \mu^\eta\big|\{x\in\Omega_{r_i}(x_i) : |\mathbf{F}|^{2p(x)/p^-} > \mu\}\big|\frac{d\mu}{\mu}\\
&= \frac{\eta}{|\Omega_{r_i}(x_i)|}\int_0^{\zeta\lambda} \mu^\eta\big|\{x\in\Omega_{r_i}(x_i) : |\mathbf{F}|^{2p(x)/p^-} > \mu\}\big|\frac{d\mu}{\mu}\\
&\quad + \frac{\eta}{|\Omega_{r_i}(x_i)|}\int_{\zeta\lambda}^\infty \mu^\eta\big|\{x\in\Omega_{r_i}(x_i) : |\mathbf{F}|^{2p(x)/p^-} > \mu\}\big|\frac{d\mu}{\mu}\\
&\le (\zeta\lambda)^\eta + \frac{\eta}{|\Omega_{r_i}(x_i)|}\int_{\zeta\lambda}^\infty \mu^\eta\big|\{x\in\Omega_{r_i}(x_i) : |\mathbf{F}|^{2p(x)/p^-} > \mu\}\big|\frac{d\mu}{\mu}.
\end{aligned}$$

Let $\delta = 4\zeta$. We derive that

$$(\zeta\lambda)^\eta \le \frac{\eta}{|\Omega_{r_i}(x_i)|}\int_{\zeta\lambda}^\infty \mu^\eta\big|\{x\in\Omega_{r_i}(x_i) : |\mathbf{F}|^{2p(x)/p^-} > \mu\}\big|\frac{d\mu}{\mu}$$

and

$$|\Omega_{r_i}(x_i)| \le \frac{\eta}{(\zeta\lambda)^\eta}\int_{\zeta\lambda}^\infty \mu^\eta\big|\{x\in\Omega_{r_i}(x_i) : |\mathbf{F}|^{2p(x)/p^-} > \mu\}\big|\frac{d\mu}{\mu}. \tag{4.7}$$

Combining (4.6) and (4.7) we have

$$\begin{aligned}
|\Omega_{r_i}(x_i)| &\le c\big|\Omega_{r_i}(x_i)\cap E\big(\frac{\lambda}{4},\Omega_{2R}\big)\big|\\
&\quad + \frac{c\eta}{(\zeta\lambda)^\eta}\int_{\zeta\lambda}^\infty \mu^\eta\big|\{x\in\Omega_{r_i}(x_i) : |\mathbf{F}|^{2p(x)/p^-} > \mu\}\big|\frac{d\mu}{\mu}.
\end{aligned} \tag{4.8}$$

Note that Ω is (δ, R_0)-Reifenberg flat. It follows from (3.13) that

$$\begin{aligned}
&\fint_{\Omega_i^j} (|\nabla u| + |P|)^{2p(z)/p^-}\,dz + \frac{1}{\delta}\Big(\fint_{\Omega_i^j} \mathbf{F}|^{\frac{2p(z)}{p^-}\eta} dz\Big)^{1/\eta} \\
&\le \frac{|\Omega_{250r_i}(z_i)|}{|\Omega_{25r_i}(0)|}\Big[\fint_{\Omega_{250r_i}(z_i)} (|\nabla u| + |P|)^{2p(z)/p^-}\,dz + \Big(\fint_{\Omega_{250r_i}(z_i)} |\mathbf{F}|^{\frac{2p(z)}{p^-}\eta} dz\Big)^{1/\eta}\Big] \\
&\le \frac{|B_{250r_i}(z_i)|}{|B^+_{25r_i}(0)|}\Big[\fint_{\Omega_{250r_i}(z_i)} (|\nabla u| + |P|)^{2p(z)/p^-}\,dz + \frac{1}{\delta}\Big(\fint_{\Omega_{250r_i}(z_i)} |\mathbf{F}|^{\frac{2p(z)}{p^-}\eta} dz\Big)^{1/\eta}\Big] \\
&\le 2\cdot 10^n\Big[\fint_{\Omega_{250r_i}(z_i)} (|\nabla u| + |P|)^{2p(z)/p^-}\,dz + \frac{1}{\delta}\Big(\fint_{\Omega_{250r_i}(z_i)} |\mathbf{F}|^{\frac{2p(z)}{p^-}\eta} dz\Big)^{1/\eta}\Big].
\end{aligned}$$

Employing (4.4) and using change of variables of (3.11), we deduce that

$$\fint_{\Omega_i^j} (|\nabla u| + |P|)^{2p(z)/p^-}\,dz + \frac{1}{\delta}\Big(\fint_{\Omega_i^j} |\mathbf{F}|^{\frac{2p(z)}{p^-}\eta} dz\Big)^{1/\eta} \le 2\cdot 10^n\lambda. \tag{4.9}$$

Step 2. We consider various comparison estimates. Since $(\mathbf{A}, \Omega)$ is (δ, R_0)-vanishing for some $R_0 > 0$, we have

$$\fint_{\Omega_{150r_i}(0)} |\mathbf{A} - \mathbf{A}_{\Omega_{150r_i}(0)}|^2 dx \le \delta^2. \tag{4.10}$$

Taking (4.9) into account, we have

$$\begin{gathered}
\fint_{\Omega_i^5} (|\nabla u| + |P|)^{2p(x)/p^-}\,dx \le c\lambda, \\
\Big(\fint_{\Omega_i^5} |\mathbf{F}|^{\frac{2p(z)}{p^-}\eta}\,dx\Big)^{1/\eta} \le c\lambda\delta.
\end{gathered} \tag{4.11}$$

Let us first show that

$$\begin{gathered}
\fint_{\Omega_i^5} \big(|\nabla u|^2 + |P|^2\big)\,dx \le c_3\lambda^{p^-/p_i^+}, \\
\fint_{\Omega_i^5} |\mathbf{F}|^2\,dx \le c_3\lambda^{p^-/p_i^+}\delta^{\gamma_1/\gamma_2},
\end{gathered} \tag{4.12}$$

for a constant $c_3 \ge 1$. We claim that

$$\Big(\fint_{\Omega_i^5} \big(|\nabla u|^2 + |P|^2\big)dx\Big)^{p_i^+ - p_i^-} \le c, \tag{4.13}$$

where $c \ge 1$ is a universal constant.

Notice that $|\mathbf{F}|^{p(x)} \in L(t,q)(\Omega)$ implies $\int_\Omega |\mathbf{F}|^{p(x)}\,dx \le c$. From (3.1), we have

$$\int_\Omega |\mathbf{F}|^2 dx \le \int_\Omega \Big(|\mathbf{F}|^{p(x)} + 1\Big)\,dx \le c + |\Omega|,$$

and

$$\int_\Omega \big(|\nabla u|^2 + |P|^2\big)dx \le c(1 + |\Omega|). \tag{4.14}$$

Given $p_i^+ - p_i^- \le \omega(250r_i)$, it yields

$$\Big(\fint_{\Omega_i^5} \big(|\nabla u|^2 + |P|^2\big)dx\Big)^{p_i^+ - p_i^-} = \Big(\frac{1}{|\Omega_i^5|}\Big)^{p_i^+ - p_i^-}\Big(\int_{\Omega_i^5} \big(|\nabla u|^2 + |P|^2\big)dx\Big)^{p_i^+ - p_i^-}$$

$$
\begin{aligned}
&\le c\Big(\frac{1}{|B_{250r_i}|}\Big)^{p_i^+-p_i^-}\Big(\int_{\Omega_i^5}\big(|\nabla u|^2+|P|^2\big)dx\Big)^{p_i^+-p_i^-}\\
&\le c\Big(\frac{1}{250r_i}\Big)^{n\omega(250r_i)}\Big(\int_{\Omega_i^5}\big(|\nabla u|^2+|P|^2\big)dx\Big)^{p_i^+-p_i^-}\\
&\le c\Big(\int_{\Omega_i^5}\Big(|\nabla u|^2+|P|^2\Big)dx\Big)^{p_i^+-p_i^-}.
\end{aligned}
$$

On the other hand, using (3.10), (4.14) as well as $\frac{1}{250r_i}\ge\frac{1}{R}\ge\frac{c^*}{R_0}\ge|\Omega|+1$, we find

$$
\begin{aligned}
\Big(\int_{\Omega_i^5}\Big(|\nabla u|^2+|P|^2\Big)dx\Big)^{p_i^+-p_i^-}&\le\Big(\int_{\Omega}\Big(|\nabla u|^2+|P|^2\Big)dx\Big)^{p_i^+-p_i^-}\\
&\le c(|\Omega|+1)^{p_i^+-p_i^-}\\
&\le c\Big(\frac{1}{250r_i}\Big)^{\omega(250r_i)}\le c.
\end{aligned}
$$

So, we arrive at (4.13) due to the log-Hölder continuity of $p(x)$. Recalling $\gamma_1\le p_i^-$ and (4.13) with $\lambda>1$, we obtain

$$
\begin{aligned}
⨍_{\Omega_i^5}\Big(|\nabla u|^2+|P|^2\Big)dx&=\Big(⨍_{\Omega_i^5}\big(|\nabla u|^2+|P|^2\big)dx\Big)^{\frac{p_i^+-p_i^-}{p_i^+}}\Big(⨍_{\Omega_i^5}\big(|\nabla u|^2+|P|^2\big)dx\Big)^{\frac{p_i^-}{p_i^+}}\\
&\le c^{1/\gamma_1}\Big(⨍_{\Omega_i^5}\Big(|\nabla u|^2+|P|^2\Big)dx\Big)^{p^-/p_i^+}\\
&\le c\Big(⨍_{\Omega_i^5}\big(|\nabla u|+|P|\big)^{\frac{2p_i^-}{p^-}}dx\Big)^{p^-/p_i^+}\\
&\le c\Big(⨍_{\Omega_i^5}\big(|\nabla u|+|P|\big)^{2p(x)/p^-}dx+1\Big)^{p^-/p_i^+}\\
&\le c\lambda^{p^-/p_i^+}.
\end{aligned}
$$

Similarly, recalling $\delta\lambda_0\ge1$ and $\lambda\ge M\lambda_0$ we find

$$
\begin{aligned}
⨍_{\Omega_i^5}|\mathbf{F}|^2dx&\le c\Big(⨍_{\Omega_i^5}|\mathbf{F}|^{2p(x)/p^-}dx+1\Big)^{p^-/p_i^+}\\
&\le c\,(\delta\lambda+1)^{p^-/p_i^+}\\
&\le c\,(\delta\lambda+\delta\lambda_0)^{p^-/p_i^+}\\
&\le c\lambda^{p^-/p_i^+}\delta^{\gamma_1/\gamma_2}.
\end{aligned}
$$

We now define

$$
\begin{gathered}
\tilde{\mathbf{A}}_i(x)=\mathbf{A}(25r_ix),\quad \tilde{u}_i(x)=\frac{u(25r_ix)}{25r_i\sqrt{c_3\lambda^{p^-/p_i^+}}},\\
\tilde{P}_i(x)=\frac{P(25r_ix)}{\sqrt{c_3\lambda^{p^-/p_i^+}}},\quad \tilde{\mathbf{F}}_i(x)=\frac{\mathbf{F}(25r_ix)}{\sqrt{c_3\lambda^{p^-/p_i^+}}}.
\end{gathered}
$$

By Lemma 3.5, we obtain that $(\tilde{u}, \tilde{P})$ is a pair of weak solution of

$$\begin{gathered}\operatorname{div}(\tilde{\mathbf{A}}_i(x)\nabla \tilde{u}_i) - \nabla \tilde{P}_i = \operatorname{div} \tilde{\mathbf{F}}_i, \quad \text{in } \Omega_5,\\ \operatorname{div} \tilde{u}_i = 0, \quad \text{in } \Omega_5,\\ \tilde{u}_i = 0, \quad \text{on } \partial\Omega_5.\end{gathered}$$

Moreover, using (4.10) and (4.12) leads to

$$\begin{gathered}\fint_{\Omega_6} |\tilde{\mathbf{A}}_i - (\tilde{\mathbf{A}}_i)_{\Omega_6}|^2 dx \le \delta^2,\\ \fint_{\Omega_5} \Big(|\nabla \tilde{u}_i|^2 + |\tilde{P}_i|^2 \Big) dx \le 1,\\ \fint_{\Omega_5} |\tilde{\mathbf{F}}_i|^2 dx \le \delta^{\gamma_1/\gamma_2}.\end{gathered}$$

In accordance with Lemmas 3.7 and 3.8, we obtain

$$\begin{gathered}\fint_{\Omega_2} \Big(|\nabla \tilde{u}_i - \nabla \tilde{h}_i|^2 + |\tilde{P}_i - \tilde{P}_{\tilde{h}_i}|^2 \Big) dx \le \epsilon^2,\\ \|\nabla \tilde{h}_i\|_{L^\infty(\Omega_1)} \le c_1, \quad \|\tilde{P}_{\tilde{h}_i}\|_{L^\infty(\Omega_1)} \le c_1.\end{gathered}$$

Scaling back with

$$\tilde{h}_i(x) = \frac{\bar{h}_i(250 r_i x)}{25 r_i \sqrt{c_3 \lambda^{p^-/p_i^+}}},$$

where $\bar{h}_i$ is the weak solution of (3.17), replacing B_4^+ and T_4 by B_i^{4+} and T_i^4 respectively, and extending B_i^{4+} to Ω_i^4, we obtain

$$\fint_{\Omega_i^2} \big(|\nabla u - \nabla \bar{h}_i|^2 + |P - P_{\bar{h}_i}|^2 \big) dx \le c_3 \lambda^{p^-/p_i^+} \epsilon^2, \tag{4.15}$$

$$\|\nabla \bar{h}_i\|_{L^\infty(\Omega_i^1)} \le c_1 \lambda^{p^-/p_i^+}, \quad \|P_{\bar{h}_i}\|_{L^\infty(\Omega_i^1)} \le c_1 \lambda^{p^-/p_i^+} \tag{4.16}$$

for $c_1 > 1$.

Similar to (4.15) and (4.16), one can deduce the interior estimates

$$\fint_{B_i^2} \big(|\nabla u - \nabla w_i|^2 + |P - P_{w_i}|^2 \big) dx \le c_4 \lambda^{p^-/p_i^+} \epsilon^2, \tag{4.17}$$

$$\|\nabla w_i\|_{L^\infty(B_i^1)} \le c_2 \lambda^{p^-/p_i^+}, \quad \|P_{w_i}\|_{L^\infty(B_i^1)} \le c_2 \lambda^{p^-/p_i^+} \tag{4.18}$$

for constants $c_2, c_4 > 1$, where w_i is the weak solution of (3.24) when replacing B_3 by B_i^3.

Step 3. We want to make an estimate of the super-level $E(A\lambda, \Omega_R)$. For any fixed point $x \in \Omega$, we select a universal constant R satisfying $R \le \min\{\frac{R_0}{2}, \frac{R_0}{|\Omega|+1}, 1\}$, and there exists a constant $\delta = \delta(\epsilon) > 0$ as given in Lemmas 3.8 and 3.10. Let

$$c_0 = \max\big\{c_1, c_2\big\}. \tag{4.19}$$

For any $x \in E(A\lambda, \Omega_R)$, we consider the collection $\mathcal{B}_\lambda$ of all subsets of $\Omega_r(x)$. By the Vitali covering argument, we extract a countable sub-collection $\{\Omega_i^0\} \in \mathcal{B}_\lambda$ such that five times larger ball $\Omega_{5r_i}(x_i)$ covers almost all $E(A\lambda, \Omega_R)$ and the balls $\{\Omega_i^0\}_{i=1}^\infty$ are pointwise disjoints with $\Omega_i^0 = \Omega_{r_i}(x_i)$ for $i \in \mathbb{N}$. This leads to

$$\Omega_i^0 \cap \Omega_j^0 = \emptyset, \quad \text{whenever } i \ne j,$$

$$E(A\lambda,\Omega_R)\subset \cup_{i\in\mathbb{N}}\Omega_{5r_i}(x_i)\cup\mathcal{N}_\lambda,$$

with $|\mathcal{N}_\lambda|=0$.

Let $A=(8c_0)^{\gamma_2/\gamma_1}$. We separate the above resulting estimation into the two cases of interior and boundary, and deduce that

$$\begin{aligned}
|E(A\lambda,\Omega_R)| &= |E((8c_0)^{\gamma_2/\gamma_1}\lambda,\Omega_R)| \\
&= \big|\big\{x\in\Omega_R:(|\nabla u|+|P|)^{2p(x)/p^-}\geq (8c_0)^{\gamma_2/\gamma_1}\lambda\big\}\big| \\
&\leq \sum_{i\geq 1}\big|\big\{x\in\Omega_{5r_i}:(|\nabla u|+|P|)^2\geq 8c_0\lambda^{p^-/p(x)}\big\}\big| \\
&\leq \sum_{i\geq 1}\big|\big\{x\in\Omega_{5r_i}:|\nabla u|^2+|P|^2\geq 4c_0\lambda^{p^-/p(x)}\big\}\big| \\
&= \sum_{\text{interiorcase}}\big|\big\{x\in\Omega_{5r_i}:|\nabla u|^2+|P|^2\geq 4c_0\lambda^{p^-/p(x)}\big\}\big| \\
&\quad +\sum_{\text{boundarycase}}\big|\big\{x\in\Omega_{5r_i}:|\nabla u|^2+|P|^2\geq 4c_0\lambda^{p^-/p(x)}\big\}\big|.
\end{aligned}\tag{4.20}$$

For the interior setting, from (4.17)-(4.19) we see that

$$\begin{aligned}
&\big|\big\{x\in\Omega_{5r_i}:|\nabla u|^2+|P|^2\geq 4c_0\lambda^{p^-/p(x)}\big\}\big| \\
&= \big|\big\{x\in B_i^1:|\nabla u|^2+|P|^2\geq 4c_2\lambda^{p^-/p(x)}\big\}\big| \\
&\leq \big|\big\{x\in B_i^1:|\nabla u-\nabla w_i|^2+|P-P_{w_i}|^2\geq c_2\lambda^{p^-/p_i^+}\big\}\big| \\
&\quad +\big|\big\{x\in B_i^1:|\nabla w_i|^2+|P_{w_i}|^2\geq c_2\lambda^{p^-/p_i^+}\big\}\big| \\
&\leq \frac{1}{c_2\lambda^{p^-/p_i^+}}\int_{B_i^1}\big(|\nabla u-\nabla w_i|^2+|P-P_{w_i}|^2\big)\,dx \\
&\leq c\epsilon^2|B_i^1|\leq c\epsilon^2|B_i^0|,
\end{aligned}\tag{4.21}$$

where we applied the weak $(1,1)$-type estimate

$$\big|\big\{x\in E:f(x)>\lambda\big\}\big|\leq\frac{1}{\lambda}\int_E f(x)dx.$$

Similarly, for the boundary setting one can derive that

$$\begin{aligned}
&\big|\big\{x\in\Omega_{5r_i}:|\nabla u(x)|^2+|P(x)|^2\geq 4c_0\lambda^{p^-/p(x)}\big\}\big| \\
&= \big|\big\{z\in\Omega_{5r_i}:|\nabla u(z)|^2+|P(z)|^2\geq 4c_0\lambda^{\frac{p^-}{p(z)}}\big\}\big| \\
&\leq \big|\big\{z\in\Omega_i^1:|\nabla u(z)|^2+|P(z)|^2\geq 4c_1\lambda^{\frac{p^-}{p(z)}}\big\}\big| \\
&\leq \big|\big\{z\in\Omega_i^1:|\nabla u-\nabla\bar{h}_i|^2+|P-P_{\bar{h}_i}|^2\geq c_1\lambda^{p^-/p_i^+}\big\}\big| \\
&\quad +\big|\big\{z\in\Omega_i^1:|\nabla\bar{h}_i|^2+|P_{\bar{h}_i}|^2\geq c_1\lambda^{p^-/p_i^+}\big\}\big| \\
&\leq \frac{1}{c_1\lambda^{p^-/p_i^+}}\int_{\Omega_i^1}\big(|\nabla u-\nabla\bar{h}_i|^2+|P-P_{\bar{h}_i}|^2\big)dz \\
&\leq c\epsilon^2|\Omega_i^6| \\
&\leq c\epsilon^2\big(\frac{16}{7}\big)^n|\Omega_i^0|.
\end{aligned}\tag{4.22}$$

This is so because

$$
\begin{gathered}
c\epsilon^2|\Omega_i^6| \le c\epsilon^2|B_{150r_i}| = c\epsilon^2|B_{r_i}|,\\
c\epsilon^2|B_{r_i}| \le c\epsilon^2\frac{|B_i^0|}{|\Omega_i^0|}|\Omega_i^0| \le c\epsilon^2\big(\frac{1}{1-\delta}\big)^n|\Omega_i^0| \le c\epsilon^2\big(\frac{16}{7}\big)^n|\Omega_i^0|.
\end{gathered}
$$

By (4.20)-(4.22), we have

$$
|E(A\lambda,\Omega_R)| \le c\epsilon^2\sum_{i\ge 1}|\Omega_i^0|. \tag{4.23}
$$

Using the Vitali covering argument and (4.8), we obtain

$$
\begin{aligned}
&|E(A\lambda,\Omega_R)|\\
&\le c\epsilon^2\sum_{i\ge1}\big|\Omega_{r_i}(x_i)\cap E(\frac{\lambda}{4},\Omega_{2R})\big|\\
&\quad + c\epsilon^2\frac{\eta}{(\zeta\lambda)^\eta}\sum_{i\ge1}\int_{\zeta\lambda}^\infty \mu^\eta\big|\{x\in\Omega_{r_i}(x_i):|\mathbf{F}|^{\frac{2p(x)}{P^-}}>\mu\}\big|\frac{d\mu}{\mu}\\
&\le c\epsilon^2\big|E(\frac{\lambda}{4},\Omega_{2R})\big| + c\epsilon^2\frac{\eta}{(\zeta\lambda)^\eta}\int_{\zeta\lambda}^\infty \mu^\eta\big|\{x\in\Omega_{2R}:|\mathbf{F}|^{\frac{2p(x)}{P^-}}>\mu\}\big|\frac{d\mu}{\mu}.
\end{aligned}
\tag{4.24}
$$

Step 4. We prove that $\|(|\nabla u|+|P|)^{p(x)}\|_{L(t,q)(\Omega_{2R})}<\infty$ in the case $0<q<\infty$. Since $t\ge 1$, we multiply both sides of inequality (4.24) by $(\frac{tp^-}{2})^{t/q}(A\lambda)^{tp^-/2}$, and integrate it with respect to the measure $\frac{d\lambda}{A\lambda}$ from $M\lambda_0$ to ∞, and then have

$$
\begin{aligned}
&\frac{tp^-}{2}\int_{M\lambda_0}^\infty\Big((A\lambda)^{tp^-/2}\big|\{x\in\Omega_R:(|\nabla u|+|P|)^{2p(x)/p^-}>A\lambda\}\big|\Big)^{q/t}\frac{d\lambda}{A\lambda}\\
&\le c\epsilon^{\frac{2q}{t}}(I_1+I_2),
\end{aligned}
\tag{4.25}
$$

where c depends on $n,\gamma_1,\gamma_2,\nu,\Lambda,q,t$ and $\omega(\cdot)$, and

$$
\begin{gathered}
I_1=\frac{tp^-}{2}\int_0^\infty\Big(\lambda^{tp^-/2}\big|\{x\in\Omega_{2R}:(|\nabla u|+|P|)^{2p(x)/p^-}>\frac{\lambda}{4}\}\big|\Big)^{q/t}\frac{d\lambda}{\lambda}\\
I_2=\frac{tp^-}{2}\int_0^\infty\lambda^{q(\frac{p^-}{2}-\frac{\eta}{t})}\Big(\int_{\zeta\lambda}^\infty\mu^\eta\big|\{x\in\Omega_{2R}:|\mathbf{F}|^{2p(x)/p^-}>\mu\}\big|\frac{d\mu}{\mu}\Big)^{q/t}\frac{d\lambda}{\lambda}
\end{gathered}
\tag{4.26}
$$

Thanks to (3.27), we have

$$
\begin{aligned}
&\|(|\nabla u|+|P|)^{p(x)}\|^q_{L(t,q)(\Omega_R)}\\
&=\|(|\nabla u|+|P|)^{2p(x)/p^-}\|^{\frac{p^-}{2}q}_{L(\frac{tp^-}{2},\frac{qp^-}{2})(\Omega_R)}\\
&=\frac{tp^-}{2}\int_0^\infty\Big(\mu^{tp^-/2}\big|x\in\Omega_R:(|\nabla u|+|P|)^{2p(x)/p^-}>\mu\big|\Big)^{q/t}\frac{d\mu}{\mu}.
\end{aligned}
\tag{4.27}
$$

Making a change of variables yields

$$
I_1=c(q)\big\|(|\nabla u|+|P|)^{p(x)}\big\|^q_{L(t,q)(\Omega_{2R})}.
$$

For estimating I_2, we consider two cases.

Case 1. If $q \geq t$, note that (3.28) is satisfied because of $|\mathbf{F}|^{2p(x)/p^-} \in L^\eta(\Omega_{2R})$. By making the change of variables $\bar\lambda = \zeta\lambda$ and $\zeta = \frac{\delta}{4}$, in view of $\alpha = q/t \geq 1$ and $r = q\big(\frac{p^-}{2} - \frac{\eta}{t}\big) > 0$, and using

$$f(\mu) = \mu^{\eta-1}\big|\big\{x \in \Omega_{2R} : |\mathbf{F}|^{\frac{2p(x)}{P^-}} > \mu\big\}\big|,$$

it follows from Lemma 3.12 that

$$\begin{aligned} I_2 &= c\frac{tp^-}{2}\int_0^\infty \bar\lambda^{q(\frac{p^-}{2}-\frac{\eta}{t})}\Big(\int_{\bar\lambda}^\infty \mu^\eta\big|\big\{x \in \Omega_{2R} : |\mathbf{F}|^{2p(x)/p^-} > \mu\big\}\big|\frac{d\mu}{\mu}\Big)^{q/t}\frac{d\bar\lambda}{\bar\lambda} \\ &\leq c\frac{tp^-}{2}\int_0^\infty \bar\lambda^{\frac{qp^-}{2}}\big|\big\{x \in \Omega_{2R} : |\mathbf{F}|^{2p(x)/p^-} > \bar\lambda\big\}\big|^{q/t}\frac{d\bar\lambda}{\bar\lambda} \\ &= c\||\mathbf{F}|^{p(x)}\|^q_{L(t,q)(\Omega_{2R})}, \end{aligned}$$

where $c = c(\gamma_1, \gamma_2, q, t)$.

Case 2. If $0 < q < t$, by Lemma 3.13 with

$$h(\mu) = \big|\big\{x \in \Omega_{2R} : |\mathbf{F}|^{2p(x)/p^-} > \mu\big\}\big|^{q/t}, \quad r = \frac{\eta q}{t},\ \alpha_1 = 1 < \frac{t}{q} = \alpha_2, \quad \varepsilon = 1,$$

we have

$$\begin{aligned} &\Big(\int_\lambda^\infty \mu^\eta\big|\big\{x \in \Omega_{2R} : |\mathbf{F}|^{2p(x)/p^-} > \mu\big\}\big|\frac{d\mu}{\mu}\Big)^{q/t} \\ &\leq \lambda^{\frac{\eta q}{t}}\big|\big\{x \in \Omega_{2R} : |\mathbf{F}|^{2p(x)/p^-} > \lambda\big\}\big|^{q/t} \\ &\quad + c\int_\lambda^\infty \mu^{\frac{\eta q}{t}}\big|\big\{x \in \Omega_{2R} : |\mathbf{F}|^{2p(x)/p^-} > \mu\big\}\big|^{q/t}\frac{d\mu}{\mu}. \end{aligned}$$

Hence, after changing the variable $\zeta\lambda \to \lambda$, it follows by Fubini's theorem that

$$\begin{aligned} I_2 &\leq c\frac{tp^-}{2}\int_0^\infty \lambda^{q(\frac{p^-}{2}-\frac{\eta}{t})}(\lambda)^{\frac{\eta q}{t}}\big|\big\{x \in \Omega_{2R} : |\mathbf{F}|^{2p(x)/p^-} > \lambda\big\}\big|^{q/t}\frac{d\lambda}{\lambda} \\ &\quad + c\frac{tp^-}{2}\int_0^\infty \lambda^{q(\frac{p^-}{2}-\frac{\eta}{t})}\int_\lambda^\infty \mu^{\frac{\eta q}{t}-1}\big|\big\{x \in \Omega_{2R} : |\mathbf{F}|^{2p(x)/p^-} > \mu\big\}\big|^{q/t}d\mu\frac{d\lambda}{\lambda} \\ &\leq c\||\mathbf{F}|^{p(x)}\|^q_{L(t,q)(\Omega_{2R})} \\ &\quad + c\frac{tp^-}{2}\int_0^\infty \lambda^{q(\frac{p^-}{2}-\frac{\eta}{t})}\Big(\int_\lambda^\infty \mu^{\frac{\eta q}{t}-1}\big|\big\{x \in \Omega_{2R} : |\mathbf{F}|^{2p(x)/p^-} > \mu\big\}\big|^{q/t}d\mu\Big)\frac{d\lambda}{\lambda} \\ &\leq c\||\mathbf{F}|^{p(x)}\|^q_{L(t,q)(\Omega_{2R})}, \end{aligned}$$

where $c = c(\gamma_1, \gamma_2, q, t)$.

Substituting the estimates of I_1 and I_2 into (4.25), for $t \geq 1$ we derive that

$$\begin{aligned} &\|(|\nabla u| + |P|)^{p(x)}\|_{L(t,q)(\Omega_R)} \\ &\leq c\Big[\frac{tp^-}{2}\int_{M\lambda_0}^\infty\Big((A\lambda)^{tp^-/2}\big|\big\{x \in \Omega_R : (|\nabla u| + |P|)^{2p(x)/p^-} > A\lambda\big\}\big|\Big)^{q/t}\frac{d(A\lambda)}{A\lambda}\Big]^{1/q} \\ &\quad + c\Big[\frac{tp^-}{2}\int_0^{M\lambda_0}\Big((A\lambda)^{tp^-/2}\big|\big\{x \in \Omega_R : (|\nabla u| + |P|)^{2p(x)/p^-} > A\lambda\big\}\big|\Big)^{q/t}\frac{d(A\lambda)}{A\lambda}\Big]^{1/q} \\ &\leq c\Big[\frac{tp^-}{2}\int_0^{M\lambda_0}\Big((A\lambda)^{tp^-/2}\big|\big\{x \in \Omega_R : (|\nabla u| + |P|)^{2p(x)/p^-} > A\lambda\big\}\big|\Big)^{q/t}\frac{d(A\lambda)}{A\lambda}\Big]^{1/q} \\ &\quad + \bar c\epsilon^{2/t}\Big(\|(|\nabla u| + |P|)^{p(x)}\|_{L(t,q)(\Omega_{2R})} + \||\mathbf{F}|^{p(x)}\|_{L(t,q)(\Omega_{2R})}\Big) \end{aligned}$$

$$\le c\lambda_0^{p^-/2}|\Omega_{2R}|^{1/t} + \bar{c}\epsilon^{2/t}\Big(\big\|(|\nabla u| + |P|)^{p(x)}\big\|_{L(t,q)(\Omega_{2R})} + \big\||\mathbf{F}|^{p(x)}\big\|_{L(t,q)(\Omega_{2R})}\Big),$$

where $\bar{c} = \bar{c}(n, \gamma_1, \gamma_2, \nu, \Lambda, q, t, \omega(\cdot))$. It suffices to choose $\epsilon > 0$ small enough such that $\bar{c}\epsilon^{2/t} \le \frac{1}{2}$. Once the selection of ϵ is fixed, we can find the corresponding constant $\delta = \delta(\epsilon, \gamma_1, \gamma_2)$ such that

$$\begin{aligned} &\big\|(|\nabla u| + |P|)^{p(x)}\big\|_{L(t,q)(\Omega_R)} \\ &\le c\lambda_0^{p^-/2}|\Omega_{2R}|^{1/t} + \frac{1}{2}\big\|(|\nabla u| + |P|)^{p(x)}\big\|_{L(t,q)(\Omega_{2R})} + c\big\||\mathbf{F}|^{p(x)}\big\|_{L(t,q)(\Omega_{2R})}. \end{aligned} \tag{4.28}$$

By a standard iteration argument, we can attain an estimate as (2.11) in the case of $t > 1$ and $0 < q < \infty$.

Step 5. We are ready to prove the claim of Step 4: $\|(|\nabla u| + |P|)^{p(x)}\|_{L(t,q)(\Omega_{2R})} < \infty$. To this end, we first refine the estimate of $(|\nabla u|+|P|)^{p(x)}$ in the scale of Lorentz spaces. Consider the truncated function:

$$\big|\big(|\nabla u| + |P|\big)^{p(x)}\big|_k = (|\nabla u| + |P|)^{p(x)} \wedge k \quad \text{for } x \in \Omega \text{ and } k \in \mathbb{N} \cap [M\lambda_0, \infty).$$

In view of $E_k(\lambda, \Omega_\rho) = \{x \in \Omega_\rho : |(|\nabla u| + |P|)^{p(x)}|_k > \lambda\}$ in line with (4.24), we have

$$E_k(A\lambda, \Omega_R) \le c\epsilon^2\big|E_k\big(\frac{\lambda}{4}, \Omega_{2R}\big)\big| + c\epsilon^2\frac{\eta}{(\zeta\lambda)^\eta}\int_{\zeta\lambda}^\infty \mu^\eta\big|\big\{x \in \Omega_{2R} : |\mathbf{F}|^{2p(x)/p^-} > \mu\big\}\big|\frac{d\mu}{\mu}$$

for $k \in \mathbb{N} \cap [M\lambda_0, \infty)$.

Indeed, for $k \le A\lambda$ we have $E_k(A\lambda, \Omega_R) = \emptyset$, which implies that the above estimate holds trivially. For $k > A\lambda$, it is also valid because

$$E_k(A\lambda, \Omega_R) = E(A\lambda, \Omega_R) = \big\{x \in \Omega_R, \ (|\nabla u| + |P|)^{p(x)} > A\lambda\big\}$$

and

$$E_k\big(\frac{\lambda}{4}, \Omega_{2R}\big) = E\big(\frac{\lambda}{4}, \Omega_{2R}\big).$$

Proceeding in the same manner as the above, one can see that (4.28) holds with $\big|(|\nabla u| + |P|)^{p(x)}\big|_k$ in place of $(|\nabla u| + |P|)^{p(x)}$. Let

$$\begin{gathered} B_0 = 0, \quad L = c\lambda_0^{p^-/2}|\Omega_{2R}|^{1/t} + c\big\||\mathbf{F}|^{p(x)}\big\|_{L(t,q)(\Omega_{2R})}, \\ \varphi(\rho) = \big\|\,|(|\nabla u| + |P|)^{p(x)}|_k\big\|_{L(t,q)(\Omega_\rho)}. \end{gathered}$$

because

$$\big\|\,|(|\nabla u| + |P|)^{p(x)}|_k\big\|_{L(t,q)(\Omega_R)} < \infty,$$

using the iteration argument we have

$$\big\|\,|(|\nabla u| + |P|)^{p(x)}|_k\big\|_{L(t,q)(\Omega_R)} \le c\lambda_0^{p^-/2}|\Omega_R|^{1/t} + c\big\|\,|\mathbf{F}|^{p(x)}\big\|_{L(t,q)(\Omega_{2R})}.$$

In what follows, we use a standard finite covering argument to achieve the global estimate. Note that Ω is a bounded domain in $\mathbb{R}^n$. There exist $N \in \mathbb{N}$ and $x_l \in \Omega$ for $l = 1, 2, \dots, N$ such that

$$\overline{\Omega} \subset \cup_{l=1}^N B_R(x_l).$$

Then we have

$$\big\|\big|(|\nabla u| + |P|)^{p(x)}\big|_k\big\|_{L(t,q)(\Omega)} \le \sum_{l=1}^N \big\|\,|(|\nabla u| + |P|)^{p(x)}|_k\big\|_{L(t,q)(\Omega_R)}$$

$$
\begin{aligned}
&\le c\sum_{l=1}^{N}\Big(\lambda_0^{p^-/2}|\Omega_R|^{1/t}+\big\||\mathbf{F}|^{p(x)}\big\|_{L(t,q)(\Omega_{2R})}\Big)\\
&\le cN\Big(\lambda_0^{p^-/2}|\Omega_R|^{1/t}+\big\||\mathbf{F}|^{p(x)}\big\|_{L(t,q)(\Omega_{2R})}\Big).
\end{aligned}
$$

Recalling the definition of λ_0, we obtain

$$
\begin{aligned}
&\big\|\,|(|\nabla u|+|P|)^{p(x)}|_k\big\|_{L(t,q)(\Omega)}\\
&\le cN|\Omega_R|^{1/t}\Big(\fint_{\Omega_{2R}}(|\nabla u|+|P|)^{2p(x)/p^-}dx\\
&\quad+\Big(\fint_{\Omega_{2R}}\Big(|\mathbf{F}|^{2p(x)/p^-}+1\Big)^{\eta}dx\Big)^{1/\eta}\Big)^{p^-/2}+cN\big\||\mathbf{F}|^{p(x)}\big\|_{L(t,q)(\Omega_{2R})}\\
&\le cN|\Omega_R|^{1/t}\Big(\fint_{\Omega_{2R}}(|\nabla u|+|P|)^{2p(x)/p^-}dx\\
&\quad+\Big(\fint_{\Omega_{2R}}\Big(|\mathbf{F}|^{2p(x)/p^-}+1\Big)^{\eta}dx\Big)^{1/\eta}\Big)^{p^-/2}+cN\big\||\mathbf{F}|^{p(x)}\big\|_{L(t,q)(\Omega)}.
\end{aligned}
\tag{4.29}
$$

Note that

$$
\frac{2p^+}{p^-}=2\Big(1+\frac{p^+-p^-}{p^-}\Big)\le 2\Big(1+\frac{\omega(2R)}{\gamma_1}\Big)\le 2(1+\sigma),
$$

where σ is the same as given in Lemma 3.4. Then, it follows from the reverse Höder's inequality of Lemma 3.4 that

$$
\begin{aligned}
\fint_{\Omega_{2R}}(|\nabla u|+|P|)^{2p(x)/p^-}dx &\le \fint_{\Omega_{2R}}(|\nabla u|+|P|)^{\frac{2p^+}{p^-}}dx+1\\
&\le c\Big(\fint_{\Omega_{4R}}\big(|\nabla u|^2+|P|^2\big)dx+1\Big)^{p^+/p^-}\\
&\quad+\fint_{\Omega_{4R}}|\mathbf{F}|^{\frac{2p^+}{p^-}}dx.
\end{aligned}
\tag{4.30}
$$

Using (3.1) and Höder's inequality we have

$$
\begin{aligned}
&\Big(\fint_{\Omega_{4R}}\big(|\nabla u|^2+|P|^2\big)dx\Big)^{p^+/p^-}\\
&\le\Big(\frac{1}{|\Omega_{4R}|}\Big)^{p^+/p^-}\Big(\int_{\Omega}\big(|\nabla u|^2+|P|^2\big)dx\Big)^{p^+/p^-}\\
&\le c\Big(\frac{1}{|\Omega_{4R}|}\Big)^{p^+/p^-}\Big(\int_{\Omega}|\mathbf{F}|^2dx\Big)^{p^+/p^-}\\
&\le c\Big(\frac{1}{|\Omega_{4R}|}\Big)^{p^+/p^-}|\Omega|^{1-\frac{p^-}{p^+}}\int_{\Omega}|\mathbf{F}|^{\frac{2p^+}{p^-}}dx\\
&\le c\Big(\frac{1}{|\Omega_{4R}|}\Big)^{p^+/p^-}\int_{\Omega}\Big(|\mathbf{F}|^{\frac{2p(x)}{p^-}\frac{p^+}{p^-}}+1\Big)dx.
\end{aligned}
\tag{4.31}
$$

From (4.30)-(4.29), we deduce that

$$
\begin{aligned}
&\| \Big|(|\nabla u| + |P|)^{p(x)}\Big|_k \|_{L(t,q)(\Omega)} \\
&\le cN|\Omega_R|^{1/t}\Big\{\Big(\frac{1}{|\Omega_{4R}|}\Big)^{p^+/p^-}\int_\Omega \Big(|\mathbf{F}|^{\frac{2p(x)}{p^-}\frac{p^+}{p^-}}+1\Big)dx \\
&\quad + \Big[\fint_{\Omega_{2R}}\big(|\mathbf{F}|^{2p(x)/p^-}+1\big)^\eta\,dx\Big]^{1/\eta}\Big\}^{p^-/2} + cN\big\||\mathbf{F}|^{p(x)}\big\|_{L(t,q)(\Omega)} \\
&\le cN|\Omega_R|^{1/t}\Big\{\Big(\frac{1}{|\Omega_{4R}|}\Big)^{p^+/p^-}\int_\Omega \big(|\mathbf{F}|^{\frac{2p(x)}{p^-}\frac{p^+}{p^-}}+1\big)dx \\
&\quad + \big(\frac{1}{|\Omega_{2R}|}\big)^{1/\eta}\Big[\int_\Omega\big(|\mathbf{F}|^{2p(x)/p^-}+1\big)^\eta\,dx\Big]^{1/\eta}\Big\}^{p^-/2} + cN\big\||\mathbf{F}|^{p(x)}\big\|_{L(t,q)(\Omega)}.
\end{aligned}
\tag{4.32}
$$

Using a standard Hardy's inequality in the Marcinkiewicz spaces [28, Lemma 2.3]) and the reverse Hölder's inequality of Lemma 3.13, we obtain

$$
\begin{aligned}
\int_\Omega |\mathbf{F}|^{\frac{2p(x)}{p^-}\frac{p^+}{p^-}}\,dx &\le \frac{t(p^-)^2}{t(p^-)^2-2p^+}|\Omega|^{1-\frac{2p^+}{t(p^-)^2}}\,\|\,|\mathbf{F}|^{2p(x)/p^-}\,\|^{p^+/p^-}_{\mathcal{M}^{tp^-/2}(\Omega)} \\
&= \frac{t(p^-)^2}{t(p^-)^2-2p^+}|\Omega|^{1-\frac{2p^+}{t(p^-)^2}} \\
&\quad\times\Big\{\sup_{h>0}\Big[h^{tp^-/2}\big|\{x\in\Omega : |\mathbf{F}|^{2p(x)/p^-} > h\}\big|\Big]^{\frac{2}{tp^-}}\Big\}^{p^+/p^-} \\
&\le c|\Omega|^{1-\frac{2p^+}{t(p^-)^2}}\,\|\,|\mathbf{F}|^{2p(x)/p^-}\,\|^{p^+/p^-}_{L(\frac{tp^-}{2},\frac{qp^-}{2})(\Omega)} \\
&\le c|\Omega|^{1-\frac{2p^+}{t(p^-)^2}}\,\||\mathbf{F}|^{p(x)}\|^{\frac{2p^+}{(p^-)^2}}_{L(t,q)(\Omega)}.
\end{aligned}
\tag{4.33}
$$

Similarly, we can derive that

$$
\begin{aligned}
\Big(\int_\Omega |\mathbf{F}|^{\frac{2p(x)}{p^-}\eta}\,dx\Big)^{1/\eta} &\le c(\gamma_1,\gamma_2,q,t)|\Omega|^{\frac{1}{\eta}-\frac{2}{tp^-}}\,\|\,|\mathbf{F}|^{2p(x)/p^-}\,\|_{L(\frac{tp^-}{2},\frac{qp^-}{2})(\Omega)} \\
&\le c(\gamma_1,\gamma_2,q,t)|\Omega|^{\frac{1}{\eta}-\frac{2}{tp^-}}\,\big\|\,|\mathbf{F}|^{p(x)}\big\|^{\frac{2}{p^-}}_{L(t,q)(\Omega)}.
\end{aligned}
$$

In the case $q<\infty$, it follows from (4.29) that

$$
\begin{aligned}
\big\|\big|(|\nabla u|+|P|)^{p(x)}\big|_k\big\|_{L(t,q)(\Omega)} &\le cN\Big\{\Big(\frac{|\Omega|}{|\Omega_{2R}|}\Big)^{\frac{p^+}{2}-\frac{1}{t}}\Big(\||\mathbf{F}|^{p(x)}\|^{p^+/p^-}_{L(t,q)(\Omega)}+1\Big) \\
&\quad + \Big(\frac{|\Omega|}{|\Omega_{2R}|}\Big)^{\frac{p^-}{2\eta}-\frac{1}{t}}\big[\||\mathbf{F}|^{p(x)}\|_{L(t,q)(\Omega)}+1\big]\Big\} \\
&\le cN\Big\{\Big(\frac{|\Omega|}{|B_{2R}|}\frac{|B_{2R}|}{|\Omega_{2R}|}\Big)^{\frac{p^+}{2}-\frac{1}{t}}\Big(\||\mathbf{F}|^{p(x)}\|^{p^+/p^-}_{L(t,q)(\Omega)}+1\Big) \\
&\quad + \Big(\frac{|\Omega|}{|B_{2R}|}\frac{|B_{2R}|}{|\Omega_{2R}|}\Big)^{\frac{p^-}{2\eta}-\frac{1}{t}}\big[\|\,|\mathbf{F}|^{p(x)}\|_{L(t,q)(\Omega)}+1\big]\Big\} \\
&\le cN\Big\{\big[\frac{|\Omega|}{|B_{2R}|}\big(\frac{2}{1-\delta}\big)^n\big]^{\frac{p^+}{2}-\frac{1}{t}}\Big(\||\mathbf{F}|^{p(x)}\|^{p^+/p^-}_{L(t,q)(\Omega)}+1\Big) \\
&\quad + \big[\frac{|\Omega|}{|B_{2R}|}\big(\frac{2}{1-\delta}\big)^n\big]^{\frac{p^-}{2\eta}-\frac{1}{t}}\Big(\||\mathbf{F}|^{p(x)}\|_{L(t,q)(\Omega)}+1\Big)\Big\}
\end{aligned}
$$

$$\begin{aligned}&\le cN\big(\||\mathbf{F}|^{p(x)}\|_{L(t,q)(\Omega)}+1\big)^{p^+/p^-}\\&\le cN\big(\||\mathbf{F}|^{p(x)}\|_{L(t,q)(\Omega)}+1\big)^{\gamma_2/\gamma_1}.\end{aligned}$$

Taking $k\to\infty$, by the lower semi-continuity of Lorentz quasi-norm we have

$$\big\|\,|(|\nabla u|+|P|)^{p(x)}|_k\big\|_{L(t,q)(\Omega)}\le cN\Big(\big\||\mathbf{F}|^{p(x)}\big\|_{L(t,q)(\Omega)}+1\Big)^{\gamma_2/\gamma_1},$$

where c depends only on $n,\gamma_1,\gamma_2,\nu,\ \Lambda,t,q,\omega(\cdot),R_0$ and $|\Omega|$.

Step 6. Finally, for the case $q=\infty$, we obtain back to the second inequality in (4.5) and split it into two parts with a small $\iota>0$ to be determined later:

$$\big(\frac{\lambda}{2}\big)^{\eta}\le\frac{1}{\delta^{\eta}}\fint_{\Omega_i^0}|\mathbf{F}|^{\frac{2p(x)}{p^-}\eta}dx\le\frac{(\iota\lambda)^{\eta}}{\delta^{\eta}}+\frac{1}{\delta^{\eta}|\Omega_i^0|}\int_{\{x\in\Omega_i^0:|\mathbf{F}|^{2p(x)/p^-}>\iota\lambda\}}|\mathbf{F}|^{\frac{2p(x)}{p^-}\eta}dx.$$

Set

$$G(\iota\lambda,\Omega_i^0)=\{x\in\Omega_i^0:|\mathbf{F}|^{2p(x)/p^-}>\iota\lambda\},\quad G(\mu,\Omega_i^0)=\{x\in\Omega_i^0:|\mathbf{F}|^{2p(x)/p^-}>\mu\}.$$

Similar to (4.33), by using Hölder's inequality we obtain

$$\begin{aligned}&\big(\frac{\lambda}{2}\big)^{\eta}-\big(\frac{\iota\lambda}{\delta}\big)^{\eta}\\&\le\frac{1}{\delta^{\eta}|\Omega_i^0|}\int_{\{x\in\Omega_i^0:|\mathbf{F}|^{2p(x)/p^-}>\iota\lambda\}}|\mathbf{F}|^{\frac{2p(x)}{p^-}\eta}dx\\&\le\frac{t}{(t-\eta)\delta^{\eta}}\frac{|G(\iota\lambda,\Omega_i^0)|^{1-\frac{\eta}{t}}}{|\Omega_i^0|}\sup_{\mu>0}\mu^{\eta}\big|\big\{x\in G(\iota\lambda,\Omega_i^0):|\mathbf{F}|^{\frac{2p(x)}{p^-}\eta}\ge\mu\big\}\big|^{\eta/t}\\&\le\frac{t|G(\iota\lambda,\Omega_i^0)|^{1-\frac{\eta}{t}}}{(t-\eta)\delta^{\eta}|\Omega_i^0|}\Big[(\iota\lambda)^{\eta}|G(\iota\lambda,\Omega_i^0)|^{\eta/t}+\sup_{\mu>\iota\lambda}\mu^{\eta}|G(\mu,\Omega_i^0)|^{\eta/t}\Big]\\&=\frac{t}{(t-\eta)\delta^{\eta}}\Big[(\iota\lambda)^{\eta}+\frac{|G(\iota\lambda,\Omega_i^0)|^{1-\frac{\eta}{t}}}{|\Omega_i^0|}\sup_{\mu>\iota\lambda}\mu^{\eta}|G(\mu,\Omega_i^0)|^{\eta/t}\Big].\end{aligned}$$

Choose $\iota>0$ appropriately small to satisfy

$$\big(\frac{\lambda}{2}\big)^{\eta}-\big(\frac{\iota\lambda}{\delta}\big)^{\eta}-\frac{t}{t-\eta}\big(\frac{\iota\lambda}{\delta}\big)^{\eta}=\big(\frac{\lambda}{2}\big)^{\eta}-\big(\frac{\iota\lambda}{\delta}\big)^{\eta}\big(1+\frac{t}{t-\eta}\big)\ge\big(\frac{\lambda}{4}\big)^{\eta}.$$

Then there exists a positive constant $c(t)$ depending only on t such that $\iota\le c(t)\delta$. Thus, we obtain

$$\begin{aligned}|\Omega_i^0|&\le\frac{ct}{t-\eta}\frac{|G(\iota\lambda^i,\Omega_i^0)|^{1-\frac{\eta}{t}}}{(\iota\lambda)^{\eta}}\Big(\sup_{\mu>\iota\lambda}\mu^t|G(\mu,\Omega_i^0)|\Big)^{\eta/t}\\&\le\frac{ct(\iota\lambda)^{-t}}{t-\eta}\Big((\iota\lambda)^t|G(\iota\lambda,\Omega_i^0)|\Big)^{1-\frac{\eta}{t}}\Big(\sup_{\mu>\iota\lambda}\mu^t|G(\mu,\Omega_i^0)|\Big)^{\eta/t}\\&\le\frac{ct(\iota\lambda)^{-t}}{t-\eta}\sup_{\mu>\iota\lambda}\mu^t|G(\mu,\Omega_i^0)|.\end{aligned}\tag{4.34}$$

Substituting (4.5) into (4.23) (namely, plugging (4.6) and (4.34) into (4.23)) yields

$$|E(A\lambda,\Omega_R)|\le c\epsilon\big|E\big(\frac{\lambda}{4},\Omega_{2R}\big)\big|+c\epsilon(\iota\lambda)^{-t}\sup_{\mu>\iota\lambda}\mu^t|G(\mu,\Omega_{2R})|.\tag{4.35}$$

Multiplying both sides of (4.35) by $(A\lambda)^{tp^-/2}$ and taking the supremum with respect to λ over $(M\lambda_0,\infty)$, we deduce that

$$\begin{aligned}
&\sup_{\lambda>M\lambda_0}(A\lambda)^{tp^-/2}\big|\big\{x\in\Omega_R:(|\nabla u|+|P|)^{2p(x)/p^-}>A\lambda\big\}\big|\\
&\le c\epsilon^2A^{tp^-/2}\Big(\sup_{\lambda>M\lambda_0}\lambda^{tp^-/2}\big|\big\{x\in\Omega_{2R}:(|\nabla u|+|P|)^{2p(x)/p^-}>\frac{\lambda}{4}\big\}\big|\\
&\quad+\sup_{\lambda>M\iota\lambda_0}\lambda^{\frac{tp^-}{2}-t}\Big(\sup_{\mu>\lambda}\mu^t|G(\mu,\Omega_{2R})|\Big)\Big)\\
&\le c\epsilon^2\Big(\sup_{\lambda>M\lambda_0}\lambda^{tp^-/2}\big|\big\{x\in\Omega_{2R}:(|\nabla u|+|P|)^{2p(x)/p^-}>\frac{\lambda}{4}\big\}\big|\\
&\quad+\sup_{\lambda>M\iota\lambda_0}\Big(\sup_{\mu>\lambda}\mu^{tp^-/2}|G(\mu,\Omega_{2R})|\Big)\Big).
\end{aligned}$$

Note that

$$\sup_{\lambda>M\iota\lambda_0}\sup_{\mu>\lambda}\mu^{tp^-/2}|G(\mu,\Omega_{2R})|\le\big\||\mathbf{F}|^{p(x)}\big\|^t_{\mathcal{M}^t(\Omega_{2R})}.$$

Let $\epsilon>0$ be so small that $c\epsilon^{2/t}\le\frac12$. Then

$$\begin{aligned}
&\big\|(|\nabla u|+|P|)^{p(x)}\big\|_{\mathcal{M}^t(\Omega_R)}\\
&\le c\epsilon^{2/t}\Big(\big\|(|\nabla u|+|P|)^{p(x)}\big\|_{\mathcal{M}^t(\Omega_{2R})}+c(\gamma_1,\gamma_2,q,t)\big\||\mathbf{F}|^{p(x)}\big\|_{\mathcal{M}^t(\Omega_{2R})}\Big)\\
&\quad+c|\Omega_{2R}|^{1/t}M\lambda_0^{p^-/2}\\
&\le\frac12\big\|(|\nabla u|+|P|)^{p(x)}\big\|_{\mathcal{M}^t(\Omega_{2R})}+c\big\||\mathbf{F}|^{p(x)}\big\|_{\mathcal{M}^t(\Omega_{2R})}\\
&\quad+c|\Omega_{2R}|^{1/t}\Big[\fint_{\Omega_{2R}}(|\nabla u|+|P|)^{2p(x)/p^-}\,dx\\
&\quad+\Big(\fint_{\Omega_{2R}}\Big(|\mathbf{F}|^{2p(x)/p^-}+1\Big)^\eta\,dx\Big)^{1/\eta}\Big]^{p^-/2}.
\end{aligned}$$

The rest of the proof is closely similar to the argument of Step 5. Consequently, we arrive at the desired result for the case $q=\infty$. □

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