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GLOBAL WELL-POSEDNESS FOR THE RADIAL DEFOCUSING CUBIC WAVE EQUATION ON $\mathbb{R}^3$ AND FOR ROUGH DATA

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ABSTRACT. We prove global well-posedness for the radial defocusing cubic wave equation

$$\begin{gathered}\partial_{tt}u - \Delta u = -u^3 \\ u(0,x) = u_0(x) \\ \partial_t u(0,x) = u_1(x)\end{gathered}$$

with data $(u_0, u_1) \in H^s \times H^{s-1}$, $1 > s > 7/10$. The proof relies upon a Morawetz-Strauss-type inequality that allows us to control the growth of an almost conserved quantity.

1. INTRODUCTION

We shall study the defocusing cubic wave equation on $\mathbb{R}^3$

$$\begin{gathered}\partial_{tt}u - \Delta u = -u^3 \\ u(0,x) = u_0(x) \\ \partial_t u(0,x) = u_1(x)\end{gathered} \tag{1.1}$$

We shall focus on the strong solutions of the defocusing cubic wave equation on some interval $[0,T]$ i.e real-valued maps $(u, \partial_t u) \in C([0,T], H^s(\mathbb{R}^3)) \times C([0,T], H^{s-1}(\mathbb{R}^3))$ that satisfy for $t \in [0,T]$ the following integral equation

$$u(t) = \cos(tD)u_0 + D^{-1}\sin(tD)u_1 - \int_0^t D^{-1}\sin((t-t')D)u^3(t')\,dt' \tag{1.2}$$

with (u_0, u_1) lying in $H^s \times H^{s-1}$. Here H^s is the usual inhomogeneous Sobolev space; i.e., H^s is the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^3)$ with respect to the norm

$$\|f\|_{H^s} := \|(1+D^s)f\|_{L^2(\mathbb{R}^3)} \tag{1.3}$$

where D is the operator defined by

$$\widehat{Df}(\xi) := |\xi|\hat{f}(\xi) \tag{1.4}$$

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and $\hat{f}$ denotes the Fourier transform

$$\hat{f}(\xi) := \int_{\mathbb{R}^3} f(x)e^{-ix\cdot\xi}\,dx \tag{1.5}$$

Here $H^s \times H^{s-1}$ is the product space of H^s and H^{s-1} endowed with the standard norm $\|(f,g)\|_{H^s\times H^{s-1}} := \|f\|_{H^s} + \|g\|_{H^{s-1}}$.

It is known [11] that (1.1) is locally well-posed in $H^s(\mathbb{R}^3)\times H^{s-1}(\mathbb{R}^3)$ for $s \geq \frac{1}{2}$. Moreover if $s > \frac{1}{2}$ the time of local existence only depends on the norm of the initial data $\|(u_0,u_1)\|_{H^s\times H^{s-1}}$.

Now we turn our attention to the global well-posedness theory of (1.1). In view of the above local well-posedness theory and standard limiting arguments it suffices to establish an a priori bound of the form

$$\|u(T)\|_{H^s} + \|\partial_t u(T)\|_{H^{s-1}} \leq C(s, (\|u_0\|, \|u_1\|)_{H^s\times H^{s-1}}, T) \tag{1.6}$$

for all times $0 < T < \infty$ and all smooth-in-time Schwartz-in-space solutions $(u, \partial_t u) : [0,T]\times\mathbb{R}^3 \to \mathbb{R}$, where the right-hand side is a finite quantity depending only on s, $\|u_0\|_{H^s}$, $\|u_1\|_{H^{s-1}}$ and T. Therefore in the sequel we shall restrict attention to such smooth solutions.

The defocusing cubic wave equation (1.1) enjoys the following energy conservation law

$$E(u(t)) := \frac{1}{2}\int_{\mathbb{R}^3} (\partial_t u)^2(x,t)\,dx + \frac{1}{2}\int_{\mathbb{R}^3} |Du(x,t)|^2\,dx + \frac{1}{4}\int_{\mathbb{R}^3} u^4(x,t)\,dx \tag{1.7}$$

Combining this conservation law to the local well-posedness theory we immediately have global well-posedness for (1.1) and for $s = 1$.

In this paper we are interested in studying global well-posedness for (1.1) and for data below the energy norm, i.e $s < 1$. It is conjectured that (1.1) is globally well-posed in $H^s(\mathbb{R}^3)\times H^{s-1}(\mathbb{R}^3)$ for all $s > \frac{1}{2}$. The global existence for the defocusing cubic wave equation has been the subject of several papers. Let us some mention some results for data lying in a slightly different space than $H^s\times H^{s-1}$ i.e $\dot{H}^s\times\dot{H}^{s-1}$. Here $\dot{H}^s$ is the usual homogeneous Sobolev space i.e the completion of Schwartz functions $\mathcal{S}(\mathbb{R}^3)$ with respect to the norm

$$\|f\|_{\dot{H}^s} = \|D^s f\|_{L^2(\mathbb{R}^3)} \tag{1.8}$$

Kenig, Ponce and Vega [9] were the first to prove that (1.1) is globally well-posed for $1 > s > \frac{3}{4}$. They used the *Fourier truncation method* discovered by Bourgain [2]. Gallagher and Planchon [7] proposed a different method to prove global well-posedness for $1 > s > \frac{3}{4}$. Bahouri and Jean-Yves Chemin [1] proved global-wellposedness for (1.1) and for $s = \frac{3}{4}$ by using a non linear interpolation method and logarithmic estimates from Klainermann and Tataru [10]. We shall consider global well-posedness for the radial defocusing cubic wave equation i.e global existence for the initial value problem (1.1) with radial data. The main result of this paper is the following one

Theorem 1.1. *The radial defocusing cubic wave equation is globally well-posed in $H^s\times H^{s-1}$ for $1 > s > \frac{7}{10}$. Moreover if T large then*

$$\|u(T)\|^2_{H^s} + \|\partial_t u(T)\|^2_{H^{s-1}} \leq C(\|u_0\|_{H^s}, \|u_1\|_{H^{s-1}})T^{\frac{16s-10}{10s-7}+} \tag{1.9}$$

for $\frac{5}{6} \geq s > \frac{7}{10}$ and

$$\|u(T)\|^2_{H^s} + \|\partial_t u(T)\|^2_{H^{s-1}} \leq C(\|u_0\|_{H^s}, \|u_1\|_{H^{s-1}})T^{\frac{2s}{2s-1}+} \tag{1.10}$$

for $1 > s > \frac{5}{6}$. *Here* $C(\|u_0\|_{H^s}, \|u_1\|_{H^{s-1}})$ *is a constant only depending on* $\|u_0\|_{H^s}$ *and* $\|u_1\|_{H^{s-1}}$.

We set some notation that appear throughout the paper. Given A, B positive number $A \lesssim B$ means that there exists a universal constant K such that $A \leq KB$. We say that K_0 is the constant determined by the relation $A \lesssim B$ if K_0 is the smallest K such that $A \leq KB$ is true. We write $A \sim B$ when $A \lesssim B$ and $B \lesssim A$. $A \ll B$ denotes $A \leq KB$ for some universal constant $K < \frac{1}{100}$. We also use the notations $A+ = A + \epsilon$, $A- = A - \epsilon$ for some universal constant $0 < \epsilon \ll 1$. Let ∇ denote the gradient operator. If J is an interval then $|J|$ is its size. If E is a set then $\operatorname{card}(E)$ is its cardinal. Let I be the following multiplier

$$\widehat{If}(\xi) := m(\xi)\hat{f}(\xi) \tag{1.11}$$

where $m(\xi) := \eta(\frac{\xi}{N})$, η is a smooth, radial, nonincreasing in $|\xi|$ such that

$$\eta(\xi) := \begin{cases} 1, & |\xi| \leq 1 \\ (\frac{1}{|\xi|})^{1-s}, & |\xi| \geq 2 \end{cases} \tag{1.12}$$

and $N \gg 1$ is a dyadic number playing the role of a parameter to be chosen. We shall abuse the notation and write $m(|\xi|)$ for $m(\xi)$, thus for instance $m(N) = 1$.

We recall some basic results regarding the defocusing cubic wave equation. Let $\lambda \in \mathbb{R}$ and u_λ denote the following function

$$u_\lambda(t, x) := \frac{1}{\lambda} u(\frac{t}{\lambda}, \frac{x}{\lambda}) \tag{1.13}$$

If u satisfies (1.1) with data (u_0, u_1) then u_λ also satisfies (1.1) but with data $(\frac{1}{\lambda} u_0(\frac{x}{\lambda}), \frac{1}{\lambda^2} u_1(\frac{x}{\lambda}))$. If u satisfies the radial defocusing cubic wave equation then u is radial.

Now we recall some standard estimates that we use later in this paper.

Proposition 1.2 (Strichartz estimates in 3 dimensions [8, 11])**.** *Let* $m \in [0, 1]$. *If* u *is a strong solution to the IVP problem*

$$\begin{gathered} \partial_{tt} u - \Delta u = F \\ u(0, x) = f(x) \in \dot{H}^m \\ \partial_t u(0, x) = g(x) \in \dot{H}^{m-1} \end{gathered} \tag{1.14}$$

then for $0 \leq \tau < \infty$ *we have*

$$\begin{aligned} &\|u\|_{L_t^q([0,\tau]) L_x^r} + \|u\|_{C([0,\tau]; \dot{H}^m)} + \|\partial_t u\|_{C([0,\tau]; \dot{H}^{m-1})} \\ &\lesssim \|f\|_{\dot{H}^m} + \|g\|_{\dot{H}^{m-1}} + \|F\|_{L_t^{\tilde{q}}([0,\tau]) L_x^{\tilde{r}}} \end{aligned}$$

under two assumptions

- (q, r) *lie in the set* $\mathcal{W}$ *of wave-admissible points; i.e.,*

$$\mathcal{W} := \{(q, r) : (q, r) \in (2, \infty] \times [2, \infty), \frac{1}{q} + \frac{1}{r} \leq \frac{1}{2}\} \tag{1.15}$$

- $(\tilde{q}, \tilde{r})$ *lie in the dual set* $\mathcal{W}'$ *of* $\mathcal{W}$*; i.e.,*

$$\mathcal{W}' := \{(\tilde{q}, \tilde{r}) : \frac{1}{\tilde{q}} + \frac{1}{q} = 1, \frac{1}{r} + \frac{1}{\tilde{r}} = 1, (q, r) \in \mathcal{W}\} \tag{1.16}$$

- $(q, r, \tilde{q}, \tilde{r})$ *satisfy the dimensional analysis conditions*

$$\frac{1}{q} + \frac{3}{r} = \frac{3}{2} - m, \tag{1.17}$$

$$\frac{1}{\tilde{q}} + \frac{3}{\tilde{r}} - 2 = \frac{3}{2} - s\,. \tag{1.18}$$

We also have the well-known estimate

Proposition 1.3 (Radial Sobolev inequality)**.** *If* $u : \mathbb{R}^3 \to \mathbb{C}$ *is radial and smooth, then*

$$|u(x)| \lesssim \frac{\|u\|_{\dot{H}^1}}{|x|^{\frac{1}{2}}} \tag{1.19}$$

The Hardy-type inequality is proved in [3].

Proposition 1.4 (Hardy-type inequality)**.** *If* $1 < p < 3$ *and* $u : \mathbb{R}^3 \to \mathbb{C}$ *is smooth, then*

$$\|\frac{u}{|x|}\|_{L^p} \leq \frac{3}{3-p}\|Df\|_{L^p} \tag{1.20}$$

Some variables appear frequently in this paper. We define them now.

We say that (q, r) is a m-wave admissible pair if $0 \leq m \leq 1$ and (q, r) satisfy the two following conditions

- $(q, r) \in \mathcal{W}$
- $\frac{1}{q} + \frac{3}{r} = \frac{3}{2} - m$

Let $J = [a, b]$ be an interval included in $[0, \infty)$. Let

$$Z_{m,s}(J) := \sup_{q,r}(\|D^{1-m}Iu\|_{L_t^q(J)L_x^r} + \|D^{-m}I\partial_t u\|_{L_t^q(J)L_x^r}) \tag{1.21}$$

where the sup is taken over m-wave admissible (q, r), and let

$$Z(J) := \sup_{m\in[0,1)} Z_{m,s}(J) \tag{1.22}$$

Let

$$R_1(J) := \int_J\int_{\mathbb{R}^3} \frac{\nabla Iu(t,x).x}{|x|}((Iu)^3(t,x) - Iu^3(t,x))\,dxdt \tag{1.23}$$

and

$$R_2(J) := \int_J\int_{\mathbb{R}^3} \tfrac{Iu(t,x)}{|x|}((Iu)^3(t,x) - Iu^3(t,x))\,dxdt \tag{1.24}$$

If $J = [0, \tau]$ we shall abuse the notation and write

$$Z(\tau) := Z(J)$$
$$R(\tau) := R(J)$$

Some estimates that we establish throughout the paper require a Paley-Littlewood decomposition. We set it up now. Let $\phi(\xi)$ be a real, radial, nonincreasing function that is equal to 1 on the unit ball $\{\xi \in \mathbb{R}^3 : |\xi| \leq 1\}$ and that that is supported on $\{\xi \in \mathbb{R}^3 : |\xi| \leq 2\}$. Let ψ denote the function

$$\psi(\xi) := \phi(\xi) - \phi(2\xi) \tag{1.25}$$

If $M \in 2^{\mathbb{Z}}$ is a dyadic number we define the Paley-Littlewood operators in the Fourier domain by

$$\widehat{P_{\leq M}f}(\xi) := \phi(\frac{\xi}{M})\hat{f}(\xi)$$

$$\widehat{P_M f}(\xi) := \psi(\frac{\xi}{M})\hat{f}(\xi)$$
$$\widehat{P_{>M} f}(\xi) := \hat{f}(\xi) - \widehat{P_{\leq M} f}(\xi)$$

Since $\sum_{M\in 2^{\mathbb{Z}}} \psi(\frac{\xi}{M}) = 1$ we have

$$f = \sum_{M\in 2^{\mathbb{Z}}} P_M f \tag{1.26}$$

We conclude this introduction by giving the main ideas of the proof of theorem 1.1 and explaining how the paper is organized. Following the proof of the global well-posedness for $s = 1$ we try to compare for every $T > 0$ the relevant quantity $\|(u(T), \partial_t u(T))\|_{H^s\times H^{s-1}}$ to the supremum of the energy conservation law $\sup_{t\in[0,T]} E(u(t))$. Unfortunately this strategy does not work if $s < 1$ since the energy can be infinite. We get around this difficulty by using the I-method designed by Colliander, Keel, Staffilani, H.Takaoka and Tao [5] and successfully applied to prove global well-posedness for semilinear Schrödinger equations and for rough data. The idea consists of introducing the following smoothed energy

$$E(Iu(t)) := \frac{1}{2}\int_{\mathbb{R}^3} \left|\partial_t Iu(x,t)\right|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} |DIu(x,t)|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} |Iu(x,t)|^4 dx$$

We prove in section 3 that $\|(u(T), \partial_t u(T))\|^2_{H^s\times H^{s-1}}$ and the supremum of the smoothed energy on $[0, T]$ are comparable. Therefore we try to estimate the quntity $\sup_{t\in[0,T]} E(Iu(t))$ in order to give an upper bound of $\|(u(T), \partial_t u(T))\|_{H^s\times H^{s-1}}$. For convenience we place the mollified energy at time zero into $[0, \frac{1}{2}]$ by choosing the right scaling factor λ. This operation shows that we are reduced to estimate $\sup_{t\in[0,\lambda T]} E(Iu_\lambda(t))$. In section 4 we prove that we can locally control a variable namely $Z(J)$ provided that the interval J satisfies some constraints that give some information about its size. $\sup_{t\in J} E(Iu_\lambda(t))$ is estimated by the fundamental theorem of calculus. The upper bound depends on the parameter N and the controlled quantity $Z(J)$. This estimate is established in section 5. Now we can iterate: the process generates a sequence of intervals (J_i) that cover the whole interval $[0, \lambda T]$ and satisfy the same constraints as J. We should be able to estimate $\sup_{t\in[0,\lambda T]} E(Iu_\lambda(t))$ provided that we can control the number of intervals J_i. This requires the establishment of a long time estimate, the so-called almost Morawetz-Strauss inequality. This estimate is proved in section 6. It depends on some remainder integrals that are estimated in section 7. Combining this inequality to the radial Sobolev inequality (1.19) we can give an upper bound of the cardinal of (J_i). The proof of theorem 1.1 is given in section 2.

2. Proof of global well-posedness for $1 > s > 7/10$

In this section we prove the global existence of (1.1) for $1 > s > 7/10$. Our proof relies on some intermediate results that we prove in later sections. More precisely we shall show the following results.

Proposition 2.1 (H^s norms and mollified energy estimates)**.** *Let* $T > 0$*. Then*

$$\|u(T)\|^2_{H^s} + \|\partial_t u(T)\|^2_{H^{s-1}} \lesssim \|u_0\|^2_{H^s} + (T^2 + 1) \sup_{t\in[0,T]} E(Iu(t)) \tag{2.1}$$

for every u.

Proposition 2.2 (Local boundedness). *Let $J = [a, b]$ be an interval included in $[0, \infty)$. Assume that $E(Iu(a)) \leq 2$ and that u satisfies* (1.1). *There exist C_1, C_2 small and positive constants such that if J satisfies*

$$\|Iu\|_{L_t^6(J)L_x^6} \leq \frac{C_1}{|J|^{\frac{1}{3}}}, \tag{2.2}$$

$$|J| \leq C_2 N^{\frac{1-s}{s-\frac{1}{2}}} \tag{2.3}$$

then $Z(J) \lesssim 1$.

Proposition 2.3 (Almost conservation law). *Let $J = [a, b]$ be an interval included in $[0, \infty)$. Assume that u satisfies* (1.1). *Then*

$$\big| \sup_{t \in J} E(Iu(t)) - E(Iu(a)) \big| \lesssim \frac{Z^4(J)}{N^{1-}} \tag{2.4}$$

Proposition 2.4 (Almost Morawetz-Strauss inequality). *Let $T \geq 0$. Assume that u satisfies* (1.1). *Then*

$$\int_0^T \int_{\mathbb{R}^3} \frac{|Iu|^4(t,x)}{|x|} dx\, dt - 2(E(Iu(0)) + E(Iu(T))) \lesssim |R_1(T)| + |R_2(T)|\,. \tag{2.5}$$

Proposition 2.5 (Estimate of integrals). *Let J be an interval included in $[0, \infty)$. Then if $i = 1, 2$ we have*

$$R_i(J) \lesssim \frac{Z^4(J)}{N^{1-}} \tag{2.6}$$

For the remainder of this section we show how propositions 2.2, 2.3, 2.4 and 2.5 imply Theorem 1.1.

Let $T > 0$ and $N = N(T) \gg 1$ be a parameter to be chosen later. There are three steps to prove Theorem 1.1.

(1) Scaling. Let $\lambda \gg 1$ to be chosen later. Then by Plancherel theorem

$$\begin{aligned} \|DIu_\lambda(0)\|_{L^2}^2 &\lesssim \int_{|\xi| \leq 2N} |\xi|^2 |\widehat{u_\lambda}(0,\xi)|^2\, d\xi + \int_{|\xi| \geq 2N} |\xi|^2 \frac{N^{2(1-s)}}{|\xi|^{2(1-s)}} |\widehat{u_\lambda}(0,\xi)|^2\, d\xi \\ &\lesssim N^{2(1-s)} \|u_\lambda(0)\|_{\dot{H}^s}^2 \\ &\lesssim N^{2(1-s)} \lambda^{1-2s} \|u_0\|_{\dot{H}^s}^2 \\ &\lesssim N^{2(1-s)} \lambda^{1-2s} \|u_0\|_{H^s}^2, \end{aligned} \tag{2.7}$$

$$
\begin{aligned}
\|\partial_t I u_\lambda(0)\|_{L^2}^2 &\lesssim \int_{|\xi|\le 2N} |\widehat{\partial_t u_\lambda}(0,\xi)|^2\,d\xi + \int_{|\xi|\ge 2N} \frac{N^{2(1-s)}}{|\xi|^{2(1-s)}} |\widehat{\partial_t u_\lambda}(0,\xi)|^2\,d\xi \\
&\lesssim N^{2(1-s)} \|\partial_t u_\lambda(0)\|_{H^{s-1}}^2 \\
&\lesssim N^{2(1-s)} \Big(\int_{|\xi|\le 1} |\widehat{\partial_t u_\lambda}(0,\xi)|^2\,d\xi + \int_{|\xi|\ge 1} |\xi|^{2(s-1)} |\widehat{\partial_t u_\lambda}(0,\xi)|^2\,d\xi \Big) \\
&\lesssim N^{2(1-s)} \Big(\frac{1}{\lambda} \int_{|\xi|\le \lambda} |\widehat{u_1}(\xi)|^2\,d\xi + \lambda^{1-2s} \int_{|\xi|\ge\lambda} |\xi|^{2(s-1)} |\widehat{u_1}(\xi)|^2\,d\xi \Big) \\
&\lesssim N^{2(1-s)} \lambda^{1-2s} \|u_1\|_{H^{s-1}}^2 .
\end{aligned} \tag{2.8}
$$

By homogeneous Sobolev embedding,

$$
\begin{aligned}
&\|I u_\lambda(0)\|_{L^4}^2 \\
&\lesssim \int_{\mathbb{R}^3} |\xi|^{\frac{3}{2}} |\widehat{I u_\lambda}(0,\xi)|^2\,d\xi \\
&\lesssim \int_{|\xi|\le 2N} |\xi|^{\frac{3}{2}} |\widehat{u_\lambda}(0,\xi)|^2\,d\xi + \int_{|\xi|\ge 2N} |\xi|^{\frac{3}{2}} \frac{N^{2(1-s)}}{|\xi|^{2(1-s)}} |\widehat{u_\lambda}(0,\xi)|^2\,d\xi \\
&\lesssim \frac{1}{\lambda^{\frac{1}{2}}} \int_{|\xi|\le 2N\lambda} |\xi|^{\frac{3}{2}} |\widehat{u_0}(\xi)|^2\,d\xi + N^{2(1-s)} \lambda^{\frac{3}{2}-2s} \int_{|\xi|\ge 2N\lambda} |\xi|^{2s-\frac{1}{2}} |\widehat{u_0}(\xi)|^2\,d\xi \\
&\lesssim \frac{\max(N^{\frac{3}{2}-2s}\lambda^{\frac{3}{2}-2s}, 1)}{\lambda^{\frac{1}{2}}} \|u_0\|_{H^s}^2 + N^{\frac{3}{2}-2s} \lambda^{1-2s} \|u_0\|_{H^s}^2 .
\end{aligned} \tag{2.9}
$$

Hence

$$
\|I u_\lambda(0)\|_{L^4}^4 \lesssim N^{2(1-s)} \lambda^{1-2s} \|u_0\|_{H^s}^4 \tag{2.10}
$$

By (2.7), (2.8) and (2.10) we see that there exists $C_0 = C_0(\|u_0\|_{H^s}, \|u_1\|_{H^{s-1}})$ such that if λ satisfies

$$
\lambda = C_0 N^{\frac{2(1-s)}{2s-1}} \tag{2.11}
$$

then

$$
E(I u_\lambda(0)) \le \frac{1}{2} \tag{2.12}
$$

(2) Boundedness of the mollified energy. Let F_T denote the set

$$
\begin{gathered}
F_T = \big\{ T' \in [0,T] : \sup_{t\in[0,\lambda T']} E(I u_\lambda(t)) \le 1 \text{ and} \\
\|I u_\lambda\|_{L_t^6([0,\lambda T'])L_x^6} \le (16 C_s^2)^{\frac{1}{6}} + 1 \big\}
\end{gathered}
$$

with C_s being the constant determined by $\lesssim$ in (1.19) and λ satisfying (2.11). We claim that F_T is the whole set $[0,T]$ for $N = N(T) \gg 1$ to be chosen later. Indeed

- $F_T \neq \emptyset$ since $0 \in F_T$ by (2.12).
- F_T is closed by continuity and by the dominated convergence theorem
- F_T is open. Let $\widetilde{T'} \in F_T$. By continuity there exists $\delta > 0$ such that for every $T' \in (\widetilde{T'} - \delta, \widetilde{T'} + \delta) \cap [0,T]$ we have

$$
\sup_{t\in[0,\lambda T']} E(I u_\lambda(t)) \le 2, \tag{2.13}
$$

$$
\|I u_\lambda\|_{L_t^6([0,\lambda T'])L_x^6} \le (16 C_s^2)^{\frac{1}{6}} + 2 . \tag{2.14}
$$

We are interested in generating a partition $\{J_j\}$ of $[0,\lambda T']$ such that (2.2) and (2.3) are satisfied for all J_j. We describe now the algorithm.

Description of the algorithm. Let $\mathcal{L}$ be the present list of intervals. Let L be the sum of the lengths of the intervals making up $\mathcal{L}$. Let n be the number of the last interval of $\mathcal{L}$. Initially there is no interval and we start from the time $t=0$. Therefore $\mathcal{L}$ is empty and we assign the value 0 to L and n. Then as long as $L<\lambda T'$ do the following

(1) consider $f_L(\tau)=\|Iu_\lambda\|_{L_t^6([L,L+\tau])L_x^6}-\frac{C_1}{\tau^{\frac{1}{3}}}$, $\tau\geq 0$ with C_1 defined in (2.2).
(2) since f_L is continuous, does not decrease and $f_L(\tau)\to-\infty$ as $\tau\to 0$, $\tau\geq 0$ there are two options
   - f_L is always negative on $[0,\lambda T'-L]$: in this case if (2.3) is satisfied by $[L,\lambda T']$ then let $J_n:=[L,\lambda T']$. If not let $J_n:=[L,L+C_2N^{\frac{1-s}{s-\frac{1}{2}}}]$.
   - f_L has one and only one root on $[0,\lambda T'-L]$: in this case let τ_0 be this root. If (2.3) is satisfied by $[L,L+\tau_0]$ then let $J_n:=[L,L+\tau_0]$. If not let $J_n:=[L,L+C_2N^{\frac{1-s}{s-\frac{1}{2}}}]$.
(3) assign the value $L+|J_n|$ to L.
(4) assign the value $n+1$ to the variable n
(5) insert J_n into $\mathcal{L}$ so that $\mathcal{L}=(J_j)_{j\in\{1,\dots,n\}}$

When we apply this algorithm it is not difficult to see that

- $\|Iu_\lambda\|_{L_t^6(J_j)L_x^6}=\frac{C_1}{|J_j|^{\frac{1}{3}}}$ or $|J_j|=C_2N^{\frac{1-s}{s-\frac{1}{2}}}$ for every $j\in\{1,\dots,\mathrm{card}(\mathcal{L})-1\}$
- $J_j\cap J_k=\emptyset$ for every $(j,k)\in\{1,\dots,\mathrm{card}(\mathcal{L})\}^2$ such that $j\neq k$
- $\bigcup_{j=1}^{\mathrm{card}(\mathcal{L})}J_j$ is a left-closed interval with left endpoint 0 and included in $[0,\lambda T']$. Moreover $\bigcup_{j=1}^{\mathrm{card}(\mathcal{L})}J_j=[0,\lambda T']$ if the process is finite.

Let

$$\mathcal{L}_1=\{J_j,J_j\in\mathcal{L},\|Iu\|_{L_t^6(J_j)L_x^6}=\frac{C_1}{|J_j|^{\frac{1}{3}}}\}, \tag{2.15}$$

$$\mathcal{L}_2=\{J_j,J_j\in\mathcal{L},|J_j|=C_2N^{\frac{1-s}{s-\frac{1}{2}}}\} \tag{2.16}$$

We have $(J_j)_{j\in\{1,\dots,\mathrm{card}(\mathcal{L})-1\}}\subset\mathcal{L}_1\cup\mathcal{L}_2$. We claim that $\mathrm{card}(\mathcal{L}_i)<\infty$, $i=1,2$. If not let us consider the m_1, m_2 first elements of $\mathcal{L}_1$, $\mathcal{L}_2$ respectively. Then

$$m_1C_2N^{\frac{1-s}{s-\frac{1}{2}}}\leq\lambda T' \tag{2.17}$$

By Hölder inequality and by (2.14) we have

$$\begin{aligned} m_2&=\sum_{j=1}^{m_2}|J_j|^{-2/3}|J_j|^{2/3}\\ &\leq(\sum_{j=1}^{m_2}\frac{1}{|J_j|^2})^{\frac{1}{3}}(\sum_{j=1}^{m_2}|J_j|)^{2/3}\\ &\leq\|Iu\|^2_{L_t^6([0,\lambda T'])L_x^6}(\lambda T)^{2/3}\\ &\lesssim(\lambda T)^{2/3} \end{aligned} \tag{2.18}$$

Letting m_1 and m_2 go to infinity in (2.17) and (2.18) we have a contradiction. Therefore $\mathrm{card}(\mathcal{L})<\infty$ and $\bigcup_{j=1}^{\mathrm{card}(\mathcal{L})}J_j=[0,\lambda T']$. Moreover by (2.11), (2.17), (2.18), we have

$$\operatorname{card}(\mathcal{L}) \lesssim (\lambda T)^{2/3} + \frac{\lambda T}{N^{\frac{1-s}{s-\frac{1}{2}}}} + 1 \lesssim N^{\frac{4(1-s)}{6s-3}} T^{2/3} + T + 1 \tag{2.19}$$

Now by (2.12), (2.13), (2.19), proposition 2.2, 2.3, 2.4 and 2.5 we get, after iterating,

$$\sup_{t \in [0,\lambda T']} E(Iu_\lambda(t)) - \frac{1}{2} \lesssim \frac{N^{\frac{4(1-s)}{6s-3}} T^{2/3} + T + 1}{N^{1-}} \tag{2.20}$$

and

$$\begin{aligned} &\int_0^{\lambda T'} \int_{\mathbb{R}^3} \frac{|Iu_\lambda(t,x)|^4}{|x|} \, dxdt - 2(E(Iu_\lambda(\lambda T')) + E(Iu_\lambda(0))) \\ &\lesssim \sum_{i=1}^{2} \sum_{j=1}^{\operatorname{card}(\mathcal{L}_i)} R_i(J_j) \\ &\lesssim \frac{N^{\frac{4(1-s)}{6s-3}} T^{2/3} + T + 1}{N^{1-}} \end{aligned} \tag{2.21}$$

By (1.19), (2.13), (2.21) and the inequality $(1+x)^{\frac{1}{6}} \leq 1 + x$, $x \geq 0$

$$\|Iu_\lambda\|_{L_t^6([0,\lambda T'])L_x^6} - (16C_s^2)^{\frac{1}{6}} \lesssim \frac{N^{\frac{4(1-s)}{6s-3}} T^{2/3} + T + 1}{N^{1-}} \tag{2.22}$$

Let C', C'' be the constant determined by $\lesssim$ in (2.20), (2.22) respectively. Since $s > \frac{7}{10}$ we can always choose for every $T > 0$ a $N = N(T) \gg 1$ such that

$$\frac{\max(C', C'') N^{\frac{4(1-s)}{6s-3}} T^{2/3}}{N^{1-}} \leq \frac{1}{6}, \tag{2.23}$$

$$\frac{\max(C', C'') T}{N^{1-}} \leq \frac{1}{6}, \tag{2.24}$$

$$\frac{\max(C', C'')}{N^{1-}} \leq \frac{1}{6}\,. \tag{2.25}$$

By (2.20), (2.22), (2.23), (2.24) and (2.25) we have $\sup_{t \in [0,\lambda T']} E(Iu_\lambda(t)) \leq 1$ and $\|Iu_\lambda\|_{L_t^6([0,\lambda T'])L_x^6} \leq (16C_s^2)^{\frac{1}{6}} + 1$. Hence $F_T = [0,T]$ with $N = N(T)$ satisfying (2.23), (2.24) and (2.25).

(3) Conclusion. Following the I-method described in [5]

$$\sup_{t \in [0,T]} E(Iu(t)) = \lambda \sup_{t \in [0,\lambda T]} E((Iu)_\lambda(t)) \lesssim \lambda \sup_{t \in [0,\lambda T]} E(Iu_\lambda(t)) \lesssim \lambda \tag{2.26}$$

Combining (2.26) and proposition 2.1 we have global well-posedness.

Now let T be large. If $\frac{5}{6} \geq s > \frac{7}{10}$ then let N such that

$$\frac{0.9}{6} \leq \frac{\max(C', C'') N^{\frac{4(1-s)}{6s-3}} T^{2/3}}{N^{1-}} \leq \frac{1}{6} \tag{2.27}$$

Notice that (2.24) and (2.25) are also satisfied. We plug (2.27) into (2.26) and we apply proposition 2.1 to get (1.9). If $1 > s > \frac{5}{6}$ then let N such that

$$\frac{0.9}{6} \leq \frac{\max(C', C'') T}{N^{1-}} \leq \frac{1}{6} \tag{2.28}$$

Notice that (2.23) and (2.25) are also satisfied. We plug (2.28) into (2.26) and we apply proposition 2.1 to get (1.10).

3. Proof of the H^s norms and mollified energy estimates

In this section we are interested in proving proposition 2.1. By Plancherel theorem

$$\|u(T)\|_{H^s}^2 \lesssim \|P_{\leq 1}u(T)\|_{H^s}^2 + \int_{1\leq|\xi|\leq 2N} |\xi|^{2s}|\hat{u}(T,\xi)|^2 d\xi + \int_{|\xi|\geq 2N} |\xi|^{2s} \left|\hat{u}(T,\xi)\right|^2 \, d\xi$$

But

$$\begin{aligned} \int_{1\leq|\xi|\leq 2N} |\xi|^{2s} \left|\hat{u}(T,\xi)\right|^2 \, d\xi &\leq \int_{|\xi|\leq 2N} |\xi|^2 \left|\hat{u}(T,\xi)\right|^2 \, d\xi \\ &\lesssim \int_{\mathbb{R}^3} \left|DIu(T,x)\right|^2 \, dx \\ &\lesssim E(Iu(T)) \end{aligned} \tag{3.1}$$

$$\begin{aligned} \int_{|\xi|\geq 2N} |\xi|^{2s}|\hat{u}(T,\xi)|^2 \, d\xi &\leq \int_{|\xi|\geq 2N} |\xi|^2 \frac{N^{2(1-s)}}{|\xi|^{2(1-s)}} \left|\hat{u}(T,\xi)\right|^2 \, d\xi \\ &\lesssim \int_{\mathbb{R}^3} \left|DIu(T,x)\right|^2 \, dx \\ &\lesssim E(Iu(T)) \end{aligned} \tag{3.2}$$

and by the fundamental theorem of calculus and Minkowski inequality

$$\begin{aligned} \|P_{\leq 1}u(T)\|_{H^s} &\lesssim \|P_{\leq 1}u_0\|_{H^s} + \int_0^T \|P_{\leq 1}\partial_t u(t)\|_{H^s} \, dt \\ &\lesssim \|u_0\|_{H^s} + T \sup_{t\in[0,T]} \|\partial_t Iu(t)\|_{L^2} \end{aligned} \tag{3.3}$$

which implies that

$$\|P_{\leq 1}u(T)\|_{H^s}^2 \lesssim \|u_0\|_{H^s}^2 + T^2 \sup_{t\in[0,T]} E(Iu(t)) \tag{3.4}$$

We also have

$$\|\partial_t u(T)\|_{H^{s-1}}^2 \lesssim E(Iu(T)) \tag{3.5}$$

Combining (3.1), (3.2), (3.4) and (3.5) we get (2.1).

4. Proof of the local boundedness estimate

We are interested in proving proposition 2.2 in this section. In what follows we also assume that $J = [0,\tau]$: the reader can check after reading the proof that the other cases can be reduced to that one.

Before starting the proof let us state the following lemma.

Lemma 4.1 (Strichartz estimates with derivative). *Let $m \in [0,1]$ and $0 \leq \tau < \infty$. If u satisfies the IVP problem*

$$\begin{aligned} \Box u &= F \\ u(t=0) &= f \\ \partial_t u(t=0) &= g \end{aligned} \tag{4.1}$$

then we have the m-Strichartz estimate with derivative

$$\|u\|_{L_t^q([0,\tau])L_x^r} + \|\partial_t D^{-1}u\|_{L_t^q([0,\tau])L_x^r} \lesssim \|f\|_{\dot{H}^m} + \|g\|_{\dot{H}^{m-1}} + \|F\|_{L_t^{\tilde{q}}([0,\tau])L_x^{\tilde{r}}} \tag{4.2}$$

for $(q,r)\in\mathcal{W}$, $(\tilde q,\tilde r)\in\widetilde{\mathcal{W}}$ and $(q,r,\tilde q,\tilde r)$ satisfying the gap condition

$$\frac{1}{q}+\frac{3}{r}=\frac{3}{2}-m=\frac{1}{\tilde q}+\frac{3}{\tilde r}-2 \tag{4.3}$$

We postpone the proof of lemma 4.1 to subsection 4.1. Assuming that is true we now show how lemma 4.1 implies proposition 2.2.

Multiplying the m-Strichartz estimate with derivative (4.2) by $D^{1-m}I$ we get

$$\begin{aligned} Z_{m,s}(\tau) &\lesssim \|DIu_0\|_{L^2}+\|Iu_1\|_{L^2}+\|D^{1-m}IF\|_{L_t^{\tilde q}([0,\tau])L_x^{\tilde r}} \\ &\lesssim 1+\|D^{1-m}IF\|_{L_t^{\tilde q}([0,\tau])L_x^{\tilde r}} \end{aligned} \tag{4.4}$$

The remainder of proof is divided into three steps.

First Step. First we assume that $m\le s$. Notice that the point $(\frac{1}{1-s},6)$ is s-wave admissible. In this case we get from the fractional Leibnitz rule the Hölder in time and the Hölder in space inequalities

$$\begin{aligned} Z_{m,s}(\tau) &\lesssim 1+\|D^{1-m}I(uuu)\|_{L_t^1([0,\tau])L_x^{\frac{6}{5-2m}}} \\ &\lesssim 1+\|D^{1-m}Iu\|_{L_t^\infty([0,\tau])L_x^{\frac{6}{3-2m}}}\|u\|^2_{L_t^2([0,\tau])L_x^6} \\ &\lesssim 1+Z_{m,s}(\tau)\Big(\tau^{\frac13}\|P_{\le N}u\|_{L_t^6([0,\tau])L_x^6}+\tau^{s-\frac12}\|P_{>N}u\|_{L_t^{\frac{1}{1-s}}([0,\tau])L_x^6}\Big)^2 \\ &\lesssim 1+Z_{m,s}(\tau)\Big(\tau^{\frac13}\|Iu\|_{L_t^6([0,\tau])L_x^6}+\tau^{s-\frac12}\frac{\|D^{1-s}Iu\|_{L_t^{\frac{1}{1-s}}([0,\tau])L_x^6}}{N^{1-s}}\Big)^2 \\ &\lesssim 1+Z_{m,s}(\tau)\Big(\tau^{\frac13}\|Iu\|_{L_t^6([0,\tau])L_x^6}+\tau^{s-\frac12}\frac{Z_{s,s}(\tau)}{N^{1-s}}\Big)^2 \end{aligned} \tag{4.5}$$

Assume $m=s$. Then if we apply a continuity argument to (4.5) we get, from the inequalities (2.2) and (2.3),

$$Z_{s,s}(\tau)\lesssim 1 \tag{4.6}$$

Now assume $m<s$. Then if we apply a continuity argument to (4.5) and the inequalities (2.2) and (4.6) we get

$$Z_{m,s}(\tau)\lesssim 1 \tag{4.7}$$

Second Step. We assume $m>s$. By (4.5), (4.6), (4.7), (2.2) and (2.3) we have

$$\|D^{1-r}I(uuu)\|_{L_t^1([0,\tau])L_x^{\frac{6}{5-2r}}}\lesssim Z_{r,s}(\tau)(\tau^{\frac13}\|Iu\|_{L_t^6([0,\tau])L_x^6}+\frac{\tau^{s-\frac12}Z_{s,s}(\tau)}{N^{1-s}})^2\lesssim 1 \tag{4.8}$$

for $r\le s$. The inequality

$$\|D^{1-m}I(uuu)\|_{L_t^1([0,\tau])L_x^{\frac{6}{5-2m}}}\lesssim\|D^{1-r}I(uuu)\|_{L_t^1([0,\tau])L_x^{\frac{6}{5-2r}}} \tag{4.9}$$

follows from the application of Sobolev homogeneous embedding. We get from (4.4), (4.8) and (4.9)

$$\begin{aligned} Z_{m,s}(\tau) &\lesssim 1+\|D^{1-m}I(uuu)\|_{L_t^1([0,\tau])L_x^{\frac{6}{5-2m}}} \\ &\lesssim 1+\|D^{1-r}I(uuu)\|_{L_t^1([0,\tau])L_x^{\frac{6}{5-2r}}}\lesssim 1 \end{aligned} \tag{4.10}$$

4.1. **Proof of lemma 4.1.** By decomposition it suffices to prove that $u_l^1(t) = e^{\pm itD}f$, $u_l^2(t) = \frac{e^{\pm itD}}{D}g$ and $u_n(t) = \int_0^t D^{-1}\sin((t-t')D)F\,dt'$ satisfy (4.2).

We have $\partial_t u_l^1(t) = \pm iDe^{\pm itD}f$ and $\partial_t u_l^2 = \pm e^{\pm itD}g$. We know from the Strichartz estimates that

$$\|D^{-1}\partial_t u_l^1\|_{L_t^q([0,\tau])L_x^r} \lesssim \|e^{\pm itD}f\|_{L_t^q([0,\tau])L_x^r} \lesssim \|f\|_{\dot{H}^m} \tag{4.11}$$

and

$$\|D^{-1}\partial_t u_l^2\|_{L_t^q([0,\tau])L_x^r} = \|e^{\pm itD}D^{-1}g\|_{L_t^q([0,\tau])L_x^r} \lesssim \|D^{-1}g\|_{\dot{H}^m} \lesssim \|g\|_{\dot{H}^{m-1}} \tag{4.12}$$

We also have

$$D^{-1}\partial_t u_n(t) = \int_0^t \cos((t-t')D)F(t')\,dt' \tag{4.13}$$

and by the Strichartz estimates

$$\begin{aligned}\|D^{-1}\partial_t u_n\|_{L_t^q([0,\tau])L_x^r} &\lesssim \|\int_0^t D^{-1}e^{i(t-t')D}F(t')dt'\|_{L_t^q([0,\tau])L_x^r}\\ &\quad + \|\int_0^t D^{-1}e^{-i(t-t')D}F(t')dt'\|_{L_t^q([0,\tau])L_x^r}\\ &\lesssim \|F\|_{L_t^{\tilde{q}}([0,\tau])L_x^{\tilde{r}}}\end{aligned} \tag{4.14}$$

Inequality (4.2) follows from (4.11), (4.12) and (4.14).

5. Proof of almost conservation law

Now we prove proposition 2.3. In what follows we also assume that $J = [0,\tau]$: the reader can check after reading the proof that the other cases can be reduced to that one.

Let $\tau_0 \in J$. It suffices to prove

$$|E(Iu(\tau_0)) - E(Iu(0))| \lesssim \frac{Z^4(\tau)}{N^{1-}} \tag{5.1}$$

In what follows we also assume that $\tau_0 = \tau$: the reader can check after reading the proof that the other cases can be reduced to this one.

The Plancherel formula and the fundamental theorem of calculus yield

$$\begin{aligned}&E(Iu(\tau)) - E(Iu(0))\\ &= \int_0^\tau \int_{\xi_1+\dots+\xi_4=0} \mu(\xi_2,\xi_3,\xi_4)\widehat{\partial_t Iu}(t,\xi_1)\widehat{Iu}(t,\xi_2)\widehat{Iu}(t,\xi_3)\widehat{Iu}(t,\xi_4)\,d\xi_2 d\xi_3 d\xi_4 dt\end{aligned}$$

with

$$\mu(\xi_2,\xi_3,\xi_4) = 1 - \frac{m(\xi_2+\xi_3+\xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)} \tag{5.2}$$

It is left to prove that

$$\begin{aligned}&\Big|\int_0^\tau \int_{\xi_1+\dots+\xi_4=0} \mu(\xi_2,\xi_3,\xi_4)\widehat{\partial_t Iu}(t,\xi_1)\hat{Iu}(t,\xi_2)\widehat{Iu}(t,\xi_3)\widehat{Iu}(t,\xi_4)\,d\xi_2 d\xi_3 d\xi_4\,dt\Big|\\ &\lesssim \frac{Z^4(\tau)}{N^{1-}}\end{aligned} \tag{5.3}$$

We perform a Paley-Littlewood decomposition to prove (5.3). Let $u_i = P_{N_i} u$ with $i \in \{1, \dots, 4\}$ and let

$$\begin{aligned} X = \Big| \int_0^\tau \int_{\xi_1+\dots+\xi_4=0} & \mu(\xi_2,\xi_3,\xi_4) \widehat{\partial_t I u_1}(t,\xi_1) \widehat{Iu_2}(t,\xi_2) \\ & \times \widehat{Iu_3}(t,\xi_3) \widehat{Iu_4}(t,\xi_4)\, d\xi_2 d\xi_3 d\xi_4 dt \Big| \end{aligned} \tag{5.4}$$

There are different cases resulting from this Paley-Littlewood analysis and we describe now the strategy to estimate (5.3). We suggest that the reader at first ignores the second and third steps of the description and the $N_j^{\pm}$ appearing in the study of these cases to solve the summation issue.

Description of the strategy

(1). We follow [6] to estimate X. First we recall the following Coifman-Meyer theorem [4], p179 for a class of multilinear operators

Theorem 5.1 (Coifman Meyer multiplier theorem)**.** *Consider an infinitely differentiable symbol $\sigma : \mathbb{R}^{nk} \to \mathbb{C}$ so that for all $\alpha \in N^{nk}$ there exists $c(\alpha)$ such that for all $\xi = (\xi_1, \dots, \xi_k) \in \mathbb{R}^{nk}$*

$$|\partial_\xi^\alpha \sigma(\xi)| \le \frac{c(\alpha)}{(1+|\xi|)^{|\alpha|}} \tag{5.5}$$

Let Λ_σ be the multilinear operator

$$\Lambda_\sigma(f_1,\dots,f_k)(x) = \int_{\mathbb{R}^{nk}} e^{ix\cdot(\xi_1+\dots+\xi_k)} \sigma(\xi_1,\dots,\xi_k) \widehat{f_1}(\xi_1)\dots\widehat{f_k}(\xi_k)\, d\xi_1\dots d\xi_k \tag{5.6}$$

Assume that $q_j \in (1,\infty)$, $j \in \{1,\dots,k\}$ are such that $\frac{1}{q} = \frac{1}{q_1} + \dots + \frac{1}{q_k} \le 1$. Then there is a constant $C = C(q_j, n, k, c(\alpha))$ so that for all Schwarz class functions $f_1, \dots, f_k$

$$\|\Lambda_\sigma(f_1,\dots,f_k)\|_{L^q(\mathbb{R}^n)} \le C \|f_1\|_{L^{q_1}(\mathbb{R}^n)} \dots \|f_k\|_{L^{q_k}(\mathbb{R}^n)} \tag{5.7}$$

Then we proceed as follows. We seek a pointwise bound on the symbol

$$|\mu(\xi_2,\xi_3,\xi_4)| \le B(N_2,N_3,N_4) \tag{5.8}$$

We factor $B = B(N_2,N_3,N_4)$ out of the right side of (5.4) and we are left to evaluate

$$B \int_0^\tau \int_{\mathbb{R}^3} \Lambda_{\frac{\mu}{B}} (\partial_t I u_1 \widehat{(t), Iu_2}(t), Iu_3(t))(\xi_4) \widehat{Iu_4}(t,\xi_4)\, d\xi_4\, dt$$

We notice that the multiplier $\frac{\mu}{B}$ satisfy the bound (5.5) and by the Plancherel theorem, Hölder inequality, theorem 5.1 and Bernstein inequalities we have

$$\begin{aligned} X &\lesssim B \|\partial_t I u_1\|_{L_t^{p_1}([0,\tau])L_x^{q_1}} \|Iu_2\|_{L_t^{p_2}([0,\tau])L_x^{q_2}} \dots \|Iu_4\|_{L_t^{p_4}([0,\tau])L_x^{q_4}} \\ &\lesssim B N_1^{m_1} N_2^{m_2-1} \dots N_4^{m_4-1} \|\partial_t D^{-m_1} I u_1\|_{L_t^{p_1}([0,\tau])L_x^{q_1}} \\ &\quad \times \|D^{1-m_2} I u_2\|_{L_t^{p_2}([0,\tau])L_x^{q_2}} \dots \|D^{1-m_4} I u_4\|_{L_t^{p_4}([0,\tau])L_x^{q_4}} \\ &\lesssim B N_1^{m_1} N_2^{m_2-1} \dots N_4^{m_4-1} Z^4(\tau) \end{aligned} \tag{5.9}$$

with (p_j, q_j) such that $p_j \in [1,\infty]$ and $q_j \in (1,\infty)$ for $j = \{1,\dots,4\}$, $\sum_{j=1}^4 \frac{1}{p_j} = 1$, $\sum_{j=1}^4 \frac{1}{q_j} = 1$, (p_j,q_j) m_j-wave admissible for some $m_j' s$ such that $0 \le m_j < 1$ and $\frac{1}{p_j} + \frac{1}{q_j} = \frac{1}{2}$. In other words $(p_j, q_j) = (\frac{2}{m_j}, \frac{2}{1-m_j})$.

(2). The series must be summable. Therefore in some cases we might create $N_k^{\pm}$ for some $k's$ by considering slight variations $(p_k\pm, q_k\pm) \in [1,\infty] \times (1,\infty)$ of (p_k, q_k) that are $m_k\pm$ - wave admissible and such that $\frac{1}{p_k\pm} + \frac{1}{q_k\pm} = \frac{1}{2}$. For instance if we create slight variations (p_2+, q_2-), (p_4-, q_4+) of (p_2, q_2), (p_4, q_4) respectively we have

$$
\begin{aligned}
\|Iu_2\|_{L_t^{p_2+} L_x^{q_2-}} &\lesssim N_2^- N_2^{m_2-1} \|D^{1-(m_2-)} Iu_2\|_{L_t^{p_2+} L_x^{q_2-}} \\
\|Iu_4\|_{L_t^{p_4-} L_x^{q_4+}} &\lesssim N_4^+ N_4^{m_4-1} \|D^{1-(m_4+)} Iu_4\|_{L_t^{p_4-} L_x^{q_4+}}
\end{aligned} \tag{5.10}
$$

and (5.9) becomes

$$
X \lesssim B N_2^- N_4^+ N_1^{m_1} N_2^{m_2-1} \dots N_4^{m_4-1} Z^4(\tau) \tag{5.11}
$$

(3). When we deal with low frequencies, i.e $N_k < 1$ for some $k \in \{1, \dots, 4\}$ we might consider generating N_k^+ by creating a variation $(2+, \infty-)$ of $(2, \infty)$. Such a task cannot be directly performed since we unfortunately have

$$
\|Iu_k\|_{L_t^{2+} L_x^{\infty-}} \lesssim N_k^- \|D^{1-(1-)} Iu_k\|_{L_t^{2+} L_x^{\infty-}} \lesssim N_k^- Z(\tau) \tag{5.12}
$$

But we can indirectly create N_k^+ by appropriately using Hölder in time inequality. Indeed if $\epsilon > 0$, $\epsilon' > 0$ and $\epsilon'' > 0$ are such that $\frac{\epsilon}{2} = \frac{\epsilon'}{2} - \frac{\epsilon''}{3}$ we get from Bernstein inequalities, Hölder in time inequality and Sobolev homogeneous embedding

$$
\begin{aligned}
\|Iu_k\|_{L_t^{\frac{2}{1-\epsilon}}(([0,\tau]))L_x^{\frac{2}{\epsilon}}} &\lesssim N_k^{\epsilon'} \|D^{-\epsilon'} Iu_k\|_{L_t^{\frac{2}{1-\epsilon}}(([0,\tau]))L_x^{\frac{2}{\epsilon}}} \\
&\lesssim N_k^{\epsilon'} \tau^{\frac{\epsilon'-\epsilon}{2}} \|D^{-\epsilon'} Iu_k\|_{L_t^{\frac{2}{1-\epsilon'}}(([0,\tau]))L_x^{\frac{2}{\epsilon}}} \\
&\lesssim N_k^{\epsilon'} \tau^{\frac{\epsilon'-\epsilon}{2}} \|D^{-\epsilon'+\epsilon''} Iu_k\|_{L_t^{\frac{2}{1-\epsilon'}}(([0,\tau]))L_x^{\frac{2}{\epsilon'}}} \\
&\lesssim N_k^{\epsilon''-\epsilon'} \tau^{\frac{\epsilon'-\epsilon}{2}} \|D^{1-(1-\epsilon')} Iu_k\|_{L_t^{\frac{2}{1-\epsilon'}}(([0,\tau]))L_x^{\frac{2}{\epsilon'}}} \\
&\lesssim N_k^{\epsilon''-\epsilon'} \tau^{\frac{\epsilon'-\epsilon}{2}} Z(\tau)
\end{aligned} \tag{5.13}
$$

We would like $\epsilon'' > \epsilon'$. A quick computation show that it suffices that $\epsilon' > 3\epsilon$. Letting $\epsilon' = 5\epsilon$ we get

$$
\|Iu_k\|_{L_t^{\frac{2}{1-\epsilon}}(([0,\tau]))L_x^{\frac{2}{\epsilon}}} \lesssim N_k^{\epsilon} \tau^{2\epsilon} Z(\tau) \tag{5.14}
$$

Now if we choose $\epsilon > 0$ so small that $|\tau|^{2\epsilon} \leq 2$ we eventually get

$$
\|Iu_k\|_{L_t^{2+}(([0,\tau]))L_x^{\infty-}} \lesssim N_k^+ Z(\tau) \tag{5.15}
$$

For the remainder of the paper we say that we directly create $N_k^{\pm}$ if we directly use Bernstein inequality like in (5.10) or (5.12) and we say that we indirectly create N_k^+ if we also use Hölder in time inequality to get (5.15). This completes the general description of the strategy.

Let us get back to the proof. By symmetry we may assume that $N_2 \geq N_3 \geq N_4$. There are several cases.

Case 1: $N \gg N_2 \geq N_3$**.** In this case $X = 0$ since $\mu = 0$.

Case 2: $N_2 \gtrsim N \gg N_3$**.** In this case we have

$$|\mu(\xi_2,\dots,\xi_4)| \lesssim \frac{|\nabla m(\xi_2)||\xi_3+\xi_4|}{m(\xi_2)} \lesssim \frac{N_3}{N_2} \tag{5.16}$$

We also get $N_1 \sim N_2$ from the convolution constraint $\xi_1+\dots+\xi_4=0$. We assume that $N_4 \geq 1$. By (5.16) and by the Bernstein inequalities we have

$$\begin{aligned}
X &\lesssim \frac{N_3}{N_2}\|\partial_t Iu_1\|_{L_t^{6-}([0,\tau])L_x^{3+}}\|Iu_2\|_{L_t^{6}([0,\tau])L_x^{3}}\|Iu_3\|_{L_t^{6}([0,\tau])L_x^{3}}\|Iu_4\|_{L_t^{2+}([0,\tau])L_x^{\infty-}}\\
&\lesssim N_1^{+}N_4^{-}\frac{N_3}{N_2}N_1^{\frac13}N_2^{-2/3}N_3^{-2/3}\|\partial_t D^{-(\frac13+)}Iu_1\|_{L_t^{6-}([0,\tau])L_x^{3+}}\\
&\quad\times\|D^{1-\frac13}Iu_2\|_{L_t^{6}([0,\tau])L_x^{3}}\|D^{1-\frac13}Iu_3\|_{L_t^{6}([0,\tau])L_x^{3}}\|D^{1-(1-)}Iu_4\|_{L_t^{2+}([0,\tau])L_x^{\infty-}}\\
&\lesssim \frac{N_2^{-}N_4^{-}}{N^{1-}}Z^4(\tau)
\end{aligned}$$

after directly creating N_1^{+} and N_4^{-}. If $N_4<1$ the proof is similar except that we indirectly create N_4^{+} to get $X \lesssim \frac{N_2^{-}N_4^{+}}{N^{1-}}Z^4(\tau)$. This makes the summation possible. We get (5.3) after summation.

Case 3: $N_3 \gtrsim N \gg N_4$**.** In this case we have

$$|\mu(\xi_2,\dots,\xi_4)| \lesssim \frac{m(\xi_1)}{m(\xi_2)m(\xi_3)m(\xi_4)} \tag{5.17}$$

There are two subcases:

Case 3.a: $N_1 \sim N_2$**.** We assume that $N_4 \geq 1$. By (5.17) we have

$$\begin{aligned}
X &\lesssim \frac{N_3^{1-s}}{N^{1-s}}\|\partial_t Iu_1\|_{L_t^{6-}([0,\tau])L_x^{3+}}\|Iu_2\|_{L_t^{6}([0,\tau])L_x^{3}}\|Iu_3\|_{L_t^{6}([0,\tau])L_x^{3}}\|Iu_4\|_{L_t^{2+}([0,\tau])L_x^{\infty-}}\\
&\lesssim N_1^{+}N_4^{-}\frac{N_3^{1-s}}{N^{1-s}}N_1^{\frac13}N_2^{-2/3}N_3^{-2/3}\|\partial_t D^{-(\frac13+)}Iu_1\|_{L_t^{6-}([0,\tau])L_x^{3+}}\\
&\quad\times\|D^{1-\frac13}Iu_2\|_{L_t^{6}([0,\tau])L_x^{3}}\|D^{1-\frac13}Iu_3\|_{L_t^{6}([0,\tau])L_x^{3}}\|D^{1-(1-)}Iu_4\|_{L_t^{2+}([0,\tau])L_x^{\infty-}}\\
&\lesssim \frac{N_2^{-}N_4^{-}}{N^{1-}}Z^4(\tau)
\end{aligned}$$

after directly creating N_1^{+} and N_4^{-}. If $N_4<1$ the proof is similar except that we indirectly create N_4^{+}. We get (5.3) after summation.

Case 3.b: $N_1 \ll N_2$**.** In this case by the convolution constraint $\xi_1+\dots+\xi_4=0$ we have $N_2 \sim N_3$. There are two subcases

Case 3.b.1: $N_1 \ll N$**.** We assume that $N_1 \geq 1$ and $N_4 \geq 1$. By (5.17) we have

$$\begin{aligned}
X &\lesssim \frac{N_2^{2(1-s)}}{N^{2(1-s)}}\|\partial_t Iu_1\|_{L_t^{6-}([0,\tau])L_x^{3+}}\|Iu_2\|_{L_t^{6}([0,\tau])L_x^{3}}\|Iu_3\|_{L_t^{6}([0,\tau])L_x^{3}}\\
&\quad\times\|Iu_4\|_{L_t^{2+}([0,\tau])L_x^{\infty-}}\\
&\lesssim N_1^{+}N_4^{-}\frac{N_2^{2(1-s)}}{N^{2(1-s)}}N_1^{\frac13}N_2^{-2/3}N_3^{-2/3}\|\partial_t D^{-(\frac13+)}Iu_1\|_{L_t^{6-}([0,\tau])L_x^{3+}}\\
&\quad\times\|D^{1-\frac13}Iu_2\|_{L_t^{6}([0,\tau])L_x^{3}}\|D^{1-\frac13}Iu_3\|_{L_t^{6}L_x^{3}}\|D^{1-(1-)}Iu_4\|_{L_t^{2+}([0,\tau])L_x^{\infty-}}\\
&\lesssim \frac{N_1^{-}N_2^{-}N_4^{-}}{N^{1-}}Z^4(\tau)
\end{aligned}$$

after directly creating N_1^+ and N_4^-. If $N_1 < 1$ and $N_4 < 1$ the proof is similar except that we indirectly create N_4^+ and we substitute N_1^- for N_1^+. The proof for the other cases [1] is a slight variant to that for the case $N_1 \geq 1$, $N_4 \geq 1$ and that for the case $N_1 < 1$, $N_4 < 1$. Details are left to the reader. We get (5.3) after summation.

Case 3.b.2: $N_1 \gg N$. We assume that $N_4 \geq 1$. By (5.17) we have

$$
\begin{aligned}
X &\lesssim \frac{N_2^{2(1-s)}}{N^{1-s}N_1^{1-s}} \|\partial_t Iu_1\|_{L_t^{6-}([0,\tau])L_x^{3+}} \|Iu_2\|_{L_t^6([0,\tau])L_x^3} \|Iu_3\|_{L_t^6([0,\tau])L_x^3} \\
&\quad \times \|Iu_4\|_{L_t^{2+}([0,\tau])L_x^{\infty-}} \\
&\lesssim N_1^+ N_4^- \frac{N_2^{2(1-s)}}{N^{1-s}N_1^{1-s}} N_1^{\frac{1}{3}} N_2^{-2/3} N_3^{-2/3} \|\partial_t D^{-(\frac{1}{3}+)} Iu_1\|_{L_t^{6-}([0,\tau])L_x^{3+}} \\
&\quad \times \|D^{1-\frac{1}{3}} Iu_2\|_{L_t^6([0,\tau])L_x^3} \|D^{1-\frac{1}{3}} Iu_3\|_{L_t^6([0,\tau])L_x^3} \|D^{1-(1-)} Iu_4\|_{L_t^{2+}([0,\tau])L_x^{\infty-}} \\
&\lesssim \frac{N_2^- N_4^-}{N^{1-}} Z^4(\tau)
\end{aligned}
$$

after directly creating N_1^+ and N_4^-. If $N_4 < 1$ the proof is similar except that we indirectly create N_4^+. We get (5.3) after summation.

Case 4: $N_4 \gtrsim N$. There are two subcases.

Case 4.a: $N_1 \sim N_2$. By (5.17) we have

$$
\begin{aligned}
X &\lesssim \frac{N_3^{1-s}}{N^{1-s}} \frac{N_4^{1-s}}{N^{1-s}} \|\partial_t Iu_1\|_{L_t^4([0,\tau])L_x^4} \|Iu_2\|_{L_t^{4+}([0,\tau])L_x^{4-}} \|Iu_3\|_{L_t^4([0,\tau])L_x^4} \\
&\quad \times \|Iu_4\|_{L_t^{4-}([0,\tau])L_x^{4+}} \\
&\lesssim N_2^- N_4^+ \frac{N_3^{1-s}}{N^{1-s}} \frac{N_4^{1-s}}{N^{1-s}} N_1^{\frac{1}{2}} \frac{1}{N_2^{\frac{1}{2}}} \frac{1}{N_3^{\frac{1}{2}}} \frac{1}{N_4^{\frac{1}{2}}} \|\partial_t D^{-\frac{1}{2}} Iu_1\|_{L_t^4([0,\tau])L_x^4} \\
&\quad \times \|D^{1-(\frac{1}{2}-)} Iu_2\|_{L_t^{4+}([0,\tau])L_x^{4-}} \|D^{1-\frac{1}{2}} Iu_3\|_{L_t^4([0,\tau])L_x^4} \|D^{1-(\frac{1}{2}+)} Iu_4\|_{L_t^4([0,\tau])L_x^4} \\
&\lesssim \frac{N_2^-}{N^{1-}} Z^4(\tau)
\end{aligned}
$$

after directly creating N_2^- and N_4^+. We get (5.3) after summation.

Case 4.b: $N_1 \ll N_2$. In this case we have $N_2 \sim N_3$. There are two subcases

Case 4.b.1: $N_1 \gtrsim N$. By (5.17) we have

$$
\begin{aligned}
X &\lesssim \frac{N_2^{2(1-s)}}{N^{2(1-s)}} \frac{N_4^{1-s}}{N^{1-s}} \frac{N^{1-s}}{N_1^{1-s}} \|\partial_t Iu_1\|_{L_t^4([0,\tau])L_x^4} \|Iu_2\|_{L_t^4([0,\tau])L_x^4} \|Iu_3\|_{L_t^4([0,\tau])L_x^4} \\
&\quad \times \|Iu_4\|_{L_t^4([0,\tau])L_x^4} \\
&\lesssim \frac{N_2^{2(1-s)}}{N^{2(1-s)}} \frac{N_4^{1-s}}{N^{1-s}} \frac{N^{1-s}}{N_1^{1-s}} N_1^{\frac{1}{2}} \frac{1}{N_2^{\frac{1}{2}}} \frac{1}{N_3^{\frac{1}{2}}} \frac{1}{N_4^{\frac{1}{2}}} \|\partial_t D^{-\frac{1}{2}} Iu_1\|_{L_t^4([0,\tau])L_x^4} \\
&\quad \times \|D^{1-\frac{1}{2}} Iu_2\|_{L_t^4([0,\tau])L_x^4} \|D^{1-\frac{1}{2}} Iu_3\|_{L_t^4([0,\tau])L_x^4} \|D^{1-\frac{1}{2}} Iu_4\|_{L_t^4([0,\tau])L_x^4}
\end{aligned}
$$

[1] i.e $N_1 \geq 1$, $N_4 \leq 1$ or $N_1 \leq 1$, $N_4 \geq 1$

$$\lesssim \frac{N_2^{-}}{N^{1-}} Z^4(\tau)$$

We get (5.3) after summation.

Case 4.b.2: $N_1 \ll N$**.** We assume that $N_1 \geq 1$. We have

$$\begin{aligned}
X &\lesssim \frac{N_2^{2(1-s)}}{N^{2(1-s)}} \frac{N_4^{1-s}}{N^{1-s}} \|\partial_t I u_1\|_{L_t^4([0,\tau])L_x^4} \|Iu_2\|_{L_t^4([0,\tau])L_x^4} \|Iu_3\|_{L_t^4([0,\tau])L_x^4} \\
&\quad \times \|Iu_4\|_{L_t^4([0,\tau])L_x^4} \\
&\lesssim \frac{N_2^{2(1-s)}}{N^{2(1-s)}} \frac{N_4^{1-s}}{N^{1-s}} N_1^{\frac{1}{2}} \frac{1}{N_2^{\frac{1}{2}}} \frac{1}{N_3^{\frac{1}{2}}} \frac{1}{N_4^{\frac{1}{2}}} \|\partial_t D^{-\frac{1}{2}} I u_1\|_{L_t^4([0,\tau])L_x^4} \|D^{1-\frac{1}{2}} I u_2\|_{L_t^4([0,\tau])L_x^4} \\
&\quad \times \|D^{1-\frac{1}{2}} I u_3\|_{L_t^4([0,\tau])L_x^4} \|D^{1-\frac{1}{2}} I u_4\|_{L_t^4([0,\tau])L_x^4} \\
&\lesssim \frac{N_1^{-} N_2^{-}}{N^{1-}} Z^4(\tau)
\end{aligned}$$

If $N_1 < 1$ the proof is similar except that we create N_1^{+} instead of N_1^{-}. We get (5.3) after summation.

6. Proof of Almost Morawetz-Strauss inequality

We prove proposition 2.4 in this section. The proof is divided into two steps

First Step: Morawetz-Strauss inequality. We recall the proof of this inequality in [12, 13]. We have the identity

$$\begin{aligned}
&\Big(\frac{x.\nabla u}{|x|} + \frac{u}{|x|}\Big)(u_{tt} - \triangle u + u^3) \\
&= \partial_t\Big(\frac{1}{|x|}(x.\nabla u + u)\partial_t u\Big) + \operatorname{div}\Big[\frac{1}{|x|}\Big(- \frac{1}{2}(\partial_t u)^2 - (x.\nabla u)\nabla u \\
&\quad + \frac{1}{2}|\nabla u|^2 x - u\nabla u - \frac{u^2}{2|x|^2} x + \frac{1}{4} u^4 x\Big)\Big] + \frac{1}{|x|}\Big(|\nabla u|^2 - \frac{(x.\nabla u)^2}{|x|^2}\Big) + \frac{u^4}{2|x|}
\end{aligned} \tag{6.1}$$

and since u satisfies (1.1) we have, after integration,

$$\begin{aligned}
&2\pi \int_0^T u^2(t,0)dt + \int_0^T \int_{\mathbb{R}^3} \frac{u^4(t,x)}{2|x|} dx\, dt \\
&= - \int_{\mathbb{R}^3} \Big(\frac{\nabla u(T,x).x}{|x|} + \frac{u(T,x)}{|x|}\Big)\partial_t u(T,x) dx \\
&\quad + \int_{\mathbb{R}^3} \Big(\frac{\nabla u(0,x).x}{|x|} + \frac{u(0,x)}{|x|}\Big)\partial_t u(0,x) dx
\end{aligned}$$

Now we apply the basic inequality $|ab| \leq \frac{|a|^2}{2} + \frac{|b|^2}{2}$ to the right hand side of the integral and we get

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}^3} \frac{u^4(t,x)}{2|x|} dx\, dt &\leq \frac{1}{2} \int_{\mathbb{R}^3} \Big(\frac{\nabla u(T,x).x}{|x|} + \frac{u(T,x)}{|x|}\Big)^2 + (\partial_t u)^2(T,x) dx \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^3} \Big(\frac{\nabla u(0,x).x}{|x|} + \frac{u(0,x)}{|x|}\Big)^2 + (\partial_t u)^2(0,x) dx
\end{aligned} \tag{6.2}$$

We also notice that

$$(\frac{\nabla u.x}{|x|}+\frac{u}{|x|})^2=\frac{(\nabla u.x)^2}{|x|^2}+\operatorname{div}(\frac{u^2}{|x|^2}x)\le|\nabla u|^2+\operatorname{div}(\frac{u^2}{|x|^2}x) \tag{6.3}$$

We substitute (6.3) into (6.2). We get the Morawetz-Strauss's inequality

$$\int_0^T\int_{\mathbb{R}^3}\frac{u^4(t,x)}{|x|}dx\,dt\le 2(E(u(T))+E(u(0))) \tag{6.4}$$

Second Step. Almost Morawetz-Strauss's inequality. We substitute u for Iu in (6.1) and we proceed similarly. We get

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^3}\frac{|Iu|^4(t,x)}{|x|}dx\,dt-2(E(Iu(T))+E(Iu(0)))\\ &\le|R_1(T)+R_2(T)|\\ &\le|R_1(T)|+|R_2(T)|\end{aligned} \tag{6.5}$$

7. Proof of the integral estimates

We are interested in proving proposition 2.5 in this section. In what follows we also assume that $J=[0,\tau]$; the reader can check after reading the proof that the other cases can be reduced to that one.

Plancherel formula yields

$$\begin{aligned}R_1(\tau)=\int_0^\tau\int_{\xi_1+\dots+\xi_4=0}&\mu(\xi_2,\xi_3,\xi_4)\widehat{\frac{\nabla Iu.x}{|x|}}(t,\xi_1)\\ &\times\widehat{Iu}(t,\xi_2)\widehat{Iu}(t,\xi_3)\widehat{Iu}(t,\xi_4)d\xi_2\dots d\xi_4\,dt\end{aligned}$$

and

$$R_2(\tau)=\int_0^\tau\int_{\xi_1+\dots+\xi_4}\mu(\xi_2,\xi_3,\xi_4)\widehat{\frac{Iu}{|x|}}(t,\xi_1)\widehat{Iu}(t,\xi_2)\widehat{Iu}(t,\xi_3)\widehat{Iu}(t,\xi_4)d\xi_2\dots d\xi_4\,dt$$

with μ defined in (5.2). It suffices to prove

$$\begin{aligned}&\Big|\int_0^\tau\int_{\xi_1+\dots+\xi_4=0}\mu(\xi_2,\xi_3,\xi_4)\widehat{\frac{\nabla Iu.x}{|x|}}(t,\xi_1)\widehat{Iu}(t,\xi_2)\widehat{Iu}(t,\xi_3)\widehat{Iu}(t,\xi_4)d\xi_2\dots d\xi_4\,dt\Big|\\ &\lesssim\frac{Z^4(\tau)}{N^{1-}}\end{aligned} \tag{7.1}$$

and

$$\begin{aligned}&\Big|\int_0^\tau\int_{\xi_1+\dots+\xi_4=0}\mu(\xi_2,\xi_3,\xi_4)\widehat{\frac{Iu}{|x|}}(t,\xi_1)\widehat{Iu}(t,\xi_2)\widehat{Iu}(t,\xi_3)\widehat{Iu}(t,\xi_4)d\xi_2\dots d\xi_4\,dt\Big|\\ &\lesssim\frac{Z^4(\tau)}{N^{1-}}\end{aligned} \tag{7.2}$$

We perform a Paley-Littlewood decomposition to prove (7.1) and (7.2). Let $u_i:=P_{N_i}u$, $i\in\{2,\dots 4\}$, $(\frac{\nabla Iu\cdot x}{|x|})_1:=P_{N_1}(\frac{\nabla Iu\cdot x}{|x|})$ and $(\frac{Iu}{|x|})_1:=P_{N_1}(\frac{Iu}{|x|})$.

$$\begin{aligned}X_1=\Big|\int_0^\tau\int_{\xi_1+\dots+\xi_4=0}&\mu(\xi_2,\xi_3,\xi_4)\\ &\times(\widehat{\frac{\nabla Iu.x}{|x|}})_1(t,\xi_1)\widehat{Iu_2}(t,\xi_2)\widehat{Iu_3}(t,\xi_3)\widehat{Iu_4}(t,\xi_4)d\xi_2\dots d\xi_4\,dt\Big|\end{aligned}$$

and

$$X_2 = \Big| \int_0^{\tau} \int_{\xi_1+\dots+\xi_4=0} \mu(\xi_2,\xi_3,\xi_4) \widehat{(\frac{Iu}{|x|})_1}(t,\xi_1) \\ \times \widehat{Iu_2}(t,\xi_2)\widehat{Iu_3}(t,\xi_3)\widehat{Iu_4}(t,\xi_4) d\xi_2 \dots d\xi_4 \, dt \Big|$$

Notice that by Bernstein inequality, Hölder inequality, Plancherel theorem and (1.20) we have

$$\begin{aligned} \big\|(\frac{\nabla Iu \cdot x}{|x|})_1\big\|_{L_t^{\infty-}([0,\tau])L_t^{2+}} &\lesssim N_1^{+} \Big\| \frac{\nabla Iu \cdot x}{|x|} \Big\|_{L_t^{\infty}([0,\tau])L_x^2} \\ &\lesssim N_1^{+} \|\nabla Iu\|_{L_t^{\infty}([0,\tau])L_x^2} \\ &\lesssim N_1^{+} \|DIu\|_{L_t^{\infty}([0,\tau])L_x^2} \end{aligned} \tag{7.3}$$

and

$$\big\|(\frac{Iu}{|x|})_1\big\|_{L_t^{\infty-}([0,\tau])L_x^{2+}} \lesssim N_1^{+} \big\|\frac{Iu}{|x|}\big\|_{L_t^{\infty}([0,\tau])L_x^2} \lesssim N_1^{+} \|DIu\|_{L_t^{\infty}([0,\tau])L_x^2} \tag{7.4}$$

If $p_j \in [1,\infty]$ and $q_j \in (1,\infty)$, $j \in \{2,\dots,4\}$ such that $\frac{1}{(\infty-)} + \sum_{j=2}^{4} \frac{1}{p_j} = 1$, $\frac{1}{(2+)} + \sum_{j=2}^{4} \frac{1}{q_j} = 1$, (p_j,q_j) -m_j wave admissible for some $m_j^{'}$ s such that $0 \leq m_j < 1$ and $\frac{1}{p_j} + \frac{1}{q_j} = \frac{1}{2}$ then we have by the methodology explained in the proof of Proposition 2.3

$$X_1 \lesssim B(N_2,N_3,N_4) \big\|(\frac{\nabla Iu \cdot x}{|x|})_1\big\|_{L_t^{\infty-}([0,\tau])L_x^{2+}} \\ \times \|Iu_2\|_{L_t^{p_2}([0,\tau])L_x^{q_2}} \dots \|Iu_4\|_{L_t^{p_4}([0,\tau])L_x^{q_4}}$$

and

$$X_2 \lesssim B(N_2,N_3,N_4) \big\|(\frac{Iu}{|x|})_1\big\|_{L_t^{\infty-}([0,\tau])L_x^{2+}} \|Iu_2\|_{L_t^{p_2}([0,\tau])L_x^{q_2}} \dots \|Iu_4\|_{L_t^{p_4}([0,\tau])L_x^{q_4}}$$

By symmetry we can assume that $N_2 \geq N_3 \geq N_4$. There are different cases

Case 1: $N >> N_2 \geq N_3$**.** In this case $X_1 = 0$ and $X_2 = 0$ since $\mu = 0$.

Case 2: $N_2 \gtrsim N >> N_3$**.** By (5.15), (5.16), (7.3) and (7.4) we have

$$\begin{aligned} X_1 &\lesssim \frac{N_3}{N_2} \big\|(\frac{\nabla Iu.x}{|x|})_1\big\|_{L_t^{\infty-}([0,\tau])L_x^{2+}} \|Iu_2\|_{L_t^{\infty}([0,\tau])L_x^2} \\ &\quad \times \|Iu_3\|_{L_t^{2+}([0,\tau])L_x^{\infty-}} \|Iu_4\|_{L_t^{2+}([0,\tau])L_x^{\infty-}} \\ &\lesssim \frac{N_3}{N_2} N_1^{+} \frac{1}{N_2} N_3^{+} N_4^{+} \|DIu\|_{L_t^{\infty}([0,\tau])L_x^2} \|DIu_2\|_{L_t^{\infty}([0,\tau])L_x^2} \\ &\quad \times \|D^{1-(1-)} Iu_3\|_{L_t^{2+}([0,\tau])L_x^{\infty}} \|D^{1-(1-)} Iu_4\|_{L_t^{2+}([0,\tau])L_x^{\infty-}} \\ &\lesssim \frac{N_2^{--} N_4^{+}}{N^{1-}} Z^4(\tau) \end{aligned}$$

and

$$\begin{aligned} X_2 &\lesssim \frac{N_3}{N_2} \big\|(\frac{Iu}{|x|})_1\big\|_{L_t^{\infty-}([0,\tau])L_x^{2+}} \|Iu_2\|_{L_t^{\infty}([0,\tau])L_x^2} \\ &\quad \times \|Iu_3\|_{L_t^{2+}([0,\tau])L_x^{\infty-}} \|Iu_4\|_{L_t^{2+}([0,\tau])L_x^{\infty-}} \end{aligned}$$

$$
\begin{aligned}
&\lesssim \frac{N_3}{N_2} N_1^{+} \frac{1}{N_2} N_3^{+} N_4^{+} \|DIu\|_{L_t^\infty([0,\tau])L_x^2} \|DIu_2\|_{L_t^\infty([0,\tau])L_x^2} \\
&\quad \times \|D^{1-(1-)} Iu_3\|_{L_t^{2+}([0,\tau])L_x^\infty} \|D^{1-(1-)} Iu_4\|_{L_t^{2+}([0,\tau])L_x^{\infty-}} \\
&\lesssim \frac{N_2^{--} N_4^{+}}{N^{1-}} Z^4(\tau)
\end{aligned}
$$

Case 3: $N_3 \gtrsim N >> N_4$**.** There are two subcases

Case 3.a: $N_1 \sim N_2$**.** By (5.15), (5.17) and (7.3)

$$
\begin{aligned}
X_1 &\lesssim \frac{N_3^{1-s}}{N^{1-s}} \|(\frac{\nabla Iu.x}{|x|})_1\|_{L_t^{\infty-}([0,\tau])L_x^{2+}} \|Iu_2\|_{L_t^\infty([0,\tau])L_x^2} \|Iu_3\|_{L_t^{2+}([0,\tau])L_x^{\infty-}} \\
&\quad \times \|Iu_4\|_{L_t^{2+}([0,\tau])L_x^{\infty-}} \\
&\lesssim \frac{N_3^{1-s}}{N^{1-s}} N_1^{+} \frac{1}{N_2} N_3^{+} N_4^{+} \|DIu\|_{L_t^\infty([0,\tau])L_x^2} \|DIu_2\|_{L_t^\infty([0,\tau])L_x^2} \\
&\quad \times \|D^{1-(1-)} Iu_3\|_{L_t^{2+}([0,\tau])L_x^\infty} \|D^{1-(1-)} Iu_4\|_{L_t^{2+}([0,\tau])L_x^{\infty-}} \\
&\lesssim \frac{N_2^{--} N_4^{+}}{N^{1-}} Z^4(\tau)
\end{aligned} \tag{7.5}
$$

Similarly we get $X_2 \lesssim \frac{N_2^{--} N_4^{+}}{N^{1-}} Z^4(\tau)$ after substituting X_1, $\|(\frac{\nabla Iu.x}{|x|})_1\|_{L_t^{\infty-}([0,\tau])L_x^{2+}}$ for X_2, $\|(\frac{Iu}{|x|})_1\|_{L_t^{\infty-}([0,\tau])L_x^{2+}}$ respectively in (7.5).

Case 3.b: $N_1 << N_2$**.** There are two subcases

Case 3.b.1: $N_1 << N$**.**

$$
\begin{aligned}
X_1 &\lesssim \frac{N_2^{2(1-s)}}{N^{2(1-s)}} \|(\frac{\nabla Iu.x}{|x|})_1\|_{L_t^{\infty-}([0,\tau])L_x^{2+}} \|Iu_2\|_{L_t^\infty([0,\tau])L_x^2} \|Iu_3\|_{L_t^{2+}([0,\tau])L_x^{\infty-}} \\
&\quad \times \|Iu_4\|_{L_t^{2+}([0,\tau])L_x^{\infty-}} \\
&\lesssim \frac{N_2^{2(1-s)}}{N^{2(1-s)}} N_1^{+} \frac{1}{N_2} N_3^{+} N_4^{+} \|DIu\|_{L_t^\infty([0,\tau])L_x^2} \|DIu_2\|_{L_t^\infty([0,\tau])L_x^2} \\
&\quad \times \|D^{1-(1-)} Iu_3\|_{L_t^{2+}([0,\tau])L_x^\infty} \|D^{1-(1-)} Iu_4\|_{L_t^{2+}([0,\tau])L_x^{\infty-}} \\
&\lesssim \frac{N_1^{+} N_2^{---} N_4^{+}}{N^{1-}} Z^4(\tau)
\end{aligned}
$$

Similarly $X_2 \lesssim \frac{N_1^{+} N_2^{---} N_4^{+}}{N^{1-}} Z^4(\tau)$.

Case 3.b.2: $N_1 \gtrsim N$**.**

$$
\begin{aligned}
X_1 &\lesssim \frac{N_2^{2(1-s)}}{N^{1-s} N_1^{1-s}} \|(\frac{\nabla Iu.x}{|x|})_1\|_{L_t^{\infty-}([0,\tau])L_x^{2+}} \|Iu_2\|_{L_t^\infty([0,\tau])L_x^2} \|Iu_3\|_{L_t^{2+}([0,\tau])L_x^{\infty-}} \\
&\quad \times \|Iu_4\|_{L_t^{2+}([0,\tau])L_x^{\infty-}} \\
&\lesssim \frac{N_2^{2(1-s)}}{N^{1-s} N_1^{1-s}} N_1^{+} \frac{1}{N_2} N_3^{+} N_4^{+} \|DIu\|_{L_t^\infty([0,\tau])L_x^2} \|DIu_2\|_{L_t^\infty([0,\tau])L_x^2} \\
&\quad \times \|D^{1-(1-)} Iu_3\|_{L_t^{2+}([0,\tau])L_x^\infty} \|D^{1-(1-)} Iu_4\|_{L_t^{2+}([0,\tau])L_x^{\infty-}} \\
&\lesssim \frac{N_2^{--} N_4^{+}}{N^{1-}} Z^4(\tau)
\end{aligned}
$$

Similarly $X_2 \lesssim \frac{N_2^{--} N_4^{+}}{N^{1-}} Z^4(\tau)$.

Case 4: $N_4 \gtrsim N$**.** There are two subcases

Case 4.a: $N_1 \sim N_2$**.**

$$\begin{aligned}
X_1 &\lesssim \frac{N_3^{1-s}}{N^{1-s}} \frac{N_4^{1-s}}{N^{1-s}} \big\|(\frac{\nabla Iu.x}{|x|})_1\big\|_{L_t^{\infty-}([0,\tau])L_x^{2+}} \|Iu_2\|_{L_t^{\infty}([0,\tau])L_x^2} \|Iu_3\|_{L_t^{2+}([0,\tau])L_x^{\infty-}} \\
&\quad \|Iu_4\|_{L_t^{2+}([0,\tau])L_x^{\infty-}} \\
&\lesssim \frac{N_3^{1-s}}{N^{1-s}} \frac{N_4^{1-s}}{N^{1-s}} N_1^{+} \frac{1}{N_2} N_3^{+} N_4^{+} \|DIu\|_{L_t^{\infty}([0,\tau])L_x^2} \|DIu_2\|_{L_t^{\infty}([0,\tau])L_x^2} \\
&\quad \times \|D^{1-(1-)} Iu_3\|_{L_t^{2+}([0,\tau])L_x^{\infty}} \|D^{1-(1-)} Iu_4\|_{L_t^{2+}([0,\tau])L_x^{\infty-}} \\
&\lesssim \frac{N_2^{-}}{N^{1-}} Z^4(\tau)
\end{aligned}$$

Similarly $X_2 \lesssim \frac{N_2^{-}}{N^{1-}} Z^4(\tau)$.

Case 4.b: $N_1 << N_2$**.** There are two subcases

Case 4.b.1: $N_1 \gtrsim N$**.** We have

$$\begin{aligned}
X_1 &\lesssim \frac{N_2^{2(1-s)}}{N^{2(1-s)}} \frac{N_4^{1-s}}{N^{1-s}} \frac{N^{1-s}}{N_1^{1-s}} \big\|(\frac{\nabla Iu.x}{|x|})_1\big\|_{L_t^{\infty-}([0,\tau])L_x^{2+}} \|Iu_2\|_{L_t^{\infty}([0,\tau])L_x^2} \\
&\quad \times \|Iu_3\|_{L_t^{2+}([0,\tau])L_x^{\infty-}} \|Iu_4\|_{L_t^{2+}([0,\tau])L_x^{\infty-}} \\
&\lesssim \frac{N_2^{2(1-s)}}{N^{2(1-s)}} \frac{N_4^{1-s}}{N^{1-s}} \frac{N^{1-s}}{N_1^{1-s}} N_1^{+} \frac{1}{N_2} N_3^{+} N_4^{+} \|DIu\|_{L_t^{\infty}([0,\tau])L_x^2} \|DIu_2\|_{L_t^{\infty}([0,\tau])L_x^2} \\
&\quad \times \|D^{1-(1-)} Iu_3\|_{L_t^{2+}([0,\tau])L_x^{\infty}} \|D^{1-(1-)} Iu_4\|_{L_t^{2+}([0,\tau])L_x^{\infty-}} \\
&\lesssim \frac{N_2^{-}}{N^{1-}} Z^4(\tau)
\end{aligned}$$

Similarly $X_2 \lesssim \frac{N_2^{-}}{N^{1-}} Z^4(\tau)$.

Case 4.b.2: $N_1 << N$**.** We have

$$\begin{aligned}
X_1 &\lesssim \frac{N_2^{2(1-s)}}{N^{2(1-s)}} \frac{N_4^{1-s}}{N^{1-s}} \big\|(\frac{\nabla Iu.x}{|x|})_1\big\|_{L_t^{\infty-}([0,\tau])L_x^{2+}} \|Iu_2\|_{L_t^{\infty}([0,\tau])L_x^2} \|Iu_3\|_{L_t^{2+}([0,\tau])L_x^{\infty-}} \\
&\quad \|Iu_4\|_{L_t^{2+}([0,\tau])L_x^{\infty-}} \\
&\lesssim \frac{N_2^{2(1-s)}}{N^{2(1-s)}} \frac{N_4^{1-s}}{N^{1-s}} N_1^{+} \frac{1}{N_2} N_3^{+} N_4^{+} \|DIu\|_{L_t^{\infty}([0,\tau])L_x^2} \|DIu_2\|_{L_t^{\infty}([0,\tau])L_x^2} \\
&\quad \times \|D^{1-(1-)} Iu_3\|_{L_t^{2+}([0,\tau])L_x^{\infty}} \|D^{1-(1-)} Iu_4\|_{L_t^{2+}([0,\tau])L_x^{\infty-}} \\
&\lesssim \frac{N_1^{+} N_2^{--}}{N^{1-}} Z^4(\tau)
\end{aligned}$$

Similarly $X_2 \lesssim \frac{N_1^{+} N_2^{--}}{N^{1-}} Z^4(\tau)$.

We get (7.1) and (7.2) after summation.

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