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EXISTENCE OF GLOBAL WEAK SOLUTIONS FOR A p-LAPLACIAN INEQUALITY WITH STRONG DISSIPATION IN NONCYLINDRICAL DOMAINS

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Abstract. In this work, we obtain global solutions for nonlinear inequalities of p-Laplacian type in noncylindrical domains, for the unilateral problem with strong dissipation

$$u'' - \Delta_p u - \Delta u' - f \geq 0 \quad \text{in } Q_0,$$

where Δ_p is the nonlinear p-Laplacian operator with $2 \leq p < \infty$, and Q_0 is the noncylindrical domain. Our proof is based on a penalty argument by J. L. Lions and Faedo-Galerkin approximations.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set with smooth boundary Γ, T be a positive real, fixed but arbitrary, and $Q_0 = \Omega \times (0,T)$ be the cylinder with side border $\Sigma_0 = \Gamma_0 \times (0,T)$. J. L. Lions [11] considered the problem

$$\begin{gathered} u'' - \Delta u - f \geq 0 \quad \text{in } Q_0, \\ u' \geq 0 \quad \text{in } Q_0, \\ u = 0 \quad \text{on } \Sigma_0, \\ u(0) = u_0, \quad u'(0) = u_1 \quad \text{in } \Omega. \end{gathered} \tag{1.1}$$

If $K = \{v \in H_0^1(\Omega); v(x) \geq 0 \text{ a.e. on } \Omega\}$, then (1.1) can be reformulated as

$$\begin{gathered} \langle u''(t), v - u'(t)\rangle + \langle -\Delta u(t), v - u'(t)\rangle \geq \langle f(t), v - u'(t)\rangle, \quad \forall v \in K, \\ u'(t) \in K \quad \text{a.e.}, \\ u(0) = u_0, \quad u'(0) = u_1. \end{gathered} \tag{1.2}$$

We consider the p-Laplacian operator $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, which can be extended to a monotone, bounded, hemicontinuos and coercive operator between the spaces $W_0^{1,p}(\Omega)$ and its dual by

$$-\Delta_p \colon W_0^{1,p}(\Omega) \to W^{-1,q}(\Omega), \quad \langle -\Delta_p u, v\rangle_p = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx.$$

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The existence of a global solution for the wave equation of p-Laplacian type

$$u_{tt} - \Delta_p u = 0 \tag{1.3}$$

without an additional dissipation term, is an open problem. For $n = 1$, Derher [5] gave the finite time existence of solution and showed, by a generic counter-example, that the global solution not can be expected. Later, adding a strong dissipation $(-\Delta u')$ in (1.3), the well-posedness and asymptotic behavior it was studied by Greenberg [9]. Nevertheless, when the strong damping is replaced by weaker damping (u_t), existence and uniqueness of a global solution are only known for $n = 1, 2$. See [4, 22]. Gao and Ma [8] proved global existence of solution and asymptotic behavior under the intermediate damping $(-\Delta)^\alpha u_t$ with $0 < \alpha \le 1$. The memory damping was analyzed by Raposo et al. [18], p-Laplacian damping was studied by Pereira et al. [16] and a thermoelastic effect was considered in [20]. For wave coupled systems of the p-laplacian type see [15]. For other works on this subject, we cite [14] and the references therein. For a brief review of the literature on non-cylindrical domain, we cite [1, 7, 11, 17]. Unilateral problem is very interesting because in general, dynamic contact problems are characterized by nonlinear hyperbolic variational inequalities. For contact problems in elasticity and finite element method see Kikuchi-Oden [10] and reference therein. For contact problem viscoelastic materials, see Rivera and Oquendo [21]. For dynamic contact problems with friction, for instance, problems involving unilateral contact with dry friction of Coulomb, see Ballard and Basseville [3]. The study of variational inequalities in bounded domains has been analyzed by several authors, for example, see [2, 6, 12, 13].

In this work we consider the following p-Laplacian unilateral problem with strong dissipation,

$$\begin{gathered} u'' - \Delta_p u - \Delta u' - f \ge 0 \quad \text{in } Q_0, \\ u' \ge 0 \quad \text{in } Q_0, \\ u = 0 \quad \text{on } \Sigma_0, \\ u(0) = u_0 \quad u'(0) = u_1 \quad \text{in } \Omega. \end{gathered} \tag{1.4}$$

We prove the existence of solutions for (1.4) by using the penalty method.

2. Penalty method

When using the penalization technique as in [11], a difficulty may appear since the term $\langle u''(t), v - u'(t)\rangle$ makes sense only when $u''(t) \in H^{-1}(\Omega)$, which is not always possible to obtain. For this reason, the result obtained is the weak formulation of (1.4), namely: if $K \subset W_0^{1,p}(\mathbb{R}^n)$ is a closed and convex subset with $0 \in K$, and

$$\begin{gathered} V = \{v \in L^2(0,T; W_0^{1,p}(\Omega_t)); v' \in L^2(0,T, W_0^{-1,p'}(\Omega_t)), v(t) \in K \text{ a.e.}\}, \\ K = \{v \in W_0^{1,p}(\Omega); v(x) \ge 0 \text{ a.e. in } \Omega\}, \end{gathered}$$

equation (1.4) can be reformulated as

$$\begin{gathered} \langle u''(t), v - u'(t)\rangle + \langle \Delta_p u(t), v - u'(t)\rangle + \langle -\Delta u'(t), v - u'(t)\rangle \ge \langle f(t), v - u'(t)\rangle, \\ u(0) = u_0, \quad u'(0) = u_1, \end{gathered} \tag{2.1}$$

for $u'(t) \in K$ a.e. and for all $v \in K$.

Then, the existence of $u : Q \to \mathbb{R}$, with

$$u(0) = u_0 \in W_0^{1,p}(\Omega_0), u'(0) = u_1 \in L^2(\Omega_0) \cap K, \quad u'(t) \in K \text{ a.e. in } (0,T)$$

and

$$\begin{aligned} &\int_0^s \langle v'(t) + \Delta_p u(t) - \Delta u'(t) - f(t), v(t) - u'(t)\rangle \\ &\geq \frac{1}{2}|v(s) - u'(s)|^2 - \frac{1}{2}|v(0) - u'(0)|^2, \forall s \in (0,T), \ \forall v \in V\,. \end{aligned} \tag{2.2}$$

It is easy to check that if $u' \in V$, then (2.1) and (2.2) are equivalent. However, we shall find a solution for (1.4) in the sense of (2.2). Thus, the objective of this work is to obtain the existence of global weak solution to (1.4) considering Q as a non-cylindrical domain, as in fact, Lions [11] provides existence and uniqueness of weak solutions and/or regular for operators of the parabolic-hyperbolic type in the noncylindrical domain.

By $\mathcal{D}(\Omega)$ we denote the space of infinitely differentiable functions with compact support contained in Ω. The inner product and norm in $L^2(\Omega)$ and $H_0^1(\Omega)$ will be represented by $(\cdot\,,\cdot)$, $|\cdot|$, $\|\cdot\|$, respectively, and by $\langle\cdot,\cdot\rangle$ the duality between $W_0^{1,p}(\Omega)$ and $W^{-1,p'}(\Omega)$.

If $T > 0$ and X is a Banach space with the norm $\|\cdot\|_X$, we denote by $L^p(0,T;X)$, $1 \leq p < +\infty$, the Banach space of vector functions $u : (0,T) \to X$ that are measurable and $\|u(t)\|_X \in L^p(0,T)$ with the norm

$$\|u\|_{L^p(0,T;X)} = \Big[\int_0^T \|u(t)\|_X^p dt\Big]^{1/p},$$

and by $L^\infty(0,T;X)$ the Banach space of vector functions $u : (0,T) \to X$ that are measurable and $\|u(t)\|_X \in L^\infty(0,T)$ with the norm

$$\|u\|_{L^\infty(0,T;X)} = \text{esssup}_{0<t<T}\|u(t)\|_X.$$

Let Ω be an open, connected and bounded subset of $\mathbb{R}^n$ with regular boundary Γ, $Q \subset Q_0$ a noncylindrical domain. We will use the following notation

$$\begin{gathered} \Omega_s = Q \cap \{t = s\} \text{ for } 0 < s < T, \quad \Omega_0 = \text{int}_{\mathbb{R}^n}(\overline{Q} \cap \{t = 0\}), \\ \Omega_T = \text{int}_{\mathbb{R}^n}(\overline{Q} \cap \{t = T\}), \quad \Gamma_s = \partial\Omega_s, \\ \Sigma = \cup_{0<s<T}\Gamma_s, \quad \partial Q = \Omega_0 \cup \Sigma \cup \Omega_T \quad \text{the boundary of } Q. \end{gathered}$$

It is clear that $\Omega_0 \neq \emptyset$. Our hypotheses on Q are:

- (H1) Ω_t is monotonically increasing, that is, $\Omega_t^* \subset \Omega_s^*$ if $t < s$, where $\Omega_t^* = \text{Proj}_{\{t=0\}}\Omega_t$.
- (H2) For each $t \in [0,T]$, Ω_t has the following regularity property: if $u \in W_0^{1,p}(\Omega)$ e $u = 0$ a.e. in $\Omega \setminus \Omega_t^*$, then $u|_{\Omega_t^*} \in H_0^1(\Omega_t^*)$.

To simplify the notation, we identity Ω_t^* with Ω_t.

Let us define

$$L^q(0,T;L^p(\Omega_t)) = \{w \in L^q(0,T;L^p(\Omega)) : w = 0 \text{ a.e. in } Q_0 \setminus Q\}.$$

When $1 \leq q < \infty$ we consider the norm

$$\|w\|_{L^p(0,T;L^p(\Omega_t))} = \Big[\int_0^T \|w(t)\|_{L^p(\Omega_t)}^q dt\Big]^{1/q},$$

which coincides with $\|w\|_{L^q(0,T;L^p(\Omega))}$. And when $q=\infty$ we consider

$$\|w\|_{L^\infty(0,T;L^p(\Omega_t))} = \operatorname{ess\,sup}_{0<t<T} \|w(t)\|_{L^p(\Omega_t)}.$$

Note that $L^q(0,T;L^p(\Omega_t))$ is a closed subspace of $L^q(0,T;L^p(\Omega))$ for $1\le q\le \infty$. Analogously we define $L^q(0,T;W_0^{1,p}(\Omega_t))$, $1\le q\le\infty$. It is also true that $L^q(0,T;W_0^{1,p}(\Omega_t))$ is a closed subspace of $L^q(0,T;W_0^{1,p}(\Omega))$.

3. Existence of a global weak solution

Theorem 3.1. *Let $f\in L^2(0,T,L^2(\Omega_t))$, $u_0\in W_0^{1,p}(\Omega_0)$, $u_1\in L^2(\Omega_0)\cap K$, with K being a convex and closed subset of $W_0^{1,p}(\Omega)$, and $0\in K$. Lets us suppose that* (H1) *and* (H2) *are satisfied. Then there exists a function $u:Q\to\mathbb{R}$ satisfying*

$$u\in L^\infty(0,T;W_0^{1,p}(\Omega_t)), \tag{3.1}$$

$$u'\in L^\infty(0,T;L^2(\Omega_t))\cap L^2(0,T;H_0^1(\Omega_t)), \tag{3.2}$$

$$u'(t)\in K \text{ a.e. in } (0,T), \tag{3.3}$$

$$u(t)\to \tilde{u_0} \text{ in } H_0^1(\Omega) \text{ if } t\to 0, \tag{3.4}$$

$$u'(t)\to \tilde{u_1} \text{ in } L^2(\Omega) \text{ if } t\to 0, \tag{3.5}$$

$$\begin{aligned}&\int_0^s \langle v'(t)+\Delta_p u(t)-\Delta u(t)-f(t), v(t)-u'(t)\rangle dt\\ &\ge \frac12 |v(s)-u'(s)|^2_{L^2(\Omega_s)} - \frac12 |v(0)-v(1)|_{L^2(\Omega_0)},\quad \forall s\in(0,T),\ \forall v\in V,\end{aligned} \tag{3.6}$$

$$u(0)=u_0,\quad u'(0)=u_1 \quad \text{in } \Omega_0. \tag{3.7}$$

with $\tilde{u_0}$ and $\tilde{u_1}$ being extensions of u_0 and u_1 to Ω that vanish outside Ω_0.

Theorem 3.1 will be proved by using the Faedo-Galerking method, penalty operator associated to the convex set and penalty method from Lions [11]. At First we find a solution of penalized problem in the cylinder Q_0 and then we show that the restriction to the noncylinder domain Q is indeed weak solution for the original problem.

To this end, let $\tilde{u_0}\in W_0^{1,p}(\Omega)$, $\tilde{u_1}\in L^2(\Omega)$, and $\tilde f\in L^2(Q_0)$ be the extensions to zero outside Ω_0 of u_0, u_1, and f, respectively. Let us also consider a penalty function for noncylindrical domains:

$$M(x,t)=\begin{cases}0 & \text{in } Q\cup\Omega_0\times\{0\},\\ 1 & \text{in } Q_0\setminus(Q\cup\Omega_0\times\{0\}).\end{cases}$$

Let $P_K: H_0^r(\Omega)\to K$ be the projection operator: for $u\in H_0^r(\Omega)$, $P_k u$ is the unique element in K such that

$$\|u-P_K u\|\le \|u-k\|,\quad \forall k\in K,$$

where r is a fixed integer with $r>1+\frac{n}{2}-\frac{n}{p}$ such that $H_0^r(\Omega)\hookrightarrow W_0^{1,p}(\Omega)$ continuously. Let J be the duality operator from $H_0^r(\Omega)$ into $H^{-1}(\Omega)$ relatively to the identity from R_+ to R_+. That is,

$$\langle Ju,u\rangle = \|Ju\|_{H^{-r}(\Omega)}\|u\|,$$
$$\|J(u)\|_{H^{-r}(\Omega)} = \|u\|.$$

We consider now $\beta : H_0^1(\Omega) \to H^{-1}(\Omega)$ defined by $\beta(u) = J(u - P_K u)$. The operator β is a penalty operator associated to K, thus satisfies

$$\begin{gathered} \beta \text{ is monotone, bounded, Hemicontinuous and} \\ K = \{v \in H_0^1(\Omega); \beta(v) = 0\}. \end{gathered} \tag{3.8}$$

The proof of Theorem 3.1 is a consequence of the following theorem.

Theorem 3.2. *Suppose the hypotheses of Theorem 3.1 are satisfied. Then for each $\mu > 0$ there exists a function $u_\mu : Q_0 \to \mathbb{R}$ satisfying*

$$u_\mu \in L^\infty(0,T; W_0^{1,p}(\Omega)), \tag{3.9}$$

$$u'_\mu \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)), \tag{3.10}$$

$$\begin{aligned} &\int_0^s [\langle v'(t) + \Delta_p u_\mu(t) - \Delta u'_\mu(t) - \tilde{f}, v(t) - u'_\mu(t) + \frac{1}{\mu}\langle M(t)u'_\mu(t), v(t)\rangle\rangle]\,dt \\ &\geq \frac{1}{2}|v(s) - u'_\mu(s)|^2 - \frac{1}{2}|v(0) - u_1|^2, \quad \forall t \in (0,T), \\ &\forall \mu,\ \forall v \in L^2(0,T; W_0^{1,P}(\Omega)) \text{ such that } v' \in L^2(0,T, W^{-1,p'}(\Omega)). \end{aligned} \tag{3.11}$$

Before prove the main theorem, we present the existence of a special basis.

4. Galerkin basis

According [19], we will show that there exists a Hilbert space $H_0^s(\Omega)$ with $0 < s$ such that $H_0^s(\Omega) \hookrightarrow W_0^p(\Omega)$ is continuous and $H_0^s(\Omega) \hookrightarrow L^2(\Omega)$ is continuous and compact.

For $v \in H^1(\mathbb{R}^n)$ we consider Fourier transform of v,

$$\hat{v}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-(\xi.x)i} v(x)\,dx$$

and

$$H^s(\mathbb{R}^n) = \{v \in L^2(\mathbb{R}^n) : (1 + \|\xi\|^{s/2}\hat{v}(\xi)) \in L^2(\mathbb{R}^n)\}.$$

Since Ω is a bounded open set with sufficiently smooth boundary, we have $H^s(\Omega)$ is the set of restrictions on Ω of the functions $v \in H^s(\mathbb{R}^n)$, then

$$\|v\|_{H^s(\Omega)} = \inf\{\|V\|_{H^s(\mathbb{R}^n)} : V = v \text{ a.e. in } \Omega\}$$

and

$$H_0^s(\Omega) = \overline{C_0^\infty(\Omega)}^{H^s(\Omega)}.$$

We need

$$W_0^{m,q}(\Omega) \hookrightarrow W_0^{m-k,q_k}(\Omega), \quad \frac{1}{q_k} = \frac{1}{q} - \frac{k}{n}.$$

Choosing $q_k = p$, $m - k = 1$ and $q = 2$ we obtain $m = 1 + \frac{n}{2} - \frac{n}{p}$. For $s > m$ we have

$$H_0^s(\Omega) \hookrightarrow W_0^{1,p}(\Omega) \hookrightarrow H_0^1(\Omega) \hookrightarrow L^2(\Omega)$$

from where our goal follows. Now, from spectral theory the problem

$$((v_j, v))_{H_0^s(\Omega)} = \lambda_j(v_j, v), \quad \text{for all } v \in H_0^s(\Omega)$$

has solution and moreover $\{v_j\}_{j\in\mathbb{N}}$ precisely, is a Schauder basis for $H_0^s(\Omega) \cap L^{r+1}(\Omega)$ with elements that are orthogonal in $L^2(\Omega)$.

5. Proof of the main theorem

The proof of Theorem 3.2 will be made in 4 steps.

5.1. **Penalty approximated problem.** Let $\{w_1, w_2, \dots\}$ be a Schauder basis of $H_0^s(\Omega)$ as demonstrate before, and for each $m \in \mathbb{N}$ let $V_m = [w_1, \dots, w_m]$ be the subspace generated by the m first vectors from this basis. Let $0 < \varepsilon < 1$ fixed. We wish to find

$$u_{\varepsilon\mu m}(x,t) := u_{\varepsilon\mu m}(t) = \sum_1^m g_{j\varepsilon\mu m}(t) w_j(x),$$

where $g_{j\varepsilon\mu m}(t)$ of the system of ODEs

$$\begin{aligned}
&(u''_{\varepsilon\mu m}(t), w_j) + \langle \Delta_p u_{\varepsilon\mu m}(t), w_j\rangle + (\nabla u_{\varepsilon\mu m}(t), \nabla w_j)\\
&+ \frac{1}{\varepsilon}(\beta(u_{\varepsilon\mu m}(t)), w_j) + \frac{1}{\mu}(M(t) u_{\varepsilon\mu m}(t), w_j)\\
&= (f\tilde{(t)}, w_j), \quad \forall w_j \in V_m,
\end{aligned} \tag{5.1}$$

$$u_{\varepsilon\mu m}(0) = u_{0m} \to \tilde{u_0} \quad \text{strongly in } W_0^{1,p}(\Omega), \tag{5.2}$$

$$u''_{\varepsilon\mu m}(0) = u_{1m} \to \tilde{u_1} \quad \text{strongly in } L^2(\Omega). \tag{5.3}$$

By Caratheodory the system (5.1)–(5.3) has a local solution $u_{\varepsilon\mu m}(t)$ defined in some interval $[0, t_m)$, $0 < t_m < T$.

5.2. **A priori estimates I.** Composing (5.1) with $u'_{\varepsilon\mu m}(t) \in V_m$ and then integrating from 0 to $t < t_m$, we obtain

$$\begin{aligned}
&\frac{1}{2}\big\{|u'_{\varepsilon\mu m}(t)|^2 + \frac{1}{p}\|u_{\varepsilon\mu m}(t)\|^p_{W_0 1,p(\Omega)}\big\} + \int_0^t \|u'_{\varepsilon\mu m}(s)\|^2_{H_0^1(\Omega)}\,ds\\
&+ \frac{1}{\varepsilon}\int_0^t (\beta(u'_{\varepsilon\mu m}(s)), u'_{\varepsilon\mu m}(s))\,ds + \frac{1}{\mu}\int_0^t (M(t) u'_{\varepsilon\mu m}(s), u'_{\varepsilon\mu m}(s))\,ds\\
&= \int_0^t (\tilde{f}(s), u'_{\varepsilon\mu m}(s))\,ds + \frac{1}{2}|u_{0m}|^2 + \frac{1}{p}\|u_{1m}\|^p_{W_0^{1,p}(\Omega)}.
\end{aligned} \tag{5.4}$$

Using (5.2) and (5.3), the monotonicity of β, the definition of M, $\tilde{f} \in L^2(Q_0)$, and Gronwall's lemma in (5.3), we obtain

$$\begin{aligned}
&\frac{1}{2}|u'_{\varepsilon\mu m}(t)|^2 + \frac{1}{p}\|u_{\varepsilon\mu m}(t)\|^p_{W_0 1,p(\Omega)} + \int_0^t \|u'_{\varepsilon\mu m}(s)\|^2_{H_0^1(\Omega)}\,ds\\
&+ \frac{1}{\varepsilon}\int_0^t (\beta(u'_{\varepsilon\mu m}(s)), u'_{\varepsilon\mu m}(s))\,ds + \frac{1}{\mu}\int_0^t (M(t) u'_{\varepsilon\mu m}(s), u'_{\varepsilon\mu m}(s))ds \le C
\end{aligned}$$

where C is a positive constant independent of ε, μ, m and $t \in [0, t_m)$. Hence we can extend the solution $u_{\varepsilon\mu m}(t)$ to the whole interval $[0, T]$, obtaining in addition

$$(u_{\varepsilon\mu m}) \text{ is bounded in } L^\infty(0, T; W_0^{1,p}(\Omega)), \tag{5.5}$$

$$(u'_{\varepsilon\mu m}) \text{ is bounded in } L^\infty(0, T; L^2(\Omega)), \tag{5.6}$$

$$(u'_{\varepsilon\mu m}) \text{ is bounded in } L^2(0, T; H_0^1(\Omega)), \tag{5.7}$$

$$(u_{\varepsilon\mu m}(T)) \text{ is bounded in } W_0^{1,p}(\Omega), \tag{5.8}$$

$$(u_{\varepsilon\mu m}(T)') \text{ is bounded in } L^2(\Omega), \tag{5.9}$$

$$\big(\frac{1}{\sqrt{\mu}}Mu'_{\varepsilon\mu m}\big) \text{ is bounded in } L^\infty(0,T;L^2(\Omega)). \tag{5.10}$$

From the definition of β one can prove that β is Lipschitz and thus from (5.7), it follows that

$$(\beta(u'_{\varepsilon\mu m})) \text{ is bounded in } L^2(0,T;H^{-1}(\Omega)). \tag{5.11}$$

In addition, the operator $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is a bounded operator from $W_0^{1,p}(\Omega)$ to $W^{-1,p'}(\Omega)$. Thus it follows from (5.5) that

$$(\Delta_p u_{\varepsilon\mu m}) \text{ is bounded in } L^\infty(0,T;W^{-1,p'}(\Omega)). \tag{5.12}$$

We can thus extract subsequences from above sequences, denoted up to subindexes, such that

$$u_{\varepsilon\mu m} \overset{*}{\rightharpoonup} u_{\varepsilon\mu} \quad \text{in } L^\infty(0,T;W_0^{1,p}(\Omega)), \tag{5.13}$$

$$u'_{\varepsilon\mu m} \overset{*}{\rightharpoonup} u'_{\varepsilon\mu} \quad \text{in } L^\infty(0,T;L^2(\Omega)), \tag{5.14}$$

$$u'_{\varepsilon\mu m} \rightharpoonup u'_{\varepsilon\mu} \quad \text{in } L^2(0,T;H_0^1(\Omega)), \tag{5.15}$$

$$u_{\varepsilon\mu m}(T) \rightharpoonup u_{\varepsilon\mu}(T) \quad \text{in } W_0^{1,p}(\Omega), \tag{5.16}$$

$$u'_{\varepsilon\mu m}(T) \rightharpoonup u'_{\varepsilon\mu}(T) \quad \text{in } L^2(\Omega). \tag{5.17}$$

Since $M \in L^\infty(Q_0)$, it follows from (5.14) that

$$\frac{1}{\sqrt{\mu}}Mu'_{\varepsilon\mu m} \overset{*}{\rightharpoonup} \frac{1}{\sqrt{\mu}}Mu'_{\varepsilon\mu} \quad \text{in } L^\infty(0,T;L^2(\Omega)), \tag{5.18}$$

$$\beta(u'_{\varepsilon\mu}) \rightharpoonup \chi_{\varepsilon\mu} \quad \text{in } L^2(0,T;H^{-1}(\Omega)), \tag{5.19}$$

$$\Delta_p u_{\varepsilon\mu m} \overset{*}{\rightharpoonup} \varphi_{\varepsilon\mu} \quad \text{in } L^\infty(0,T;W^{-1,p'}(\Omega)). \tag{5.20}$$

5.3. **A priori estimate II.** Now we obtain an estimate for $u''_{\varepsilon\mu m}$. It is done through a standard argument on projections. Consider the projection operator $P_m : H_0^m(\Omega) \to V_m$ defined by

$$P_m[h] = \sum_{j=1}^m ((h,w_j))w_j, \quad h \in H_0^r(\Omega)$$

where $((\cdot,\cdot))$ stands for the inner product in $H_0^r(\Omega)$. Let $P_m^* \in \mathcal{L}(H^{-r}(\Omega), H^{-r}(\Omega))$ the self-adjoint extension of P_m. Since $P_m^*[h] = P_m[h] = h$, $\forall h \in V_m$, we conclude from (5.1) that

$$\begin{aligned}(u''_{\varepsilon\mu m}(t),w) &= (P_m^*[\tilde f(t)],w) - \langle P_m^*[\Delta_p u_{\varepsilon\mu m}(t)],w\rangle + \langle P_m^*[\Delta u'_{\varepsilon\mu m}(t)],w\rangle\\ &\quad - \frac{1}{\varepsilon}(P_m^*[\beta(u'_{\varepsilon\mu m}(t))],w) - \frac{1}{\mu}(P_m^*[M(t)u'_{\varepsilon\mu m}(t)],w) \quad \forall w \in V_m.\end{aligned}$$

Thereby, using argument of denseness it follows from (5.7), (5.10), (5.11) and (5.12) that

$$(u''_{\varepsilon\mu m}) \text{ is bounded in } L^2(0,T;H^{-r}(\Omega)) \text{ for each } \varepsilon,\mu. \tag{5.21}$$

Taking into account the convergence obtained above, we can pass to the limit when $m \to \infty$ in the approximated equation and obtain

$$\begin{gathered}u''_{\varepsilon\mu} + \varphi_{\varepsilon\mu} - \Delta u'_{\varepsilon\mu} + \frac{1}{\varepsilon}\chi_{\varepsilon\mu} + \frac{1}{\mu}Mu'_{\varepsilon\mu} = \tilde f \quad \text{in } L^2(0,T;W^{-1,p'}(\Omega)),\\ u_{\varepsilon\mu}(0) = \tilde u_0,\\ u'_{\varepsilon\mu}(0) = \tilde u_1.\end{gathered}$$

It can be shown through the same arguments as in Ferreira-Ma [7] that $\varphi_{\varepsilon\mu} = \Delta_p u_{\varepsilon\mu}$ and reasoning likewise as in Rabello [17] that $\chi_{\varepsilon\mu} = \beta(u'_{\varepsilon\mu})$. Therefore, we obtain

$$\begin{gathered} u''_{\varepsilon\mu} + \Delta_p u_{\varepsilon\mu} - \Delta u'_{\varepsilon\mu} + \frac{1}{\varepsilon}\beta(u'_{\varepsilon\mu}) + V\frac{1}{\mu}Mu'_{\varepsilon\mu} \\ = V\tilde{f} \text{ in } L^2(0,T;W^{-1,p'}(\Omega)), \\ u_{\varepsilon\mu}(0) = \tilde{u}_0, \\ u'_{\varepsilon\mu}(0) = \tilde{u}_1. \end{gathered} \tag{5.22}$$

We observe that the bounds obtained are independently on ε, μ and t, thus there exist subsequences from previous sequences such that

$$u_{\varepsilon\mu} \xrightarrow[\varepsilon\to 0]{*} u_{\varepsilon\mu} \quad \text{in } L^\infty(0,T;W_0^{1,p}(\Omega)), \tag{5.23}$$

$$u'_{\varepsilon\mu} \xrightarrow[\varepsilon\to 0]{*} u'_{\varepsilon\mu} \quad \text{in } L^\infty(0,T;L^2(\Omega)), \tag{5.24}$$

$$u'_{\varepsilon\mu} \xrightarrow[\varepsilon\to 0]{} u'_{\varepsilon\mu} \quad \text{in } L^2(0,T;H_0^1(\Omega)), \tag{5.25}$$

$$\frac{1}{\sqrt{\mu}}Mu'_{\varepsilon\mu} \xrightarrow[\varepsilon\to 0]{*} \frac{1}{\sqrt{\mu}}Mu'_{\varepsilon\mu} \quad \text{in } L^\infty(0,T;L^2(\Omega)), \tag{5.26}$$

$$\int_0^T (\beta(u'_{\varepsilon\mu}), u'_{\varepsilon\mu})dt \xrightarrow[\varepsilon\to 0]{} 0. \tag{5.27}$$

From (5.22) we obtain

$$\beta(u'_{\varepsilon\mu}) = \varepsilon\big[f - \Delta_p u_{\varepsilon\mu} - u''_{\varepsilon\mu} + \Delta u'_{\varepsilon\mu} - \frac{1}{\mu}Mu'_{\varepsilon\mu}\big] \quad \text{in } \mathcal{D}'(0,T;W^{-1,p'}).$$

Thus, from the convergences (5.20), (5.23)-(5.26), it follows that

$$\beta(u'_{\varepsilon\mu}) \xrightarrow[\varepsilon\to 0]{} 0 \quad \text{in } \mathcal{D}'(0,T;H^{-r}(\Omega)).$$

In addition, from (5.26), since β is Lipschitz,

$$\beta(u'_{\varepsilon\mu}) \xrightarrow[\varepsilon\to 0]{} \chi \quad \text{in } L^2(0,T;H^{-1}(\Omega)).$$

Thereby we have $\chi = 0$. On the other hand, thanks to the monotonicity and hemicontinuity of β and (5.27), we prove that $\chi = \beta(u_\mu)$ and therefore we conclude that

$$\beta(u'_\mu(t)) = 0 \text{ a.e. or } u'_\mu \in K \text{ a.e.} \tag{5.28}$$

Let $v \in L^2(0,T;W_0^{1,p}(\Omega))$ such that $v' \in L^2(0,T;W^{-1,p'}(\Omega))$. Therefore,

$$
\begin{aligned}
&\frac{1}{2}|v(s)-u'_{\varepsilon\mu}(s)|^2-\frac{1}{2}|v(0)-u'_{\varepsilon\mu}(0)|^2\\
&=\frac{1}{2}\int_0^s\frac{d}{dt}|v(t)-u'_{\varepsilon\mu}(t)|^2dt\\
&=\int_0^s\langle v'(t)-u''_{\varepsilon\mu}(t),v(t)-u'_{\varepsilon\mu}(t)\rangle dt\\
&=\int_0^s\langle v'(t)-\Big[\tilde f(t)-\Delta_p u_{\varepsilon\mu}(t)+\Delta u'_{\varepsilon\mu}(t)-\frac{1}{\varepsilon}\beta(u'_{\varepsilon\mu})\\
&\quad-\frac{1}{\mu}Mu'_{\varepsilon\mu}(t)\Big],v(t)-u'_{\varepsilon\mu}(t)\rangle dt\\
&=\int_0^s\langle v'(t)-\tilde f(t)+\Delta_p u_{\varepsilon\mu}(t)-\Delta u'_{\varepsilon\mu}(t),v(t)\rangle dt\\
&\quad+\int_0^s\langle v'(t)-\tilde f(t),-u'_{\varepsilon\mu}(t)\rangle dt\\
&\quad+\int_0^s\langle +\Delta_p u_{\varepsilon\mu}(t),u'_{\varepsilon\mu}(t)\rangle dt-\int_0^s\langle -\Delta u'_{\varepsilon\mu}(t),u'_{\varepsilon\mu}(t)\rangle dt\\
&\quad+\underbrace{\frac{1}{\epsilon}\int_0^s\langle\beta(u'_{\varepsilon\mu}(t))-\beta(v),v(t)-u'_{\varepsilon\mu}(t)\rangle dt}_{\le 0}+\int_0^s\frac{1}{\mu}\langle M(t)u'_{\varepsilon\mu}(t),v(t)\rangle dt\\
&\quad+\underbrace{\int_0^s\frac{1}{\mu}\langle M(t)u'_{\varepsilon\mu}(t),-u'_{\varepsilon\mu}(t)\rangle dt}_{\le 0}.
\end{aligned}
\tag{5.29}
$$

Let $\Psi=\{\varphi\in C^0[0,T],\varphi(t)\ge 0\ \forall t\in[0,T]\}$. Multiplying (5.29) by $\varphi\in\Psi$ and integrating from 0 to T, we obtain

$$
\begin{aligned}
&\int_0^T\Big[\frac{1}{2}|v(s)-u'_{\varepsilon\mu}(s)|^2+\frac{1}{p}\|u_{\varepsilon\mu}(s)\|^2_{W_0^{1,p}}+\int_0^s\|u'_{\varepsilon\mu}(t)\|^2dt\Big]\varphi(s)ds\\
&\le\int_0^T\varphi(s)\int_0^s\langle v'(t)-\tilde f(t)+\Delta_p u_{\varepsilon\mu}(t)-\Delta u'_{\varepsilon\mu}(t),v\rangle\,dt\,ds\\
&\quad+\int_0^T\varphi(s)\int_0^s\langle v'(t)-\tilde f(t),-u'_{\varepsilon\mu}(t)\rangle\,dt\,ds\\
&\quad+\frac{1}{p}\|u_{\varepsilon\mu}(0)\|^2_{W_0^{1,p}(\Omega)}\int_0^T\varphi(s)ds+\int_0^T\varphi(s)\int_0^s\frac{1}{\mu}\langle M(t)u'_{\varepsilon\mu}(t),v(t)\rangle\,dt\,ds\\
&\quad+\frac{1}{2}\int_0^T|v(0)-u'_{\varepsilon\mu}(0)|^2\varphi(s)ds.
\end{aligned}
\tag{5.30}
$$

Taking the limit inferior, it follows from (5.23)–(5.26) and from Banach-Steinhauss' Theorem that

$$
\begin{aligned}
&\int_0^T \varphi(s)\left[\frac{1}{2}|v(s)-u'_\mu(s)|^2+\frac{1}{p}\|u_\mu(s)\|^2+\int_0^s \|u'_\mu(t)\|^2 dt\right] ds\\
&\le \int_0^T \varphi(s)\int_0^s \langle v'(t)-\tilde{f}(t)+\Delta_p u'_\mu(t)-\Delta u'_\mu(t), v(t)\rangle\, dt\, ds\\
&\quad+\int_0^T \varphi(s)\int_0^s \langle v'(t)-\tilde{f}(t), -u'_\mu(t)\rangle\, dt\, ds\\
&\quad+\frac{1}{p}\|u(0)\|^2_{W_0^{1,p}}\int_0^T \varphi(s)ds+\int_0^T \varphi(s)\int_0^s \frac{1}{\mu}\langle M(t)u'_\mu(t), v(t)\rangle\, dt\, ds\\
&\quad+\int_0^T \varphi(s)\frac{1}{2}|v(0)-u_1|^2\, ds, \quad \forall \varphi\in\Psi.
\end{aligned}
\tag{5.31}
$$

Thus considering

$$
\varphi=\begin{cases}1 \text{ if } t=s,\\ \text{linear in } (s-\delta,s) \text{ and } (s,s+\delta),\end{cases}
$$

$0\le s\le 1$, $\varphi\in C^0[0,T]$, splitting the inequality (5.31) by $\delta>0$, taking the limit with $\delta\to 0$, we obtain from the Lebesgue points Theorem for integrable functions

$$
\begin{aligned}
&\int_0^s [\langle v'(t)-\tilde{f}(t)+\Delta_p u_\mu(t)-\Delta u'_\mu(t), v(t)-u'_\mu(t)\rangle]\\
&+\frac{1}{\mu}\langle M(t)u'_\mu(t), v(t)\rangle dt\\
&\ge \frac{1}{2}|v(s)-u'_\mu(s)|^2-\frac{1}{2}|v(0)-u_1|^2, \quad \forall\mu, \text{ a.e.}
\end{aligned}
\tag{5.32}
$$

We obtain, therefore, the penalized inequality in cylinder domain Q_0, what proves Theorem 3.2.

5.4. **Passage to the limit.** It remains now passing to the limit when $\mu\to 0$ to obtain the inequality in the noncylindrical domain Q and thus to have Theorem 3.1 proved.

From (5.23)-(5.26), Banach-Stainhaus' Theorem and boundedness provided by (5.5), (5.6),(5.7) and (5.10) independently on ε and μ, there exist subsequences such that

$$u_\mu \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty(0,T;W_0^{1,p}(\Omega)), \tag{5.33}$$

$$u'_\mu \overset{*}{\rightharpoonup} u' \quad \text{in } L^\infty(0,T;L^2(\Omega)), \tag{5.34}$$

$$u'_\mu \rightharpoonup u' \quad \text{in } L^2(0,T;H_0^1(\Omega)), \tag{5.35}$$

$$\frac{1}{\sqrt{\mu}}Mu'_\mu \rightharpoonup \chi_1 \quad \text{in } L^2(0,T;L^2(\Omega)). \tag{5.36}$$

From (5.35) we obtain

$$Mu'_\mu \rightharpoonup \chi_2 \quad \text{in } L^2(0,T;H_0^1(\Omega)). \tag{5.37}$$

We also have the convergence

$$\beta u'_\mu \overset{*}{\rightharpoonup} \chi_3 \quad \text{in } L^\infty(0,T;L^2(\Omega)). \tag{5.38}$$

Since $(Mu'_\mu, w) = (u'_\mu, Mw)$, it follows that $\chi_2 = Mu'$, thus

$$Mu'_\mu \rightharpoonup Mu' \quad \text{in} \quad L^2(0,T;H_0^1(\Omega)).$$

Since $\frac{1}{\mu}\int_0^T |M(t)u'_\mu(t)|^2dt \le C \ \ \forall\mu$, it follows that $Mu'_\mu \rightharpoonup 0$ in $L^2(Q_0)$. Hence,

$$Mu' = 0 \text{ a.e. in } Q_0.$$

From the definition of M we obtain

$$u' = 0 \quad \text{a.e. in } Q_0 \setminus Q \text{ or}$$
$$u' = 0 \quad \text{a.e. in } \Omega \setminus \Omega_t$$

in $[0,T]$, which combined with (5.35) yields $u' \in L^2(0,T;H_0^1(\Omega_t))$. Since $u' = 0$ in $Q_0 \setminus Q$ and $u(x,0) = \tilde{u}_0 = 0$ in $\Omega \setminus \Omega_0$, it follows that $u = 0$ in $Q_0 \setminus Q$, which jointly with (5.33),

$$u \in L^\infty(0,T;W_0^{1,p}(\Omega_t)). \tag{5.39}$$

Again, from de monotonicity and hemicontinuity of β, and owing to the fact that $\beta(u_\mu) = 0$ in $L^2(0,T;H^{-1}(\Omega))$, we conclude that

$$\beta(u') = 0 \quad \text{a.e. or } u'(t) \in K \text{ a.e.} \tag{5.40}$$

We have $v \in L^2(0,T;W_0^{1,p}(\Omega_t)) \hookrightarrow L^2(0,T;H_0^1(\Omega_t))$. Let $v' \in L^2(0,T;W^{-1,p'}(\Omega_t))$. Hence: for almost every $t \in (0,T)$, $v = 0$ in $\Omega \setminus \Omega_t$. Thus

$$\begin{aligned}\int_0^s (M(t)u'_\mu(t), v)dt &= \int_0^s \int_\Omega M(t)u'_\mu(t)v(t)dxdt\\ &= \int_0^s \int_{\Omega_t} M(t)u'_\mu(t)v(t)dxdt = 0, \quad \forall\mu\end{aligned}$$

because $M = 0$ in Ω_t.

Taking the limit inferior in (5.32) in first member of the equation and the limit in the second member when $\mu \to 0$ and using the convergence obtained up to here, it follows that

$$\begin{aligned}&\int_0^T \varphi(s)\Big[\frac{1}{2}|v(s) - u'(s)|^2 + \frac{1}{p}\|u(s)\|^p_{W_0^{1,p}(\Omega)} + \int_0^s \|u'(t)\|^2dt\Big]\\ &\le \int_0^T \varphi(s)\int_0^s \langle v' - f + \Delta_p u - \Delta u', v\rangle\,dt\,ds + \int_0^T \varphi(s)\int_0^s \langle v' - f, -u'\rangle\,dt\,ds\\ &\quad + \int_0^T \varphi(s)\frac{1}{p}\|u(0)\|^2\,ds + \int_0^T \varphi(s)\frac{1}{2}|v(0) - u_1|^2ds.\end{aligned}$$

Thus, for almost s we have

$$\begin{aligned}&\frac{1}{2}|v(s) - u'(s)|^2_{L^2(\Omega_s)} - \frac{1}{2}|v(0) - u'(0)|^2_{L^2(\Omega_0)}\\ &\le \int_0^s \langle v' - f + \Delta_p - \Delta u', v\rangle dt + \int_0^s \langle v' - f, -u'\rangle\,ds\\ &\quad + \frac{1}{p}\|u(0)\|^p_{W_0^{1,p}(\Omega_0)} - \frac{1}{p}\|u(s)\|^p_{W_0^{1,p}(\Omega_s)}ds - \int_0^s \|u'(t)\|^2dt\\ &= \int_0^s \langle v' - f + \Delta_p u - \Delta u', v\rangle dt + \int_0^s \langle v' - f, -u'\rangle\,ds\\ &\quad - \int_0^s \langle \Delta_p u, u'\rangle dt - \int_0^s \langle -\Delta u', u'\rangle dt\end{aligned}$$

$$
\begin{aligned}
&= \int_0^s \langle v' - f + \Delta_p u - \Delta u', v\rangle dt + \int_0^s \langle v' - f + \Delta_p u - \Delta u', -u'\rangle dt \\
&= \int_0^s \langle v' - f + \Delta_p - \Delta u', v - u'\rangle dt
\end{aligned}
$$

for all $s \in (0,T)$, and all $v \in V$.

To show the continuity of u' we can use the same arguments as in [11].

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