THE EFFECT OF MATHEMATICS RESEARCH ON MATHEMATICS MAJORS' MATHEMATICAL BELIEFS

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THE EFFECT OF MATHEMATICS RESEARCH ON MATHEMATICS MAJORS'

MATHEMATICAL BELIEFS

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To my patient and ever loving wife,
Kenyatta Y. Dawson

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ABSTRACT

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This is a dissertation about the beliefs that mathematics majors have about mathematics and how their beliefs are affected by the introduction of mathematics research. The mathematics research presented to the students dealt with counting regular orbits of an action. Research has shown that the beliefs that teachers hold about mathematics influences how they teach their classes. For example, it has been observed that some teachers who believe mathematics as a static field might teach a drill and memorization type of class. Four sections of an Introduction to Advanced Mathematics course were split into a two treatment and two control groups. There were a total of 40 students in the study; 22 in the treatment group and 18 in the control group. The treatment group received a three weeks intervention at the end of the semester in which basic modern algebra content was presented to the student to prepare them for presentations pertaining to the mathematics research. The study incorporated a pre/mid/post-survey format. The pre-survey was given at the beginning of the semester; the mid-survey was given before the beginning of the three week intervention; and the post-survey was given at the end of the semester after the intervention. Additional data in the form of written work and student interviews were collected to supplement survey data. There was no statistical difference between the treatment and control groups for each administration of the survey. However, there was a statistical difference between the post and mid-surveys at the .05 level and a statistical difference between the post and pre-surveys at the .02 level for the treatment group while the control group had significance levels of .12 and .09 respectively. Additionally the survey subscale Status of Mathematics had statistically significant differences at the .00 level between the post and mid-surveys and between the post and pre-surveys in the treatment group while there was no significant difference for the control group (p-values > .3). Supplemental data from student interviews and reflections showed mathematical growth and gave indications of students becoming interested in the idea of mathematics research. These results reveal that the introduction of mathematics research had a positive impact on what the students believe about mathematics and their appreciation for the subject.

CHAPTER 1

INTRODUCTION

This is a dissertation about the beliefs held by mathematics majors and how they are influenced by the presentation of mathematics research. An experimental study was done with surveys administered to treatment and control groups to determine if the beliefs of the mathematics majors were influenced by the introduction to mathematics research.

Background

The National Council of Teachers of Mathematics (NCTM) 2007 standards for teaching math state that teachers should engage their students in "worthwhile mathematical tasks." These tasks should engage students' intellect, develop their mathematical skills, help them make connections and develop a mathematical framework, and represent mathematics as an ongoing human endeavor. Worthwhile mathematical tasks should discourage students from simply memorizing mathematics and promote the need to develop deep understanding of mathematics. However, certain beliefs of teachers can get in the way of choosing worthwhile mathematical tasks.

Many teachers view mathematics as a static body of knowledge and facts (Brendefur & Frykholm, 2000; Cuoco, 2001; Stipek, Givvin, Salmon, & MacGyvers, 2001). This mindset leads teachers towards a teaching method that is not reform-oriented and emphasizes performance and memorization rather than understanding (Stipek et al., 2001). This belief about mathematics is not confined to a small subset of mathematics teachers but is held by teachers throughout the K-12 educational system (Ball, 1990).

Table 1

Description of What a Positive or Negative Belief is for Each Subscale

	Negative	Positive
Composition of Mathematics	Procedural	Conceptual
Structure of Mathematics	Isolated Ideas	Connected Ideas
Status of Mathematics	Dead	Alive
Doing Mathematics	Procedural	Conceptual
Validating in Mathematics	Told from Authority	Self-Proof
Learning Mathematics	Memorization	Understanding
Usefulness of Mathematics	Not Useful	Useful
Mathematicians are	Unimportant/Not Good	Important/Good

As teachers' beliefs about mathematics directly affect the presentation of material and student engagement, it is important to explore how teachers view mathematics. Hence, this study dealt with the beliefs teachers have about mathematics. Beliefs about mathematics were categorized as either positive or negative. Table 1 lists the subscales that are represented in the survey that was administered to the students and what it means for a belief to be positive or negative in each subscale. It is understood that many of the beliefs in the Negative column of Table 1 inform beliefs in Positive column. For example there is a degree of memorization that is needed to understand mathematics.

Thus in this case the distinction, between negative and positive beliefs, is whether individuals believe that memorization is learning mathematics or understanding is learning mathematics.

While Table 1 mentions eight subscales, the literature tends to focus on a few of them. Typically these include Doing, Learning, Usefulness, and Mathematicians in the context of student beliefs. In the context of teacher beliefs the subscales often discussed include Composition, Status, Doing, and Learning.

There are many differences between the traditional and reform views of teaching mathematics. The traditional view of teaching mathematics is based on a lecture style of teaching. Attention is given to individual practice, the correctness and validity of student work, and efficient use of mathematical rules and algorithms. Students are grouped homogeneously by ability. The reform view of teaching mathematics is based on guided discovery. Students are encouraged to invent and discuss mathematical techniques, find patterns, make connections, and engage in real-life problem solving. Students are grouped heterogeneously by ability (Goldin, 2008).

From the definition of the reform method of teaching and comparing it to Table 1 it can be seen that the reform method aims to combat the negative beliefs of mathematics. According to Schoenfeld (2002) “on tests of conceptual understanding and problem solving, students who learn from reform curriculum consistently outperform students who learn from traditional curricula by a wide margin” (p. 16). However it should also be noted that when it came to being tested on basic mathematic skills there was no significant difference between the two curriculums. Thus a higher score on the survey corresponds to a reform-oriented belief system.

This is not to say that someone who has a traditional style of teaching does not have positive beliefs concerning mathematics. It is assumed that mathematics professors might have different beliefs about the mathematics they teach and the mathematics that they do professionally (i.e. research). Many articles discuss what mathematicians believe about mathematics and about how mathematics should be taught (Beswick, 2005; Cooney & Shealy, 1997; Cuoco, Goldenberg, and Mark, 1996; Cuoco, 2001; NCTM, 2007). However, they generally do not cite articles or provide research on how they came to their conclusions about what mathematicians believe.

NCTM (2007) states, “through the experiences of learning mathematics, prospective and practicing teachers develop many of their core beliefs about how mathematics is learned and, therefore, how it should be taught” (p. 119). Cuoco, Goldenberg, and Mark (1996) say that students should be taught mathematics through the “habits of mind” of the mathematician. In other words, the mathematics curriculum should be written from the perspective of how mathematicians conduct research and the methods they use rather than presenting students with mathematical results. NCTM and Cuoco et al. are in agreement that teachers should be taught from a mathematics research perspective.

Students have their beliefs about mathematics influenced by their teachers (Muis, 2004; Wilkins & Ma, 2003). One might think that as students enter high school and are taught by teachers who have majored in mathematics, who were themselves taught by mathematicians, they would develop a more positive conception of mathematics. However, many students' beliefs tend to become more negative the more they make their way through school (Wilkins & Ma, 2003). Many students’ beliefs about mathematics have a negative impact on their academic performance as well as their educational futures (Gilroy, 2002).

For example, students have a tendency to believe that understanding mathematics is unnecessary and knowing the rules is the only important thing about mathematics (Mason 2003; Muis, 2004). However, this view neglects the problem solving aspect of mathematics that Cuoco, Goldenberg and Mark (1996) and NCTM (2000) point out is an important tool not only for mathematicians, but also professionals in other fields.

Also, NCTM (2007) calls for teachers to have an appreciation for the rigor and inquiry of mathematics. This is consistent with Cuoco et al.'s (1996) assertion that mathematics should be taught in a way that emphasizes mathematical thought and research. This dissertation will use mathematical research as the catalyst for changing the beliefs of mathematics majors.

Purpose

There is a need for both students and society to develop more positive beliefs about mathematics. This will aid in producing more mathematics majors and mathematics teachers, improving students' achievement scores, and contribute to the functioning of society. Students form their beliefs from a variety of places, including family, friends, the media, and their teachers. Of these influences, the teacher is the person with whom students spend the most time dealing with topics of an explicit mathematical nature. Thus, it is important that students be influenced by a teacher who has positive views and beliefs about mathematics.

Mathematics teachers are responsible for student growth in mathematics and influence the way in which their students think about mathematics as a subject.

Elementary teachers, the first to teach students mathematics professionally, often do not major in mathematics in college and generally have negative beliefs about mathematics and what it means to do mathematics (Undergraduate Degree Programs, n.d.; Szydlik, Szydlik, & Benson, 2003). Therefore, many students begin their academic careers with teachers who possess a negative view of mathematics. Moreover, secondary mathematics teachers, who major in mathematics in college, have beliefs about mathematics that are similar to their primary teacher counterparts (Ball, 1990).

In addition, teachers who believe mathematics is computation and getting the correct answers are more likely to teach mathematics in a traditional way. However, teachers who believe mathematics is a creative subject dealing with problem solving and investigation will more likely teach a reform-oriented class (Cross, 2009).

With teachers' views of mathematics and teaching style in mind, this study aims to provide resources for supporting pre-service secondary mathematics teachers and contribute to the growing body of literature by providing an alternative way of introducing advanced mathematics to mathematics majors. As Leatham (2006) states, research on teacher beliefs "has great potential to inform educational research and practice and is therefore worth the effort" (p. 91). Additionally, Pajares (1992) states that "the investigation of teachers' beliefs is a necessary and valuable avenue of educational inquiry" (p. 326).

Thus the purpose of this dissertation is to determine if the presentation of mathematics research in a course that introduces advanced mathematics increases the positive mathematical beliefs of pre-service secondary mathematics teachers. This includes raising pre-service teachers' awareness of mathematics research and increasing their appreciation of mathematics and mathematical rigor.

Research Questions

The following research questions were investigated in this study.

1. Do pre-service secondary mathematics teachers, pure mathematics majors, applied mathematics majors, and mathematics minors have different beliefs about mathematics?
2. Do mathematics majors who participate in MATH 3330 with a mathematics research component demonstrate greater changes to their beliefs about mathematics than mathematics majors who participate in a standard MATH 3330 course?
3. Do pre-service secondary mathematics teachers who participate in MATH 3330 with a mathematics research component demonstrate greater changes to their beliefs about mathematics than pre-service secondary mathematics teachers who participate in a standard MATH 3330 course?

Null Hypotheses

The null hypotheses associated with the quantitative research questions in respective order are:

1. Pre-service secondary mathematics teachers, pure mathematics majors, applied mathematics majors, and mathematics minors have the same beliefs about mathematics.
2. Mathematics majors who participate in MATH 3330 from the perspective of mathematics research demonstrate the same changes to their beliefs about mathematics as mathematics majors who participate in a standard MATH

3330 course.

3. Pre-service secondary mathematics teachers who participate in MATH 3330 from the perspective of mathematics research demonstrate the same changes to their beliefs about mathematics as pre-service secondary mathematics teachers who participate in a standard MATH 3330 course.

Mathematics Research

The intervention involves the presentation of mathematics research. Thus, in order to accurately convey research experience to the students, the researcher was involved in an abstract algebra research project. This project involved the action of extra-special groups acting on faithful irreducible modules. Therefore, since mathematics research plays an important role in this research project it will have its own chapter in the dissertation.

CHAPTER 2

REVIEW OF THE LITERATURE

This study looks into whether the beliefs of mathematics majors about mathematics and mathematicians are affected by taking the Introduction to Advanced Mathematics course with the addition of a mathematics research component. This literature review addresses the problems associated with defining beliefs, explains the beliefs of students and teachers, and explores teachers' influence on the beliefs of students. The review examines different aspects of presenting advanced mathematics concepts, especially modern algebra, to students. This includes problems students have with the abstraction of algebra and different ways of introducing abstract algebra to students. The final section is a review dealing with background information used for the mathematics research.

Beliefs

In reviewing the literature, the difficulty in defining beliefs becomes apparent. Many studies tend to have different views on what beliefs are. In a commonly cited article, Pajares (1992) discusses beliefs in educational research. Pajares (1992) points out that difficulty arises from the poor definitions and conceptualizations as well as differing understandings of beliefs within the literature. He explains that the reason for this confusion is the lack of differentiation between beliefs and knowledge. For example, is there a difference between believing the sun rises in the east and knowing the sun rises in the east. An acceptable distinction that Pajares identifies in the literature is that beliefs are about evaluation and judgment of one's experiences while knowledge is about objective fact. Thus, Pajares (1992) concludes that the construct of beliefs is not as ambiguous as some might believe it to be. He concludes:

> When they are clearly conceptualized, when their key assumptions are examined, when precise meanings are consistently understood and adhered to, and when specific belief constructs are properly assessed and investigated, beliefs can be, as Fenstermacher predicted, the single most important construct in educational research. (p. 329)

Students' Beliefs

Three aspects of student beliefs about mathematics are explored in this section: beliefs about mathematicians, beliefs about the nature of mathematics, and beliefs about the value of mathematics.

In an investigation of the images students have of mathematicians, Picker and Berry (2000) found that students tend to have a negative view of mathematicians. For example, in drawings by 12-13 year olds mathematicians were depicted as coercive, foolish, nervous, and having special powers. While the students in Picker and Berry's study were not high school students, it has been observed that student beliefs about mathematics either do not change or tend to become less positive as they make their way through secondary school (McLeod, 1992; Wilkins & Ma, 2003).

Scientists and mathematicians are seen as frightening individuals who intimidate their students into doing their work correctly (McNarry & O'Farrell, 1971; Picker & Berry, 2000). Many of these beliefs about mathematicians come from the way society perpetuates stereotypes. Picker and Berry point out that through teachers and the media, 7th grade students perceive that a privileged few can do mathematics, mathematics is a special language for a selected few, and that mathematics should be done quickly. This leads students to the conclusion that mathematicians are authoritarian figures (Picker & Berry 2000).

However, Rock and Shaw (2000) found that younger students show a decidedly different view of mathematicians. They administered a qualitative survey asking students in grades K-8 the following questions: “(1) What do mathematicians do? (2) What types of problems do mathematicians solve? (3) What tools do mathematicians use?” (p. 551). Rock and Shaw asked for drawings from students in kindergarten to fourth grade (5 to 10 year olds) showing the students' depictions of mathematicians. Their drawings of mathematicians were more pleasing than the ones depicted by the seventh graders in the Picker and Berry study, with mathematicians smiling and working with other people. Clearly there is a shift in students’ opinions regarding mathematicians between the fourth and seventh grades.

Many students also have misconceptions about the work that mathematicians do and what it entails. Rock and Shaw (2000) found that elementary students believed that mathematicians do work that is similar to the work that they are doing but with larger numbers, or simply do problems that other people do not know how to solve. Many middle school students, when asked when they would hire a mathematician, either did not know when someone would hire a mathematician, did not know what a mathematician does, or think that people do not need mathematicians (Picker & Berry, 2000). When the students did mention jobs for which you would hire a mathematician, the majority named teaching, illustrating a general lack of knowledge about the work of mathematicians (Picker & Berry, 2000). In one of several studies reviewed by Muis (2004), around a third of students thought mathematicians worked with symbols rather than ideas and believe that new discoveries are seldom made.

The NCTM Standards for teaching (1991) state that, “mathematics is a changing and evolving domain, one in which ideas grow and develop over time” (p. 26). However, Muis (2004) states that students view mathematical knowledge as unchanging. Muis' review of research shows that many students believe that mathematics is a set of fragmented rules and procedures, not a subject in which ideas grow and develop.

Students also believe that understanding mathematics is unnecessary and that the only thing that matters is knowing the rules to get to the correct answer (Mason, 2003; Muis 2004). Furthermore, students who believe that all mathematics can be solved using rules and procedures rely on memorization as their main way of learning (Muis, 2004). This is not surprising as the literature shows that mathematics has traditionally been taught in a way that emphasizes facts and procedures that must be memorized in order to do well (NAEP, 1983; Schoenfeld, 1989; Wilkins & Ma, 2003).

Additionally, students believe that being able to do well in mathematics is a natural ability (Muis 2004; Schoenfeld, 1989). This belief has a negative impact on motivation: a person who does not believe that he or she is good at mathematics will be less inclined to try to understand the subject (Gilroy, 2002). Gilroy further points out that studies have shown that “student achievement is often related to factors that are controllable, such as effort and persistence, rather than innate ability” (p. 41), countering this widely held belief.

Researchers have also investigated students' perceptions of the usefulness of mathematics. In the early grade levels, students either believe mathematics is useful as a means of moving on to the next grade level or they express knowing that mathematics is useful but cannot give an example or reason why (Kloosterman, Raymond, & Emenaker, 1996). When students are able to express the usefulness of mathematics and as they move up through the grade levels, students give examples of jobs where mathematics is useful such as being an accountant or architect, or employed in other notable calculation based jobs (Kloosterman et al., 1996; Mason, 2003; Picker & Berry, 2000; Rock & Shaw, 2000). However, even many older students fail to see how mathematics can be useful in their lives (Gilroy, 2002). This shortsighted view of the usefulness of mathematics, coupled with a dislike for mathematics, can affect the number of mathematics classes a student takes, or worse, convince a student to take mathematics classes at the last moment in college (Gilroy, 2002). This in turn, can result in the student forgetting much of what they had learned in high school (Gilroy, 2002; Reyes, 1984).

Wilkins and Ma (2003) provide further insight into high school and college bound students in their analysis of data from the Longitudinal Study of American Youth (LSAY), which followed and surveyed 3116 students from grades 7-12. Topics in the survey included attitude toward mathematics, the social importance of mathematics and the nature of mathematics. The results showed that as high school students get older, their beliefs become less positive about mathematics and its social importance. Additionally, Wilkins and Ma found that students who want to attend college generally have more negative beliefs about mathematics than those who do not wish to go to college. They believe that this is due to the added pressure of college entrance requirements.

Studies also show that there is a link between what students believe about mathematics and their performance in mathematics (Gilroy, 2002). As mentioned previously, students tend to believe that mathematical competency is an innate ability rather than an acquired one (Schoenfeld, 1989). To compound the problem, society has developed the idea that intelligence and mathematics go hand in hand (Gilroy, 2002). Students believe that mathematical learning should take place in a matter of minutes (Muis, 2004). If they are not able to answer a mathematics problem quickly, then they begin to believe that they will never solve the problem. If a student believes that they are not good in mathematics, then they believe they are not smart, even though achievement in mathematics is linked more with effort than natural ability (Gilroy, 2002).

Dweck's 2007 article highlights some of the research she has done concerning student motivation in a broad educational setting. She compared students who believe that intelligence is innate, which she describes as a fixed mind-set, with those who believe intelligence is about effort, which she describes as a growth mind-set. Dweck provides insight into the effect that these different mindsets have on the motivation of the students. A student who believes that intelligence is a natural ability is afraid to put forth effort when solving a problem because the student feels that if he or she is able to do something then no effort is needed. In this case when material in the curriculum becomes challenging the student stops working. Dweck provides the following example of what is happening to this student when the mathematics curriculum becomes more difficult:

> Up until then, he has breezed through math. Even when he barely paid attention in class and skimped on his homework, he always got *As.* But this is different.

> It's hard. The student feels anxious and thinks, “What if I'm not as good at math as I thought? What if other kids understand it and I don't?” At some level, he realizes that he has two choices: try hard, or turn off. His interest in math begins to wane, and his attention wanders. He tells himself, “Who cares about this stuff? It's for nerds. I could do it if I wanted to, but it's boring. You don't see CEOs and sports stars solving for x and y. (p. 35)

These students, when asked to choose between problems that are described as challenging but educational or problems that are easy with minimal opportunities for mistakes, will choose the easy ones.

On the other hand, students of the growth mind-set believe that intelligence can be developed through effort and education (Dweck, 2007). These students care about learning, and when they make mistakes they try to make corrections. For these students effort is not a thing to be scared of but to embrace. Dweck offers a scenario for this type of student:

> She finds it new, hard, and confusing, unlike anything else she has ever learned. But she's determined to understand it. She listens to everything the teacher says, asks the teacher questions after class, and takes her textbook home and reads the chapter over twice. As she begins to get it, she feels exhilarated. A new world of math opens up for her. (p. 35)

These students, when presented with the same choice between problems, will choose the challenging educational problems.

Students considering a career in mathematics may compare themselves to professional mathematicians by looking for similarities (Brush, 1980). However, high school students generally do not liken themselves to mathematicians. In 1980, Brush conducted a questionnaire of 510 high school juniors and seniors. The questionnaire had two parts, one about mathematicians and the other about writers, with identical sets of 21 items. Survey results showed that mathematicians were considered less creative, independent, intuitive, sensitive, and gentle than writers, but were viewed as more competitive, stable, and rational, as well as calmer and wiser. The students were also asked to rate themselves on the same 21 items and their ratings tended to agree more with how they rated the writers than with the mathematicians.

Research has shown that there is a positive correlation between positive beliefs about mathematics and mathematical achievement (Antonnen, 1969; Bouchey & Harter, 2005; Fennema & Sherman, 1978; Furinghetti & Morselli, 2009; Grootenboer & Hemmings, 2007;). Similarly, beliefs about the usefulness of mathematics have a positive correlation with mathematical achievement (Fennema & Sherman, 1977, 1978). Also, Schoenfeld (1983) points out that purely cognitive behavior is rare and that that it closely related with that of affective behaviors. To further complicate matters, Grootenboer and Hemmings (2007) state:

> It appears that there is a cyclical or reciprocal relationship between beliefs and attitudes, and success in learning mathematics. While it is more complex than a simple relationship, in general terms, success in mathematical learning seems to lead to more positive affective views about mathematics, which then lead to greater success in learning mathematics, and so forth, with the converse also being the case. (p. 6)

Teacher Beliefs

According to NCTM (2007), one of the many roles of the teacher is fostering positive attitudes and values about mathematics, presenting mathematics as a human endeavor, and helping students understand the uses of mathematics in life. Also, the mathematics teacher should foster “positive attitudes about the aesthetic and utilitarian values of mathematics” (NCTM, 2007, p. 6). Research suggests that teachers' beliefs and values affect their teaching practices and hence students' mathematical learning (Clark & Peterson, 1986; Fang, 1996; Kagan, 1992; Peterson, Fennema, Carpenter, & Loef, 1989; Thompson, 1992). Additionally, Muis (2004) points out that “scholars in mathematics education generally agree that the formal mathematics education students receive has a major influence on the development of their beliefs about mathematics” (p. 334). Much of the research has been done with pre-secondary teachers: however, there has been some research involving secondary teachers as well as research comparing pre-secondary and secondary teachers.

Most teachers believe mathematics to be a static body of knowledge, with set rules and procedures to attain a correct answer and promote this idea in their teaching (Brendefur & Frykholm, 2000; Cuoco, 2001; Stipek et al., 2001). A comparison of elementary teachers from 1968 and 1998 showed this perception of mathematics had not changed in over 30 years (Seaman, Szydlik, Szydlik, & Beam, 2005). According to Battista (1994), teachers who believe that math is following set procedures will have trouble making sense of mathematics and thus will have difficulty effectively teaching and guiding students into inventing mathematical ideas. In other words, in order for pre-service teachers to become effective mathematics teachers they must develop an understanding of the nature of mathematics (Steele, 1994). NCTM (2007) further states that when a teacher presents mathematics as a static subject in which answers are derived from symbolic representation, it illustrates that that teacher does not have an acceptable view of the nature of mathematics. Driving the point further, NCTM (2007) points out, "the very essence of studying mathematics is itself an exercise in exploring, conjecturing, examining, and testing, and in building new mathematical knowledge" (p. 85), and "mathematics is a dynamic discipline that continues to grow and expand in its uses in our culture" (p. 121). Thus, teachers need to develop beliefs about mathematics that incorporate these aspects of the nature of mathematics.

Many of the beliefs that teachers have about mathematics are not limited to elementary and middle school teachers. The idea of mathematics as a procedure is emphasized throughout all levels of mathematics education, including the college level (Battista, 1994). Ball (1990) states:

> We are seeing few differences in ideas about mathematics between the elementary candidates and the mathematics majors. Most of the prospective teachers in both groups tended to see mathematics as a body of rules and facts, a set of procedures to be followed step by step, and they considered rules as explanations. Clearly, some secondary candidates did not see mathematics this way, while all the elementary candidates did. However, the overlap between the two groups was larger than one might expect, given that the secondary candidates had had far more opportunity to be immersed in the discipline than had the elementary candidates. (p. 464)

The only major difference between elementary and secondary teacher candidates was that the secondary teacher candidates were more confident in their abilities in mathematics than their elementary counterparts (Ball, 1990).

Research shows that a teacher's beliefs play a crucial role in how they teach and interact with students (Buehl, Alexander, & Murphy, 2002; Hofer & Pintrich, 1997). NCTM (2007) states, "their conceptions of mathematics shape their choices of worthwhile mathematical tasks, the kinds of learning environments they create, and the nature of the discourse in their classrooms" (p. 119). Teachers with beliefs about mathematics as a set of operations and rules have a positive correlation with an emphasis on performance rather than learning and a negative correlation with an emphasis on understanding (Stipek et al., 2001). Additionally, teachers who believe mathematics is a static body of knowledge are more likely to teach a drill and memorization class (Charalambous, Panaoura, & Philippou, 2009).

Evidence of the causal relationship between teacher beliefs and teaching style is found in a study conducted by Cross (2009) of five secondary Algebra I teachers. The study involved interviewing the teachers about their beliefs about mathematics and then observing their classroom environments. Three of the teachers viewed mathematics as computation, calculation, and formulas. The other two viewed mathematics as problem solving, learning how to think, and navigating one's way through problem situations. The views that these teachers held manifested themselves in their classroom practices.

The three teachers that viewed mathematics as rules and procedures ran classrooms devoid of collaborative activities. Classes tended to be teacher-centered lectures in which the teachers would elicit numeric or algebraic answers from students. Competence in mathematics was defined by performing accurate calculations and appropriately applying procedures while stating correct answers and detailing procedures were considered evidence of skill mastery (Cross, 2009).

In contrast, the teachers who viewed mathematics as thinking and problem solving conducted their classes more in line with current reform-oriented strategies. They asked more probing questions rather than computational questions, encouraged their students to be in charge of their own learning, and asked them to examine their thinking. These teachers believe that mathematics is a way of thinking and that students should be making their own sense and ideas about mathematics (Cross, 2009). This finding supports Dweck's (2007) work. Teachers who believe mathematics is about thinking and problem solving encourage their students to be in charge of their own learning and to put forth effort, and therefore promote a growth mind-set within the student.

Frank (1990) surveyed 131 pre-service teachers concerning twelve statements that he considers to be myths about mathematics. He defines a math myth "as a belief about mathematics that is (potentially) harmful to the person holding that belief because belief in math myths can result in false impressions about how mathematics is done" (p. 10). Of the pre-service teachers that Frank surveyed, 63% thought being good at mathematics is a natural ability, which agrees with Foss and Kleinsasser's (1996) study.

Consider the teachers who believe that being good in mathematics is a natural ability and view them in light of Dweck's (2007) article. Recall that what teachers believe effects their teaching and how they interact with students (Buehl, Alexander, & Murphy, 2002; Hofer & Pintrich, 1997). So a teacher who believes that mathematical intelligence is a natural ability would promote a fixed mind-set within the students.

Pre-service teachers come to college with preconceived beliefs about mathematics, shaped by thousands of hours in classrooms as students, and instead of changing their beliefs by the time they leave, they become more comfortable with those beliefs (Book, Byers, & Freeman, 1983; Feiman-Nemser & Buchmann, 1987; Feiman-Nemser, McDiarmid, Melnick, & Parker, 1988; Handal, 2003; Tabachnick & Zeichner, 1984; Weinstein, 1989). This is consistent with two assumptions made by teachers teaching grade school mathematics: that grade school provides most of the information that teachers need to know about mathematics, and that majoring in mathematics ensures knowledge of the subject matter (Ball, 1990; NCTM, 2007). These ideas are founded on the assumption that remembering and doing mathematics are tantamount to understanding mathematics (Ball, 1990). As Ball (1990) writes, “assuming that the content of first-grade mathematics is something any adult understands is to doom school mathematics to a continuation of the dull, rule-based curriculum that is so widely criticized” (p. 462).

Cuoco (2001) agrees with Ball that teachers are under the impression that what they learn in college is all they will need in order to teach mathematics effectively. This places the teacher in a situation where a student could ask a question that the teacher might not be able to answer. In order to teach effectively, Cuoco says that secondary mathematics teachers should become lifelong mathematics learners to help prepare them for the unexpected.

Mathematics educators agree that the way in which mathematics is presented to students has an effect on the students' beliefs (Muis, 2004). As Carter and Norwood (1997) state:

It is evident that what the teacher does in the classroom influences students'

> beliefs about mathematics. It is also evident that what teachers believe about mathematics and the teaching of mathematics influence what they do in the classroom and that their beliefs may be translated into students' beliefs. (p. 63)

Additionally, the ways in which prospective teachers and in-service teachers are taught plays a crucial role in their beliefs (NCTM, 2007). Therefore, it is vital that pre-service teachers learn about mathematics in a way that promotes problem solving and intellectual explorations in meaningful ways (NCTM, 2007). Thus, changing teachers' beliefs is essential in teacher development (Cooney, Shealy, & Arvold, 1998).

Changing Beliefs

Mason and Scrivani (2004) point out that studies that try to improve beliefs about mathematics are scarce. They were only able to find two studies dealing with changing student beliefs, Higgins (1997) and De Corte, Verschaffel, and Eynde (2000), both of which had positive results in improving mathematical beliefs among students. Mason and Scrivani were able to add to this body of literature with their study. However, all three of these studies dealt with students in the fifth grade or middle school, not with college students or, more specifically, pre-service secondary mathematics teachers.

With regard to pre-service teachers, there is also a scarcity of studies dealing with changing beliefs about mathematics. Szydlik, Szydlik, and Benson (2003) illustrate how they were able to alter the beliefs of pre-service elementary school teachers by focusing their problem solving class on different strategies, process reflection, and engagement in the process of exploration, conjecture, and argument. Focusing on these processes would be a good design for an introductory to advanced mathematics course.

Teaching Advanced Mathematics

Cuoco et al. (1996) suggest a mathematics curriculum for teaching future teachers in which the teaching of mathematics emphasizes the methods that mathematicians use when conducting research, or implementing a mathematician's "habits of mind." In this way, the students learn about the ways that mathematicians think and how to use that thought process to solve problems. In this curriculum students would be active in the creating, inventing, conjecturing, and experimenting processes of mathematics. Students would be encouraged to experiment, to explore special cases, and to learn that false starts are part of the mathematical experience. The mathematics of today fuels the technology of tomorrow, and so the goal of Cuoco et al. is to empower students for life after school and to "prepare them to be able to use, understand, control, modify, and make decisions about a class of technology that does not yet exist" (p. 401).

A mathematics course that revolves around ways of thinking rather than results benefits not only those who wish to pursue mathematics but also those who wish to pursue other fields of knowledge. For example mathematical research techniques can be used in fields such as automotive repair, medicine, business, or any other profession that involves problem solving. If students should learn to think like mathematicians, then teachers should learn to think like mathematicians (Cuoco et al., 1996).

One advantage of Cuoco and associates' suggestion is that it encourages students to continue to try to solve a problem that they find difficult. In this way, the curriculum will promote a growth mind-set within the students. Also, this curriculum places an emphasis on problem solving; Cross (2009) showed that teachers who view mathematics in this way are more likely to teach in a reform-oriented way.

Many prospective mathematics teachers are unable to make a connection between advanced mathematics classes and the courses that they took in high school and question how their college coursework in mathematics will help them in teaching mathematics (Cuoco, 2001; Fernandez & Jones, 2006). "This is especially true in algebra, where abstract algebra is seen as a completely different subject from school algebra" (Cuoco, 2001, p. 169). For example, teachers feel that abstract algebra has nothing to do with teaching because they do not talk about groups and rings in their classes (Hill, 2003). To exacerbate the problem, university faculty rarely make the connections for their students (Hill, 2003). Wilkins and Ma (2003) state:

> If American society hopes to reverse the negative trends related to attitudes and beliefs about mathematics, we evidently must provide positive experiences in classrooms and in the home that portray positive values related to mathematics and its importance in a quantitatively complex society. Teachers and school leaders have an important role in creating context that promote these ideals and values. (p. 61)

So pre-service secondary teachers need an experience with abstract algebra that will provide them an understanding of abstract structures, a structure high school algebra students should learn, that will help them make connections between high school and college mathematics (Cullinane, 2005; NCTM, 2000). There are resources for instructors of future teachers that can help make the necessary connections to school mathematics and provide teachers with a more positive experience of advanced mathematics. Cullinane (2005), Hill (2003), Grassl and Mingus (2007), and Fernandez and Jones (2006) provide examples of how to make connections between advanced mathematics and school mathematics that will be discussed later.

Abstract algebra plays an important role in learning advanced mathematics and therefore is important to any person learning advanced mathematics (Hazzan, 1999). However, abstract algebra is a difficult course for students; Grassl & Mingus (2007) list five reasons for the difficulty. First, they suggest that students lack the "necessary strong foundation on which to build" (p. 582). However, even when a student possesses a strong foundation they may still struggle with the course. This may be because most of a student's courses before abstract algebra have "a strong computational component and seem to lack an emphasis on proof" (Grassl & Mingus, 2007, p. 582).

Second, the "timetable in abstract algebra is ferocious" (p. 582). Students may cover several different topics each day and are expected to cover a plethora of content in only a few weeks. Too much of this content is new to the students and thus represents an obstacle for them.

Third, the "concepts in abstract algebra are by their nature abstract" (p. 583). In other words, students cannot simply solve a problem the same way the instructor solved it in class. There is not an algorithm that will help do a proof that the student can utilize. The students also have little opportunity "to practice communicating, either verbally or in writing, mathematical ideas prior to taking abstract algebra" (p. 583). In sum, students are underprepared for the abstractness of abstract algebra.

Finally, Grassl and Mingus (2007) point out that "the times they are a changing and so is the audience for the course…. [and] the methods of teaching and the content of the course have essentially remained constant" (p. 583). They note that it is no longer just the best mathematics students taking abstract algebra, as there are students taking the course who wish to pursue a career in public education and not necessarily mathematics. For this reason, the traditional form of lecturing might not be the best platform for teaching pre-service teachers.

Teaching Abstract Algebra

Mathematics teachers should interact more with other mathematicians to gain more understanding of the complex and changing subject that is mathematics (NCTM, 2007). Pre-service mathematics teachers should be presented with the struggles that accompany the search for an elegant proof (NCTM, 2007). Advanced mathematics courses, such as abstract algebra, can address these goals.

There are many ways to help pre-service mathematics teachers understand abstract algebra and to make the connections between abstract algebra and high school mathematics. As previously mentioned, Fernandez and Jones (2006), Hill (2003), Grassl and Mingus (2007), and Cullinane (2005) provide information on relating advanced mathematics to high school mathematics. Hazzan (1999) gives insight into the way students think about abstract algebra that can aid an instructor in teaching. These articles are highlighted because they show that abstract algebra can be taught in a way that makes connections to high school mathematics and can be beneficial to students.

Mathematics teachers of algebra should "know how to apply the major concepts of abstract algebra to justify algebraic operations and formally analyze algebraic structures" (NCTM, 2007). Fernandez and Jones (2006) provide recommendations for connecting undergraduate mathematics to the NCTM standards; they provide examples of implementing these techniques in a Methods in Teaching Mathematics course. While the authors had plans to cover teaching abstract algebra topics, they were unable to do so because some students had not yet had abstract algebra. They do, however, provide a list of topics covered in abstract algebra and their connections to the standards. The topics include permutation groups, homomorphisms, isomorphisms, the Euclidean algorithm, and Euler's function.

Hill (2003) describes the key to making connections between college mathematics and high school mathematics for pre-service teachers as making advanced mathematics relevant for them. Hill discusses two case studies of methods that he uses to address the disconnect. The first case study dealt with one of Hill's students doing student teaching. He had his student incorporate aspects of abstract algebra into her teaching. She did this by having the students develop the complex number system using distance from the origin and the angle made with the positive x-axis of the plane. Aspects of abstract algebra incorporated into the lesson include investigation, generalization, and use of axioms. The second case study dealt with a workshop in which teachers explored the symmetries of a regular hexagon. Through this exploration, teachers saw how symmetries are composed with each other, and then related symmetries to concepts such as permutation, function, one-to-one, onto, and commutativity, which ultimately led to a discussion of matrices of trigonometric functions to represent symmetries.

Grassl and Mingus (2007) took a different approach to helping students with abstract algebra. They decided to team teach a course with one of the instructors serving as a mathematician and the other a mathematics educator. The mathematics educator was able to provide leading questions regarding current material, make connections to other branches of mathematics, make worksheets to further cement ideas, and serve as a "watchdog for student speaking opportunities" (p. 584) (the mathematician was used to a lecture style of teaching). Through team teaching they were also able to employ cooperative learning groups and other constructivist methods that acted as examples for pre-service mathematics teachers in addition to producing better mathematicians.

Cullinane (2005) employed a method that made abstract algebra relevant to his students at the beginning of the course. Cullinane outlines his method for deriving the ideas of binary operation and the group axioms through solving linear equations. Cullinane also used a form of constructivist teaching by utilizing class discussion and in-class investigation.

Hazzan's (1999) article attempts to document the mental processes of students as they solve abstract algebra problems. More specifically, Hazzan investigates how students take the abstract concepts and reduce the abstraction to make them accessible. For example, Hazzan explains that students tend to use numbers to help themselves interpret the abstractions. He points out that this method is effective for the students but can be inappropriate in some situations and lead students astray, for example, when students think that a specific case or example is a proof. Thus, pre-service secondary mathematics teachers need advanced mathematics classes that provide them with insight into how advanced mathematics relates to what they will be teaching and makes the advanced mathematics more accessible for them.

Many of these examples are used to show how to make traditional abstract algebra relevant to the students and future teachers. However none of them consider using new mathematics as a way of implementing Cuoco’s suggestion of emphasizing the methods that mathematicians use to conduct research. Cuoco’s suggested method has the benefit of allowing the students (and future teachers) to have a connection to advanced mathematical work rather than just connecting to mathematics they did in high school.

The Mathematics

Foulser (1969) discusses Huppert's theorem in relation to finite solvable primitive permutation groups of rank greater than two. However, the mathematics that is of interest to this research is in Foulser's discussion in the third section of his paper. In this section he addresses many properties of extra-special groups of prime exponent, their actions on faithful irreducible modules, and the orbits resulting from these actions.

Foulser spends much of the section providing basic information about extra-special groups including their structure and their number of subgroups. He then describes the action of the extra-special groups on faithful irreducible modules by discussing which elements are fixed or stabilized. Ultimately, the section culminates by counting the number of regular orbits of the actions. However, he did not extend his result to encompass all types of extra-special groups. More information on the background of the problem of interest will be provided in Chapter 4, where the mathematics research is discussed.

Orbits play an important role in the proofs of many results. Their importance in proving major results is illustrated in articles from Moretó and Wolf (2004) and Dolfi and Pacifici (2011). However, in both of the articles, and in Foulser's as well, orbits take a backseat to the results that they help prove.

Summary

Students should be viewed as potential new mathematicians and if they do not have a positive outlook of what a mathematician is and what a mathematician does then it can be expected that there are going to be fewer mathematicians in the future. Sadly, students appear to have negative beliefs about mathematics and mathematicians as well as misconceptions about the work of mathematicians.

As Pajares states, beliefs are about a person's evaluation and judgment of experiences. Thus, teachers should teach mathematics in such a way that students are provided with worthwhile mathematical experiences that lead to an appreciation of mathematics. Studies show that when teachers have positive beliefs about mathematics they tend to teach accordingly. However, it has been shown that many teachers have negative beliefs about mathematics, which is also evidenced in their teaching. These beliefs are a product of years of education that teachers receive in school.

As educators of prospective teachers, it is important to provide these students with experiences that emphasize the problem solving, conjecturing, and creative thinking processes that go into doing mathematics. Cuoco and associates' suggestion of presenting mathematics from the prospective of a researcher is an example of way to emphasize many such experiences.

CHAPTER 3

RESEARCH DESIGN

The purpose of this study is to investigate the introduction of advanced mathematics with a mathematics research component in the MATH 3330, "Introduction to Advanced Mathematics" course at Texas State University-San Marcos. The study examines the effects that the introduction to mathematics research has on the beliefs that mathematics majors enrolled in this course hold of mathematicians and the nature of mathematics. Beliefs in this study follow Pajares' suggestion of being the evaluation and judgment of one's experiences. Thus we have provided a new experience to the students to see how they were affected. The research questions investigated are:

1. Do pre-service secondary mathematics teachers, pure mathematics majors, applied mathematics majors, and mathematics minors have different beliefs about mathematics?
2. Do mathematics majors who participate in MATH 3330 with a mathematics research component demonstrate greater changes to their beliefs about mathematics than mathematics majors who participate in a standard MATH 3330 course?
3. Do pre-service secondary mathematics teachers who participate in MATH 3330 with a mathematics research component demonstrate greater changes to their beliefs about mathematics than pre-service secondary mathematics

teachers who participate in a standard MATH 3330 course?

The study uses a quantitative-methods approach to answer the above questions with supplemental data in the form of interviews and written papers from the experimental class to enhance the findings by providing insight into the survey responses. The study is a quasi-experimental design and uses Likert type pre/mid/post-surveys with two convenient samples of mathematics majors that enrolled in different sections of MATH 3330. A higher score on the survey corresponds with a more reform-oriented belief system, in the context of education, about mathematics.

Authorization and Informed Consent

A letter (see appendix A) was attached to the pre-survey in order to inform the students of the study and notifying them that participation in the study was voluntary. Consent was considered given if the student completed and turned in the survey. Students were asked in the survey if they wished to opt out of having their reflections used in the study. Students who participated in the interviews were provided with another consent form (Appendix C) dealing specifically with the interviews.

Participants

Data were gathered over the course of two long semesters, Fall 2010 and Spring 2011. Since all mathematics majors at the university must take MATH 3330, a sample from this course could be considered an accurate representation of the mathematics major population of the university. The participants in this study were a convenient sample of the students that were in the two MATH 3330 classes in each semester. Table 2 summarizes the sampling information as well as includes the number of students who took all three surveys that were administered.

Table 2

Students Who Enrolled In Course and Students Who Took the Survey Three Times

	Treatment		Control	
Semester	In Course	All Surveys	In Course	All Surveys
Fall	22	10	29	10
Spring	20	12	23	8

Four students were suppose to be chosen to participate in interviews in each semester. They were chosen based on their pre-surveys and their major classification. Those four students who were chosen to participate in the interviews in the Fall semester included one pre-service teacher and one other mathematics major for both the highest pre-survey scores and the lowest pre-survey scores. Students were to be selected in a similar fashion during the Spring semester but student unwillingness to reply made it not possible. This resulted in three students being selected for interviews; all of which were pre-service teachers. Two of these had high pre-test scores. The initial selection process was designed so comparisons could be made between interviews of pure mathematics majors and pre-service secondary mathematics teachers as well as students with high and low pre-survey scores representatives.

Introduction to Advanced Mathematics

The course MATH 3330, Introduction to Advanced Mathematics, is a class that all mathematics majors are required to take. The university's website states that the course is “an introduction to the theory of sets, relations, functions, finite and infinite sets, and other selected topics. Algebraic structure and topological properties of Euclidean Space, and an introduction to metric spaces” (http://www.math.txstate.edu/degrees-programs/undergrad/ucourses.html). The department syllabus (Appendix H) provides more information about the course. The class is meant to give students an idea of what it means to do pure mathematics and how to begin to structure proofs. Generally this involves starting with certain axioms and definitions and proving elementary theorems. Students learn how to begin their proofs, to justify their reasoning, and to appreciate mathematical rigor. However, some of the context for the proofs is somewhat up to the instructor and could take on a flavor of the instructor's field of specialty. For example, if an instructor is an analyst, then the class curriculum may include an analysis slant or if the instructor is an algebraist, then the class curriculum may include more algebra. Instructors do provide fundamental mathematical concepts such as set-theory and relations in the course.

Data Sources

Quantitative data. All students in the experimental and control classes were asked to take a survey (see Appendix B) three times during the semester. The survey is a Likert-style instrument consisting of 62 statements with which the students will indicate their level of agreement. The levels of agreement include strongly disagree, partly disagree, disagree, agree, partly agree, and strongly agree and will have point values of 1 through 6 respectively. The highest score a student can get on the survey is 372 and the lowest is 62. A higher score is interpreted to mean the student holds a reform-oriented belief system about mathematics.

This survey has 56 items adapted from the Conceptions of Mathematics Inventory (CMI) from Grouws (1994). The statements are the same as those on the CMI; however, the original survey had a five-point Likert scale. This was changed so that neutral was not an option and the students had to choose between agreeing and disagreeing. These items are divided into seven subscales, described in Table 3, with eight items each. Half of the items are worded positively and half are worded negatively. In analyzing the responses, positive items will receive the scale value corresponding to the level of agreement and negative items will be reversed by subtracting the initial survey value from seven.

Table 3

CMI Subscales (Star and Hoffmann, 2005, p. 28)

Composition of Mathematical Knowledge. Mathematical knowledge is either concepts, principles, and generalizations or facts, formulas, and algorithms.

Structure of Mathematical Knowledge. Mathematics is structured either as a coherent system or a collection of isolated pieces.

Status of Mathematical Knowledge. Mathematics as either a dynamic field or a static entity.

Doing Mathematics. Doing mathematics is either a process of sense-making or a process of obtaining results.

Validating Ideas in Mathematics. Validating ideas in mathematics occurs either through logical thought or via mandate from an outside authority.

Learning Mathematics. Learning mathematics is either a process of constructing and understanding or a process of memorizing intact knowledge.

Usefulness of Mathematics. Mathematics is viewed as either a useful endeavor or as a school subject with little value in everyday life or future work.

Star and Hoffmann (2005) point out that Grouws does not report any statistical reliability of the survey; however, they state that their study had Cronbach's alpha reliability values ranging from 0.26 to 0.87 while another study had Cronbach's alpha reliability values ranging from 0.45 to 0.91. There was no mention of factor analysis being performed on the data. From the pre-survey it was found that the whole survey had a Cronbach's alpha of 0.91, the subscale Status had 0.65, Usefulness had 0.74, Composition had 0.72, Learning had 0.63, Doing had 0.63, Validation had 0.66, and Structure had 0.58.

The final 6 items on the survey measured students' beliefs about mathematicians and developed for the purposes of this study. These items consist of two positively worded questions and four negatively worded questions. They were scored in the same fashion as the items from the CMI. This part of the survey had a Cronbach's alpha reliability score of 0.48 from the pre-survey.

Supplemental data. Supplemental data were collected in the form of written papers and interviews. There were four written papers collected from every student in the treatment group participating in the study. The first paper was assigned on the first day of class and is a narrative about each student's mathematical journey. It was designed to supplement and inform their survey responses. Topics that the students were asked to include were how they became interested in mathematics, why they decided to major in mathematics or why they want to teach mathematics, and their beliefs on the nature of mathematics. The other three papers were assigned at the end of each week of the treatment for the students to reflect on the presentation of mathematics research. These papers are included to provide insight into how the students' beliefs changed during the intervention. The students were asked to include any insights they gathered about mathematics, how the presentations related to topics discussed earlier in the week, and what they learned from the presentations.

Interviews were conducted throughout the semester to explain missing data from their written papers, responses to the survey, and how their conceptions have changed by the intervention. Four students were interviewed over the course of the Fall semester and three over the course of the Spring semester. Three of the four students in the Fall semester were interviewed once at the beginning and once at the end of the semester. The fourth student did not reply to inquiries at the end of the semester and so was only interviewed at the beginning of the semester. In the Spring semester only one student was interviewed at the beginning and end of the semester. One of the other two students was interviewed at the beginning of the semester but did not respond to contact at the end of the semester. The last student was added at the end of the semester to have at least two students interviewed at the end of the semester.

Each interview lasted between 30 to 45 minutes depending on how the student responded to questions. An informal interview guide (Appendix D) was used to help conduct the first interviews. More questions were added for each individual student based on their responses on the pre-survey and the mathematics biography paper assigned on the first day of class. An interview guide for each student was developed for the second interviews that were based on all their responses to all three surveys, all their reflection papers and the first interview.

Procedure

The course used for the study was MATH 3330, Introduction to Advanced Mathematics. This course was chosen because it is one of the first courses where students encounter advanced mathematics topics. The course was team taught with a mathematics professor at Texas State University, who was the instructor of record.

Figure 1 illustrates how the semesters were structured. The control and treatment groups had different instructors of record and they switched roles after the Fall semester. So each instructor had a treatment class and a control class. The first day of class was very typical as students were introduced to the class syllabus, the researcher, and the instructor of record. At this point the pre-survey was administered with a letter of consent that describes the study and informed the students of their voluntary participation in the study. The students were then told of their first writing assignment, the mathematics biography narrative, and given a week to complete it. The pre-survey was also administered in the control class on their first day.

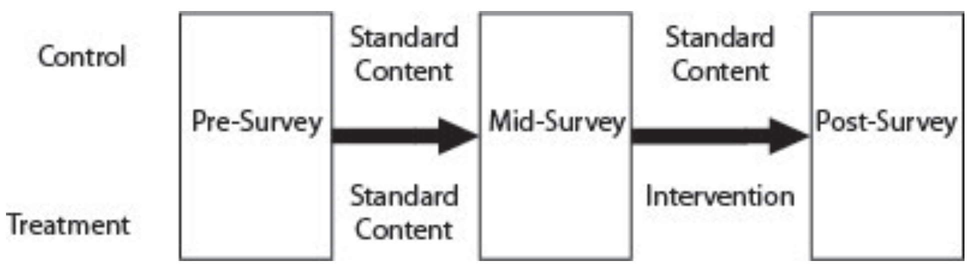

Figure 1. Semester Structure.

The data from the pre-survey were entered into the computer, the scores for negative worded questions were changed according to the method stated above, and a total score for each student's survey was calculated. Two students, one of which was seeking secondary certification, whose pre-survey scores were among the highest and two students, one of which was seeking secondary certification, whose scores were among the lowest, were selected during the Fall semester for interviews. In the Spring a similar process was attempted but either students dropped the course without notification (before the first interview), never showed up for the interview, or never responded to the inquiry. In the Fall semester four students were interviewed within a month of the beginning of school and two students in the Spring.

The instructor of record taught the first 11 weeks of the 14-week course. The last three weeks were used for the intervention. The intervention was put at the end of the course because the students would need the content from the course to help them with the advanced nature of the mathematics research that was presented. At the start of the last three weeks of the semester the mid-survey was administered in the treatment and control classes. This was done to control for any changes that might be attributed to the instructor of record teaching the course.

Table 4

Weekly Topics

	Monday	Friday
Week 1	Groups, Subgroups	Extra-Special p-Groups, Elementary p-Groups
Week 2	Counting	Number of Abelian Subgroups of an Extra-Special p-Group
Week 3	Isomorphisms	Number of Regular Orbits of the Action and Research Process

The study then followed a weekly format that included a content component at the beginning of the week and a research presentation component on the last day of the week. The content component provided students with fundamental advanced mathematical knowledge relevant to the mathematics research that was presented at the end of the week. The presentations at the end of the week were of the mathematics research that the researcher had done. The homework following the presentation consisted of students writing a reflection related to the information from the past week. Table 4 provides a detailed list of topics for each week. Chapter 5 discusses what was in the lesson plans and presentations, how they were implemented in the classes, and changes made between the semesters. To see the final lesson plans see Appendix E.

For example, on the first day of the week the lesson covered the definitions of terms such as a group, abelian, and subgroup along with other relevant information, like examples. The research that was presented to the students on Friday consisted of the definition of an extra-special group, exponent, and the various structures of the groups that were relevant to the mathematics research.

This purpose for using this type of intervention was based on Cuoco and associates' (1996) suggestion of including a mathematician's habit of mind through the inclusion of mathematics research in mathematical teaching. Mathematics research can illustrate the hard work that is done in order to do mathematics, leading the students to a growth mind-set described in Dweck's (2007) article. Also, the students will get a glimpse of mathematics that they might not have seen before and thus enriching their mathematical beliefs.

On the last day of the semester the post-survey was administered to the treatment and control classes. In the following days the students who were interviewed at the beginning of the semester were contacted for a second interview. Three of the students in the Fall responded and were interviewed while one of the students in the Spring responded. An attempt was made to reach out to other students in the Spring treatment class for interviews with one student responding.

Data Analysis

Quantitative data. Data from the pre-survey were analyzed first using an independent samples t-test to determine whether there was any statistical difference between the control and treatment groups at the beginning of each semester and in the study as a whole. The same was done with each subscale defined above. Additionally averages and standard deviations for each question were found to give an idea where the students lie for each question and to see how close the treatment and control groups are for each question.

The data from the three administrations of the survey were used to answer each of the research questions outlined at the beginning of this chapter. The null hypotheses associated with the research questions in respective order are:

1. Pre-service secondary mathematics teachers, pure mathematics majors, applied mathematics majors, and mathematics minors have the same beliefs about the mathematics.
2. Mathematics majors who participate in MATH 3330 from the perspective of mathematics research demonstrate the same changes to their beliefs about mathematics as mathematics majors who participate in a standard MATH 3330 course.
3. Pre-service secondary mathematics teachers who participate in MATH 3330 from the perspective of mathematics research demonstrate the same changes to their beliefs about mathematics as pre-service secondary mathematics teachers who participate in a standard MATH 3330 course.

Question 1 was investigated using ANOVA on the data from the pre-surveys. The response variable was the pre-survey score and the predicting factor was the major. The majors were coded categorically and students are either a pre-secondary mathematics teachers, pure mathematics majors, applied mathematics majors, or mathematics minors.

Questions 2 and 3 were first investigated using an independent samples t-test to determine if there was a significant difference between survey scores of the treatment group and the control group. Then a paired t-test between each administration of the survey as a total score and the subscales was performed. All pairs of administrations were analyzed in both groups to determine if there were any statistically significant changes at the .05 level.

Additional analysis was done to determine if there were any significant differences between an instructor's treatment and control class. An independent samples t-test was used to determine if there is a significant difference between their treatment and control groups across each survey and subscale.

Supplemental data. All papers were copied and then returned to the students. All interviews were recorded and then transcribed by an outside source. The interviews and papers were read twice and coded using the subscales from the survey. For example if a student mentioned something about how mathematics is done then that quote would be marked as "Doing" so that it could be used later when analyzing survey data.

The data were then reread looking for any illuminating or unexpected information. Several new themes presented themselves during the reading. Some of these include connections to Dweck's work, information on how the treatment affected future plans, and curiosity about what the students' professors are researching. After each new theme presented itself the interviews and papers were read again to see if that new theme presented itself somewhere else.

Summary

Using the quantitative data along with the supplemental data allowed for a richer examination of the students' beliefs. Unfortunately, several obstacles prevented the interview data from being as robust as it might have been. Nevertheless, the surveys, papers, and interviews actually obtained did allow the research questions to be thoroughly investigated.

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CHAPTER 4

THE MATHEMATICAL RESEARCH ON EXTRA-SPECIAL GROUPS

In this chapter we will discuss the mathematics research and proofs of the results that I presented to students in the MATH 3330 course. It is assumed that the reader has an Algebra knowledge base of at least a first semester graduate Algebra course in Group Theory. It should be noted that in the presentation to the students the faithful irreducible $GF(q^k)$-module discussed here is referred to as an elementary abelian group. These two descriptions of the group are equivalent, but the elementary abelian description was used to simplify the discussion for the benefit of the students. However it will be referenced as a $GF(q^k)$-module in this chapter to keep it more in line with the texts used in the research literature. Formal definitions will follow the introduction in the main sections below.

Let P be an extra-special p-group of order p^{2m+1} acting on a faithful irreducible $GF(q^k)$-module of dimension p^m, where p and q are prime, m is a natural number, and k is the smallest natural number for which p divides $q^k - 1$. The parameter k will be revisited later in the chapter.

Foulser (1969) counted the regular orbits of the action of P on such a $GF(q^k)$-module, when P is of exponent p. However, he did not consider the case where the extra-special p-group has exponent p^2 for odd p. In this chapter, I will extend Foulser's results to the exponent p^2 case, when p is odd.

All extra-special p-groups have the same basic structure. When $m = 1$, the order is p^3, and we get the following structures as derived by Doerk and Hawkes (1992, p. 79):

- If P is of exponent p, then $P = \langle x, y : x^p = y^p = 1, [x, y] \in Z(P) \rangle$

- If P is of exponent p^2, then $P = \langle x, y : x^p = y^{p^2} = 1, [x, y] = y^p \in Z(P)\rangle$.

They then use these to explain the structure of P for any m. When P has exponent p^2, then P is a central product of m extra-special p-groups of order p^3 of which $m - 1$ have exponent p and one has exponent p^2. When P is of exponent p, then it is a central product of m extra-special p-groups of order p^3 and exponent p. Thus the structure of the extra-special p-group of exponent p^2 is very similar to that of the extra-special groups of exponent p. This is the basis for my approach of adapting Foulser's formula to the exponent p^2 case.

To summarize, the main result of this chapter is to derive an exact formula for the number of regular orbits of an extra-special p-group of exponent p^2 acting on a faithful irreducible $GF(q^k)$-module, which is formally stated as Corollary 42 toward the end of the chapter. Most definitions pertaining to this result will be provided in the text. However, action and orbit will be discussed briefly now because of the prominent role they play in the result. When for each g in a group G and u in a set U there is a unique element $u \cdot g \in U$ such that $u \cdot 1 = u$ and $(u \cdot g) \cdot h = u \cdot (gh)$ for all $u \in U$ and $g, h \in G$ we say that G *acts* on U. If G acts on U then an *orbit* is the set $\{u \cdot g | g \in G\}$ which is a subset of U. It turns out that the orbits partition U into disjoint subsets. In my case I am interested in *regular orbits*, orbits whose cardinality is equal to the order of the acting group.

Orbits play an important role in mathematics by assisting in the proving of many results. For example, the formula for the number of regular orbits that Foulser derived was not the final result, but was a result used in the process of proving the main result. In addition to the examples given in the Chapter 2, other examples of recent work involving orbits include Keller's (2005) work on the $k(GV)$-problem and Wilson's (2009) work involving p-groups and Jordan algebras.

Elementary Abelian Subgroups

Before we consider the action of interest we will look at the number of elementary abelian subgroups in an extra-special p-group of length m and exponent p^2. These subgroups were shown by Foulser (1969) to be important to counting the orbits of the action in the exponent p case and are just as important in the exponent p^2 case. Before we go any further we will formally define some terms that have already been used and terms that will be used. Also, note that $p > 2$ and q are always distinct prime numbers. In the second definition $\Phi(P)$ is known as the *Frattini subgroup* of P and $Z(P)$ is the *center* of P. The set $P' = \langle [x, y] | x, y \in P \rangle$ is the *commutator subgroup* of P where $[x, y] = x^{-1}y^{-1}xy$. It is a well-known fact that for extra-special groups even $P' = \{[x, y] | x, y \in P\}$.

Definition 1. If G is a finite group, then the smallest positive integer n such that $g^n = 1$ for all $g \in G$ is called the *exponent* of G.

Definition 2. Let P be a p-group. P is called *extra-special* if $\Phi(P) = P' = Z(P)$ and $Z(P)$ has order p. Thus $Z(P) = \langle z \rangle$ for any $1 \neq z \in Z(P)$. Extra-special p-groups have order p^{2m+1} and either exponent p or p^2 (Doerk & Hawkes, 1992, p. 78). Following the work of Foulser, m will be defined as the *length* of the extra-special p-group.

Definition 3. Let G be a group. Then G is a *central product* of normal subgroups H_i, where i is a natural number with $1 \leq i \leq \alpha$, for some natural number α if $G = \Pi_{i=1}^{\alpha} H_i$, $\bigcap_{i=1}^{\alpha} H_i = Z(G)$, and $h_i h_j = h_j h_i$ for all $h_i \in H_i$ and $h_j \in H_j$ when $i \neq j$. Write $G = H_1 \circ H_2 \circ ... \circ H_m$ in this situation.

The first three definitions set up many of the important constructs that will be discussed. However informal definitions will be provided as well throughout and the reader is encouraged to seek out other material for more descriptive definitions. We now restate information from the introduction as a formal lemma so that it can be referenced throughout the section.

Lemma 4. Let P be an extra-special p-group of length 1. If P is of exponent p, then

$$P = \langle x, y : x^p = y^p = 1, 1 \neq [x, y] \in Z(P) \rangle$$

and if P is of exponent p^2, then

$$P = \langle x, y : x^p = y^{p^2} = 1, 1 \neq [x, y] = y^p \in Z(P) \rangle$$

(Doerk & Hawkes, 1992, p. 79). Note that when it is understood which particular extra-special p-group of length 1 we are talking about we can simply write $P = \langle x, y \rangle$.

The following two lemmas, which again formally give information that was in the introduction, will provide important facts about the structures of these extra-special p-groups, mainly that they are central products. Lemma 6 shows that in the case of the group being of exponent p^2 we can make the central product look almost identical to the exponent p group but change one subgroup in the central product to an extra-special p-group of exponent p^2.

Lemma 5. Let P be an extra-special p-group of exponent p and length m. Then P is the central product of m extra-special p-groups of exponent p of length 1. Thus, we can write $P = P_1 \circ P_2 \circ ... \circ P_i \circ ... \circ P_{m-1} \circ P_m$, where $P_i = \langle x_i, y_i \rangle$ is an extra-special p-group of exponent p with length 1 for $1 \leq i \leq m$.

Proof. See Foulser (1969, p. 7-8) Lemma 3.4. □

It is important to understand where the center of P, $Z(P)$, is in the central product described in Lemma 5. We know from Definition 2 that the center is of order p. From

Definition 3 we know that for all $z \in Z(P_i)$, for any i, and $g \in P$ such that $g \notin P_i$, $gz = zg$. Then we see that $Z(P_i)$ is contained in $Z(P)$ and thus $Z(P_i) = Z(P)$. This same idea is also true in the central product that we see in the following lemma, even though by Lemma 4 it might appear that P_m has a different center than P_i for $i < m$ in Lemma 6.

Lemma 6. Let P be an extra-special p-group of exponent p^2 and length m. Then P is the central product of $m-1$ extra-special p-groups of exponent p of length 1 and one extra-special p-group of exponent p^2 of length 1 (Doerk & Hawkes, 1992, p. 79).

As mentioned earlier, an extra-special p-group P of exponent p^2 and length m can be written as the following central product $P = P_1 \circ P_2 \circ ... \circ P_i \circ ... \circ P_{m-1} \circ P_m$, where $P_i = \langle x_i, y_i \rangle$ is an extra-special p-group of exponent p with length 1 for $1 \leq i \leq m-1$ and $P_m = \langle x_m, y_m \rangle$ is an extra-special p-group of exponent p^2 with length 1. It should be noted that for any P of exponent p^2 the generators for P_m, x_m and y_m, will be fixed. This enables us to simplify the counting process by referencing x_m and y_m without them being arbitrary suitable generators. Thus when discussing extra-special p-groups of exponent p^2, y_m is the generator of order p^2 for the remainder of the chapter. Also, when $m = 1$ the central product $P_1 \circ P_2 \circ ... \circ P_{m-1}$ is an empty central product, as $m - 1 = 0$, and will be defined for our purposes as $Z(P)$.

Notation 7. Let P be an extra-special p-group of length m. Let the $P_i \leq P$ be as in Lemma 5 and 6. Let $r \in \{0, ..., m\}$. When P has exponent p, let $\mathscr{S}_{r,m}$ denote the set of elementary abelian subgroups of P such that if $S \in \mathscr{S}_{r,m}$, then $|S| = p^{r+1}$ and $Z(P) \subseteq S$. When P has exponent p^2 and $S > 1$ we say that $S \leq P$ is of type $j = 1$ when $S \subseteq P_1 \circ ... \circ P_{m-1}$, of type $j = 2$ when $S \not\subseteq P_1 \circ ... \circ P_{m-1}$ and $x_m \in S$, and of type $j = 3$ when $S \not\subseteq P_1 \circ ... \circ P_{m-1}$ and $x_m \notin S$. Furthermore, let $\mathscr{S}^j_{r,m}$ denote the set of elementary abelian subgroups of P such that if $S \in \mathscr{S}^j_{r,m}$, then $|S| = p^{r+1}$ and $Z(P) \subseteq S$ and S is of type $j \in \{1, 2, 3\}$. We call r the length of any such S. Let $C_{r,m} = |\mathscr{S}_{r,m}|$ and $C^j_{r,m} = |\mathscr{S}^j_{r,m}|$.

Notation 7 sets up many of the different sets that will be referenced often throughout the chapter. Whenever $\mathscr{S}_{r,m}$ is referenced then an extra-special p-group of exponent p is being discussed. When $\mathscr{S}^j_{r,m}$ is referenced then we are discussing extra-special p-groups of exponent p^2. Furthermore, if S is an element of either set, then S contains the center of the respective extra-special p-group, P, for which it is a subgroup. Thus, as $\langle z \rangle = Z(P)$, S has z as one of its generators.

Lemma 8. Let $S \in \mathscr{S}_{r,m}$ for $1 \leq r \leq m$. Then:

1. There exist elements $s_1, ..., s_r \in P - Z(P)$ such that $S = \langle s_1, ..., s_r, z \rangle$ where the order of s_i is $p, 1 \leq i \leq r$.

2. Moreover, there exist subgroups $P_1, ..., P_m$ of P (each P_i is an extra-special p-group of exponent p and of length 1) such that $s_i \in P_i$ for $1 \leq i \leq r$, and $P = P_1 \circ ... \circ P_m$.

Proof. See Foulser (1969, p. 11-12) Lemma 3.17. □

Corollary 9. Let $S \in \mathscr{S}^1_{r,m}$ for $1 \leq r \leq m$. Then:

1. There exist elements $s_1, ..., s_r \in P - Z(P)$ such that $S = \langle s_1, ..., s_r, z \rangle$ where the order of s_i is $p, 1 \leq i \leq r$.

2. Moreover, there exist subgroups $P_1, ..., P_{m-1}$ of P (each P_i is an extra-special p-group of exponent p) such that $s_i \in P_i$ for $1 \leq i \leq r$, and $P = P_1 \circ ... \circ P_m$. Thus $\mathscr{S}^1_{m,m} = \emptyset$.

Proof. As $P_1 \circ ... \circ P_{m-1}$ is an extra-special p-group of exponent p, this follows from Lemma 8. Also there does not exist a subgroup $S \leq P_1 \circ ... \circ P_{m-1}$ of order p^{m+1}. Thus $\mathscr{S}^1_{m,m} = \emptyset$. □

Lemma 8 and Corollary 9 might look the same but remember that we are talking about extra-special p-groups with different exponents. In Lemma 8 Foulser shows that these subgroups exist and how the generators relate to the larger group. Corollary 9 is part of an extension to the exponent p^2 case.

Lemma 10. Let P be an extra-special p-group of exponent p^2 of length m. Then for any integer α we have $(y_m)^{\alpha p} \in Z(P)$, and $(x_m)^{\alpha_1}(y_m)^{\alpha_2}$ is of order p^2 for all positive integers α_1 $(1 \leq \alpha_1 \leq p)$ and α_2 $(1 \leq \alpha_2 \leq p^2 - 1)$ such that p does not divide α_2.

Proof. First consider that $[x_m, y_m] = z \in Z(P)$. Thus $(x_m)^{-1}(y_m)^{-1}x_m y_m = z$ and so $x_m y_m = y_m x_m z$. Thus, by induction, and the fact that the order of z is p by Definition 2, it can be shown that $x_m(y_m)^p = (y_m)^p x_m(z)^p = (y_m)^p x_m$. Thus, $(y_m)^p \in Z(P)$ and so $(y_m)^{\alpha p} \in Z(P)$.

Suppose p does not divide α_2. Then, either $(x_m)^{\alpha_1} = 1$ or $(x_m)^{\alpha_1} \neq 1$. If $(x_m)^{\alpha_1} = 1$ then $(x_m)^{\alpha_1}(y_m)^{\alpha_2} = (y_m)^{\alpha_2}$. Now consider $((y_m)^{\alpha_2})^p = (y_m)^{\alpha_2 p}$. Since p does not divide α_2, p^2 does not divide $\alpha_2 p$. Thus as the order of y_m is p^2, $(y_m)^{\alpha_2 p} \neq 1$. Thus the order of $(y_m)^{\alpha_2}$ is p^2.

Suppose $(x_m)^{\alpha_1} \neq 1$ and consider $((x_m)^{\alpha_1}(y_m)^{\alpha_2})^p$. By induction it can be shown that

$$((x_m)^{\alpha_1}(y_m)^{\alpha_2})^p = ((x_m)^{\alpha_1})^p((y_m)^{\alpha_2})^p(z)^{\frac{p(p-1)}{2}\alpha_1\alpha_2} = (y_m)^{\alpha_2 p} \neq 1.$$

Thus the order of $(x_m)^{\alpha_1}(y_m)^{\alpha_2}$ is p^2. □

Definition 11. Let P be a p-group. Define $\Omega_1(P)$ to be the group generated by all elements $g \in P$ such that $g^p = 1$.

It is easy to see that if P is of exponent p then $\Omega_1(P) = P$. However when P is of exponent p^2 this is not true. Lemma 12 shows what $\Omega_1(P)$ is in this case. For further information on $\Omega_1(P)$ the reader is directed to Isaacs' *Finite Group Theory* book (2008, p.

120). Also, in Lemma 13 we see $C_P(x_m)$, which is the *centralizer* of $\langle x_m\rangle$ in P.

Lemma 12. Let P be an extra-special p-group of exponent p^2 and length m. Then $\langle x_1, y_1\rangle \circ ... \circ \langle x_{m-1}, y_{m-1}\rangle \times \langle x_m\rangle = \Omega_1(P)$ for $m \geq 2$ and $\langle x_1\rangle \times Z(P) = \Omega_1(P)$ for $m = 1$.

Proof. Let $m \geq 2$. Clearly $\langle x_1, y_1\rangle \circ ... \circ \langle x_{m-1}, y_{m-1}\rangle \times \langle x_m\rangle \subseteq \Omega_1(P)$. Now suppose $\alpha \in \Omega_1(P)$ such that $\alpha \notin \langle x_1, y_1\rangle \circ ... \circ \langle x_{m-1}, y_{m-1}\rangle \times \langle x_m\rangle$. Since $\alpha \notin \langle x_1, y_1\rangle \circ ... \circ \langle x_{m-1}, y_{m-1}\rangle \times \langle x_m\rangle$, $\alpha = sx_m^i y_m^j$ for some $s \in \langle x_1, y_1\rangle \circ ... \circ \langle x_{m-1}, y_{m-1}\rangle$ and j not a multiple of p (if j was a multiple of p then $y_m^j \in Z(P)$). Recall that $[x_m, y_m] = z$, so $x_m^{-1} y_m^{-1} x_m y_m = z$. Then $x_m y_m = y_m x_m z$. This, with $\alpha \in \Omega_1(P)$, means that $1 = \alpha^p = (sx_m^i y_m^j)^p$. Since s commutes with x_m and y_m and as we saw in Lemma 10 we get $(sx_m^i y_m^j)^p = s^p \cdot x_m^{ip} \cdot y_m^{jp} \cdot z^{\frac{p(p-1)}{2} ij} = y_m^{jp}$. This is a contradiction since the order of y_m is p^2 and j is not a multiple of p. Hence such an α does not exist.

In the case that $m = 1$ we see that $\langle x_1\rangle \times Z(P) \subseteq \Omega_1(P)$. Also, if we assume that $\alpha \in \Omega_1(P)$ such that $\alpha \notin \langle x_1\rangle \times Z(P)$ then $\alpha = zx_1^i y_1^j$ for some j not a multiple of p and $z \in Z(P)$ by an argument identical to the case when $m \geq 2$. Such an α does not exist, again using an argument identical to the $m \geq 2$ case and the lemma is proved. □

Lemma 13. Let $S \in \mathscr{S}_{r,m}^2$ for some $1 \leq r \leq m$. Then there exist elements $s_1, ..., s_{r-1} \in (P_1 \circ ... \circ P_{m-1}) - Z(P)$ such that $S = \langle s_1, ..., s_{r-1}, x_m, z\rangle$. Furthermore there exist subgroups $P_1^*, ..., P_{m-1}^*$ of P (each P_i^* is an extra-special p-group of exponent p and of length 1) such that $s_i \in P_i^*$ for $1 \leq i \leq r - 1$ and $P = P_1^* \circ ... \circ P_{m-1}^* \circ P_m$.

Proof. Since $x_m \in S$, then x_m will be used as a generator for the subgroup. Thus the remaining generators can be chosen from $P_1 \circ ... \circ P_{m-1}$ and the result follows from Lemma 8. □

Lemma 14. Let $S \in \mathscr{S}_{r,m}^3$ for some $1 \leq r \leq m$. Then there exist elements

$s_1, ..., s_r \in (P_1 \circ ... \circ P_{m-1}) - Z(P)$ with $\langle s_1, ..., s_r, z\rangle \in \mathscr{S}^1_{r,m}$ such that $S = \langle s_1(x_m)^{j_1}, ..., s_r(x_m)^{j_r}, z\rangle$ where j_i is a positive integer with $1 \leq j_i \leq p$ for all i with at least one $j_i \neq p$.

Proof. First, since $S \in \mathscr{S}^3_{r,m}$, then $x_m \notin S$. Thus, if $m = 1$, then x_m has to be in S. Thus this lemma applies only when $m > 1$.

Note that for non-negative integers α_1 and α_2, if $(y_m)^{\alpha_2} \notin Z(P)$, then $(x_m)^{\alpha_1}(y_m)^{\alpha_2} \notin S$ as it has order p^2 by Lemma 10. Since $S \nsubseteq P_1 \circ ... \circ P_{m-1}$, S must contain elements that do not commute with y_m (because $C_P(y_m) = P_1 \circ ... \circ P_{m-1} \cdot \langle y_m\rangle$). Hence S contains elements of the form sx_m^j for some $1 \neq s \in P_1 \circ ... \circ P_{m-1}$ and $1 \leq j \leq p-1$. Also, as S is elementary abelian, $S \leq \Omega_1(P) = P_1 \circ ... \circ P_{m-1} \times \langle x_m\rangle$ and thus all of the elements of S have the form sx_m^j for some $1 \neq s \in P_1 \circ ... \circ P_{m-1}$ and $1 \leq j \leq p$. Take $s_1, ..., s_r \in (P_1 \circ ... \circ P_{m-1}) - Z(P)$ so that $S = \langle s_1(x_m)^{j_1}, ..., s_r x_m^{j_r}, z\rangle$ with $0 \leq j_i \leq p$ and at least one $j_i \neq p$. As $s_i(x_m)^{j_i} s_h(x_m)^{j_h} = s_h(x_m)^{j_h} s_i(x_m)^{j_i}$ for all integers i, h we get $s_i s_h (x_m)^{j_i + j_h} = s_h s_i (x_m)^{j_i + j_h}$ because $C_P(x_m) = P_1 \circ ... \circ P_{m-1} \times \langle x_m\rangle$. This implies $s_i s_h = s_h s_i$ for all i and h. Thus $\langle s_1, ..., s_r, z\rangle$ is abelian and so $\langle s_1, ..., s_r, z\rangle \in \mathscr{S}^1_{r,m}$. Thus the Lemma is proved. □

Lemmas 13 and 14 together with Corollary 9 for the exponent p^2 case is analogous to Lemma 8 for the exponent p case. To summarize what is said in Corollary 9, Lemma 13, and Lemma 14, an elementary abelian p-group S of an extra-special p-group, $P = P_1 \circ ... \circ P_m$, of exponent p^2 is either contained in $P_1 \circ ... \circ P_{m-1}$ or it is not. If it is, then we are in we are in the setting of Corollary 9 and we can use Lemma 8. If S is not in $P_1 \circ ... \circ P_{m-1}$ then it is contained in $P_1 \circ ... \circ P_{m-1} \times \langle x_m\rangle$, or $\langle z\rangle \times \langle x_1\rangle$ when $m = 1$, since we know that if y_m is used we will have elements of order p^2. So Corollary 9, Lemma 13, and Lemma 14 account for all the elementary abelian p-subgroups that contain the center of an extra-special p-group of exponent p^2, as Lemma 8 does for an extra-special p group of exponent p.

Lemma 15. Let $S = \langle s_1(x_m)^{j_1}, ..., s_i(x_m)^{j_i}, ..., s_r(x_m)^{j_r}, z\rangle$ and $S^* = \langle s_1^*(x_m)^{j_1^*}, ..., s_i^*(x_m)^{j_i^*}, ..., s_r^*(x_m)^{j_r^*}, z\rangle$ be in $\mathscr{S}^3_{r,m}$, where j_i is a positive integer with $1 \leq j_i \leq p$ for all i with at least one $j_i \neq p$, where j_i^* is a positive integer with $1 \leq j_i^* \leq p$ for all i with at least one $j_i^* \neq p$, $\langle s_1, ..., s_r, z\rangle \in \mathscr{S}^1_{r,m}$, and $\langle s_1^*, ..., s_r^*, z\rangle \in \mathscr{S}^1_{r,m}$. Then the following hold:

1. If $\langle s_1, ..., s_r, z\rangle \neq \langle s_1^*, ..., s_r^*, z\rangle$, then $S \neq S^*$.

2. If $s_i = s_i^*$ for all i and $j_i \neq j_i^*$ for at least one i, then $S \neq S^*$.

Proof. For Part 1 assume that $S = S^*$. Then there is a generator $s_i^* x_m^{j_i^*}$, for some i, $1 \leq i \leq r$, of S^* such that $s_i^* \notin \langle s_1, ..., s_r, z\rangle$ and $s_i^*(x_m)^{j_i^*} = z^l \prod_{\alpha=1}^r (s_\alpha(x_m)^{j_\alpha})^{l_\alpha} = z^l(x_m)^{L_1} \prod_{\alpha=1}^r (s_\alpha)^{l_\alpha}$ for some non-negative integers l, l_α and $L_1, 1 \leq L_1 \leq p$. Then $s_i^* = z^l(x_m)^{L_2} \prod_{\alpha=1}^r (s_\alpha)^{l_\alpha}$ for some natural number L_2 with $1 \leq L_2 \leq p$.

Either $L_2 = p$ or $L_2 \neq p$. If $L_2 = p$ then $s_i^* \in \langle s_1, ..., s_r, z\rangle$, a contradiction. If $L_2 \neq p$ then $s_i^* \notin P_1 \circ ... \circ P_{m-1}$, a contradiction. Thus $S \neq S^*$.

For Part 2 again assume $S = S^*$. Note that

$$\langle s_1(x_m)^{j_1}, ...s_i(x_m)^{j_i}, ..., s_r(x_m)^{j_r}, z\rangle = \langle s_1(x_m)^{j_1^*}, ..., s_i(x_m)^{j_i^*}, ..., s_r(x_m)^{j_r^*}, z\rangle$$

and $j_\alpha \neq j_\alpha^*$ for some α such that $1 \leq \alpha \leq r$. Without loss, assume that $j_\alpha < j_\alpha^*$ so $j_\alpha^* = j_\alpha + \beta$ for some positive integer β such that $0 < \beta < p$. Then $s_\alpha(x_m)^{j_\alpha^*} = s_\alpha(x_m)^{j_\alpha}(x_m)^\beta$. Thus $(s_\alpha(x_m)^{j_\alpha})^{-1}s_\alpha(x_m)^{j_\alpha}(x_m)^\beta = (x_m)^\beta \in S$, which implies $x_m \in S$, a contradiction. Thus $S \neq S^*$. □

In the following Lemmas we are interested in counting the number of subgroups discussed in Lemma 8, Corollary 9, Lemma 13, and Lemma 14. If you look at Lemma 14 and 15 you can pick two things to change in the construction of a subgroup S, either you change the subgroup from $P_1 \circ ... \circ P_{m-1}$ used in the construction (Part 1) or change the j_i for at least one of the $x_m^{j_i}$ (Part 2). In either case Lemma 15 shows that the result is a different subgroup.

Lemma 16, Lemma 17, Lemma 18, and Lemma 19 are counting the number of subgroups from Lemma 8, Corollary 9, Lemma 13, and Lemma 14 respectively. Since Lemma 8 was from Foulser's work, Lemma 16 is as well. Just as Corollary 9 followed from Lemma 8, so too does Lemma 17 follow from Lemma 16. However Lemma 17 was not as straightforward as Corollary 9 and thus is a lemma. Lemma 17, Lemma 18, and Lemma 19 use Foulser's result as well as basic counting principles in their results.

Lemma 16.

$$C_{r,m} = \prod_{i=0}^{r-1} \frac{(p^{2(m-i)} - 1)}{(p^{i+1} - 1)}, \qquad 0 \le r \le m \quad \text{and} \quad m \ge 0.$$

Proof. For $1 \le r \le m$ see Foulser (1969, p. 12-13) Lemma 3.20. For $r = 0$ we get the empty product, so $C_{0,m} = 1$. This agrees with there being only one elementary abelian subgroup of length zero that contains the center and that is the center. □

Lemma 17.

$$C^1_{m,m} = 0 \qquad \text{and}$$

$$C^1_{r,m} = \prod_{i=0}^{r-1} \frac{(p^{2(m-1-i)} - 1)}{(p^{i+1} - 1)} = C_{r,m-1}, \qquad 0 \le r < m \quad \text{and} \quad m \ge 1.$$

Proof. First, it is obvious that $C^1_{m,m} = 0$ since $\mathscr{S}^1_{m,m} = \emptyset$. Now let $r < m$ and $S \in \mathscr{S}^1_{r,m}$. Then $S \le P_1 \circ ... \circ P_{m-1}$ which is an extra-special p-group with length $m - 1$ of exponent p. Thus the result follows from Lemma 16. □

Lemma 18.

$$C^2_{r,m} = C^1_{r-1,m}, \qquad 1 \le r \le m \quad \text{and} \quad m \ge 1.$$

Proof. We will find the cardinality of $\mathscr{S}^2_{r,m}$ by counting the number of ways to select the generators for any $S \in \mathscr{S}^2_{r,m}$. If $x_m \in S$, then x_m can be selected as a generator. If $m = 1$, then $r = 1$ and $S \in \mathscr{S}^2_{1,1} = \{\langle x_m, z\rangle\}$. Then $C^2_{1,1} = 1$, and the formula yields $C^2_{1,1} = C^1_{1-1,1} = 1$, as desired.

Suppose $m \ge 2$, then $C_P(\langle x_m\rangle) = \langle x_m, P_1 \circ ... \circ P_{m-1}\rangle$ and the remaining $r-1$ generators of S can be selected from $\langle P_1 \circ ... \circ P_{m-1}\rangle$. Thus we want the number of ways to choose $r-1$ generators from an extra-special p-group of exponent p and length $m-1$. Thus there are $C_{r-1,m-1}$ such ways which by Lemma 17 is $C^1_{r-1,m}$. □

Lemma 19.

$$C^3_{r,m} = (p^r - 1) \cdot C^1_{r,m}, \qquad 1 \le r \le m \quad \text{and} \quad m \ge 1.$$

Proof. As stated in Lemma 14, a subgroup in $\mathscr{S}^3_{r,m}$ is of the form $S = \langle s_1 x_m^{j_1}, ...s_i x_m^{j_i}, ..., s_r x_m^{j_r}, z\rangle$ where j_i is a natural number for $1 \le j_i \le p$ for all i such that at least one $j_i \ne p$ and $\langle s_1, ..., s_r, z\rangle \in \mathscr{S}^1_{r,m}$. Thus by Lemma 15 the number of ways to choose S is the number of ways to choose $\langle s_1, ..., s_r, z\rangle \in \mathscr{S}^1_{r,m}$ times the number of ways to choose a sequence $j_1, ..., j_r$ where at least one $j_i \ne p$. By definition, the number of ways to choose $\langle s_1, ..., s_r, z\rangle$ is $C^1_{r,m}$. The number of ways to choose a sequence $j_1, ..., j_r$ where at least one $j_i \ne p$ is $p^r - 1$. Thus, the number of ways to choose S is $C^1_{r,m} \cdot (p^r - 1)$. □

The Action of P

Up until this point we have been setting up information about the subgroups that we will need to count the orbits. We now begin to look at the group action of interest in this section. We will be looking at how an extra-special p-group of length m and exponent p^2 relates to an extra-special p-group of length m and exponent p. Since we know the number of orbits in the exponent p case from Foulser (1969), this relation becomes key to counting the orbits of the exponent p^2 case. Notation 20, Definition 21, and Definition 22 define many of the terms in the statement of the problem in the introduction of this chapter.

Notation 20. A finite field is known as a *Galois Field* and is denoted by $GF(q^w)$ with order q^w where q is a prime number and w is a positive integer.

Definition 21. A group $G \leq GL(V)$ acts *irreducibly* on a vector space V if the only subspaces of V fixed by G are the trivial subspace 0 and V.

Definition 22. An action of a group G on a set Ω is *faithful* if and only if the identity is the only element $g \in G$ such that $\alpha \cdot g = \alpha$ for all $\alpha \in \Omega$.

To help visualize the difference and similarities between extra-special p-groups of exponent p and p^2 one can look at their matrix representations. Foulser (1969, p. 8-9, Lemma 3.7) gives an explanations on the construction of these representations for the exponent p case which the reader is encouraged to examine. A brief explanation of the construction will be given here along with the contruction for the exponent p^2 case. First, let P be an extra-special p-group of length 1, then P has a faithful irreducible representation over $GF(q^k)$. If $P = \langle x, y \rangle$ has exponent p^2, then let $\omega \in GF(q^k)$ have multiplicative order p. Let $\bar{x}$ and $\bar{y}$ be the following $p \times p$ matrices:

$$\bar{y} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & & \vdots & \\ \omega & 0 & 0 & \dots & 0 \end{pmatrix} \qquad \bar{x} = \begin{pmatrix} 1 & & & 0 \\ & \omega & & \\ & & \ddots & \\ 0 & & & \omega^{p-1} \end{pmatrix}.$$

One can see that P is isomorphic to $\langle \bar{x}, \bar{y} \rangle$ with x mapped to $\bar{x}$ and y mapped to $\bar{y}$ either by hand or using the computer program GAP (The GAP Group, 2008) to verify. If $P = \langle x, y\rangle$ has exponent p, then we can use the same ω that has order p. Again, let $\bar{x}$ and $\bar{y}$ be the following $p \times p$ matrices:

$$\bar{y} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & & \vdots & \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix} \qquad \bar{x} = \begin{pmatrix} 1 & & & 0 \\ & \omega & & \\ & & \ddots & \\ 0 & & & \omega^{p-1} \end{pmatrix}.$$

One can again see that P is isomorphic to $\langle \bar{x}, \bar{y}\rangle$ with the same corresponding mapping.

So we can always look at extra-special p-groups in terms of these matrices and when we compare the two cases we see that $\bar{x}$ is the same. Additionally, when we view the extra-special p groups of length m in terms of their central product and matrices, we see that $P_1, P_2, ..., P_{m-1}$ can be exactly the same matrix groups in the exponent p^2 case as they are in the exponent p case. Also, in P_m we see that in either case x_m is the same. So the only generator whose matrix is different between the exponent p case and the exponent p^2 case is y_m.

Now we can begin to see how the action is being done in this context. Note that the set that is being acted upon is a vector space. In this case the action must respect vector space operations. Let P be an extra-special p-group of exponent p or p^2 acting on a faithful irreducible $GF(q^w)$ module V. Let $g \in P$, $v \in V$ and the action of P on V be written as $v^g = v \cdot g$. In this action, v can be thought of as a row vector being multiplied by the matrix representation of g.

In the following lemma there occurs what is known as a *tensor product* which we will define informally with a brief discussion. A tensor product of two vector spaces V and W over the same field F is denoted $V \otimes W$ and is itself a vector space over F. If $\{v_1, ..., v_n\}$ and $\{w_1, ...w_m\}$ are the bases for V and W respectively, then $\{v_i \otimes w_j | i = 1, ..., n, j = 1, ..., m\}$ is a basis for $V \otimes W$. Thus an element in $V \otimes W$ has the form $\sum \sum a_{ij}(v_i \otimes w_j)$ where each $a_{ij} \in F$. For a more formal discussion see Doerk and Hawkes Chapter B (1992, p. 90).

Also, the parameter k, mentioned in the introduction to this chapter, has not been discussed. We mention the following facts from the representation theory of finite groups and refer interested readers to Huppert and Blackburn's *Finite Groups II*, Chapter VII (1982, p. 1). Consider P an extra-special p-group of length m acting on a faithful irreducible $GF(q)$-module, V for some prime $q \neq p$. If the action on V is restricted to the center, Z, of P then we have the vector space $V_Z = V_1 \oplus ... \oplus V_{p^m}$ where each V_i is a faithful irreducible $GF(q)$-module of Z and $|V_i| = q^k$ where k is minimal such that $p|q^k - 1$, since V_i is irreducible. We can view V_i as a one-dimensional Z-module over $GF(q^k)$, and V, viewed as a $GF(q^k)$-module, is an absolutely irreducible P-module, in other words it is still irreducible under a field extension.

Lemma 23. Let $P^* = \langle x_1^*, y_1^* \rangle \circ ... \circ \langle x_m^*, y_m^* \rangle$ and $P = \langle x_1, y_1 \rangle \circ ... \circ \langle x_m, y_m \rangle$ be extra-special p-groups of exponent p and p^2 respectively. Let $V^* = V_1^* \otimes ... \otimes V_m^*$ be a faithful irreducible P^*-module over $GF(q^k)$ and $V = V_1 \otimes ... \otimes V_m$ a faithful irreducible P-module over $GF(q^k)$. Then,

$$\left(\langle x_1^*, y_1^* \rangle \circ ... \circ \langle x_{m-1}^*, y_{m-1}^* \rangle \times \langle x_m^* \rangle\right) V^*$$
$$\cong \left(\langle x_1, y_1 \rangle \circ ... \circ \langle x_{m-1}, y_{m-1} \rangle \times \langle x_m \rangle\right) V.$$

Proof. It is obvious that
$\langle x_1^*, y_1^* \rangle \circ ... \circ \langle x_{m-1}^*, y_{m-1}^* \rangle \times \langle x_m^* \rangle \cong \langle x_1, y_1 \rangle \circ ... \circ \langle x_{m-1}, y_{m-1} \rangle \times \langle x_m \rangle$.
Since the matrix representation of the elements in both groups can be chosen the same as discussed in the introduction to this section, it follows that the

actions on the corresponding modules V^* and V are also equivalent; thus

$$\left(\langle x_1^*, y_1^*\rangle \circ ... \circ \langle x_{m-1}^*, y_{m-1}^*\rangle \times \langle x_m^*\rangle\right) V^*$$
$$\cong (\langle x_1, y_1\rangle \circ ... \circ \langle x_{m-1}, y_{m-1}\rangle \times \langle x_m\rangle) V. \quad \square$$

Lemma 23 is setting up important information needed for the main result. It shows that when we restrict the acting group to $\langle x_1, y_1\rangle \circ ... \circ \langle x_{m-1}, y_{m-1}\rangle \times \langle x_m\rangle$ in both the exponent p and p^2 case, the actions are equivalent. We want this construction because it contains all the subgroups that we discussed earlier and we will see that y_m does not affect the number of elements in regular orbits in the exponent p^2 case.

Definition 24. Let V be a vector space. A *point* is a one-dimensional subspace of V.

Lemma 25. Let P be an extra-special p-group, of exponent p and length m, acting faithfully and irreducibly on the vector space $V = V_1 \otimes ... \otimes V_m$ of order q^{kp^m}.

1. For any elementary abelian subgroup $S \subseteq P$ with $Z(P) \leq S$ there exists a $v \in V$ such that the corresponding point, $GF(q^k)v$, is fixed by S (as a set).

2. There exist exactly $p^r(q^{kp^{m-r}} - 1)/(q^k - 1)$ fixed points for an elementary abelian subgroup S of length r with $Z(P) \leq S$ in P. These fixed points are distributed equally in p^r vector subspaces of dimension p^{m-r}. These p^r vector spaces are permuted transitively by P.

Proof. See Foulser (1969, p. 14) Lemma 3.21. $\square$

Lemma 26. Let P, an extra-special p-group of length m, act on a faithful irreducible $GF(q^k)$-module V. Let $S = \langle s_1, ..., s_r, z\rangle$ be an elementary abelian p-subgroup of P of length r that contains $Z(P)$. Let $S_0 = \langle s_1, ..., s_r\rangle$ and $W = C_V(S_0)$. Then $N_P(W) = N_P(S_0) = C_P(S_0)$.

Proof. Let $T = \langle t_1, ..., t_r \rangle$ such that $s_i t_i \neq t_i s_i$ for each i. Clearly $C_P(S_0) \leq N_P(S_0)$. Working toward a contradiction, let $x \in N_P(S_0)$ and assume $x \notin C_P(S_0)$. Then there is an element $s \in S_0$ such that $s^x = sz$ for some $1 \neq z \in Z(P)$. This implies $z \in S_0$, a contradiction. Thus, $N_P(S_0) = C_P(S_0)$.

To show that $N_P(S_0) \subseteq N_P(W)$, let $g \in N_P(S_0)$. Then $W^g = C_V(S_0)^g = C_V(S_0^g) = C_V(S_0) = W$. Thus $N_P(S_0) \subseteq N_P(W)$. Again, working toward a contradiction, let $g \in N_P(W)$ and assume $g \notin N_P(S_0)$. Then g does not centralize S_0, hence there is an element $s \in S_0$ such that $s^g = sz$ for some $1 \neq z \in Z(P)$. Then $W = W^g = C_V(S_0)^g = C_V(S_0^g)$. Thus for all $w \in W$, w is fixed by s and $s^g = sz$. Hence w is fixed by $z \in Z(P)$, a contradiction. Thus $N_P(W) = N_P(S_0) = C_P(S_0)$. □

Lemma 27. Let P, an extra-special p-group of length m, act on a faithful irreducible $GF(q^k)$-module V. Let $S = \langle s_1, ..., s_r, z \rangle$ be an elementary abelian p-subgroup of P of length r that contains $Z(P)$. Let $B = \langle s_1 z^{j_1}, ..., s_r z^{j_r} \rangle$ where $1 \leq j_i \leq p$ for all i and $W = C_V(B)$. Then there are p^r such B and if we let S_i $(i = 1, ..., p^r)$ be all such B and define $W_i = C_V(S_i)$ for all i, then the W_i have mutually trivial intersection.

Proof. Note that $z \notin B$ and as the order of z is p, the number of different such B is p^r, per the proof of Part 2 for Lemma 15. Thus, let S_i $(i = 1, ..., p^r)$ be all such B's.

Let $W_i = C_V(S_i)$ for all i. Working toward a contradiction, assume there exist integers n_1, n_2 such that $1 \leq n_1, n_2 \leq p^r$, $n_1 \neq n_2$, and $0 \neq w \in W_{n_1}$ such that $w \in W_{n_2}$. Then $z^{-j_\alpha} x_\alpha^{-1} w x_\alpha z^{j_\alpha} = w = z^{-j_\alpha^*} x_\alpha^{-1} w x_\alpha z^{j_\alpha^*}$ for some $x_\alpha z^{j_\alpha} \in S_{n_1}$ and $x_\alpha z^{j_\alpha^*} \in S_{n_2}$ with $j_\alpha \neq j_\alpha^*$. This implies that $w^{z^\beta} = w$ for some $0 < \beta < p$, a contradiction as $Z(P)$ cannot fix a non-zero element of V. Thus

$W_{n_1} \cap W_{n_2} = \{0\}$ for all integers n_1, n_2 such that $n_1 \neq n_2$ and
$1 \leq n_1, n_2 \leq p^r$. □

Lemma 28. Let P be an extra-special p-group, of exponent p^2 and length m, acting faithfully and irreducibly on the vector space $V = V_1 \otimes ... \otimes V_m$ of order q^{kp^m}.

1. For any elementary abelian subgroup $S \subseteq P$ with $Z(P) \leq S$ there exists a $v \in V$ such that the corresponding point, $GF(q^k)v$, is fixed by S (as a set).

2. There exist exactly $p^r(q^{kp^{m-r}} - 1)/(q^k - 1)$ fixed points for an elementary abelian subgroup S of length r with $Z(P) \leq S$ in P. These fixed points are distributed equally in p^r vector subspaces of dimension p^{m-r}. These p^r vector spaces are permuted transitively by P.

Proof. For the most part Foulser (1969) did not use the fact that P was of exponent p in the previous lemma. In either case the modules are the same and the subgroups are the same, see Lemma 23. Thus, this follows mostly from Lemma 25. The part that does not is the p^r vector spaces that are permuted transitively by P.

To prove this let $S = \langle s_1, ..., s_r, z \rangle$ be an elementary abelian subgroup of P that contains $Z(P)$. Observe that the p^r vector spaces mentioned above are just the W_i for $i = 1, ..., p^r$ as defined in Lemma 27. Then $|W_i^P| = |P : N_P(W_i)|$. By Lemma 26 $|P : N_P(W_i)| = |P : C_P(S_i)| = p^r$. Hence all W_i are in a single orbit and thus permuted transitively by P. □

In Lemma 25 Foulser shows what the action of any elementary abelian subgroup of an extra-special p-group of exponent p does to the one-dimensional subspaces. In order to fully prove a corresponding result for the exponent p^2 case we needed Lemma 26 and Lemma 27 to understand how these points behave within the action. Lemma 28 is then the p^2 case of Lemma 25 from Foulser. Foulser then goes on to describe what a stabilizer subgroup is, Definition 29, and then shows that all elementary abelian subgroups that contain the center are stabilizer subgroups in this setting, Lemma 30.

Definition 29. Let P be an extra-special p-group that acts faithfully and irreducibly on the vector space $V = V_1 \otimes ... \otimes V_m$. Call $S \leq \Omega_1(P)$ a *stabilizer subgroup* if it satisfies the following:

1. There exists a $v \in V$ such that the point $GF(q^k)v$ is fixed by S and
2. if $S^* \leq \Omega_1(P)$ fixes $GF(q^k)v$, then $S^* \leq S$.

In other words, $S \leq \Omega_1(P)$ is a stabilizer group if and only if S is the exact setwise stabilizer in $\Omega_1(P)$ of a point $GF(q^k)v$ for some $v \in V$.

Lemma 30. Let P and V be as in Lemma 25. Then the stabilizer subgroups of P are all the subgroups contained in $\bigcup_{r=0}^{m} \mathscr{S}_{r,m}$.

Proof. See Foulser (1969, p. 14-15) Lemma 3.22. □

Definition 31. Let a finite group G act on a set Ω, let O be an orbit of this action, and $|O| = |G|$. Then we call O a *regular* orbit.

In Lemma 32 the notation $C_P(v)$, which was used as the centralizer before, is being used in this case to denote the set of *stabilizers* of an element. In this case $C_P(v)$ is the set of elements in P that, in the action, send v to itself.

Lemma 32. Let P be an extra-special p-group of exponent p^2 acting on a faithful irreducible $GF(q^k)$-module V. An element $v \in V$ is in a regular orbit of P on V if and only if v is in a regular orbit of $\Omega_1(P)$ on V.

Proof. If v is in a regular orbit of P acting on V, then $C_P(v) = \{1\}$. Thus, $C_{\Omega_1(P)}(v) = \{1\}$ and v is in a regular orbit of $\Omega_1(P)$.

Suppose $v \in V$ is in a regular orbit of $\Omega_1(P)$, then $C_{\Omega_1(P)}(v) = \{1\}$. Let $1 \neq g \in C_P(v)$. Either $o(g) = p$ or $o(g) \neq p$. If $o(g) = p$, then $g \in \Omega_1(P)$, so $g \in C_{\Omega_1(P)}(v) = 1$. If $o(g) \neq p$ then $1 \neq g^p \in C_P(v)$ and thus $g^p \in \Omega_1(P)$.

Thus $g^p \in C_{\Omega_1(P)}(v) = \{1\}$ which is a contradiction. Thus $C_P(v) = \{1\}$ and v is in a regular orbit of P. □

The previous lemma shows what was mentioned earlier, that we do not have to consider y_m when counting the orbits of the action in question. It shows that the elements that are in regular orbits $\Omega_1(P)$ are still in regular orbits when we extend the action to P (note that the orbits are not necessarily the same between the two). In light of Lemma 32 and Lemma 23 we can use Foulser's original formula to count the orbits with a modification to the number of subgroups. We first need to identify the stabilizer subgroups in the exponent p^2 case. This is done in Lemma 33, which is analogous to Lemma 30, which is from Foulser's work.

Lemma 33. Let P act on a faithful irreducible $GF(q^k)$-module as in Lemma 32. The stabilizer subgroups of $\Omega_1(P)$ of the action are the elements of $\bigcup_{r=1}^{m}(\mathscr{S}^1_{r,m} \cup \mathscr{S}^2_{r,m} \cup \mathscr{S}^3_{r,m})$.

Proof. By Lemma 23 the elementary abelian subgroups of $\Omega_1(P)$ that contain $Z(P)$ are the same elementary abelian subgroups in the exponent p case. Thus by Lemma 30 the elementary abelian subgroups of $\Omega_1(P)$ that contain $Z(P)$, the elements in $\bigcup_{r=1}^{m}(\mathscr{S}^1_{r,m} \cup \mathscr{S}^2_{r,m} \cup \mathscr{S}^3_{r,m})$, are stabilizer subgroups. □

In Lemma 34 and Lemma 35 we are again looking at how these points are behaving in the action. However you will notice that we do not explicitly discuss the points in these lemmas. This is because we are looking at subspaces of V that contain the points and this provides insight into what happens to the points in the action.

Lemma 34. Let P, an extra-special p-group of exponent p and length m, act on a faithful irreducible $GF(q^k)$-module V. Let $S = \langle s_1, ..., s_r, z\rangle$ be an elementary abelian p-subgroup of P of length r that contains $Z(P)$. Let $S_0 = \langle s_1, ..., s_r\rangle$ and $W = C_V(S_0)$. Then $|W| = q^{kp^{m-r}}$.

Proof. See Dornhoff (1969, p. 207) Lemma 3. □

Lemma 35. Let P, an extra-special p-group of exponent p^2 and length m, act on a faithful irreducible $GF(q^k)$-module V. Let $S = \langle s_1, ..., s_r, z\rangle$ be an elementary abelian p-subgroup of P. Let $S_0 = \langle s_1, ..., s_r\rangle$ and $W = C_V(S_0)$. Then $|W| = p^{kp^{m-r}}$.

Proof. In Dornhoff's (1969) proof for the previous lemma he did not use the fact that P was of exponent p. In either case the V from Lemma 30 and V from this lemma are the same, and the subgroups are the same, see Lemma 23. Thus, this result follows from Lemma 34. □

Definition 36. Let P, an extra-special p-group of exponent p and length m, act on a faithful irreducible $GF(q^k)$-module. Define g_m to be the number of points whose exact stabilizer is $Z(P)$.

Definition 37. Let P, an extra-special p-group of exponent p^2 and length m, act on a faithful irreducible $GF(q^k)$-module. Define g_m^* to be the number of points whose exact stabilizer is $Z(P)$.

Lemma 38 and Lemma 39 tell us how many points are stabilized by the particular subgroups that we counted earlier. By knowing what points are stabilized by these subgroups we will know that the elements associated with these points are not in a regular orbit, which we will see later. Lemma 38 is associated with the exponent p case while Lemma 39 is associated with the exponent p^2 case.

Lemma 38. Let P, an extra-special p-group of exponent p and length m, act on a faithful irreducible $GF(q^k)$-module V. Let S be a stabilizer subgroup of P of length r. The number of points whose exact stabilizer is S is $p^r g_{m-r}$, where r is the length of S.

Proof. By Lemma 8 there exists

$E_{m-r} = \langle x_{r+1}, y_{r+1}\rangle \circ \langle x_{r+2}, y_{r+2}\rangle \circ ... \circ \langle x_m, y_m\rangle$, an extra-special p group of exponent p, such that $S \cap \langle x_{r+1}, y_{r+1}\rangle \circ ... \circ \langle x_m, y_m\rangle$ = Z(P). Then

$C_P(S_i) = S \circ E_{m-r}$. Hence, by Lemma 26, $N_P(W_i) = S \circ E_{m-r}$. The points of W_i stabilized by exactly S are the points whose exact stabilizer in E_{m-r} is $Z(P)$. W_i is a faithful irreducible module of E_{m-r}. To see this, assume W_i is not faithful and let K be the kernel of the action of E_{m-r} on W_i. Then $1 < K \trianglelefteq E_{m-r}$, so $z \in K$, a contradiction. Thus the action is faithful. We know that $|W_i| = (q^k)^{p^{m-r}}$ and $|E_{m-r}| = p^{2(m-r)+1}$. Hence if $W_i = X_1 \oplus X_2$ where X_1 is an irreducible module of E_{m-r}, then with the same argument, X_1 is also faithful, but then X_1 is too small by Theorem 9.16 in Doerk and Hawkes (1992). Therefore, W_i is indeed a faithful irreducible module of E_{m-r}.

By Definition 36 the number of points of W_i whose exact stabilizer in E_{m-r} is $Z(P)$ is g_{m-r}. Therefore the number of points of W_i stabilized exactly by S is g_{m-r}. Since by Lemma 27 there are p^r different W_i, the number of points of V whose exact stabilizer is S is $p^r g_{m-r}$. □

Lemma 39. Let P, an extra-special p-group of exponent p^2 and length m, act on a faithful irreducible $GF(q^k)$-module V. Let S be a stabilizer subgroup of P of length r. If $S \in \mathscr{S}^1_{r,m}$, or if $S \in \mathscr{S}^3_{r,m}$, then the number of points stabilized exactly by S is $p^r g^*_m$. If $S \in \mathscr{S}^2_{r,m}$, then the number of points stabilized exactly by S is $p^r g_{m-r}$, where r is the length of S.

Proof. The proof for this lemma in the cases of $S \in \mathscr{S}^1_{r,m}$ and $S \in \mathscr{S}^3_{r,m}$ is the same as Lemma 35 with slight modifications where appropriate. When $S \in \mathscr{S}^1_{r,m}$, then E_{m-r} as defined in the proof of Lemma 38 is an extra-special p-group of exponent p^2, so we must use g^*_{m-r} in place of g_{m-r}. When $S \in \mathscr{S}^2_{r,m}$, then E_{m-r} is an extra-special p-group of exponent p so we must use g_{m-r} from Foulser's work.

For the case where $S \in \mathscr{S}^3_{r,m}$ we need to determine if there is an element of order p^2 in E_{m-r}. From Lemma 14 $S = \langle s_1 x_m^{j_1}, ..., s_r x_m^{j_r}, z\rangle$ (where $\langle s_1, ..., s_r, z\rangle \in \mathscr{S}^1_{r,m}$, $1 \leq j_i \leq p$ for each i, and at least one $j_i \neq p$) and

$S_0 = \langle s_1 x_m^{j_1}, ..., s_r x_m^{j_r} \rangle$. By Corollary 9 and Lemma 14 there exist $t_1, ..., t_r \in P$ such that $\langle s_1, t_1 \rangle \circ ... \circ \langle s_r, t_r \rangle$ is an extra-special p-group of exponent p. Consider that $[s_1 x_m^{j_1}, y_m t_1^{j_1^*}] = [x_m^{j_1}, y_m][s_1, t_1^{j_1^*}] = z^{j_1} z^{j_1^*}$ (as $[x_m, y_m] = z$), see Lemma 4 and Definition 2. By a suitable selection of j_1^*, $z^{j_1} z^{j_1^*} = 1$. Thus, accordingly let $y = y_m t_1^{j_1^*} t_2^{j_2^*} ... t_r^{j_r^*}$, which is an element of order p^2. We see that $[s_i x_m^{j_i}, y] = z^{j_i} z^{j_i^*} = 1$ since t_{n_1} commutes with x_m and s_{n_2} when $n_1 \neq n_2$. Thus, $y \in C_P(S_0)$ and so $y \in E_{m-r}$. Therefore E_{m-r} has exponent p^2 and so we must use g^*_{m-r}. □

Lemma 40. Let P, an extra-special p-group of exponent p and length $m \geq 0$, act on a faithful irreducible $GF(q^k)$-module. Then

$$g_m = \frac{q^{kp^m} - 1}{q^k - 1} - \sum_{r=1}^{m} C_{r,m} p^r g_{m-r} \quad \text{with } g_0 = 1.$$

Proof. See Foulser (1969, p. 16). □

Foulser did not discuss orbits but points whose exact stabilizer is the center of the extra-special p-group of exponent p with length m. He uses g_m for the number of points whose exact stabilizer is the center. In Lemma 38 g^*_m is used for this number in the exponent p^2 case. We will see later that g_m and g^*_m are directly related to the number of regular orbits of their respective actions.

Theorem 41. Let P, an extra-special p-group of exponent p^2 and length $m \geq 1$, act on a faithful irreducible $GF(q^k)$-module. Then $g^*_0 = 0$, and for $m \geq 1$ we have

$$g^*_m = \frac{q^{kp^m} - 1}{q^k - 1} - \sum_{r=1}^{m} p^r (C^1_{r,m} g^*_{m-r} + C^2_{r,m} g_{m-r} + C^3_{r,m} g^*_{m-r}).$$

Proof. First, $g^*_0 = 0$ since there are no extra-special p-groups of length 0 and exponent p^2 (in the case of an extra-special p-group of exponent p the center can be defined as having length 0, but the center has exponent p; thus the center will not serve in the exponent p^2 case). By Lemma 33 $C^1_{r,m}$, $C^2_{r,m}$, and $C^3_{r,m}$ are the number of stabilizer subgroups in P of length r of their respective type. By Lemma 39 the number of points having a given stabilizer subgroup of length r as their exact stabilizer is $p^r g_{m-r}$ for type 2 and $p^r g^*_{m-r}$ for types 1 and 3. Thus the number of points having a stabilizer subgroup of length r as their exact stabilizer is $C^1_{r,m}p^r g^*_{m-r} + C^2_{r,m}p^r g_{m-r} + C^3_{r,m}p^r g^*_{m-r}$. Since $q^k - 1$ is the number of elements in a point, the total number of points in V is $(q^{kp^m} - 1)/(q^k - 1)$. Thus the total number of points whose exact stabilizer is $Z(P)$ is

$$g^*_m = \frac{q^{kp^m} - 1}{q^k - 1} - \sum_{r=1}^{m} (C^1_{r,m}p^r g^*_{m-r} + C^2_{r,m}p^r g_{m-r} + C^3_{r,m}p^r g^*_{m-r}).$$

$\square$

Theorem 41 is subtracts out all the points whose exact stabilizer is an elementary abelian p-group that contains the center. We see that the formula is recursive which makes it complicated to calculate individual cases by hand as m becomes greater. It should be noted that it does not give the regular orbits. Corollary 42 explains how to get the number of regular orbits from the number found in Theorem 41. Note that this part was not shown in Foulser's article as he stopped with the number of points whose exact stabilizer is the center. However the proof for Corollary 42 can be easily adapted to Foulser's work. Additionally, the examples that have been done with this formula have shown that $g^*_m \geq g_m$, which intuitively was to be expected. However it is not immediately clear how the proof for this might be done.

Corollary 42. Let P be an extra-special p-group of exponent p^2 of length m acting on a faithful irreducible $GF(q^k)$-module V. The number of regular orbits of this action is

$$\frac{(q^k - 1)g^*_m}{p^{2m+1}}.$$

Proof. By Lemma 32 the elements in regular orbits of $\Omega_1(P)$ on V are exactly the elements in regular orbits of P on V. Thus the number of elements in regular orbits of $\Omega_1(P)$ is equal to the number of elements in regular orbits of P. Thus we will count the number of elements in regular orbits of the $\Omega_1(P)$ action first.

Let $x \in V$ be in a point not fixed by any stabilizer subgroup and let $X = GF(q^k)v$ be the corresponding point. Assume x is not in a regular orbit of the $\Omega_1(P)$ action. Then there is a $1 \neq g \in \Omega_1(P)$ ($g \notin Z(P)$) such that $x \cdot g = x$. Then x is stabilized by all elements in $\langle g \rangle$. Also, X is stabilized by $Z(P) = \langle z \rangle$. So X is stabilized by $\langle g, z \rangle$. So that point is fixed by $\langle g, z \rangle > Z(P)$, a contradiction. Thus, x is in a regular orbit. Thus all nonzero elements in points that are fixed only by $Z(P)$ are in regular orbits of $\Omega_1(P)$. Thus the number of elements in regular orbits in the $\Omega_1(P)$ action, and thus the P action, is $(q^k - 1)g_m^*$.

The number of elements in a regular orbit of the P action on V is p^{2m+1}. Thus the number of regular orbits of P on V is

$$\frac{(q^k - 1)g_m^*}{p^{2m+1}}.$$

$\square$

Concluding Remarks

At the onset of this project calculations of small examples were done using GAP (The GAP Group, 2008), a computer program for computational discrete algebra. Additionally GAP scripts written by Brian Doring and Dr. Frank Lübeck (see Appendix G) were used to help perform the action and separate out the orbits in these examples. Table 6 summarizes the calculations from GAP. The values in the table agree with the formula derived in this chapter.

Table 6

Regular Orbit Counts from GAP

				regular orbits	
p	q	m	k	exponent p	exponent p^2
3	2	1	2	1	2
3	2	2	2	1061	1070
3	5	1	2	568	576
3	7	1	1	10	12
3	7	2	1	165922	166012
3	11	1	2	65560	65600
3	19	1	1	246	252
3	31	1	1	1090	1100
5	2	1	4	8385	8388
5	11	1	1	1286	1288
7	2	1	3	6113	6114

To illustrate how the formula in Theorem 41 works and how to compute the number of regular orbits, Corollary 42, an example will be shown now that can be compared to Table 6. This example will be done for $p = 3$, $q = 7$, $m = 2$, and $k = 1$ (because 3 divides $7 - 1$).

$$C^1_{1,2} = 4 \qquad C^1_{2,2} = 0$$

$$C^2_{1,2} = 1 \qquad C^2_{2,2} = 4$$

$$C^3_{1,2} = 8 \qquad C^3_{2,2} = 0$$

$$g^*_2 = \frac{7^{3^2} - 1}{7 - 1} - ((4 \cdot 3 \cdot 54 + 1 \cdot 3 \cdot 45 + 8 \cdot 3 \cdot 54) + (0 \cdot 9 \cdot 0 + 4 \cdot 9 \cdot 1 + 0 \cdot 9 \cdot 0))$$

$$g^*_2 = 6725601 - (2079 + 36) = 6723486$$

$$\text{number of regular orbits} = \frac{6 \cdot 6723486}{243} = 166012$$

A natural question to arise out of this work is "how many orbits are there of other sizes?" Probably the answer to this question can be found within the formula in Theorem 41. More specifically, $C^1_{r,m}p^r g^*_{m-r} + C^2_{r,m}p^r g_{m-r} + C^3_{r,m}p^r g^*_{m-r}$ for a particular r, $1 \leq r \leq m$, appears to be the number of points associated with a particular orbit size. This has been verified for a couple of the examples using the GAP results.

CHAPTER 5

DESCRIPTION OF THE INTERVENTION

The purpose of the study was to introduce mathematics research to students in the *Introduction to Advanced Mathematics* course. I conducted and presented the mathematics research, which focused on the area of Modern Algebra, specifically group theory, which is not a standard topic covered in this course. As seen in Chapter 4 the definitions would be difficult for a typical student in MATH 3330 to understand. So there needed to be lessons catered to the mathematical level of the course to lay a foundation of information so the students could begin to have an understanding of the mathematics research that would be presented.

Another challenge when developing the intervention was designing a format that would not be too demanding of the students in regard to the mathematical research but demanding enough for the course they were taking. If all the course level material was presented to the students first followed by all of the mathematics research then the students could become overwhelmed with all the research that was presented. So a format that broke the mathematics research into three parts and each part paired with material that could be in the course was used. Each part had a week dedicated to it and so the intervention lasted three weeks.

Each week consisted of a launch, the lesson, and the presentation. The launch was an activity or question, related to the lesson material, which the students could relate to, to get them interested in the topic to be discussed. The lesson that followed included definitions, theorems, and examples that were at the level of the course. The presentations were slide presentations that provided the students with the information about the mathematics research.

The first presentation introduced students to the problem and various definitions, such as an extra-special group, needed to work on the problem. Thus the first lesson was written to introduce students to the definition of a group and some examples of groups in order to prepare them for the definition of extra-special groups and elementary abelian groups. The second presentation dealt with the counting of the various subgroups that were of interest. Therefore the lesson for the second week dealt with fundamental counting principles, the basis for the counting of the subgroups in the second presentation. The last presentation discussed the major connection with work done by Foulser and a discussion on my overall experience of doing mathematics research. As a result, the lesson for the final week discussed the topic of isomorphism which is the central idea connecting my work to Foulser's work.

Table 6

Major Topics Covered

	Launch	Proofs	Presentation
Week 1	Solving a linear equation	Identity of Subgroups and Groups; Cancellation	Introduction to problem
Week 2	How the Texas Lottery got their odds.	Permutation and Combination Formulas	Counting Subgroups
Week 3	Tic-Tac-Toe Game	Isomorphisms	Solution and Research Process

The purpose of this chapter is to give the reader information about how the lesson plans were developed, the rationale for certain activities and examples, how well the lessons went, and how the lessons were modified between semesters. This chapter is an outline of the lesson plans used in class and any specific details about the lesson can be found in the final lesson plans for the Spring semester in Appendix E. Table 6 gives an outline of topics discussed in each lesson and presentation.

Week One Lesson and Presentation

The launch for the first week was from Cullinane (2005), which uses the act of solving a linear equation to develop the group axioms. The purpose of presenting groups in this way was to give the students something familiar to make the idea of a group more relevant to them. The lesson called for the equation $4 + x = 10$ to be written on the board and the students to be asked to describe everything they know about solving this equation. All equations of the form $a +x = b$ where a, b, and x belong to a specific set have a unique solution if that set is a group. The properties of a group are the minimal list that insures that we have a unique solution. Integers and addition were used in the equation because the integers under addition are a group that the students can relate to easily. The 4 was positive to elicit a discussion about using one operation and the need for associativity in solving this equation. From this equation we see the need for the associative, inverse, and identity properties of groups.

After the launch concluded, the lesson asked the instructor to write the equation $3x = 18$ on the board followed by a discussion about solving this equation to illustrate that the set that the numbers come from is important, since it cannot be solved in the traditional way using only integers and multiplication. This illustrates the need for the closure property of groups. Again integers were used because the integers under multiplication are an example of something that is not a group and were recognizable to the students to help them understand why. Next the lesson asked for the formal definition of a group to be put on the board and how the group properties related to solving the previous equation.

The lesson then had the instructor put an example on the board. The example was the integers modulo six under addition and was used to illustrate a multiplication table for the students. The set of integers modulo six was used because it is an example of the smallest finite group in which the order has at least two prime factors and the operation has a straightforward explanation. The multiplication table was used to illustrate one way in which a person can see how the operation of a group works. Also since the group has two prime factors the subgroups can be easily identified.

After the example was verified to be a group, the subgroups with three elements and two elements were identified, and then a discussion about what it means to be a generator followed. Subgroups and generators were discussed in the lesson because they play an important role in the mathematics research.

Since the course focuses on doing proofs, the lesson concluded with the class proving that the identity of a subgroup is the same identity as the larger group and that right and left cancelations hold in groups. These proofs were chosen because they were basic theorems about groups.

Slide presentations were used to present the mathematics research to the students. The beginning of the presentation has the presenter inform the students that while they would have to reflect on the information presented, they were not expected to understand it completely and would not be tested on the information. Next, the presentation introduced the problem to the students as well as definitions of terminology that they needed to understand for the two presentations that followed. The last part in the presentation called for a discussion on the importance of Foulser's work in the mathematics research presented.

For the most part the first week went according to the plan outlined above during the Fall semester except for the first activity. During the activity I set up the equation that I wanted them to solve, but instead of asking them to tell me everything they knew about solving the equations I began by giving them restrictions on how I wanted them to solve the equation. For example, since in a group we have only one operation I told them not to use subtraction, instead of letting it come up naturally in discussion. After I gave the instructions the students appeared to be confused about what they were supposed to do. Instead of continuing the discussion, I did the activity at the board, while still eliciting information from the students. There was also the addition of the dihedral group of order eight into the presentation to give the students an example of a non-commutative group.

Based on the experience in the Fall some modifications to the lesson plans were made for the Spring. The instructor of record pointed out that the uniqueness of an element's inverse was assumed when it should have been proven. So after the definition of a group was presented to the students a theorem about the uniqueness of an element's inverse was given for the students to prove on their own. The new Spring lesson asked that the students do the proof on their own to conserve time. At the end of class the dihedral group that was provided in the Fall presentation was added to the lesson.

The presentation in the Spring had each part sectioned off as a) understand the problem, b) know what happened in the past, and c) toy with an example. The purpose of adding this information was to give an outline for the topic of the presentation and how it fits into the larger picture of solving the problem. Understanding the problem pertains to the definitions that one needs to know before tackling a problem. Knowing what was done in the past is in reference to Foulser’s work on the topic. Toying with an example, a topic not discussed in the Fall semester, is a brief description of a small example that was done to aid in understanding the problem. A brief description of a small example was added to show the students a concrete example.

For the most part, the Spring lesson progressed as intended. For example, the activity closely followed the vignette that was developed for the lesson, although the students were quick to catch on to where we were going. As a result, the activity went quicker than originally intended. The students were responsive throughout the lesson and participated in the writing of the proofs. The presentation went well and the students asked questions about things that they did not understand.

Week Two Lesson and Presentation

The second lesson in the Fall began by having the instructor introduce an interesting fact about the odds of winning the Texas Lottery and ask the students how one might come up with those odds. After a brief discussion of the lottery question the lesson had the instructor distribute a worksheet with examples of the sum rule, product rule, combinations, and permutations for the students to explore and count on their own or in groups. The purpose of the worksheet was to familiarize the students with how the different types of counting problems were worded and the difference between them before they were given formulas for combinations and permutations. This worksheet was followed by more examples that were done as a class.

The research presentation for the second week continued the theme of counting established at the beginning of the week. The presentation began with a reminder to the students about what was expected of them, a restatement of the problem that was being presented, and a brief recap of information that was important to the discussion for the presentation. Next, Foulser's formula for counting the number of a particular type of subgroup of the extra-special groups was discussed and how he arrived at the formula. Then a description of the types of subgroups of interest to the research, how they relate to Foulser's work, and the formula derived for counting them were provided.

Given what happened at the beginning of the first lesson, when I gave the students confusing and incorrect instructions for solving the equation, I decided to do the first worksheet as a class. We then discussed definitions of the different counting methods and their formulas. Originally there were no proofs written into the lesson, which was an oversight, so, the proofs for the permutation and combination formulas were added at the end of the lesson. The Fall presentation also did not go as intended. I was able to include the information about my own work but left out much of the explanation of how Foulser derived his formula.

For the Spring, the amount of information that was included in the lesson plan increased but the second worksheet was not carried over from the Fall semester. Theorems were included in the lesson to take the place of definitions or were added to the definitions of the four methods from the first worksheet. These theorems are the ones that were proved in the Fall semester but were added to the lesson plan late.

More information was included on many of the slides in the presentation in order to give students a better idea of the topic. For example more information about what each variable is in the formula was included. A slide that summarized the formula that was found for the total number of subgroups was presented at the end of the presentation.

I delivered the lesson as intended, with the exception that I showed the students two different proofs for the combination formula. The presentation for this week went as intended aside from clarification questions from the students.

Week Three Lesson and Presentation

The third lesson in the Fall began with the instructor having the students play a game. The game was to be played with two players who take turns picking an integer between one and nine until each number has been picked or someone obtains the sum of 15 with exactly 3 of their numbers. The lesson called for the students to get into pairs and play the game for a few minutes. After the game was played the lesson required the instructor to explain that the game was equivalent to playing tic-tac-toe and to draw a magic square with the digits one through nine in each box in the tic-tac-toe grid in such a way that each row, column, and diagonal sums to 15. The lesson called for the instructor to show that there are only eight ways to sum to 15 with the digits 1 through 9. The purpose of the game was to show that two objects, namely tic-tac-toe and magic squares in this example, could be equivalent even though they appear to be different.

Next, the lesson called for a sum of logarithms with the same base to be put on the board and to ask the students for another way that it could be written. Logarithms were used because they are an example of a homomorphism that the students are familiar with and they are able to see how the operations are preserved. Then the lesson had the instructor discuss the formal definition of a homomorphism followed by introducing an example using the groups integers modulo six under addition and integers modulo seven without zero under multiplication. These two groups were discussed in terms of homomorphisms, which were also shown to be one-to-one and onto. It was explained that the groups are isomorphic and the definition of an isomorphism was provided. The two groups were used because the students were already familiar with the integers modulo six from the first week of the intervention.

Next there was a discussion about what one gets when groups are isomorphic; the groups have the same number of elements and their operations behave in a similar manner. Last, a theorem about how if we have an isomorphism from one group to another and one of the groups is abelian then the other must be abelian was proved as a class.

The presentation began the same as the first two in regard to expectations. Next the rest of the research was to be presented along with the final result. Then discussion on the experiences of doing mathematics research and how it can be related to students doing homework followed. This was done to show the students that mathematics is done in similar ways that homework may be done. For example, a student doing homework or a researcher doing research might try several approaches before finding one that works or look for examples to help shed light on how the proof might be done.

The Fall lesson went as it was intended except for minor adjustments based on student questions. The presentation similarly went as intended. Thus, the changes for the Spring semester were minimal. Proofs regarding how the identity and inverses are mapped with a homomorphism were added to the end of the lesson. This was done to increase the number of proofs done in class and to provide the students with the opportunity to understand what happens with homomorphisms, i.e. operations are preserved.

In the Spring semester the students were having a hard time understanding how to play the game so I decided that we would first play together with the class playing against me. By the time we were done playing the first game everyone understood the rules and someone had already made the connection to tic-tac-toe. There was no reason to have them play in pairs and the remainder of the lesson proceeded as intended.

The first section of the last presentation went as it was intended. When I began the section on my experience doing research the students asked many questions about getting grants, working on grants, collaborating in mathematics, and other questions dealing with mathematics research. I tried to answer all of their questions and questions that I could not answer I deferred to Dr. Morey or Dr. Keller, the instructors of record for the observed classes. I tried to get the students' attention back on the presentation but they continued to ask more questions. I was only able to get through half of the second section of the presentation. Initially this was thought of as an unpleasant experience but upon further reflection and discussions with colleagues, I decided that it was a positive experience because the students had become interested in mathematics research to the point that they wanted to know more even though it went beyond the scope of the presentation. This experience will be discussed further in Chapter 7.

Summary

It can be seen that the treatment groups from the Fall and Spring semesters received essentially the same lessons and presentations. In some cases the Spring students received instruction that was more polished than the Fall students. However, this happens in any educational experience as an instructor repeatedly teaches a topic and refines their approach.

CHAPTER 6

RESULTS

The following are the results obtained from two semesters of gathering survey data and supplemental data in the form of interviews and written work. Four classes participated in the research project. The two *Introduction to Advanced Mathematics* classes in the Fall 2010 semester and the two in the Spring 2011 semester were used for the study. There were two instructors of record, each had a class in the Fall and one in the Spring and each had the treatment class once.

Survey data were collected three times during each semester. First, at the beginning of the semester (pre-survey), the second time with three weeks remaining in the semester before the intervention was administered to the treatment (mid-survey), and a third time at the end of the three-week intervention (post-survey). While there were anywhere from 20 to 29 students in each class at the beginning of the semester (16 to 25 students in each class at the end of the semester) only data from students who took all three surveys were used in the results. There were 11 students surveyed in the Fall treatment, 12 in the Spring treatment, 10 in the Fall control, and 7 in the Spring control. Table 7 summarizes the sample of students who took all three surveys.

Supplemental data in the form of 11 interviews from seven students and written work given predominantly during the intervention (one paper was assigned at the beginning of the semester) were gathered from the treatment group. Four students were interviewed in the Fall semester by the end of the first month of classes. A follow-up interview was done with three of the students at the conclusion of the semester; contact could not be made with the fourth interviewee. In the Spring semester two students were interviewed at the beginning of the semester. Although four was preferred, students did not respond to being contacted or did not show up for the interview. At the end of the Spring semester only one of the two returned for a follow-up interview. Additional students were contacted again for interviews but only one responded; so two students were interviewed at the end of the Spring. The names "Student 1" through "Student 7" will be used when quoting interviews.

A mathematical biography of two pages, including thoughts about the meaning of mathematics, was assigned on the first day of the semester to be turned in by the end of the following week. A one-page reflection paper was assigned at the end of each mathematics research presentation. Student names with numbers higher than "Student 7" will be used when quoting written work.

Table 7

Sample of Students

	Treatment		Control		
Major	Fall	Spring	Fall	Spring	Total
Pure Math	2	4	1	0	7
Applied Math	2	1	0	0	3
Math w/Cert	4	6	9	6	25
Other	2	1	0	2	5
Total	10	12	10	8	40

This chapter will be divided into three main sections. The first section will describe the students' conceptions of mathematics according to the pre-survey and supplemental data from the beginning of the semester. The second section will discuss any changes in the students' conceptions as measured by the mid-survey after the instructors of record had completed their sections and before the intervention had started. The last section will discuss any changes due to the intervention as well as any other relevant findings from the post-survey and written assignment.

For the purposes of this study there are three levels of significance used to analyze the data. There is implied significance (p-values between 0.2 and 0.15) and practical significance (p-values between 0.15 and 0.05), which in either case means there is an influence happening. Statistical significance is at the traditional p-value of 0.05.

The purpose of the supplemental data was to enrich and complement the quantitative data. So the subscales from the quantitative data (the surveys) were used as themes for analyzing the supplemental data as well as themes that were developed from interesting information that was an unforeseen consequence of the study.

Students and Classes at the Beginning of the Semesters

The survey administered to the classes measured eight subscales in regard to their beliefs about mathematics: Composition, Structure, Status, Doing, Validating, Learning, Usefulness, and Mathematicians. First, an analysis of the differences between the classes will be discussed. Then, each subscale will be introduced along with an analysis of where the classes were at the beginning of the semester. Students had six levels of agreement to choose from for each item on the survey and thus scores range from 1 to 6. Scores above 3.7 will be considered high, below 3.3 considered low, and between 3.3 and 3.7 will be considered neutral scores. Scores less than 5 will be considered having the potential to increase. The supplemental data will be included in relevant places to add perspective on where students were at the beginning of the semester. The supplemental data in this section was taken from the papers written in the first week of class and interviews given within the first month of class.

Difference between the classes. An independent t-test was performed on the first survey data, including each subscale separately, between the control and treatment groups to determine if the two groups started at the same level of mathematical conception. No significant difference was found between the two groups.

Similarly, an analysis of variance (ANOVA) was also performed on the survey, including each subscale of the four classes, the two control and two treatment classes, to determine if all four classes started at the same level of mathematical conception. In the case of the survey as a whole, there was no significant difference between the four classes. Of the eight subscales only one, composition of mathematics, was significant. A Tukey post-hoc test was conducted which showed a significant difference between the control class in the first semester and the treatment of the second semester. It is believed that the difference between these two classes arose because of the high number of mathematics majors seeking secondary certification in the first semester control class who took all three surveys.

Where students were by subscale. In this section a description of where the students are by subscales will be given. Students in the treatment groups were grouped together when deriving the statistical data in this section. Table 8 provides the range that the average scores for the items fell within for each subscale.

A high score in the Composition subscale means students think of mathematics as concepts and generalizations where a low score means students think of mathematics as formulas and algorithms. For six out of the eight items in this subscale the students generally scored high, between 4.01 and 4.74, on the survey. In one of the two items the students scored 2.82, indicating that the students believe that there is always a rule to follow when doing a mathematics problem. This was illustrated in the interview with student 4, when she was asked if she expected to take a course like MATH 3330 she said, "No, or at least I expected the class, if I was, to be a little bit more mathematical." Asked to explain what was meant the student said, "… And I'm like, 'Um, is there a formula that I follow, like this is what I need to find so I find that and I find this,'" Similarly, on the topic of what mathematics is, student 7 says, "My concept of math in general before this class was basically formulas and calculations and that kind of thing." On the last item the students' average score of 3.64 points to a mix of beliefs about whether mathematics is made up of mostly procedures and facts. Since all of the scores in this subscale fall below 5 there was room for increasing the scores. This subscale was expected at the beginning of the study to be affected by either the course in general or the intervention.

High scores in the Structure subscale mean students believe that mathematics is a coherent system and a low score means the students think of mathematics as unrelated topics. Students generally scored high in this subscale, between 4.22 and 5.64. Only one item had a score above 5 and five of the eight had a score of 4.77 or less. This suggests that the students already believed that topics in mathematics were interconnected before taking the class; however, there was room for increasing some scores. In fact, student 5 put the interconnectedness of mathematics this way, "Algebra, it's part of everything; geometry, modern geometry, proofs, it's part of modern algebra. Algebra is everything."

This subscale was expected at the onset of the study to be affected by either the course or the intervention.

Table 8

Mean Scores for the Lowest, Median, and Highest Item Averages for Each Subscale

	Lowest Average	Median Average	Highest Average
Composition	2.82	4.24	4.74
Structure	4.22	4.76	5.64
Status	3.25	4.41	4.99
Doing	4.12	5.00	5.38
Validating	3.32	4.82	5.27
Learning	3.41	4.61	5.47
Usefulness	5.04	5.12	5.58
Mathematicians	3.44	4.58	5.11

High scores in the Status subscale mean that students see mathematics as an evolving field where a low score means students view mathematics as a never changing field. The average of student responses on seven of the eight items was between 4.32 and 4.99 indicating that students generally believe mathematics is a growing field. However, the interviews indicate that while students might think mathematics is a growing field, they do not know how it is growing. For example, student 2 said, "I feel like it's changing. I feel like it has to." One item scored an average of 3.25, which means many students believe that once you learn some thing in mathematics, then that thing is not going to change. For example a student might think that a new or a more efficient method for solving a problem would or could not be developed. There was room to increase scores for at least six of the eight items that have scores between 3.25 and 4.63. At the beginning of the study this subscale was expected to be affected by the intervention.

A high score in the subscale Doing means that students believe that doing mathematics is a process of sense making and a low score means students believe mathematics is about getting an answer. The average scores across all items in this subscale were 4.12 or higher and half were at least 5.04. This indicates that there was a little room for increasing scores in this subscale. The intervention and the course were expected to affect this subscale at the start of the study.

High scores in the Validating subscale indicate that students believe that validating mathematics work occurs through logical thought and low scores indicate that students believe that validation comes from an outside authority like an instructor or book. While five of the eight items had average scores above 4, three items had average scores below 4, one of which was 3.32, a neutral score. These scores suggest there are mixed beliefs about what it means to validate something in mathematics. This mixed reaction can be seen in two quotes, one from student 4 and one from student 5. Student 4, who is in the process of explaining why the class is not mathematical, says, “So, I wish that this class was taken later because then I could prove what I already know instead of trying to prove what I don’t know to be true.” In contrast, student 5, while discussing how one should learn mathematics, says, “I think you learn best when you figure it out yourself.” There was room for increasing scores in this subscale. The control was expected to be just as affected in this subscale as the treatment group.

High scores in the Learning subscale point to a belief that math is a subject about understanding and a low score means students believe that learning mathematics is about memorization. A majority of the items had average scores of at least 4.14, four of whom were between 4.88 and 5.47; however, two items, dealing with mathematics being about memorization, had scores of 3.49 and 3.41. These neutral scores indicate that many students at the beginning of the semester value memorizing information to help them learn. For example, student 1 says that memorizing is a "time saver" and student 3 says, "like for younger kids I feel like it is memorizing and I don't really see like there's just any other way… like explain one plus one equals two… you memorize that." Other students felt differently about it, for example student 2 said, "I think it might go a bit deeper than just memorization," and that "instead of just memorizing it if you take the time to actually understand it, then it would be a lot better." Students 4 and 7 say that understanding is an important concept to mathematics because then you are able to "apply" what you have learned or you can "figure out" a formula if you have forgotten it. In regard to the survey, at least half of the items had room for improving scores. The control was expected to be just as affected in this subscale as the treatment group.

High scores in the subscale Usefulness indicate that students believe that mathematics is a worthwhile subject for anyone and a low score means they believe that it is not valuable to everyone. Since everyone in this class is either a math major or minor it was no surprise that all items had an average score above a 5. There is little room, if any, for scores to go up in this subscale. It was anticipated that this subscale would not to be affected by either the course or the intervention when the study began.

High scores in the subscale Mathematicians mean that students think highly of mathematicians and what mathematicians do. The average score for all items except one was at least 4.33 with the final item having an average of 3.44. These scores point out that there was still some opportunity for increasing the scores for this subscale. However the course and the intervention was not expected to influence this subscale at the beginning of the study.

Insights from supplemental data. Many of the subscales discussed above were present in the students' first paper of the semester, which included their personal mathematics biography and their thoughts about the nature of mathematics. The Usefulness of Mathematics and Doing Mathematics was apparent in nearly all of the students' papers. However, Structure, Status, and Learning were discussed less often, and Composition and Validation were rarely if ever discussed. Also, some students showed signs of a fixed mind-set or a growth mind-set as described by Dweck (2007).

In the previous section it was shown that the Usefulness subscale scored high among every class. While it was expected that mathematics majors would find mathematics useful, when providing example of its usefulness students would provide calculation-based or vague answers. For example, student 21, "Math is not just theoretical, but practical. You use it in everyday life," and student 24, "Math is used in everyday life and in almost every job," provide the theme that was apparent in most papers, mathematics is used in "everyday life" and "every job." Few students took the concept of mathematical usefulness a step further like student 18 who said, "Learning mathematics is about learning to think, and more importantly to think logically," or the example from student 21 of how he used his problem solving skills from mathematics to increase efficiency at his job.

Doing Mathematics scored high on the survey but from the papers it appears that there might be some mixed feelings. When the students discussed why they liked mathematics they indicated that when they did mathematics they knew they would always get an answer and that answer was either right or wrong. As student 12 says, "Because no matter who the teacher is, there will always be a right or wrong answer, and there is not a lot of room for subjectivity in a math class." However, some students included both the idea of coming up with a right answer and a process of sense making as student 21 who said, "The idea that I can explore concepts and solve problems using logic, patterns, and reasoning is exciting. There is nothing better than coming up with the right solution to a problem."

I believe these mixed feelings manifest themselves in some students because of competitions such as Number Sense, a competition described in the following quote. Student 20 shares some insight into Number Sense when describing her teacher, and first number sense team coach, in her paper, "She was a U.I.L. coach for an event that starts in the fourth grade – number sense. (Number sense is a math event in which the contestants must complete mathematical questions mentally without being able to write anything down except the answer on the line provided, and contestants only have ten minutes to answer as many questions correctly as they can without skipping any.)." This teacher, coach, was a person the student looked up to who encouraged her to tryout for the number sense team. The full impact of this teacher, coach, is realized when the student mentions that she learned "many tricks and short-cuts" that she used in the rest of her mathematics classes. Competitions such as these places an emphasis on finding correct answers rather than the process, influencing the contestants, individuals who are good at mathematics and potential mathematics majors to think that mathematics is a process of obtaining an answer. However, as seen from the survey in the previous section, it is not an overwhelming influence. This might also have something to do with students valuing memorization as discussed in the previous section.

One statement from the survey asked the students if mathematics was still being "invented." Many students agreed with this statement, however, many students had philosophical problems with the wording of this statement. For example, student 8 said, "Contrary to some people's thoughts, I am a firm believer that math was and is still being discovered, not invented," and student 13 said, "We do not invent new math, but discover it. All the axioms we have and have yet to discover were there before we found them and are still there now, along with many more waiting to be uncovered." This was an interesting development and will be explored more in Chapter 7.

The Structure of Mathematics was not mentioned often in the papers of the students, but when they did discuss it the students were explicit in their beliefs. As student 10 put it, "I like to learn math because I find the concepts amazing in how well they mesh with the math I already know, as if its [*sic*] all part of some beautifully woven tapestry." Learning was also not a subscale that was discussed much in the papers; however, students would mention the importance of understanding what you are doing in mathematics. As student 8 explained, "It is not as important to just getting the right answer in math, but instead understanding the concept and how it works, and also applies to us in our lives today."

One interesting thing that came out of the papers is the number of students that began their mathematical careers either not liking mathematics or did not do well in grade school mathematics classes. These students tended to have a growth mind-set as described by Dweck (2007). Student 30 in particular struggled with mathematics at an early age but with encouragements from his father and hard work he was able to pass his classes. When he got to college he knew he was going to have to put in extra work, as he explained:

> With that shift I did take a different approach towards my math classes. When I began College Algebra I made an active attempt to do all the homework that was assigned for the course. Anything that I didn't understand I went to the tutoring center for until I got it. It paid off; I made an A in College Algebra. I took the same approach with trigonometry and again got the same result.

Student 28 is an example of a student who is somewhere between a fixed mind-set and a growth mind-set. She is an example of one of the students that Dweck (2007) would describe as a student that always did well in mathematics and then when she got to a class that makes her struggle she gets frustrated. However, the student does express the need to work hard in mathematics when she says, "You must be able to put forth the energy to study every day and must have a lot of motivation to do so," and continues to explain that "Math is a subject that requires people to be attentive, motivated, and focused." What this says is that people who have made it this far, or those that will continue after this course, are probably developing a growth mind-set.

Where the Students Were Before the Treatment

Three weeks before the end of each semester I took over the class to administer the intervention. A survey was given before I started in order to determine any change in the students' conception that was due to the intervention as opposed to the instructors. An ANOVA was used to determine if there were any differences between the four classes in regard to the conception survey, a t-test to test for differences between the control and treatment, and a paired t-test to look for changes between the pre and mid-surveys as well as each subscale.

An ANOVA was performed on the mid-survey, including each subscale, between all four classes as was done with the pre-survey. This time there was no significant difference on the survey as a whole or on any of the subscales. In the pre-survey the Composition subscale had a significance of 0.018 and the mid-survey had a significance level of 0.177, indicating that while the two classes are still different they have moved close enough together that their differences are no longer statistically significant. Again, the independent sample t-test revealed no difference between the control or treatment groups on the mid-survey as a whole or any of the subscales.

Table 9

Significance Values of Paired t-tests Between Pre and Mid-Surveys and Their Subscales

	Treatment		Control	
Subscales	MD	Sig	MD	Sig
Surveys	0.121	0.085	0.064	0.356
Composition	0.131	0.233	0.278	0.052
Structure	0.176	0.175	-0.014	0.907
Status	-0.057	0.583	-0.021	0.878
Doing	0.080	0.487	-0.028	0.779
Validation	0.119	0.157	-0.028	0.799
Learning	0.114	0.235	0.118	0.478
Usefulness	0.148	0.121	0.069	0.520
Mathematicians	0.303	0.039	0.157	0.179

Note. MD = Mean Difference

A paired t-test was performed between the pre and mid-survey as a whole and between each subscale. The results of the paired t-tests are summarized in Table 9. In the treatment group there is a significance level of 0.085 for the difference between the pre and mid-surveys. While not statistically significant, it shows that the class as a whole effected the students' beliefs about mathematics to a small degree. However, when we consider each subscale individually we see that the only subscale that is statistically significant, at 0.039, is the students' beliefs of Mathematicians. This could be attributed to this being their first class that is dominated with mathematics majors taking the class, excluding electives. For example, calculus is a mixture of mathematics majors and other science majors. The students are able to gain a different perspective of a mathematics professor when the class is composed of what is assumed to be mostly mathematics majors. Another possibility is that the students are growing in their self identification of a mathematician and thus affecting their beliefs about Mathematicians.

However, when we look at the paired t-tests for the control group we see a significance level of 0.356 for the difference between the pre and mid-surveys, which is not significant or close to being significant. The treatment group being close to statistically significant might mean that the knowledge that they were part of the treatment group had an effect on the student's beliefs of mathematics. Also, the significance level for the Mathematician subscale for the control group was 0.179, again not significant. However, it is reasonable to believe that the class was having an effect on the students' beliefs of mathematicians, just not at a statistically significant level, especially considering the significance with the treatment group. The Composition subscale does have practical significance in the control with a significance level of 0.052. This is interesting as the treatment group has a significances level of 0.233, not significant. I cannot think of a reason for why there is such a drastic different between the two groups, especially since it was thought that the Composition subscale would be effected by the course in general.

Where the Students Were at the End of the Treatment

At the end of the intervention, also the end of the semester, the post-survey was administered to the students. This survey was compared to both the pre-survey and mid-survey using a paired t-test. Table 10 summarizes the results from the paired t-tests for the surveys as a whole and each subscale in the survey. Also, at the end of each mathematics research presentation a one-page reflection paper was assigned to attain the students' perspectives concerning the presentations. This section will be divided into a quantitative section and a supplemental data section for their respective data that were collected.

Quantitative analysis. An independent samples t-test showed no significant difference between the treatment and control groups for any survey or subscale. However, the paired t-tests summarized in the first row of Table 10 shows that in the surveys as a whole there was a statistically significant change between the pre to post-surveys and mid to post-surveys for the treatment group and not the control group. Note that the change in the survey in the control group does show a practical significance, suggesting that the course does have an influence on the students' beliefs of mathematics. However, the addition of the mathematics research provided the extra change to make it statistically significant. Also, in Tables 9 and 10, the scores for the Usefulness of Mathematics did not change significantly across all cases. Additionally, the scores for students' beliefs of Mathematicians continued to increase with similar results as above in the section discussing results before the treatment.

Interestingly, there was no significant change between the pre and mid-survey for the control group in the Composition subscale, yet there is a small significance between the mid to post-surveys and the pre to post-surveys. The subscale that had the most statistically significant change between all surveys in the control group is Validation. However, no significant change was evident in these subscales in the treatment group. It can be assumed that something must have happened between the mid and post-surveys in the control that did not happen in the treatment to account for these changes.

Table 10

Significance Values of Paired t-tests of Pre and Mid-Surveys with the Post-Survey

	Treatment				Control			
	Post – Mid		Post – Pre		Post – Mid		Post – Pre	
Subscales	MD	Sig	MD	Sig	MD	Sig	MD	Sig
Survey	.09	**.05**	.21	**.02**	.06	.12	.12	.09
Composition	.05	.51	.18	.21	-.03	.76	.24	**.02**
Structure	.05	.63	.22	.07	.08	.44	.06	.65
Status	.35	**.00**	.29	**.00**	.12	.30	.10	.34
Doing	.08	.24	.16	.12	-.03	.69	-.06	.52
Validation	.03	.74	.15	.27	.24	**.02**	.21	**.02**
Learning	.12	.13	.23	.08	.02	.82	.14	.32
Usefulness	-.01	.95	.14	.29	.02	.75	.09	.44
Mathematicians	.02	.90	.32	.07	.07	.32	.23	.06

Note. MD = Mean Difference

The course schedules (Appendix F) for the control classes indicate that during this window (between the mid and post-surveys) they addressed real analysis topics in the Fall and modern algebra topics in the Spring. These topics present more abstract information to the students than information presented earlier in the semester, which accounts for the change in the Composition subscale. Also, many of the topics discussed in the first part of the semester deal with topics that the students were familiar with but were presented in the context of proof. The information presented toward the end of the course in the control group is different than most of the other information presented at the beginning in that it requires a more rigorous form of validation. For example, something like proving that the sum of two even numbers is even might seem tedious and pointless to the students because it is something that they have known most of their mathematical lives. However, proving something about a boundary point or a group, things that they might not be familiar with, is going to require using a logical thought process relying on definitions and other known properties to be done correctly.

In Table 9 the significance level for the subscale Status under both control and treatment is not contained in any defined significance level. However, in Table 10 we see that the students' beliefs of the Status of Mathematics in the treatment group has made a statistically significant change from both the pre-survey to the post-survey and the mid-survey to the post-survey with significance values of .00 for both. As student 8 said, "I had no idea that people were still discovering new things about mathematics and adding to the books. I thought all the math there was ended with calculus. But I was wrong. There is so much after that." In the control group there is still no significant change in the students' beliefs about the Status of Mathematics. Recall that the intervention, and not the course, was expected to influence this subscale. Thus the expectation was met for this subscale.

The course was expected to influence the subscales Composition of Mathematics, Structure of Mathematics, Doing Mathematics, Validation of Mathematics, and Learning Mathematics, with the Structure of Mathematics and Doing Mathematics being influenced by the intervention as well. I expected there to be at least a practical significant change in these subscales in both the treatment and the control. However, expectations in these subscales were not validated with Validation and Composition already discussed above.

As was shown in Tables 9 and 10 the scores for the Structure of Mathematics did not show significance in any of the paired t-tests for the control group. However, there was small level of significance from the pre-survey to the mid-survey and a practical level of significance from the pre-survey to the post-survey for the scores of the treatment group. A possible explanation for these results is that the students were in the process of changing their conception about this subscale when the second survey was administered and continued to change to the point when the third survey was administered. One explanation for why there was a change in the treatment group and not the control group is that the treatment group was more attuned to what was happening in class because they knew they were in the treatment group and more attention was going to be paid to them.

There was a practical level of significance of .08 from the pre-survey to post-survey and of .13 from the mid-survey to the post-survey for the Learning subscale in the treatment group. However, there was not any level of significance from the pre-survey to the mid-survey. Again, this change is probably due to the students knowing that they are in the treatment group.

When comparing Tables 9 and 10 we see that the scores for Doing Mathematics changed enough between the mid and post-surveys for a practical significant change. Since there was no significance from the first to second survey with a significance level of .487, and there was no significant difference observed in the control group, the intervention must have influenced the students' beliefs. This change could be attributed to the presentations showing the students mathematics being done differently from their past experiences.

There was not an adequate number of pre-service secondary mathematics teachers to get meaningful results. Nevertheless independent samples t-tests, comparing the treatment and control groups across all surveys and subscales, and paired tests, between each survey as a whole and for each subscale, were performed and analyzed on the survey data. Table 11 summarizes significant data from the paired t-tests.

Table 11

Significance Values of Paired t-tests for Data Restricted to Pre-Service Teachers

	Treatment		Control	
	MD	Sig	MD	Sig
Post – Mid	.10	.14	.08	**.05**
Composition: Mid – Pre	.04	.73	.35	**.03**
Structure: Mid – Pre	.35	**.02**	.03	.83
Structure: Post – Pre	.46	**.03**	.10	.53
Status: Post – Mid	.29	**.07**	.13	.33
Validation: Post – Mid	.03	.88	.32	**.00**
Mathematicians: Post – Pre	.52	.14	.26	**.06**

Note. MD = Mean Difference

The independent samples t-tests showed that the subscale Validation was statically significant with a value of 0.044 on the second survey between the treatment and control groups but nothing else was significant. The treatment classes had the higher score in the Validation subscale on the mid-survey but by post-survey there was no longer a significant difference with a significant value of 0.400. When we consider the mean scores the treatment group went from 4.80 on the pre-survey, to 4.91 on the mid-survey, and 4.93 again on the post-survey. The control group went from 4.43 on the pre-survey, to 4.39 on the mid-survey, and 4.70 on the post-survey. So most of the changing occurred in the control group between the mid and post-surveys. A paired t-test showed there was a significant difference, with a significant value of 0.00, between the post-survey and mid-survey for the control group and no significant change for the treatment group. So the material at the end of the course affected the pre-service teachers as well.

The paired t-test also showed significant changes for the control group between the mid and post-surveys with a significant value of 0.05. It seems that this change occurred because of the subscales Validation, Composition, and Mathematicians. The subscale Composition had a significant change between the pre and mid-surveys with a significant value of 0.03 and Mathematicians had change between the fist and post-surveys with a significant value of 0.06. So the course affects the pre-service teachers beliefs about the Composition of Mathematics early in the course while their beliefs of Mathematicians changes gradually over the course of the semester.

When we consider the Status subscale we see that there is no statistically significant results for either group but there is a practical significant value of .07 for the treatment group. So again the intervention affected this subscale by increasing the survey scores. Perhaps the pre-service teachers are more resistant to changing their beliefs in regard to this subscale.

An independent samples t-test was done on the data between each instructor's treatment and control group. The only significance found was in the subscale Composition for the instructor whose class was the control group during the Fall.

However, the classes were significantly different or close to significantly across all surveys. The pre-survey had the most significantly different scores with a significant score of 0.01. The difference became less significant for each survey. So the students' scores starting out different was the main factor for the scores being different throughout the semester.

Supplemental data analysis. There were two types of supplemental data that were collected during and after the intervention. First, every student had to turn in a reflection paper on the presentation of the mathematics research. The other was a follow-up interview of students interviewed at the beginning of the semester. Three students returned for a follow-up interview during the Fall semester and one student in the Spring returned for a follow-up along with a student that was not interviewed at the beginning of the semester.

Only the follow-up interview with Student 4 had substance that would add to the study. Recall that Student 4 believed that the course was not very mathematical. The student mentioned something that I thought related to this idea of the class not being mathematical and the following discussion occurred (R = researchers, S = Student).

> R: So last time you said you expected the class to be a bit more mathematical. So do you think it's more mathematical now, is that what you are saying?
>
> S: The last half of it was more mathematical.
>
> R: Can you explain that? How was the last half more mathematical than the first half?
>
> S: There were more numbers involved.
>
> R: There were more numbers involved?

S: Yeah, and it was more mathematical ideas that I'm used to rather than trying to prove something is even or odd.

R: Can you give me a definition of what it means to be mathematical?

S: To me, when I think of mathematics and other stuff, it's more of, kind of step-by-step kind of deal, but it, like right now, cause we're dealing with numbers rather than ideas and stuff like that. Then eventually it transfers from ideas to math. It's just more stuff, more formulas.

At that the topic was concluded. It appears that the student does not believe that something is mathematics unless it involves numbers. The student believes that abstract mathematics is not mathematics until you use numbers. It is interesting that the student thinks that a mathematics class is not mathematical. The most likely explanation for why the student believes this is that the student has had the computational mathematics of grade school so engrained in her mind that she does not want to change what is comfortable.

There appeared to be three schools of thought about the definition of a reflection paper. First were those who thought that you think about what was talked about, relate it to what you know, and share any thoughts about the information. Second were those that thought they should tell me what I told them. The last were those who would give a review of the presentation along with some negative tones about how they did not understand the material. In general, with the exception of the second type, those definitions would be acceptable, but I told the students what I wanted in the reflection and also that they were not expected to completely understand the material. For example, I wanted them to describe some connections to the lessons from the beginning of the week and most of them did not provide that in their reflections.

In the end, I received a mix of positive and negative comments for the first two reflections. It seems that the students did not fully appreciate the mathematics presentations until after the third presentation. For example, let us consider student 32 and the student's growth as seen through the reflections after each presentation. After the first presentation the student is feeling down about not understanding the research material, "I have pretty much realized that I just don't have what it takes to continue my mathematics degree to the graduate level." Following the second presentation we see that the student has had a change of heart about graduate school after realizing that I am in a doctoral program and not a masters program. The student also confides with a friend in medical school about some apprehensions about graduate school and "Her response was, 'Well, I can't perform surgery right now either, it's baby steps.'" After the third presentation it can be seen that, while the student is still overwhelmed with the mathematics research, she has a new appreciation for it. The student shares, "The presentations that you have given have intrigued me to do a little research into my professors to look at the research work that they have done." This illustrates that this student achieved one of the purposes of the study, increasing appreciation of mathematics and mathematical rigor.

The presentations convinced some students that mathematics in academia would not be for them. Student 35 states in the third reflection, "the research itself didn't seem too interesting to me, but the end result seems really cool to be a part of, which makes me realize I would most likely do better in industry rather than in the academic world." His statement is odd and leads me to believe that he still does not have a full understanding of mathematics, because he based this feeling on a presentation that represents a small portion of mathematics. In other words, he might not be interested in algebra but perhaps in a different topic. On the other hand, student 27 felt inspired by the presentations to go to graduate school. She says, "Overall I enjoyed the presentations. I felt that it, plus my interest in mathematics, had compelled me to pursue mathematics in graduate school."

"How can this be applied?" was a common theme among many of the students' reflections. Some of these students attempted to find their own applications for the research that was presented. For example, student 25 wonders, "whether this knowledge might help to factorize large prime numbers." It should be noted that the student probably meant to say "large numbers" instead of "large prime numbers" or perhaps the student has a lack of understanding of prime numbers. While I do not believe that the research can be used in this way it shows that the students have gained an appreciation of mathematics research that provides them with a curiosity that helps them understand the material in their own way.

Summary

The results show that the intervention had the expected effect on what the students believe about the Status of Mathematics. On the other hand there were mixed results when looking at the other subscales. The course was expected to affect the subscales Composition, Structure, Doing, Validating, and Learning. The subscales Composition and Validating were the two that had a statistically significant change due to the course. Additionally the intervention and the course were not expected to affect the subscale Mathematician however was close to being statistically significant for both the treatment and control.

Furthermore the supplemental data provided insight into survey as well as unexpected information about the students. In particular it provided insight into the growth that the students experienced as they struggled with the information that they were provided within the mathematics research presentations.

CHAPTER 7

DISCUSSION

The purpose of this research study was to measure how the introduction of mathematics research in the course *Introduction to Advanced Mathematics* influences the mathematical beliefs of students, raises their awareness of mathematics research, and increases their appreciation of mathematics and mathematical rigor. A review of the literature shows that the way teachers conceptualize mathematics influences the way they teach mathematics.

These ideas were investigated using quantitative data from three surveys that were enhanced with supplemental data from interviews and writing assignments. The research questions will be revisited in this chapter along with conclusions based on the data. However, the study was not without imperfections, which are listed along with possible ways to improve the study.

Notice in Table 7 of Chapter 6 that 25 of the 40 students (about 63%) involved in the study were Mathematics majors seeking secondary certification. So any conclusions drawn from the results say more about these particular students than the students in other mathematics majors.

Conclusions and Interpretations

Conclusions are broken up into those directly related to the original research questions and those that go beyond the original research questions. The conclusions beyond the research questions were predominately from the supplemental data. These conclusions are based on data that was found to be interesting or unexpected.

Conclusions related to the research questions. The first question was: Do pre-service secondary mathematics teachers, pure mathematics majors, applied mathematics majors, and mathematics minors have different beliefs about mathematics? There were a disproportionate number of students seeking secondary mathematics certification compared to the other majors so this question cannot be adequately answered. However an analysis of variance (ANOVA) was performed on the data and showed that there was no significant difference between the majors on the pre-survey.

The literature suggests that mathematics teachers believe that mathematics is a static body of knowledge, a set of rules and facts, and algorithmic processes (Ball, 1990; Brendefur & Frykholm, 2000; Stipek et al., 2001). This suggested that their beliefs scores on the survey would be low and possibly lower than those of mathematics majors not seeking certification. Since the scores were above a strict neutral level of 3.5 on the survey, their scores were high compared to what was expected from the literature. The study also showed that there might not be a difference between the scores of those that will be teaching mathematics and those that will not. The pre-survey was administered at the beginning of the semester before the students took their first course on advanced mathematics and proofs, and at that point of the students' mathematical career there was no difference in the beliefs scores of pre-service secondary mathematics teachers and those that are not pre-service teachers. However, there may be a point in their mathematical careers that the two groups begin to deviate in their beliefs.

Pajares (1992) says that beliefs become harder to change the more they are reinforced over time. Conversely, the newer the belief the more vulnerable it is to change. Perhaps the beliefs that future teachers acquire from their advanced mathematics courses were still so new and vulnerable to change. Perhaps the mathematics majors that go on to become mathematicians, rather than K-12 teachers, will have their initial beliefs reinforced through the extra schooling required for advanced degrees.

Alternatively, the scores could be high because the education the students received in K-12 has changed to allow these beliefs to manifest themselves. This would seem to disagree with what the literature says. If true this implies that the research needs to be updated in this respect.

The second question was: Do mathematics majors who participate in MATH 3330 with a component on mathematics research demonstrate greater changes to their beliefs about mathematics than mathematics majors who participate in a standard MATH 3330 course? The third question is similar but deals specifically with pre-service secondary mathematics teachers. Several conclusions can be made in regard to these questions.

There was a statistically significant increase in survey scores for students' beliefs related to the Status of Mathematics for the treatment group and not the control group, as expected at the beginning of the study. The literature, as stated above, suggests that many mathematics teachers believe that mathematics is a static body of knowledge (Ball, 1990; Stipek et al., 2001). This study implies that it is possible to change the beliefs by introducing students, and perhaps teachers, to mathematics research. However, when we restrict to the secondary mathematics teachers we see that the intervention did not change their beliefs in a statistically significant way, although it did have a practically significant change. This is related to what Pajares (1992) says about some beliefs being harder to change than others. These pre-service teachers might need more of the intervention for the beliefs to become more ingrained in them.

Also, there was a statistically significant increase in overall survey scores between the mid and post-surveys for the treatment group and not the control group, although there was a practical significant increase in scores for the control group. This implies that the intervention was successful in increasing the students' scores on the surveys. However, when we restrict the data to the pre-service teachers the results flip; the control group has a statistically significant change and the treatment group does not. This could again be related to the information that was presented to the students in the control group toward the end of the semester. Nevertheless, MATH 3330 appears to have improved the beliefs of the pre-service secondary mathematics teachers. This again seems to contradict the literature, that argues teachers believe mathematics is a static body of knowledge, a set of rules and facts, and algorithmic (Ball, 1990; Brendefur & Frykholm, 2000; Stipek et al., 2001). So we again consider the possibilities mentioned above when discussing the first research question. The pre-service secondary mathematics teachers' beliefs were new and vulnerable to change.

In regard to Mathematicians it appears that it does not matter whether the student was in the treatment group or the control group to have had their beliefs changed in a significant way. Although there was not a statistical significance in the change, there was practical significance and the change for both groups was similar. While many of these students could be future mathematicians, it appears that they can continue to grow and learn about the community they chose to join. This growth in their beliefs probably stems from their limited exposure to advanced mathematics and the people who do advanced mathematics. The more they learn about the field they will be a part of the more they will understand that being a mathematician is not as limiting socially or professionally as they might have thought.

There was no significant change in the students' conception of the Usefulness of Mathematics across both groups, which was expected. It is possible mathematics majors think that mathematics is useful or they would not major in the subject. This agrees with the literature that reports students who believe mathematics is useful perform better in mathematics (Fennema & Sherman, 1977, 1978). When we look at the supplemental data we see that, while a few of the students mentioned the importance of mathematics in learning to think, most students provided examples that were calculation-based. This concurs with the literature, which stated that students tend to provide examples of calculation based jobs (Kloosterman et al., 1996; Mason, 2003; Picker & Berry, 2000; Rock & Shaw, 2000).

When we look at the rest of the subscales there seem to be some differences from what was expected. Subscales that were expected to change due to the course should have showed some change in both the treatment and the control groups. There were significant positive changes in the control group only for the Composition and Validation subscales while there were significant positive changes for Structure, Doing, and Learning in the treatment group only. Since the significant change occurred between the pre and post-surveys, and not the mid and post-surveys, it is assumed that the course itself played a large role in effecting the students' beliefs in the treatment group for these subscales.

Mason and Scrivani (2004) state that there is a scarcity of studies that discuss improving beliefs about mathematics. The literature that exists indicates that it is possible to improve the beliefs of students and teachers in regard to mathematics (De Corte, Verschaffel, & Eynde, 2000; Mason & Scrivani, 2004; Szydlik, Szydlik, & Benson, 2003). This study adds to this body of knowledge by showing that exposure to current mathematics research influences the beliefs of mathematics majors as does a class that introduces advanced mathematical thinking and proofs.

Conclusions beyond the research questions. The treatment had unforeseen effects on the students that were nonetheless related to the purpose of the study. For many students the intervention increased their appreciation of mathematics. This is evident in the curiosity that the students expressed in their reflection papers. Students sought out information about what their professor was up to in terms of research. Others expressed fascination with the material and would try to understand it by thinking of ways that the mathematics research could be applied to concepts with which they were familiar. This shows that if a student is interested in the material, no matter how much they do not understand it, they can have an appreciation for it and try to make sense of it in their own way. This is related to the dilemma every teacher has of getting their students interested in the subject they are teaching so that the students would be interested enough to seek out information.

The presentations provided some students with further encouragement to attend graduate school. Conversely, one student said that the presentations showed him that he did not want to go to graduate school. While this can be considered a negative reaction to the intervention it is not necessarily a bad thing. The student has found out that mathematics is probably not what he thought it was and so it might be good that he decides to stop at a certain point before becoming disappointed in the future after spending time and money towards a major that might not be fulfilling in the long term. However, he should be encouraged to seek out understanding of other branches of mathematics that might suit his needs.

One of the interesting things that happened in the treatment group for the Spring semester was the number of questions that the students began to ask when the last presentation turned from the mathematics research to the mathematics research *process*. It seems that the students were more active in this portion because they felt like they could relate to and understand it. The students appeared to be genuinely interested in the mathematics research process, wanting to know what it means to do collaborative work in mathematics, or wanting to understand why someone, government or otherwise, would want to fund mathematics research. Thus the intervention was successful in providing the students with an opportunity to gain a greater appreciation for mathematics and mathematics research.

This greater appreciation can be attributed to a mixture of the intervention and the students' natural curiosity. The students, who did not have insight about what professors of mathematics do outside of teaching, probably would have been curious about getting paid to think and do mathematics or traveling and giving talks about mathematics. In other words, the thought of getting paid to do something you enjoy doing and to have your travel expenses paid for presenting something you enjoy sounds like an appealing proposition.

The intervention provided the students with the context in which mathematics work is done. They were first presented with a complicated problem that they did not understand and assumed that the problem was done in a brief amount of time. However when they learned in the third presentation that the problem had been worked on for over a year using many of the same strategies that they use when doing their homework they became more interested in the idea of mathematics research.

It is as if the students went through a cycle of belief about what it means for mathematicians to do mathematics. First, many of them believed that mathematics was computation and mathematicians' research was finding ways of improving applications. Next, the students were shocked when they were presented with a problem that might not have any real world applications in the near future and that seemed to be complicated. Finally, during the third presentation, the students realized that the mathematics research that was presented was done in much the same way that they do their work with more advanced mathematics, and that with hard work they too could do mathematics research. Most of the students then saw mathematics as a growing field that is not confined entirely to the realm of computations and applications.

Unfortunately not every student experienced the intervention in this way. Student 32 in particular was a good example of this as well as a good example of what Dweck described as a fixed mind-set. This student believed that graduate mathematics is too complicated and decided that he would not attempt graduate school. As stated above, this is not necessarily a bad thing since the student probably will save himself time, money, and frustration by not attending graduate school.

Limitations and Suggestions for Improvement

There were many limitations over the course of this study. They fall into three main categories: sampling, intervention, and honesty. Sampling limitations include sample size, method of selection, and student willingness. The sampling method used was a convenient sampling of students that happened to take the *Introduction to Advanced Mathematics* course. Therefore, it was not a random sample, which could cast doubt on the validity of the statistical results. Unreliable survey participation was another limitation. There were a total 40 students who took the survey three times, as shown in Table 7 of Chapter 6. There were a total of 94 students between the four classes used in the study but, for a variety of reasons, not all took the survey three times. A total of 18 students withdrew from the class during the semester and the others either chose not to participate in the study or were not in class the day the survey was administered. The students were given a week to finish the first survey; however, because the intervention only lasted for three weeks and began with three weeks left in the semester, the second and third surveys were administered on one day. Each student not in class was a missed survey.

Another problem with sampling arose with the control group. A majority of the students that participated were mathematics majors seeking secondary certification (with one pure mathematics major and no applied mathematics majors). Therefore there were a disproportionate number of secondary certification students in the control group compared to the treatment group. I believe the reason for this is that the pure and applied mathematics majors felt compelled to participate because they were in the treatment group receiving the intervention while the ones in the control group were indifferent about the study. The secondary certification students in the control group may have felt compelled to participate in the study because the study was a mathematics education study, something that related to their chosen career path.

There are several ways to deal with these sampling limitations. One is to do the study multiple times to provide a greater sample size. Another way to increase the sample size is to provide better incentives for participating, especially in the control groups. In the control group the incentive used was a homework grade for turning something in, either a letter stating they did not want to participate or the completed survey. Only four students wrote letters stating they did not want to participate in the study and it was observed that a majority of the students that did not participate did not care about getting the homework grade. A gift card as a gift for participating would be an example of a greater incentive for participating.

Intervention limitations include the time needed for implementation and the length of the survey. Many students expressed in their reflections that they thought the class was going at a faster pace than they were used to for the early part of the semester. This included not only the weekly lessons during the first part of the week but also the presentations during the second part of the week. The students wished that there were more time for class discussion, as opposed to lecturing, during the research presentations. A possible solution to this would be to have a one-hour weekly seminar outside of the class that would run the whole semester. It could also be offered as an elective one-hour credit for mathematics majors. This approach would be limited to the students who had to take a particular class and the control group would be students who had not taken the course. There would be problems with recruiting students to attend the seminar but this design is a legitimate alternative to the current study.

The length of the survey was 62 statements in which the students had to choose strongly agree, agree, partially agree, partially disagree, disagree, or strongly disagree.

The survey took at least ten minutes for most students to complete. The students, in their haste to finish the survey, may have not read the statements, not completely understood the statements, or become fatigued while taking the survey to the point that it affected their answers. It is possible to take out some of the subscales, especially Usefulness, since it was considered a subscale that would not change.

Another problem with the survey is that it is a self-report survey, which puts into question the accuracy of the students' answers. The same can be said of the interviews and the written work. A larger sample size might lessen the effect of this limitation.

Suggestions for Future Studies

This study chose to look into how beliefs change for students after an intervention in MATH 3330. The natural question of what beliefs mathematicians have of mathematics arises from this research. Additional questions center on what beliefs practicing secondary mathematics teachers and graduating seniors have of mathematics. These groups could be studied separately, or one could study the similarities and difference between the three groups.

Another study could focus on high school students, with a topic that is reasonable for high school students to understand. The researcher could go to the school for several weeks to give guest lectures on the topic. In this study the researcher would not teach the class leading up to the presentation of the mathematics research.

The literature shows as students enter into college they attempt to avoid mathematics as much as possible (Gilroy, 2002). Students could be choosing majors or changing majors because of their avoidance of mathematics. It would be interesting to study how their belief about mathematics inform their choices, besides the fact that they do not think they are good. In this case the survey used for this dissertation could be too limiting. It would probably be better to do a qualitative study to get the most information about their mathematical beliefs. Interview topics could include but are not limited to mathematics history, best mathematics moment, worst mathematics moment, how those moments influence their life, what constitutes doing mathematics. This study would help inform practitioners of how they could cater their instruction in order to alleviate some of the tension that students have toward mathematics.

Similarly pre-service elementary and middle school teachers could be studied to see how their beliefs about mathematics influenced their decision to go into teaching. From experience many of these students are shocked when they find out the amount of mathematics that they are required to take in order to get their degree. At Texas State University-San Marcos elementary pre-service teacher have to take at least three mathematics courses and four for middle school certification students. They include College Algebra and two or three seemingly "remedial" classes after College Algebra. That is, the content was at the elementary and middle school levels, but the focus is on understanding the structure of the lower grades content more so than a redo of that content. These courses have the potential to present a new picture of mathematics for the students that is less about calculation and more about explaining relationships, particularly between procedure and concepts. It would be interesting to know if students enrolled in these courses have changes in their beliefs, if so in what ways, and how these hypothetical changes compare to secondary certification seeking students.

Suggestions for Practitioners

One of the natural curiosities that students have is wanting to know what their professors do besides teach. One suggestion for instructors that teach MATH 3330 is to try to emphasize connections between their research and the material in the course. In fact this suggestion can be extended to all upper level mathematics classes in college. It provides the students with a glimpse of what it means to do research in mathematics and what they might be doing if they choose to go to graduate school as well as how what they are learning in the course relates to these issues.

Another suggestion for instructors of MATH 3330 is to have the students find research from current faculty and have them present it to the class. The students could connect a face with the research and perhaps would be more interested in it, since they would have access to the individual who did the research, and the researcher could answer any questions the students might have. The instructor might need to vet research papers to ensure that the topic was not too difficult and that the students would be able to complete assigned tasks.

Texas State has a robust Honors Math Camp in the summer, which periodically leads to campers publishing articles. Instead of presenting on faculty research the students could present the research of the Math Campers. This ensures that the topics are at a level that the students can understand.

Concluding Thoughts

The purpose of this study was to see if the presentation of mathematics research would manifest positive beliefs about mathematics as described in Table 12, copied from Table 1 in the first chapter. The results and conclusions of the study showed that the desired change in student beliefs was attained, especially in regard to the changing nature of mathematics. While students in the beginning believed that mathematics was a growing field, it was more of a feeling of "it has to be" than of really knowing that it is growing.

The hope is that this study encourages more college instructors to include aspects of current mathematics research in their advanced mathematics courses. This idea of introducing mathematics research will hopefully trickle down into the high school ranks. As was discussed, the Math Camp students at that level are able to carry out mathematics research. Admittedly the students that come through Math Camp are extraordinary, but it would be possible to find topics that the average high school student could understand.

Table 12

Description of What a Positive or Negative Belief is for Each Subscale

	Negative	Positive
Composition of Mathematics	Procedural	Conceptual
Structure of Mathematics	Isolated Ideas	Connected Ideas
Status of Mathematics	Dead	Alive
Doing Mathematics	Procedural	Conceptual
Validating in Mathematics	Told from Authority	Self-Proof
Learning Mathematics	Memorization	Understanding
Usefulness of Mathematics	Not Useful	Useful
Mathematicians are	Unimportant/Not Good	Important/Good

Graph theory is an excellent example of a mathematics topic that students at the high school level can understand at a basic level, especially because of its graphic nature.

It is also new enough to the students that any subject discussed could provide the student with a sense that mathematics is a growing field. Graph theory also provides example of a topic that is pure mathematics but can be easily applied in the real world in certain situations. It also shows a part of mathematics that is a little less calculation based than the algebra and geometry that students are exposed to in high school.

While some students had a less than stellar experience every student got something out of the treatment. For example one student decided he might not want to attend graduate school. He now does not have to waste his money to find out he does not like it. Another student, who has taken the class several times has expressed renewed interest after the treatment, and is considering going to graduate school.

I am encouraged though by the reactions of the students on the last day of the treatment, particularly in the Spring semester. The students seemed genuinely interested in the idea of mathematics research and how it is conducted. With any luck these students will carry the experience that they had in the treatment group with them as they continue their mathematical careers.

APPENDIX A

CONSENT LETTER

Research and Survey Consent Letter – Experimental Group

Dear student enrolled in MATH 3330:

We are seeking your help in a research dissertation project investigating the appreciation that different groups of people have of mathematics, including beliefs about the value of mathematics and mathematicians.

The project involves the administration of surveys that consist of a series of statements with which you will be asked to rate your level of agreement. Participation is completely voluntary: your grade in this class will not be affected by your participation.

Also, you will have written assignments periodically throughout the semester for homework. The survey will ask for your permission to use your reflections for the purposes of the study. However, the reflections will still be collected as a homework grade for the course.

Your responses will remain confidential and will only be known to myself and the dissertation committee. Identifiable information will be destroyed at the end of the project. You have the option of dropping out of the study at anytime. Simply inform me in person or by email of your wish to be dropped from the study.

You indicate your voluntary agreement to participate by completing and returning this survey to me. Thank you for volunteering to participate in this research project. **Keep this letter for your own reference.**

Sincerely,

Joshua Goodson
Texas State University – Mathematics Department
jg1356@txstate.edu
(512) 245-4740

Research and Survey Consent Letter – Control Group

Dear student enrolled in MATH 3330:

We are seeking your help in a research dissertation project investigating the appreciation that different groups of people have of mathematics, including beliefs about the value of mathematics and mathematicians.

The project involves the administration of surveys that consist of a series of statements with which you will be asked to rate your level of agreement. Participation is completely voluntary: your grade in this class will not be affected by your participation.

Your responses will remain confidential and will only be known to myself and the dissertation committee. Identifiable information will be destroyed at the end of the project. You have the option of dropping out of the study at anytime. Simply inform me in person or by email of your wish to be dropped from the study.

You indicate your voluntary agreement to participate by completing and returning this survey to me. Thank you for volunteering to participate in this research project. **Keep this letter for your own reference.**

Sincerely,

Joshua Goodson
Texas State University – Mathematics Department
jg1356@txstate.edu
(512) 245-4740

APPENDIX B

SURVEY

Name: ___________________________

Directions: Circle the appropriate response that corresponds to your overall level of agreement for each statement. SD (Strongly Disagree), D (Disagree), PD (Partly Disagree), PA (Partly Agree), A (Agree), and SA (Strongly Agree). **CIRCLE ONLY ONE.**

		SD	D	PD	PA	A	SA
1 .	There is always a rule to follow when solving a mathematical problem.	SD	D	PD	PA	A	SA
2 .	Learning mathematics involves more thinking than remembering information.	SD	D	PD	PA	A	SA
3 .	Mathematics has very little to do with students' lives.	SD	D	PD	PA	A	SA
4 .	Memorizing formulas and steps is not that helpful for learning how to solve mathematics problems.	SD	D	PD	PA	A	SA
5 .	When two students don't agree on an answer in mathematics, they need to ask the teacher or check the book to see who is correct.	SD	D	PD	PA	A	SA
6 .	Finding solutions to one type of mathematics problem cannot help you solve other types of problems.	SD	D	PD	PA	A	SA
7 .	You know something is true in mathematics when it is in a book or an instructor tells you.	SD	D	PD	PA	A	SA
8 .	In mathematics there are many problems that can't be solved by following a given set of steps.	SD	D	PD	PA	A	SA
9 .	When learning mathematics, it is helpful to analyze your mistakes.	SD	D	PD	PA	A	SA

10	.	Being able to use formulas well is enough to understand the mathematical concept behind the formula.	SD	D	PD	PA	A	SA
11	.	When you learn something in mathematics, you know the mathematics learned will always stay the same.	SD	D	PD	PA	A	SA
12	.	Diagrams and graphs have little to do with other things in mathematics like operations and equations.	SD	D	PD	PA	A	SA
13	.	The field of mathematics is always growing and changing.	SD	D	PD	PA	A	SA
14	.	Mathematics will not be important to students in their life's work.	SD	D	PD	PA	A	SA
15	.	Concepts learned in one mathematics class can help you understand material in the next mathematics class.	SD	D	PD	PA	A	SA
16	.	Asking questions in mathematics class means you didn't listen to the instructor well enough.	SD	D	PD	PA	A	SA
17	.	If you cannot solve a mathematics problem quickly, then spending more time on it won't help.	SD	D	PD	PA	A	SA
18	.	Learning computational skills, like addition and multiplication, is more important than learning to solve problems.	SD	D	PD	PA	A	SA
19	.	Learning to do mathematics problems is mostly a matter of memorizing the steps to follow.	SD	D	PD	PA	A	SA
20	.	Most mathematical ideas are related to one another.	SD	D	PD	PA	A	SA
21	.	When you learn mathematics, it is essential to compare new ideas to mathematics you already know.	SD	D	PD	PA	A	SA
22	.	When two classmates don't agree on an answer, they can usually think through the problem together until they have a reason for what is correct.	SD	D	PD	PA	A	SA
23	.	Students can make new mathematical discoveries, as well as study mathematicians' discoveries.	SD	D	PD	PA	A	SA

24	.	There is little in common between the different mathematical topics you have studied, like measurement and fractions.	SD	D	PD	PA	A	SA
25	.	Mathematicians work with symbol rather than ideas.	SD	D	PD	PA	A	SA
26	.	Sometimes when you learn new mathematics, you have to change ideas you have previously learned.	SD	D	PD	PA	A	SA
27	.	New mathematics is always being invented.	SD	D	PD	PA	A	SA
28	.	Solving a problem in mathematics is more a matter of understanding than remembering.	SD	D	PD	PA	A	SA
29	.	When working mathematics problems, it is important that what you are doing makes sense to you.	SD	D	PD	PA	A	SA
30	.	Knowing mathematics will help students earn a living.	SD	D	PD	PA	A	SA
31	.	Taking mathematics is a waste of time for students.	SD	D	PD	PA	A	SA
32	.	Essential mathematical knowledge is primarily composed of ideas and concepts.	SD	D	PD	PA	A	SA
33	.	When one's method of solving a mathematics problem is different from the instructor's method, both methods can be correct.	SD	D	PD	PA	A	SA
34	.	One can be quite successful at doing mathematics without understanding it.	SD	D	PD	PA	A	SA
35	.	Mathematics is a worthwhile subject for students.	SD	D	PD	PA	A	SA
36	.	Students should expect to have little use for mathematics when they get out of school.	SD	D	PD	PA	A	SA
37	.	Knowing why an answer is correct in mathematics is as important as getting a correct answer.	SD	D	PD	PA	A	SA
38	.	Mathematics today is the same as it was when your parents were growing up.	SD	D	PD	PA	A	SA

39 .	Students need mathematics for their future work.	SD	D	PD	PA	A	SA
40 .	In mathematics, the instructor has the answer and it is the student's job to figure it out.	SD	D	PD	PA	A	SA
41 .	While formulas are important in mathematics, the ideas they represent are more useful.	SD	D	PD	PA	A	SA
42 .	Understanding the statements a person makes is an important part of mathematics.	SD	D	PD	PA	A	SA
43 .	New discoveries are seldom made in mathematics.	SD	D	PD	PA	A	SA
44 .	Often a single mathematical concept will explain the basis for a variety of formulas.	SD	D	PD	PA	A	SA
45 .	Mathematicians enjoy working in collaboration with others.	SD	D	PD	PA	A	SA
46 .	When you do an exploration in mathematics, you can only discover something already known.	SD	D	PD	PA	A	SA
47 .	If you knew every possible formula, then you could easily solve any mathematical problem.	SD	D	PD	PA	A	SA
48 .	Mathematicians have a large range of career opportunities available to them.	SD	D	PD	PA	A	SA
49 .	Learning mathematics involves memorizing information presented to you.	SD	D	PD	PA	A	SA
50 .	Mathematics consists of many unrelated topics.	SD	D	PD	PA	A	SA
51 .	Many of the things that mathematicians do are being taken over by computers.	SD	D	PD	PA	A	SA
52 .	Justifying the statements a person makes is an important part of mathematics.	SD	D	PD	PA	A	SA
53 .	The field of mathematics is for the most part made up of procedures and facts.	SD	D	PD	PA	A	SA

54	.	The work that mathematicians do is the same work that students do in grade school but with larger numbers.	SD	D	PD	PA	A	SA
55	.	Students will use mathematics in many ways as adults.	SD	D	PD	PA	A	SA
56	.	Mathematics involves more thinking about relationships among things such as numbers, points, and lines than working with separate ideas.	SD	D	PD	PA	A	SA
57	.	Mathematicians are hired mainly to make precise measurements and calculations for scientists.	SD	D	PD	PA	A	SA
58	.	You can only find out that an answer to a mathematics problem is wrong when it is different from the book's answer or when the instructor tells you.	SD	D	PD	PA	A	SA
59	.	You can only learn mathematics when someone shows you how to work a problem.	SD	D	PD	PA	A	SA
60	.	It is important to convince yourself of the truth of a mathematical statement rather than to rely on the word to others.	SD	D	PD	PA	A	SA
61	.	Mathematicians do not appreciate other fields of knowledge.	SD	D	PD	PA	A	SA
62	.	Computation and formulas are only a small part of mathematics.	SD	D	PD	PA	A	SA

Directions: Answer the following by circling the response that described you the most.

Gender: M F

Major (circle one):

Pure Mathematics Applied Mathematics

Mathematics with Secondary Teacher Certification Mathematics Minor

Other:________________________________

Number of college credit hours completed:

0-25 hours 26-50 hours 51-75 hours more than 75 hours

Do you plan on getting a graduate degree in mathematics?

Yes No

Do you plan on getting a graduate degree in mathematics education?

Yes No

Do you give permission for the use of your written assignments to be used in the research study?

Yes No

Would you be willing to participate in a series of interviews throughout the semester?

Yes No

APPENDIX C

INTERVIEW CONSENT FORM

Interview Research Consent Form

Texas State University IRB Number:
Principal Investigator: Joshua Goodson, Texas State University – Mathematics Department

Introduction

You are being interviewed today because you indicated on the in class survey that you would be willing to participate in a series of interviews. You need to know:

- Your participation is entirely voluntary
- You may choose not to take part in this study or you may withdraw from the study at any time without fear of jeopardizing your standing within the course or the University.
- You have the right to refuse to answer any questions for any reason.

Purpose

The purpose of this research project is to investigate the beliefs that mathematics major have about mathematics and mathematicians. The interview is expected to last one hour and should not exceed one hour and 30 minutes.

Risks of the Study

There are no foreseeable risks to participating in the study.

Benefits of the Study

You may receive no benefit from taking part in the study. The research may give us knowledge to help mathematics education in the future.

Participation

You do not have to participate in this study if you do not want to. To opt out, simply say so and leave. You may also stop participating at any time by contacting Joshua Goodson to inform him of your decision.

Other Information

Confidentiality. When the results of this research study are reported in Joshua Goodson's dissertation, academic journals, or other scholarly activities, your identity will remain confidential. Your thoughts, ideas, and answers to the questions will be attributed to an alias (student 1, students 2, etc.). All identifiable information will be kept digitally and will be destroyed upon conclusion of the study.

Taking you off the study. The investigators can decide to withdraw you from the study at any time. The investigators reserve the right to decide what data (in the form of comments or information obtained from interviews) will be included in the study or in publications written based on the findings of the study. Your comments and/or interview may or may not be fully or partially included in publications based on this study.

Questions and Concerns. Any questions regarding your rights as a participant should be directed to the IRB chair, Dr. Jon Lasser (512-245-3413 – lasser@txstate.edu), or Ms. Becky Northcut, Compliance Specialist (512-245-2102).

Consent Form. Please keep a copy of this consent form in case you wish to read it again.

If you wish to participate in this interview, please sign below:

Document Consent

I understand the information included in this form. I have asked any questions that I have about this study, its risks and potential benefits, and my options for participating in it with Joshua Goodson, the primary contact. My questions so far have been answered. By signing this document I indicate my understanding that I can withdraw at any time, and my assertion that I am at least eighteen years old. Further, my signature below indicates my willingness to participate in an interview and my understanding that I can stop this interview at any time.

____

Print Name - Interviewee

____

Sign Name - Interviewee **Date**

____

Joshua Goodson – Interviewer Obtaining Consent **Date**

APPENDIX D

INFORMAL INTERVIEW GUIDE

1. Describe your mathematical background that led you to your current major.
2. What is the most pivotal moment your mathematics career and what does it mean to you?
3. If you were to defining mathematics to someone who does not know what mathematics is, what would you say?
4. Explain what you think is the most important concept in mathematics?
5. When you think of a mathematician, what picture comes to your mind?

APPENDIX E

LESSONS

Week 1 Lesson

Lesson Plan 1

Title: What Group Am I In

Author: Joshua Goodson

Objectives:

- The students will investigate properties of groups

Book: Smith, D., Eggen, M., & Andre, R. *A Transition to Advanced Mathematics: 6th Ed.* Thomson Brooks/Cole (2006).

Procedures:

In this lesson students will investigate the properties that define a group. This lesson is adapted from Cullinane's (2005) article about making abstract algebra relevant to future teachers. The lesson is constructed in such a way so that students are able to make a connection between abstract algebra and the algebra they teach, or took, in high school. While the definition of a group is explored in a way that different from the way that the group structure is normally thought as, it never the less makes it relevant to the student.

Vocabulary:

1. Group
    a. A set G together with an associative binary operation "." defined on G such that there exist $e \in G$ with the following properties
        1. For each $x \in G, x \cdot e = e \cdot x = x$
        2. For each $x \in G$ there exists $y \in G$ such that $x \cdot y = y \cdot x = e$
2. Subgroup
    a. A subset $H \subseteq G$ is a subgroup of G if H is closed under "." in G and forms a group with respect to ".".
3. Abelian
    a. A group G is abelian if and only if the operation is commutative.
4. Generators

Activities*:*

The first activity begins by asking the students to solve the linear equation x+4=10. From here we try to guide the students into naming some of the group properties from solving this simple linear equation and getting them to understand which properties are needed and which are not needed to solve the equation.

Instructor: Solve this linear equation
Board: x+4=10?

See the Likely Vignette at the end of the lesson to see how this activity might happen. You can lead a discussion with you at the board or lead the discussion with a student at the board. We want to use as few properties as possible. Allow them to solve and ask them questions such as "Do we really need to use the commutative property?" We want to minimize the number of properties that we use.

- Examples of things we do not want students to do
  - To go straight to $x=6$, because we need to use the properties
  - Do subtraction of 4 on both sides because we only want to use addition
  - Do vertical addition of -4 because in this case we do not know where the -4 is being added to
    - Discuss if it matters (Not in this case because of the commutative property but what about matrices)
    - Remember we want to limit the number of properties as much as we can
    - Do $x+4+(-4)=x+(-4)$ (Add in the same position on both sides)

After they have outlined the procedure for solving the equation:

Instructor: Which properties of integers are needed in order to solve the equation?

We are trying to guide the students to some of the group properties. **Desired answers** we want them to give include the associative property, the identity 0, and an integer's additive inverse (the negative of the integer). Be sure to note the above properties on the board. Note that at this point students might think that the commutative property is needed when it is not. Others might wonder about the need for the associative property.

After these properties have been found and discussed:

Instuctor: Is the number set that we are working in important? For example, suppose we had 3x=18 and I asked you to solve in the integers using only multiplication in the traditional way, could you solve it?
Board: 3x=18

Here we are trying to show them that if we only used multiplication we cannot solve this equation unless we were using rational numbers. **Desired answer** is no because 1/3 is not an integer.

Taking many of these properties that we found for solving these equations, we can form the definition of what we call a Group.

Board:
A group is a set G, together with an operation "*" defined on the set, and denoted (G,*), such that:
1. For all x and y in G, x*y is in G
2. For all x, y, and z in G, (x*y)*z = x*(y*z)
3. There exist an element e in G such that x*e = e*x=x for all x in G.
4. For all x in G there exist a y in G such that x*y=y*x=e.

(Note to instructor: This definition is equivalent to the one above.)

Provide some simple examples of groups on the board. Inform the students that we use multiplication as an abstract operation a lot of the time when we are talking about groups in general.

Instructor: Another example of a group is the rationals without zero under multiplication
Board: (Q-{0}, *)
Instructor: Let us quickly run down the group axioms but we will not do a formal proof.
Verify each axiom with students informally.

Theorem not to be proved in class.
Instructor: The following theorem will not be proved in class. However it is necessary to know and understand, it will also help shorten what we need to write when doing a proof. I encourage you to attempt the proof on your own. If you need help on any proof I ask you to try, feel free to come by my office hours.
Board: Theorem: Let G be a group. Then the inverse of any element in G is unique.

Instructor: Thus, we can denote the inverse of an element, x, in a group G as x^{-1}.
Board: The inverse of an element, x "in" G is denoted x^{-1}
Instructor: Thus when we write x^{-1} it is understood to be the inverse of x so in our proof we need not say let y be an inverse of x and we do not need to worry about there being multiple inverses for elements.

Prove the following theorem with the class helping.
Cancellation Theorem: Let G be a group and a,b,x be in G.
a) If ax = bx then a = b.
b) If xa = xb then a = b.

Abridged Proof: a) Let G be a group and a,b,x be in G such that ax = bx. Then x^{-1} is in

G and $a = a1 = ax\,x^{-1} = bx\,x^{-1} = b1 = b$. Proof for b is similar.

Instructor: One way to display information about a group is with a multiplication table (also called operation table) for finite groups. Consider the integers modulo six with addition.

Construct the following table with the students.

Example: ($\mathbb{Z}6$, +)

	0	**1**	**2**	**3**	**4**	**5**
0	0	1	2	3	4	5
1	1	2	3	4	5	6
2	2	3	4	5	6	0
3	3	4	5	6	0	1
4	4	5	6	0	1	2
5	5	6	0	1	2	3

Check the group axioms informally for the above set with the students to find that it is a group.

Instructor: We will come back to this example in just a minute. If I said that a group had a subgroup, what would that mean to you?
Desired Answer: A subgroup is a subset of the group and is itself a group.
Instructor: Is it possible that for a subgroup to have a different identity element than the larger group?
Desired Answer: No.

Have them attempt a proof.

Theorem: If H is a subgroup of G then the identity in H is the identity in G.

Abridged Proof: Let 1_H be the identity in H and 1_G be the identity in G. Let x be in H. Then $x1_H = x = x1_G$. By the cancelation theorem $1_H = 1_G$.

Instructor: Let us return to the table of $\mathbb{Z}6$. Can you find any subgroups in this group?
Desired Answer: The set 0, 2 and 4 or the set 0 and 3.
Board: {0, 2, 4} {0, 3}

Check the group properties informally as a class.

Instructor: Note that the identity must be in the set. What do you notice about these subgroups? In other words, what happens when you continuously add 2 to itself?
Desired Answer: We get the subgroup 0, 2, 4.
Instructor: We call 2 a generator for the subgroup. In fact, notice that 1 is a

generator for the whole group.

At this point we are going to foreshadow the idea of isomorphism

Instructor: Do these subgroups look like they could be an integers mod x set or do they behave like different integer mod sets? Do not answer this question, just think about it. You might notice that in $\mathbb{Z}6$the operation is still commutative. When a group is commutative we call it an abelian group or we say the group is abelian. This type of group is named after one of the "godfathers" of algebra Abel.

We can construct groups that are not abelian. For example you might recall that multiplication with matrices does not always commute. We can construct a non-abelian group that way but I think it would be more interesting to discuss what is known as the dihedral group of order 8. The order of a group is the number of elements in the group. One of the reasons I think this group is more interesting is because it is derived not from numbers but from the different positions of a square.

Derive the dihedral group with the students using the procedure below if needed. Be sure to illustrate to the students how the operation is performed after deriving the group.

First we start with a square and label each corner whatever we want, just to keep track of how the square is positioned. (If they do not come up with anything just label it a,b,c,d counter clockwise). What are some different positions that this square can be in, for example it could look like this b,c,d,a? What are some other ones? (Draw all 8 of them on the board). This was our original square (point to the first one a,b,c,d) and we call this the identity square). Lets look at this one, what must we do to the identity square to get this one (do this with each square. Afterward show them that some are a combination of the others. For example the 180 degree turn is two 90 degree turns. This will give us names for each element in our set. Then show them how to multiply to of them and then encourage them to make the operation table for the group). We will not be proving that this is a group but I encourage you to make the operation table and investigate the group properties on your own. The purpose of this exercise is that I wanted to show a group that was not commutative, in other words not abelian. If we do a 90 degree rotation times a flip then it is different than a flip and then a 90 degree rotation.

Assessment:
Homework:
Turn in
6.2 – 3, 13 & 6.3 – 3
For extra practice
6.2 – 9, 16 & 6.3 – 4, 7, 8

Reference:
Cullinane, M. J. (2005). Motivating the notions of binary operation and group in an abstract algebra course. *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies, 15*(4), 339-348.

Likely Vignette
of first activity

T: Solve x+4 = 10 and show your work

S: $x + 4 = 10$
$\quad -4 \quad -4$
$\quad x = 6$

T: What exactly does this mean?

S: Subtract 4 from both sides

T: The -4s look like they're floating, what do they mean in terms of the "=" sign. Why not subtract 4 from each side in a separate line.

S: $x + 4 – 4 = x – 4$

T: Now finish from there.

S: I did, x = 6.

T: Why is x by itself?

S: $4 – 4 = 0$

T: For what we are doing we want to show each step. This is actually what we started with since we are subtracting 4 from the quantity that is x + 4:
$(x + 4) = 10$
$(x + 4) – 4 = 10 – 4$
How do we know we can "do" 4 – 4 if we have this? (act out associative property with hands)

S: Associative

T: We need to be careful, is subtraction associative?
(Consider asking them if the following is true: $5 – 4 – 2 = (5 – 4) – 2 = 5 – (4 – 2)$)
We do know that addition is, so is there a way to do this problem with only addition.

S: yes, $x + 4 + (-4) = 10 + (-4)$

T: Now we can use the associative property $x + (4 + -4) = 10 + -4$
How do we know $4 + -4 = 0$

S: -4 is the opposite of 4; or -4 same magnitude different sign; or additive inverses

T: So -4 is the additive inverse of 4 (define if necessary)

T: x + 0 = 6. Student wrote x = 6, why does this follow?

S: Anything plus 0 is itself.

T: So adding 0 does not change the number, does not change its identity, so we call 0 the additive identity. So here is what the problem looks like now:

x + 4 = 10	
(x + 4) + -4 = 10 + -4	equality property
x + (4 + -4) = 10 + -4	associative
x + 0 = 6	additive inverse (need this to exist to cancel the 4)
x = 6	additive identity (if didn’t exist we would have a problem solving)

Week 2 Lesson

Lesson Plan 2

Title: How many what…?

Author: Joshua Goodson

Objectives:

- The students will be able to count permutations and combinations, use the product rule and sum rule,
- The students will be able to apply counting techniques to different aspects of groups.

Book: Smith, D., Eggen, M., & Andre, R. *A Transition to Advanced Mathematics: 6th Ed.* Thomson Brooks/Cole (2006).

Procedure:

Instructor: Who in here is familiar with counting? For example, on the Texas Lottery homepage it tells us that there are 25,827,165 different number combinations in Lotto Texas. (Choose six from 54 numbers). Who knows how they came up with this number? This is the type of counting that we will be discussing today.

This lesson begins with a worksheet in which the students will work alone or in small groups to count some elementary problems. After a few minutes gather the class up and spend a few minutes discussing the problems.

Instructor: The first problem is an example of what we call the sum rule. What do you think the second problem is an example of?

Desired Answer: The product rule.

Instructor: The third is an example of a permutation and the fourth a combination. What were your strategies for solving the problems (Aside from writing them all out)?

Desired Answer: ***Answers may vary.***

Instructor: How would you define the sum rule?

Theorem 2.16: Sum Rule: If A and B are disjoint sets with $|A|=m$ and $|B|=n$ then $|A\cup B|=m+n$

Here students might just say that it is the sum of the number of two things or collection of things. We need express to them that the numbers are from two sets or a collection of things or objects. Also, they might not say that the sets need to be disjoint. If they do not, provide them with an example in which two set are not disjoint.

Instructor: How would you define the product rule?

Theorem 2.19: Product Rule: If two independent tasks T1 and T2 are to be performed, and T1 can be performed in m ways and T2 in n ways, then the two tasks can be performed in mn ways.

Just as disjoint sets were important for the definition of the sum rule, the two tasks here need to be independent. Again provide an example of tasks that are not independent.

Instructor: How would you define a permutation?

Permutation: A permutation of a set with n elements is an arrangement of the elements of the set in a specific order.

Theorem 2.22: The number of permutations of any r distinct objects taken from a set of n objects is n!/(n-r)!

Proof: Let S be a set with n elements. We will pick r elements of S in any arbitrary order. There are n elements in S, so there are n options for the first element. Then there are (n-1) elements left in S that have not been picked, so there are (n-1) options for the 2^{nd} element. Then there are (n-2) for the 3^{rd} element and so on. Thus there are (n-(r-1)) options for the rth element. Each selection is independent of the each other so by the product rule

n(n-1)(n-2)…(n-r+1) = n!/(n-r)!

Instructor: How would you define a combination?

Combination: A combination of n elements taken r at a time is the selection of an r-element subset from an n-element set. The number of combinations is denoted (n over r).

Theorem 2.23: Let n be a positive integer and r be an integer such that 0<=r<=n, n!/r!(n-1)!

Proof: Let A be a set with n elements. By Theorem 2.22, the number of permutations of all n objects is n!

The n objects can also be arranged by first selecting r objects, arranging them, and arranging the remaining n-r objects. There are (n over r) ways to select r objects, r! ways to arrange them, and (n-r)! ways to arrange the rest. By the product rule, the number of permutations of the n objects is (n over r)*r!*(n-r)!.

Instructor: So if we were to go back to the lottery example, what type of problem is that?

Desired Answer: Combination.

Do a couple more problems as a class.

Example: How many subsets are there for a set with n elements?

Example: How many 3 element subsets are there for a set of $n \geq 3$ elements?

Assessment:
Homework:
Turn in
4, 14
For extra practice
5, 8, 13

Week 3 Lesson

Lesson Plan 3

Title: Are we the same?

Author: Joshua Goodson

Objectives:

- The students will become familiar with the ideas of homomorphisms and isomorphisms.
- The students will become familiar with how two groups are "the same" up to isomorphisms.

Book: Smith, D., Eggen, M., & Andre, R. *A Transition to Advanced Mathematics: 6th Ed.* Thomson Brooks/Cole (2006).

Procedure:

Instructor: Today we are going to begin the day by playing a game. The game is a two player game in which you take turns picking an integer between 1 and 9. Once a number has been picked it cannot be picked again. The first player to have exactly three numbers add to fifteen is the winner. If, after all the numbers have been picked, no one has exactly three that add to fifteen, then there is no winner.

Make sure the students understand the game and allow them a few minutes to play in pairs.

Instructor: Did you gain any insights into how to play the game? Believe it or not the game that you were playing can be set up to look like this…

Board:

8	3	4
1	5	9
6	7	2

Instructor: If you notice, picking the numbers is like picking a square in this table. And having a sum of 15 is like completing a row, column, or diagonal in this table. What game is this like playing?

Desired Answer: Tic-tac-toe.

Instructor: The point of this exercise is to illustrate that two things that look seemingly different can turn out to be considered the same thing. This is the topic that we are going to be discussing today. Consider the function f(x) = log(x).

Board: f(x) = log(x).

Instructor: Suppose we had f(5)+f(6) = log(5) + log(6).

Board: f(5)+f(6) = log(5) + log(6)

Instructor: How can we rewrite this (Or write it the other way and ask how can we rewrite this)?

Desired Answer: log(5)+log(6) = log(5*6).

Board: log(5)+log(6) = log(5*6)

Instructor: This is a function that maps the positive rationals under multiplication to the reals under addition and is what is called a homomorphism. What do you think that means?

Board:

A **homomorphism** from a group (A,*) to a group (B,+) is a mapping f such that for all x,y in A, f(a*b) = f(a)+f(b).

Instructor: Thus, as (Q+, *) and (R, +) are groups, logarithms are homomorphisms. Is this function 1-1?

Desired Answer: Yes.

Instructor: Be sure to prove that on your own. Is this function onto?

Desired Answer: No, because the image is countable and real numbers are not.

Write the groups (Z6,+) and this on the board (Z7-0,*).

Instructor: Here are two groups. If you do not believe me prove the axioms to yourself. Is there a homomorphism from one to the other?

Give the students time to think and consider any ideas that they have.

Instructor: There is, consider the mapping from 1 in Z6 to 3 in Z7. Recall the generators we discussed a couple of weeks ago.

Write all the mappings on the board based on the mapping from 1 in Z6 to 3 in Z7

Instructor: There is something else about this example. Is it 1-1?

Desired Answer: Yes

Instructor: Is it onto?

Desired Answer: Yes

Instructor: In this case, when a homomorphism is 1-1 and onto we call it an isomorphism.

Board: A homomorphism that is 1-1 and onto is called an isomorphism.

Instructor: What do we know about the sets when we have 1-1 and onto functions from one to the other?

Desired Answer: They have the same order.

Instructor: We also get something extra when a function is an isomorphism, we can also say that the groups behave in the same way. Also, recall the subgroups of Z6 discussed a couple of weeks ago. What do you think the two subgroups would be isomorphic to?

Desired Answer: Z2 and Z3.

Put the groups on the board and compare.

The following are proofs to be done as a class while time permits.

Proofs:

Theorem 6.15b: Suppose we had two groups (A,*) and (B,+), A is abelian, and we know that f:A->B is an isomorphism. In this case we can show that B is abelian.
Do the following proof together.
Proof: f(a) + f(b) = f(a*b) = f(b*a) = f(b) + f(a)

Theorem 6.14b: Let f be a homomorphism from (A,*) to (B,+). If e is the identity in A then f(e) is the identity in B.

Proof: Let e_A be in A, then f(e) is in B. Then f(e)=f(e*e)=f(e)+f(e). So, by cancellation, e_b = f(e).

Theorem 6.14c: Let f be a homomorphism from (A,*) to (B,+). If x^-1 is the inverse for x in A, then f(x^-1) is the inverse for f(x) in B.

Proof: Let x be in A and x^-1 be the inverse for x in A. e_B = f(e_A) = f(x * x^-1) = f(x) + f(x^-1). Thus, f(x^-1) = f(x)^-1.

Assessment:
Homework: 13, 19

Extra practice:

6.4: 12, 14, 20

APPENDIX F

CLASS TOPICS SCHEDULES

Fall 2010 Treatment

Day	Topic
1	Introduction
2	Set theory
3	Set theory
4	Set theory
5	Set theory
6	The natural numbers
7	The natural numbers
8	The natural numbers
9	The natural numbers
10	Relations
11	Relations
12	Relations
13	**Test 1**
14	Relations
15	Functions
16	Functions
17	Functions
18	Cardinality
19	Cardinality
20	Cardinality
21	Cardinality
22	Cardinality
23	Cardinality
24	Cardinality
25	Aspects of real analysis
26	Aspects of real analysis
27	Aspects of real analysis
28	**Test 2**
29	Algebraic Structures
30	Algebraic Structures
31	Algebraic Structures
32	Joshua Goodson: Groups
33	Joshua Goodson: Groups
34	Joshua Goodson: Groups
35	Joshua Goodson: Groups
36	Joshua Goodson: Counting and Homomorphisms
37	Joshua Goodson: Counting and Homomorphisms
38	Joshua Goodson: Counting and Homomorphisms
39	Joshua Goodson: Orbits and extraspecial groups
40	Joshua Goodson: Orbits and extraspecial groups
41	Joshua Goodson: Orbits and extraspecial groups
42	Survey/Review
	Final Exam

Class Day	Section	Descriptive Comments
1	1.1/1.2	Propositions and Connectives, Conditionals and Biconditionals
2	1.3/1.4	Quantifiers and Basic Proof Methods
3	1.4	Basic Proof Methods
4	1.5	Basic Proof Methods II
5	1.6	Proofs Involving Quantifiers
6	Chapter1	More Proofs
7	2.1/2.2	Basics of Set Theory and Set Operations
8	2.2	Set Operations
9	2.3	Indexed Families of Sets
10	2.4	Induction
11	2.4	Induction Continued
12	2.5	Equivalent Forms of Induction
13		Exam 1 (Thursday)
14	2.6	Principles of Counting
15	3.1	Cartesian Products and Relations
16	3.1	Cartesian Products and Relations
17	3.2	Equivalence Relations
18	3.3	Partitions
19	3.4	Ordering Relations
20	4.1/4.2	Functions, Inverse Relations, Inverse Functions
21	4.3/4.4	1-1 and Onto Functions, Images of Sets
22	4.4	Images of Sets
23		Exam 2 (Thursday)
24	Chapter 5	Equivalent Sets and Counting
25	7.1	Upper and Lower Bounds on Subsets of the Reals
26	7.2	Open and Closed Sets in the Reals
27	7.3/4.5	Boundary and Accumulation Points and Sequences
28		Review and Presentations

Spring 2011 Treatment

Class Day	Section	Descriptive Comments
1	1.1/1.2	Mathematical language and notation, Logic
2	1.3/1.4	Quantifiers , Direct proofs
3	1.5	Basic Proof Methods: Contrapositive, Contradiction, Biconditionals
4	1.6	Basic Proof Methods: Quantifiers
5	1.5/1.6	Proofs :Strategies for selecting appropriate methods of proof
6	1.7	Additional Proofs and examples
7	2.1/2.2	Basics of Set Theory and Set Operations, Proofs involving sets (set equality, DeMorgan's laws, etc.)
8	2.3	Indexed Families of Sets
9	2.4	Principles of Mathematical Induction
10	2.5	Equivalent Forms of Induction(strong induction, well-ordering principal)
11		Exam 1 (Wednesday)
12	3.1	Cartesian Products and Relations
13	3.2	Equivalence Relations
14	3.3	Partitions
15	3.4	Ordering Relations
16	7.1	Upper and Lower Bounds on Subsets of the Reals
17	4.1/4.2	Functions, Inverse Relations, Inverse Functions
18	4.3/4.4	1-1 and Onto Functions, Images of Sets, Inverse Images, Proofs involving these concepts
19	Chapter 5	Equivalent Sets and Counting
20	Chapter 7	Open and Closed Sets in the Reals, Boundary and Accumulation Points and Sequences
21		Exam 2 (Wednesday)
22		Joshua Goodson: Groups
23		Joshua Goodson: Groups
24		Joshua Goodson: Counting
25		Joshua Goodson: Counting
26		Joshua Goodson: Homomorphisms
27		Joshua Goodson: Homomorphisms
28		Review and Presentations

Spring 2011 Control

Day	Topic
1	Set theory
2	Set theory
3	The natural numbers
4	The natural numbers
5	The natural numbers
6	Relations
7	Relations
8	Functions
9	Functions
10	Cardinality
11	Cardinality
12	**Test 1**
13	Cardinality
14	Cardinality
15	Cardinality
16	Algebraic Structures
17	Algebraic Structures
18	Algebraic Structures
19	Groups
20	Groups
21	Groups
22	**Test 2**
23	Groups
24	Groups
25	Rings and Fields
26	Rings and Fields
27	Rings and Fields
28	Review
	Final Exam

APPENDIX G

GAP SCRIPTS

Brian Doring's
MaxOrbitSimp.txt

```
cl:=ConjugacyClasses(G);
cl1:=List(cl,x->AsSet(cl1),OnSets);
O:=OrbitLengths(A,AsSet(cl1),OnSets);
Print(“For the subgroup A of Aut(G) of size “,Size(A),”, the maximum size of
Maximum(O), “\n”);
```

Frank Lübeck's

```
g := ExtraspecialGroup(3^5,'+');
r := IrreducibleRepresentations(g,GF(7));;
List(r, h-> DimensionOfMatrixGroup(Image(h)));
g1 := Image(r[1]);
one := One(g1);
numorbs := 1/Size(g1) * Sum(ConjugacyClasses(g1), c-> Size(c) *
                                    7^(9RankMat(Representative(c)-one)));
vsp := GF(7)^9;
enum := Enumerator(vsp);
found := BlistList([1..Size(vsp)], []);;
sizes := 0*[1..Size(g1)];;
for i in [1..Size(vsp)] do
 if not found[i] then
    orb := Orbit(g1, enum[i]);
    for v in orb do
      found[NumberFFVector(v,7)+1] := true;
    od;
    sizes[Length(orb)] := sizes[Length(orb)] + 1;
  fi;
  # show progress and stop when numorb orbits were found
  if i mod 100000 = 0 then
    if Sum(sizes) = numorbs then break; fi;
    Print(i/100000, " (",Sum(sizes),") \c");
  fi;
od;
# result, numbers of orbits of size 1, 3, 9, 27, 81, 243
sizes{List([0..5],i->3^i)};
```

APPENDIX H

MATH 3330 DEPARTMENT SYLLABUS

Course Information

Semester –
Course – MATH 3330
Section –
Class Time –
Class Room –

Instructor Information

Name –
Office –
Telephone –
Email –
Office Hours –

Course Title – Introduction to Advanced Mathematics

Course Description – An introduction to the theory of sets, relations, functions, finite and infinite sets, and other selected topics. Algebraic structure and topological properties of Euclidean Space, and an introduction to metric spaces. Prerequisite: MATH 2471 with a grade of "C" or higher.

Objectives – The goal of Introduction to Advanced Mathematics is to provide students an opportunity to learn to prove mathematical theorems. This course provides an introduction to higher level abstraction in mathematics. This is achieved within the following framework:

- Logic
- Set theory
- Number Theory
- Properties of real numbers
- Functions

Textbook –

Brief Course Outline –

Attendance Policy -

Important Dates:
Exams - *see course calendar*
Final Exam –

Drop Dates -
Drop with no record -
Drop with an automatic W – by 5:00 pm on
Last day to withdraw from the University – at the office of the Registrar by 5 pm on

Grading –

Academic Honor Code

As members of a community dedicated to learning, inquiry and creation, the students, faculty and administration of our university live by the principles in this Honor Code. These principles require all members of this community to be conscientious, respectful and honest.

We are conscientious.

We complete our work on time and make every effort to do it right. We come to class and meetings prepared and are willing to demonstrate it. We hold ourselves to doing what is required, embrace rigor, and shun mediocrity, special requests, and excuses.

We are respectful.

We act civilly toward one another and we cooperate with each other. We will strive to create an environment in which people respect and listen to one another, speaking when appropriate, and permitting other people to participate and express their views.

We are honest.

We do our own work and are honest with one another in all matters. We understand how various acts of dishonesty, like plagiarizing, falsifying data, and giving or receiving assistance to which one is not entitled, conflict as much with academic achievement as with the values of honesty and integrity.

The Pledge for Students

Students at our university recognize that, to ensure honest conduct, more is needed than an expectation of academic honesty, and we therefore adopt the practice of affixing the following pledge of honesty to the work we submit for evaluation:

Honor Code web site   http://txstate.edu/effective/upps/upps-07-10-01.html

Electronic Devices - *Cellular Telephones, Pagers, Palm Pilots or any device* *that may distract from the class should be turned off before class begins and may not be on the desk during class or tests.*

Special Needs – *Students with special needs, as documented by the Office of Disability Services, should identify themselves at the beginning of the semester.*

Resources -

Texas State Endorses Wingspread Journal's Seven Principles for Good Practice in Undergraduate Education:

1. *Student-faculty intellectual interaction*
2. *Intellectual interaction with fellow students, except when it interferes with assignments to be completed on an independent basis*
3. *Active Learning*
4. *Prompt feedback*
5. *Timely completion of tasks*
6. *High expectations, and*
7. *Respect for diverse talents and ways of learning*

Notes:

1. The instructor reserves the right to deviate from the syllabus in a short term basis to better serve the students enrolled in the course.
2. Due to diverse background of students, instructor may be required to devote more time on reviews and consequently deviate from the following calendar.
3. The instructor may select a different textbook or other ancillary material, however, the same concepts will be covered.
4. The instructor may deviate from the sequential order presented below, however the outlying concepts will be covered.
5. Some concepts, like logic, may be integrated within other contexts and therefore covered accordingly.
6. Instructor may deviate in scheduling tests and reviews depending on the pace. Moreover, some instructors review material in an ongoing basis and thereby the following schedule will be adjusted accordingly.
7. Some instructors give a daily/weekly test in an ongoing basis, and therefore the following test schedule will not necessarily be applicable. The sequential order of the material will be adjusted accordingly.

Course Calendar

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